CHAPTER

8 Problem Solving

This chapter reviews key problem-solving skills and concepts that you need to know for the SAT. Throughout the chapter are sample questions in the style of SAT questions. Each sample SAT question is followed by an explanation of the correct answer.

Translating Words into Numbers

To solve word problems, you must be able to translate words into mathematical operations. You must analyze the language of the question and determine what the question is asking you to do.

The following list presents phrases commonly found in word problems along with their mathematical equivalents:

- A **number** means a **variable.**

Example

17 minus a number equals 4.

$17 - x = 4$

- **Increase** means **add**.

Example

a number increased by 8

$x + 8$

- **More than** means **add**.

Example

4 more than a number

$4 + x$

- **Less than** means **subtract**.

Example

8 less than a number

$x - 8$

- **Times** means **multiply**.

Example

6 times a number

$6x$

- **Times the sum** means to **multiply a number by a quantity**.

Example

7 times the sum of a number and 2

$7(x + 2)$

- Note that variables can be used together.

Example

A number y exceeds 3 times a number x by 12.

$y = 3x + 12$

- **Greater than** means $>$ and **less than** means $<$.

Examples

The product of x and 9 is greater than 15.

$x \times 9 > 15$

When 1 is added to a number x, the sum is less than 29.

$x + 1 < 29$

- **At least** means $\geq$ and **at most** means $\leq$.

Examples

The sum of a number x and 5 is at least 11.

$x + 5 \geq 11$

When 14 is subtracted from a number x, the difference is at most 6.

$x - 14 \leq 6$

- To **square** means to **use an exponent of 2**.

Example

The square of the sum of m and n is 25.

$(m + n)^2 = 25$

Practice Question

If squaring the sum of y and 23 gives a result that is 4 less than 5 times y, which of the following equations could you use to find the possible values of y?

a. $(y + 23)^2 = 5y - 4$
b. $y^2 + 23 = 5y - 4$
c. $y^2 + (23)^2 = y(4 - 5)$
d. $y^2 + (23)^2 = 5y - 4$
e. $(y + 23)^2 = y(4 - 5)$

Answer

a. Break the problem into pieces while translating into mathematics:
squaring translates to *raise something to a power of 2*
the sum of y *and 23* translates to $(y + 23)$
So, *squaring the sum of* y *and 23* translates to $(y + 23)^2$.
gives a result translates to =
4 less than translates to *something* − 4
5 times y translates to 5y
So, *4 less than 5 times* y means 5y − 4.
Therefore, *squaring the sum of* y *and 23 gives a result that is 4 less than 5 times* y translates to: $(y + 23)^2 = 5y - 4$.

▶ Assigning Variables in Word Problems

Some word problems require you to create and assign one or more variables. To answer these word problems, first identify the *unknown* numbers and the *known* numbers. Keep in mind that sometimes the "known" numbers won't be actual numbers, but will instead be expressions involving an unknown.

Examples

Renee is five years older than Ana.
Unknown = Ana's age = x
Known = Renee's age is five years more than Ana's age = $x + 5$
Paco made three times as many pancakes as Vince.
Unknown = number of pancakes Vince made = x
Known = number of pancakes Paco made = three times as many pancakes as Vince made = $3x$
Ahmed has four more than six times the number of CDs that Frances has.
Unknown = the number of CDs Frances has = x
Known = the number of CDs Ahmed has = four more than six times the number of CDs that Frances has = $6x + 4$

Practice Question

On Sunday, Vin's Fruit Stand had a certain amount of apples to sell during the week. On each subsequent day, Vin's Fruit Stand had one-fifth the amount of apples than on the previous day. On Wednesday, 3 days later, Vin's Fruit Stand had 10 apples left. How many apples did Vin's Fruit Stand have on Sunday?

a. 10
b. 50
c. 250
d. 1,250
e. 6,250

Answer

d. To solve, make a list of the knowns and unknowns:

Unknown:

Number of apples on **Sunday** = x

Knowns:

Number of apples on **Monday** = one-fifth the number of apples on Sunday = $\frac{1}{5}x$

Number of apples on **Tuesday** = one-fifth the number of apples on Monday = $\frac{1}{5}(\frac{1}{5}x)$

Number of apples on **Wednesday** = one-fifth the number of apples on Tuesday = $\frac{1}{5}[\frac{1}{5}(\frac{1}{5}x)]$

Because you know that Vin's Fruit Stand had 10 apples on Wednesday, you can set the expression for the number of apples on Wednesday equal to 10 and solve for x:

$\frac{1}{5}[\frac{1}{5}(\frac{1}{5}x)] = 10$

$\frac{1}{5}[\frac{1}{25}x] = 10$

$\frac{1}{125}x = 10$

$125 \times \frac{1}{125}x = 125 \times 10$

$x = 1{,}250$

Because x = the number of apples on Sunday, you know that Vin's Fruit Stand had 1,250 apples on Sunday.

