Using the Lagrangian Method to Solve Optimization Problems

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The Optimization Problem

Assume that we want to maximize (minimize) a function $f(x_1, x_2)$ (e.g. a utility function that we want to maximize, or a cost function that we want to minimize), subject to the constraint $g(x_1, x_2) = c$ (e.g. a budget constraint, or a utility level constraint):

$$\max_{x_1, x_2} \; f(x_1, x_2) \qquad \text{s.t.} \qquad g(x_1, x_2) = c.$$

The Lagrangian function is then defined as

$$L(x_1, x_2, \lambda) = f(x_1, x_2) - \lambda[g(x_1, x_2) - c].$$

The Lagrangian equals the objective function $f(x_1, x_2)$ minus the Lagrange mulitiplicator λ multiplied by the constraint (rewritten such that the right-hand side equals zero). It is a function of three variables, x_1, x_2 and λ. By calculating the partial derivatives with respect to these three variables, we obtain the first-order conditions of the optimization problem:

$$\begin{aligned}
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_1} &= \frac{\partial f(x_1, x_2)}{\partial x_1} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_1} = 0, \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial x_2} &= \frac{\partial f(x_1, x_2)}{\partial x_2} - \lambda \frac{\partial g(x_1, x_2)}{\partial x_2} = 0, \\
\frac{\partial L(x_1, x_2, \lambda)}{\partial \lambda} &= -[g(x_1, x_2) - c] = 0.
\end{aligned}$$

We thus have three equations with three unknowns. By solving this system of three equations we obtain the optimal solutions x_1^*, x_2^*, λ^*. (Actually one should check the second-order conditions as well to see if the obtained solutions are optimal. Here, we will take for granted that the obtained solutions are optimal.)

Note that the Langrangian is constructed such that $L(x_1^*, x_2^*, \lambda^*) = f(x_1^*, x_2^*)$, because $\lambda^*[g(x_1^*, x_2^*) - c] = \lambda^* \cdot 0 = 0$.

Why Is this Method Applied?

The Lagrange method is frequently used in economics, mainly because the Lagrange multiplicator(s) has an interesting interpretation. The Lagrange multiplicator represents the shadow price of the constraint that it is multiplied with; it measures how much the optimal value of the objective function $f(x_1^*, x_2^*)$ would change if the constraint would be relaxed marginally (i.e. if the constant c would increase marginally).

Example: Utility Maximization

We want to maximize $u(x_1, x_2) = x_1 x_2$ subject to the budget constraint $p_1 x_1 + p_2 x_2 = m$:

$$\max_{x_1, x_2} \; x_1 x_2 \qquad \text{s.t.} \qquad p_1 x_1 + p_2 x_2 = m.$$

The Lagrangian is thus given by

$$L(x_1, x_2, \lambda) = x_1 x_2 - \lambda[p_1 x_1 + p_2 x_2 - m].$$

The optimal solutions are given by

$$\begin{aligned} x_1^* &= \frac{m}{2p_1}, \\ x_2^* &= \frac{m}{2p_2}, \\ \lambda^* &= \frac{m}{2p_1 p_2}. \end{aligned}$$

In this case λ^* measures the marginal utility of income, i.e. λ^* measures how much utility would increase at the optimal values x_1^* and x_2^* if the individual's income were increased marginally:

$$\begin{aligned} u(x_1^*, x_2^*) &= x_1^* x_2^* = \frac{m^2}{4p_1 p_2} \equiv u^*(p_1, p_2, m) \\ \Rightarrow \frac{du^*}{dm} &= \frac{m}{2p_1 p_2} = \lambda^*. \end{aligned}$$

Example: Cost Minimization

The utility function is given by $u(x_1, x_2) = x_1 x_2$. We want to minimize the expenditures, given by $E(x_1, x_2) = p_1 x_1 + p_2 x_2$, for attaining utility level $\overline{u}$:

$$\min_{x_1, x_2} \; p_1 x_1 + p_2 x_2 \qquad \text{s.t.} \qquad x_1 x_2 = \overline{u}.$$

The Lagrangian is thus given by

$$M(x_1, x_2, \mu) = p_1 x_1 + p_2 x_2 - \mu[x_1 x_2 - \overline{u}].$$

The optimal solutions are given by

$$\begin{aligned} x_1^h &= \sqrt{\frac{p_2 \overline{u}}{p_1}}, \\ x_2^h &= \sqrt{\frac{p_1 \overline{u}}{p_2}}, \\ \mu^h &= \sqrt{\frac{p_1 p_2}{\overline{u}}}. \end{aligned}$$

In this case μ^h measures the marginal cost of $\overline{u}$, i.e. μ^h measures how much expenditures would increase at the optimal values x_1^h and x_2^h if the individual's utility level $\overline{u}$ were increased marginally:

$$\begin{aligned} E(x_1^h, x_2^h) &= p_1 x_1^h + p_2 x_2^h = 2\sqrt{p_1 p_2 \overline{u}} \equiv E^h(p_1, p_2, \overline{u}) \\ \Rightarrow \frac{dE^h}{d\overline{u}} &= \sqrt{\frac{p_1 p_2}{\overline{u}}} = \mu^h. \end{aligned}$$