► Percentage Problems

There are three types of percentage questions you might see on the SAT:

1. finding the percentage of a given number
   Example: What number is 60% of 24?
2. finding a number when a percentage is given
   Example: 30% of what number is 15?
3. finding what percentage one number is of another number
   Example: What percentage of 45 is 5?

To answer percent questions, write them as fraction problems. To do this, you must translate the questions into math. Percent questions typically contain the following elements:

- The **percent** is a number divided by 100.
  $75\% = \frac{75}{100} = 0.75$ $4\% = \frac{4}{100} = 0.04$ $0.3\% = \frac{0.3}{100} = 0.003$
- The word **of** means to multiply.
  English: 10% **of** 30 equals 3.
  Math: $\frac{10}{100} \times 30 = 3$
- The word **what** refers to a variable.
  English: 20% of **what** equals 8?
  Math: $\frac{20}{100} \times a = 8$
- The words **is**, **are**, and **were**, mean equals.
  English: 0.5% of 18 **is** 0.09.
  Math: $\frac{0.05}{100} \times 18 = 0.09$

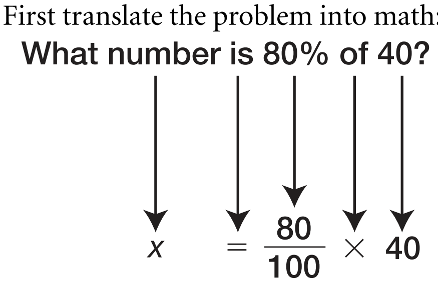
When answering a percentage problem, rewrite the problem as math using the translations above and then solve.

- finding the percentage of a given number

Example

What number is 80% of 40?

First translate the problem into math:



Now solve:

$x = \frac{80}{100} \times 40$

$x = \frac{3{,}200}{100}$

$x = 32$

Answer: 32 is 80% of 40

- finding a number that is a percentage of another number

Example

25% of what number is 16?

First translate the problem into math:

0.25% of what number is 16?

$$\frac{0.25}{100} \times x = 16$$

Now solve:

$\frac{0.25}{100} \times x = 16$

$\frac{0.25x}{100} = 16$

$\frac{0.25x}{100} \times 100 = 16 \times 100$

$0.25x = 1{,}600$

$\frac{x}{0.25} = \frac{1{,}600}{0.25}$

$x = 6{,}400$

Answer: 0.25% of 6,400 is 16.

- finding what percentage one number is of another number

Example

What percentage of 90 is 18?

First translate the problem into math:

What precentage of 90 is 18?

$$\frac{x}{100} \times 90 = 18$$

Now solve:

$\frac{x}{100} \times 90 = 18$

$\frac{90x}{100} = 18$

$\frac{9x}{10} = 18$

$\frac{9x}{10} \times 10 = 18 \times 10$

$9x = 180$

$x = 20$

Answer: 18 is 20% of 90.

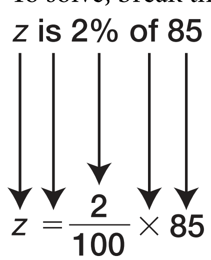
Practice Question

If z is 2% of 85, what is 2% of z?

a. 0.034
b. 0.34
c. 1.7
d. 3.4
e. 17

Answer

a. To solve, break the problem into pieces. The first part says that z is 2% of 85. Let's translate:



Now let's solve for z:

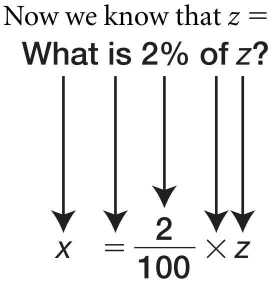
$z = \frac{2}{100} \times 85$

$z = \frac{1}{50} \times 85$

$z = \frac{85}{50}$

$z = \frac{17}{10}$

Now we know that $z = \frac{17}{10}$. The second part asks: What is 2% of z? Let's translate:



Now let's solve for x when $z = \frac{17}{10}$.

$x = \frac{2}{100} \times z$ Plug in the value of z.

$x = \frac{2}{100} \times \frac{17}{10}$

$x = \frac{34}{1{,}000} = 0.034$

Therefore, 0.034 is 2% of z.

▶ Ratios

A **ratio** is a comparison of two quantities measured in the same units. Ratios are represented with a colon or as a fraction:

$x:y$	$\frac{x}{y}$
$3:2$	$\frac{3}{2}$
$a:9$	$\frac{a}{9}$

Examples

If a store sells apples and oranges at a ratio of 2:5, it means that for every two apples the store sells, it sells 5 oranges.

If the ratio of boys to girls in a school is 13:15, it means that for every 13 boys, there are 15 girls.

Ratio problems may ask you to determine the number of items in a group based on a ratio. You can use the concept of multiples to solve these problems.

Example

A box contains 90 buttons, some blue and some white. The ratio of the number of blue to white buttons is 12:6. How many of each color button is in the box?

We know there is a ratio of 12 blue buttons to every 6 white buttons. This means that for every batch of 12 buttons in the box there is also a batch of 6 buttons. We also know there is a total of 90 buttons. This means that we must determine how many batches of blue and white buttons add up to a total of 90. So let's write an equation:

$12x + 6x = 90$, where x is the number of batches of buttons

$18x = 90$

$x = 5$

So we know that there are 5 batches of buttons.

Therefore, there are $(5 \times 12) = 60$ blue buttons and $(5 \times 6) = 30$ white buttons.

A **proportion** is an equality of two ratios.

$\frac{x}{6} = \frac{4}{7}$ $\qquad$ $\frac{1}{35} = \frac{2}{a}$

You can use proportions to solve ratio problems that ask you to determine how much of something is needed based on how much you have of something else.

Example

A recipe calls for peanuts and raisins in a ratio of 3:4, respectively. If Carlos wants to make the recipe with 9 cups of peanuts, how many cups of raisins should he use?

Let's set up a proportion to determine how many cups of raisins Carlos needs.

$\frac{3}{4} = \frac{9}{r}$

This proportion means that 3 parts peanuts to 4 parts raisins must equal 9 parts peanuts to *r* parts raisins. We can solve for *r* by finding cross products:

$\frac{3}{4} = \frac{9}{r}$

$3r = 4 \times 9$

$3r = 36$

$\frac{3r}{3} = \frac{36}{3}$

$r = 12$

Therefore, if Carlos uses 9 cups of peanuts, he needs to use 12 cups of raisins.

Practice Question

A painter mixes red, green, and yellow paint in the ratio of 6:4:2 to produce a new color. In order to make 6 gallons of this new color, how many gallons of red paint must the painter use?

a. 1
b. 2
c. 3
d. 4
e. 6

Answer

c. In the ratio 6:4:2, we know there are 6 parts red paint, 4 parts green paint, and 2 parts yellow paint. Now we must first determine how many total parts there are in the ratio:

6 parts red + 4 parts green + 2 parts yellow = 12 total parts

This means that for every 12 parts of paint, 6 parts are red, 4 parts are green, and 2 parts are yellow. We can now set up a new ratio for red paint:

6 parts red paint:12 total parts = 6:12 = $\frac{6}{12}$

Because we need to find how many gallons of red paint are needed to make 6 total gallons of the new color, we can set up an equation to determine how many parts of red paint are needed to make 6 total parts:

$\frac{r \text{ parts red paint}}{6 \text{ parts total}} = \frac{6 \text{ parts red paint}}{12 \text{ parts total}}$

$\frac{r}{6} = \frac{6}{12}$

Now let's solve for *r*:

$\frac{r}{6} = \frac{6}{12}$ Find cross products.

$12r = 6 \times 6$

$\frac{12r}{12} = \frac{36}{12}$

$r = 3$

Therefore, we know that 3 parts red paint are needed to make 6 total parts of the new color. So 3 gallons of red paint are needed to make 6 gallons of the new color.

▶ Variation

Variation is a term referring to a constant ratio in the change of a quantity.

- A quantity is said to **vary directly** with or to be **directly proportional to** another quantity if they both change in an equal direction. In other words, two quantities vary directly if an increase in one causes an increase in the other or if a decrease in one causes a decrease in the other. The ratio of increase or decrease, however, must be the same.

Example

Thirty elephants drink altogether a total of 6,750 liters of water a day. Assuming each elephant drinks the same amount, how many liters of water would 70 elephants drink?

Since each elephant drinks the same amount of water, you know that elephants and water vary directly. Therefore, you can set up a proportion:

$\frac{\text{water}}{\text{elephants}} = \frac{6{,}750}{30} = \frac{x}{70}$

Find cross products to solve:

$\frac{6{,}750}{30} = \frac{x}{70}$

$(6{,}750)(70) = 30x$

$472{,}500 = 30x$

$\frac{472{,}500}{30} = \frac{30x}{30}$

$15{,}750 = x$

Therefore, 70 elephants would drink 15,750 liters of water.

- A quantity is said to **vary inversely** with or to be **inversely proportional** to another quantity if they change in opposite directions. In other words, two quantities vary inversely if an increase in one causes a decrease in the other or if a decrease in one causes an increase in the other.

Example

Three plumbers can install plumbing in a house in six days. Assuming each plumber works at the same rate, how many days would it take nine plumbers to install plumbing in the same house?

As the number of plumbers increases, the days needed to install plumbing decreases (because more plumbers can do more work). Therefore, the relationship between the number of plumbers and the number of days varies inversely. Because the amount of plumbing to install remains constant, the two expressions can be set equal to each other:

$3 \text{ plumbers} \times 6 \text{ days} = 9 \text{ plumbers} \times x \text{ days}$

$3 \times 6 = 9x$

$18 = 9x$

$\frac{18}{9} = \frac{9x}{9}$

$2 = x$

Thus, it would take nine plumbers only two days to install plumbing in the same house.

Practice Question

The number a is directly proportional to b. If $a = 15$ when $b = 24$, what is the value of b when $a = 5$?

a. $\frac{8}{5}$

b. $\frac{25}{8}$

c. 8

d. 14

e. 72

Answer

c. The numbers a and b are directly proportional (in other words, they vary directly), so a increases when b increases, and vice versa. Therefore, we can set up a proportion to solve:

$\frac{15}{24} = \frac{5}{b}$ Find cross products.

$15b = (24)(5)$

$15b = 120$

$\frac{15b}{15} = \frac{120}{15}$

$b = 8$

Therefore, we know that $b = 8$ when $a = 5$.

▶ Rate Problems

Rate is defined as a comparison of two quantities with different units of measure.

Rate $= \frac{x \text{ units}}{y \text{ units}}$

Examples

$\frac{\text{dollars}}{\text{hour}}$ $\frac{\text{cost}}{\text{pound}}$ $\frac{\text{miles}}{\text{hour}}$ $\frac{\text{miles}}{\text{gallon}}$

There are three types of rate problems you must learn how to solve: cost per unit problems, movement problems, and work-output problems.

▶ Cost Per Unit

Some rate problems require you to calculate the cost of a specific quantity of items.

Example

If 40 sandwiches cost \$298, what is the cost of eight sandwiches?

First determine the cost of one sandwich by setting up a proportion:

$\frac{\$238}{40 \text{ sandwiches}} = \frac{x}{1}$ sandwich

$238 \times 1 = 40x$ Find cross products.

$238 = 40x$

$\frac{238}{40} = x$

$5.95 = x$

Now we know one sandwich costs $5.95. To find the cost of eight sandwiches, multiply:

$5.95 \times 8 = \$47.60$

Eight sandwiches cost $47.60.

Practice Question

A clothing store sold 45 bandanas a day for three days in a row. If the store earned a total of $303.75 from the bandanas for the three days, and each bandana cost the same amount, how much did each bandana cost?

a. $2.25
b. $2.75
c. $5.50
d. $6.75
e. $101.25

Answer

a. First determine how many total bandanas were sold:

45 bandanas per day × 3 days = 135 bandanas

So you know that 135 bandanas cost $303.75. Now set up a proportion to determine the cost of one bandana:

$\frac{\$303.75}{135 \text{ bandanas}} = \frac{x}{1 \text{ bandana}}$

$303.75 \times 1 = 135x$ Find cross products.

$303.75 = 135x$

$\frac{303.75}{135} = x$

$2.25 = x$

Therefore, one bandana costs $2.25.

▶ Movement

When working with movement problems, it is important to use the following formula:

(Rate)(Time) = Distance

Example

A boat traveling at 45 mph traveled around a lake in 0.75 hours less than a boat traveling at 30 mph. What was the distance around the lake?

First, write what is known and unknown.

Unknown = time for Boat 2, traveling 30 mph to go around the lake = x
Known = time for Boat 1, traveling 45 mph to go around the lake = $x - 0.75$
Then, use the formula (Rate)(Time) = Distance to write an equation. The distance around the lake does not change for either boat, so you can make the two expressions equal to each other:
(Boat 1 rate)(Boat 1 time) = Distance around lake
(Boat 2 rate)(Boat 2 time) = Distance around lake
Therefore:
(Boat 1 rate)(Boat 1 time) = (Boat 2 rate)(Boat 2 time)
$(45)(x - 0.75) = (30)(x)$
$45x - 33.75 = 30x$
$45x - 33.75 - 45x = 30x - 45x$
$-\frac{33.75}{15} = -\frac{15x}{15}$
$-2.25 = -x$
$2.25 = x$
Remember: x represents the time it takes Boat 2 to travel around the lake. We need to plug it into the formula to determine the distance around the lake:
(Rate)(Time) = Distance
(Boat 2 Rate)(Boat 2 Time) = Distance
(30)(2.25) = Distance
67.5 = Distance
The distance around the lake is 67.5 miles.

Practice Question

Priscilla rides her bike to school at an average speed of 8 miles per hour. She rides her bike home along the same route at an average speed of 4 miles per hour. Priscilla rides a total of 3.2 miles round-trip. How many hours does it take her to ride round-trip?

a. 0.2
b. 0.4
c. 0.6
d. 0.8
e. 2

Answer

c. Let's determine the time it takes Priscilla to complete each leg of the trip and then add the two times together to get the answer. Let's start with the trip from home to school:
Unknown = time to ride from home to school = x
Known = rate from home to school = 8 mph
Known = distance from home to school = total distance round-trip ÷ 2 = 3.2 miles ÷ 2 = 1.6 miles
Then, use the formula (Rate)(Time) = Distance to write an equation:
(Rate)(Time) = Distance
$8x = 1.6$

$\frac{8x}{8} = \frac{1.6}{8}$
$x = 0.2$
Therefore, Priscilla takes 0.2 hours to ride from home to school.
Now let's do the same calculations for her trip from school to home:
Unknown = time to ride from school to home = y
Known = rate from home to school = 4 mph
Known = distance from school to home = total distance round-trip ÷ 2 = 3.2 miles ÷ 2 = 1.6 miles
Then, use the formula (Rate)(Time) = Distance to write an equation:
(Rate)(Time) = Distance
$4x = 1.6$
$\frac{4x}{4} = \frac{1.6}{4}$
$x = 0.4$
Therefore, Priscilla takes 0.4 hours to ride from school to home.
Finally add the times for each leg to determine the total time it takes Priscilla to complete the round trip:
$0.4 + 0.2 = 0.6$ hours
It takes Priscilla 0.6 hours to complete the round-trip.

▶ Work-Output Problems

Work-output problems deal with the rate of work. In other words, they deal with how much work can be completed in a certain amount of time. The following formula can be used for these problems:

(rate of work)(time worked) = part of job completed

Example

Ben can build two sand castles in 50 minutes. Wylie can build two sand castles in 40 minutes. If Ben and Wylie work together, how many minutes will it take them to build one sand castle?

Since Ben can build two sand castles in 60 minutes, his rate of work is $\frac{\text{2 sand castles}}{\text{60 minutes}}$ or $\frac{\text{1 sand castle}}{\text{30 minutes}}$. Wylie's rate of work is $\frac{\text{2 sand castles}}{\text{40 minutes}}$ or $\frac{\text{1 sand castle}}{\text{20 minutes}}$.

To solve this problem, making a chart will help:

	RATE	TIME	=	PART OF JOB COMPLETED
Ben	$\frac{1}{30}$	x	=	1 sand castle
Wylie	$\frac{1}{20}$	x	=	1 sand castle

Since Ben and Wylie are both working together on one sand castle, you can set the equation equal to one:
(Ben's rate)(time) + (Wylie's rate)(time) = 1 sand castle
$\frac{1}{30}x + \frac{1}{20}x = 1$

Now solve by using 60 as the LCD for 30 and 20:

$\frac{1}{30}x + \frac{1}{20}x = 1$

$\frac{2}{60}x + \frac{3}{60}x = 1$

$\frac{5}{60}x = 1$

$\frac{5}{60}x \times 60 = 1 \times 60$

$5x = 60$

$x = 12$

Thus, it will take Ben and Wylie 12 minutes to build one sand castle.

Practice Question

Ms. Walpole can plant nine shrubs in 90 minutes. Mr. Saum can plant 12 shrubs in 144 minutes. If Ms. Walpole and Mr. Saum work together, how many minutes will it take them to plant two shrubs?

a. $\frac{60}{11}$

b. 10

c. $\frac{120}{11}$

d. 11

e. $\frac{240}{11}$

Answer

c. Ms. Walpole can plant 9 shrubs in 90 minutes, so her rate of work is $\frac{9 \text{ shrubs}}{90 \text{ minutes}}$ or $\frac{1 \text{ shrub}}{10 \text{ minutes}}$. Mr. Saum's rate of work is $\frac{12 \text{ shrubs}}{144 \text{ minutes}}$ or $\frac{1 \text{ shrub}}{12 \text{ minutes}}$.

To solve this problem, making a chart will help:

	RATE	TIME	=	PART OF JOB COMPLETED
Ms. Walpole	$\frac{1}{10}$	x	=	1 shrub
Mr. Saum	$\frac{1}{12}$	x	=	1 shrub

Because both Ms. Walpole and Mr. Saum are working together on two shrubs, you can set the equation equal to two:

(Ms. Walpole's rate)(time) + (Mr. Saum's rate)(time) = 2 shrubs

$\frac{1}{10}x + \frac{1}{12}$

$x = 2$

Now solve by using 60 as the LCD for 10 and 12:

$\frac{1}{10}x + \frac{1}{12}x = 2$

$\frac{6}{60}x + \frac{5}{60}x = 2$

$\frac{11}{60}x = 2$

$\frac{11}{60}x \times 60 = 2 \times 60$

$11x = 120$

$x = \frac{120}{11}$

Thus, it will take Ms. Walpole and Mr. Saum $\frac{120}{11}$ minutes to plant two shrubs.

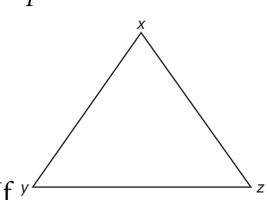
▶ Special Symbols Problems

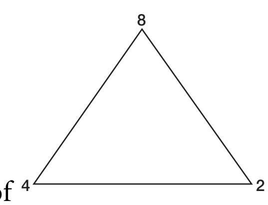
Some SAT questions invent an operation symbol that you won't recognize. Don't let these symbols confuse you. These questions simply require you to make a substitution based on information the question provides. Be sure to pay attention to the placement of the variables and operations being performed.

Example

Given $p \lozenge q = (p \times q + 4)^2$, find the value of $2 \lozenge 3$.
Fill in the formula with 2 replacing p and 3 replacing q.
$(p \times q + 4)^2$
$(2 \times 3 + 4)^2$
$(6 + 4)^2$
$(10)^2$
$= 100$
So, $2 \lozenge 3 = 100$.

Example

If 

$= \frac{x+y+z}{x} + \frac{x+y+z}{y} + \frac{x+y+z}{z}$, then what is the value of 

Fill in the variables according to the placement of the numbers in the triangular figure: $x = 8$, $y = 4$, and $z = 2$.

$\frac{8+4+2}{8} + \frac{8+4+2}{4} + \frac{8+4+2}{2}$

$\frac{14}{8} + \frac{14}{4} + \frac{14}{2}$ LCD is 8.

$\frac{14}{8} + \frac{28}{8} + \frac{56}{8}$ Add.

$\frac{98}{8}$ Simplify.

$\frac{49}{4}$

Answer: $\frac{49}{4}$

Practice Question

The operation $c \, \Omega \, d$ is defined by $c \, \Omega \, d = d^{c+d} \times d^{c-d}$. What value of d makes $2 \, \Omega \, d$ equal to 81?

a. 2
b. 3
c. 9
d. 20.25
e. 40.5

Answer

b. If $c \, \Omega \, d = d^{c+d} \times d^{c-d}$, then $2 \, \Omega \, d = d^{2+d} \times d^{2-d}$. Solve for d when $2 \, \Omega \, d = 81$:

$d^{2+d} \times d^{2-d} = 81$
$d^{(2+d)+(2-d)} = 81$
$d^{2+2+d-d} = 81$
$d^4 = 81$
$\sqrt{d^4} = \sqrt{81}$
$d^2 = 9$
$\sqrt{d^2} = \sqrt{9}$
$d = 3$
Therefore, $d = 3$ when $2 \, \Omega \, d = 81$.

▶ The Counting Principle

Some questions ask you to determine the number of outcomes possible in a given situation involving different choices.

For example, let's say a school is creating a new school logo. Students have to vote on one color for the background and one color for the school name. They have six colors to choose from for the background and eight colors to choose from for the school name. How many possible combinations of colors are possible?

The quickest method for finding the answer is to use **the counting principle.** Simply multiply the number of possibilities from the first category (six background colors) by the number of possibilities from the second category (eight school name colors):

$6 \times 8 = 48$

Therefore, there are 48 possible color combinations that students have to choose from.

Remember: When determining the number of outcomes possible when combining one out of x choices in one category and one out of y choices in a second category, simply multiply $x \times y$.

Practice Question

At an Italian restaurant, customers can choose from one of nine different types of pasta and one of five different types of sauce. How many possible combinations of pasta and sauce are possible?

a. $\frac{9}{5}$
b. 4
c. 14
d. 32
e. 45

Answer

e. You can use the counting principle to solve this problem. The question asks you to determine the number of combinations possible when combining one out of nine types of pasta and one out of five types of sauce. Therefore, multiply $9 \times 5 = 45$. There are 45 total combinations possible.

▶ Permutations

Some questions ask you to determine the number of ways to arrange n items in all possible groups of r items. For example, you may need to determine the total number of ways to arrange the letters *ABCD* in groups of two letters. This question involves four items to be arranged in groups of two items. Another way to say this is that the question is asking for the number of **permutations** it's possible make of a group with two items from a group of four items. Keep in mind when answering permutation questions that *the order of the items matters.* In other words, using the example, both *AB* and *BA* must be counted.

To solve permutation questions, you must use a special formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

P = number of permutations
n = the number of items
r = number of items in each permutation

Let's use the formula to answer the problem of arranging the letters *ABCD* in groups of two letters.

the number of items $(n) = 4$
number of items in each permutation $(r) = 2$

Plug in the values into the formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$
$${}_4P_2 = \frac{4!}{(4-2)!}$$
$${}_4P_2 = \frac{4!}{2!}$$

$_4P_2 = \frac{4 \times 3 \times 2 \times 1}{2 \times 1}$ Cancel out the 2×1 from the numerator and denominator.
$_4P_2 = 4 \times 3$
$_4P_2 = 12$

Therefore, there are 12 ways to arrange the letters *ABCD* in groups of two:

AB	*AC*	*AD*	*BA*	*BC*	*BD*
CA	*CB*	*CD*	*DA*	*DB*	*DC*

Practice Question

Casey has four different tickets to give away to friends for a play she is acting in. There are eight friends who want to use the tickets. How many different ways can Casey distribute four tickets among her eight friends?

a. 24
b. 32
c. 336
d. 1,680
e. 40,320

Answer

d. To answer this permutation question, you must use the formula $_nP_r = \frac{n!}{(n-r)!}$, where n = the number of friends = 8 and r = the number of tickets that the friends can use = 4. Plug the numbers into the formula:

$_nP_r = \frac{n!}{(n-r)!}$
$_8P_4 = \frac{8!}{(8-4)!}$
$_8P_4 = \frac{8!}{4!}$
$_8P_4 = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1}$ Cancel out the $4 \times 3 \times 2 \times 1$ from the numerator and denominator.
$_8P_4 = 8 \times 7 \times 6 \times 5$
$_8P_4 = 1{,}680$

Therefore, there are 1,680 permutations of friends that she can give the four different tickets to.

▶ Combinations

Some questions ask you to determine the number of ways to arrange n items in groups of r items without repeated items. In other words, *the order of the items doesn't matter.* For example, to determine the number of ways to arrange the letters *ABCD* in groups of two letters in which the order doesn't matter, you would count only *AB*, not both *AB* and *BA*. These questions ask for the total number of **combinations** of items.

To solve combination questions, use this formula:

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{(n-r)!r!}$$

C = number of combinations
n = the number of items
r = number of items in each permutation

For example, to determine the number of three-letter combinations from a group of seven letters (*ABCDEFGH*), use the following values: $n = 7$ and $r = 3$.

Plug in the values into the formula:

$${}_7C_3 = \frac{n!}{(n-r)!r!} = \frac{7!}{(7-3)!3!} = \frac{7!}{4!3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3!)} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = \frac{210}{6} = 35$$

Therefore there are 35 three-letter combinations from a group of seven letters.

Practice Question

A film club has five memberships available. There are 12 people who would like to join the club. How many combinations of the 12 people could fill the five memberships?

a. 60
b. 63
c. 792
d. 19,008
e. 95,040

Answer

c. The order of the people doesn't matter in this problem, so it is a combination question, not a permutation question. Therefore we can use the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$, where n = the number of people who want the membership = 12 and r = the number of memberships = 5.

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

$${}_{12}C_5 = \frac{12!}{(12-5)!5!}$$

$${}_{12}C_5 = \frac{12!}{7!5!}$$

$${}_{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1)5!}$$

$${}_{12}C_5 = \frac{12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2 \times 1}$$

$${}_{12}C_5 = \frac{95{,}040}{120}$$

$${}_{12}C_5 = 792$$

Therefore, there are 792 different combinations of 12 people to fill five memberships.

▶ Probability

Probability measures the likelihood that a specific event will occur. Probabilities are expressed as fractions. To find the probability of a specific outcome, use this formula:

Probability of an event $= \frac{\text{number of specific outcomes}}{\text{total number of possible outcomes}}$

Example

If a hat contains nine white buttons, five green buttons, and three black buttons, what is the probability of selecting a green button without looking?

Probability $= \frac{\text{number of specific outcomes}}{\text{total number of possible outcomes}}$

Probability $= \frac{\text{number of green buttons}}{\text{total number of buttons}}$

Probability $= \frac{5}{9 + 5 + 3}$

Probability $= \frac{5}{17}$

Therefore, the probability of selecting a green button without looking is $\frac{5}{17}$.

Practice Question

A box of DVDs contains 13 comedies, four action movies, and 15 thrillers. If Brett selects a DVD from the box without looking, what is the probability he will pick a comedy?

a. $\frac{4}{32}$
b. $\frac{13}{32}$
c. $\frac{15}{32}$
d. $\frac{13}{15}$
e. $\frac{13}{4}$

Answer

b. Probability is $\frac{\text{number of specific outcomes}}{\text{total number of possible outcomes}}$. Therefore, you can set up the following fraction:

$\frac{\text{number of comedy DVDs}}{\text{total number of DVDs}} = \frac{13}{13 + 4 + 15} = \frac{13}{32}$

Therefore, the probability of selecting a comedy DVD is $\frac{13}{32}$.

Multiple Probabilities

To find the probability that one of two or more mutually exclusive events will occur, add the probabilities of each event occurring. For example, in the previous problem, if we wanted to find the probability of drawing either a green or black button, we would add the probabilities together.

The probability of drawing a green button $= \frac{5}{17}$.
The probability of drawing a black button $= \frac{\text{number of black buttons}}{\text{total number of buttons}} = \frac{3}{9+5+3} = \frac{3}{17}$.
So the probability for selecting either a green or black button $= \frac{5}{17} + \frac{3}{17} = \frac{8}{17}$.

Practice Question

At a farmers' market, there is a barrel filled with apples. In the barrel are 40 Fuji apples, 24 Gala apples, 12 Red Delicious apples, 24 Golden Delicious, and 20 McIntosh apples. If a customer reaches into the barrel and selects an apple without looking, what is the probability that she will pick a Fuji or a McIntosh apple?

a. $\frac{1}{6}$
b. $\frac{1}{3}$
c. $\frac{2}{5}$
d. $\frac{1}{2}$
e. $\frac{3}{5}$

Answer

d. This problem asks you to find the probability that two events will occur (picking a Fuji apple or picking a McIntosh apple), so you must add the probabilities of each event. So first find the probability that someone will pick a Fuji apple:

the probability of picking a Fuji apple =

$\frac{\text{number of Fuji apples}}{\text{total number of apples}} =$

$\frac{40}{40+24+12+24+20} =$

$\frac{40}{120}$

Now find the probability that someone will pick a McIntosh apple:
the probability of picking a McIntosh apple =

$\frac{\text{number of McIntosh apples}}{\text{total number of apples}} =$

$\frac{20}{40+24+12+24+20} =$

$\frac{20}{120}$

Now add the probabilities together:

$\frac{40}{120} + \frac{20}{120} = \frac{60}{120} = \frac{1}{2}$

The probability that someone will pick a Fuji apple or a McIntosh is $\frac{1}{2}$.

Helpful Hints about Probability

- If an event is certain to occur, its probability is 1.
- If an event is certain *not* to occur, its probability is 0.
- You can find the probability of an unknown event if you know the probability of all other events occurring. Simply add the known probabilities together and subtract the result from 1. For example, let's say there is a bag filled with red, orange, and yellow buttons. You want to know the probability that you will pick a red button from a bag, but you don't know how many red buttons there are. However, you do know that the probability of picking an orange button is $\frac{3}{20}$ and the probability of picking a yellow button is $\frac{14}{20}$. If you add these probabilities together, you know the probability that you will pick an orange or yellow button: $\frac{3}{20} + \frac{16}{20} = \frac{19}{20}$. This probability, $\frac{19}{20}$, is also the probability that you *won't* pick a *red* button. Therefore, if you subtract $1 - \frac{19}{20}$, you will know the probability that you *will* pick a red button. $1 - \frac{19}{20} = \frac{1}{20}$. Therefore, the probability of choosing a red button is $\frac{1}{20}$.

Practice Question

Angie ordered 75 pizzas for a party. Some are pepperoni, some are mushroom, some are onion, some are sausage, and some are olive. However, the pizzas arrived in unmarked boxes, so she doesn't know which box contains what kind of pizza. The probability that a box contains a pepperoni pizza is $\frac{1}{15}$, the probability that a box contains a mushroom pizza is $\frac{2}{15}$, the probability that a box contains an onion pizza is $\frac{16}{75}$, and the probability that a box contains a sausage pizza is $\frac{8}{25}$. If Angie opens a box at random, what is the probability that she will find an olive pizza?

a. $\frac{2}{15}$
b. $\frac{1}{5}$
c. $\frac{4}{15}$
d. $\frac{11}{15}$
e. $\frac{4}{5}$

Answer

c. The problem does not tell you the probability that a random box contains an olive pizza. However, the problem does tell you the probabilities of a box containing the other types of pizza. If you add together all those probabilities, you will know the probability that a box contains a pepperoni, a mushroom, an onion, or a sausage pizza. In other words, you will know the probability that a box does NOT contain an olive pizza:

pepperoni + mushroom + onion + sausage

$= \frac{1}{15} + \frac{2}{15} + \frac{16}{75} + \frac{8}{25}$ Use an LCD of 75.

$= \frac{5}{75} + \frac{10}{75} + \frac{16}{75} + \frac{24}{75}$

$= \frac{5}{75} + \frac{10}{75} + \frac{16}{75} + \frac{24}{75}$

$= \frac{55}{75}$

The probability that a box does NOT contain an olive pizza is $\frac{55}{75}$.

Now subtract this probability from 1:

$1 - \frac{55}{75} = \frac{75}{75} - \frac{55}{75} = \frac{20}{75} = \frac{4}{15}$

The probability of opening a box and finding an olive pizza is $\frac{4}{15}$.