2 Number systems and sets

Objectives

- To understand and use **set notation**, including the symbols $\in$, $\subseteq$, $\cup$, $\cap$, $\emptyset$ and ξ.
- To be able to identify sets of numbers, including the natural numbers, integers, rational numbers, irrational numbers and real numbers.
- To interpret subsets of the real numbers defined using the **modulus function**.
- To know and apply the rules for working with **surds**, including:
  - simplification of surds
  - rationalisation of surds.
- To know and apply the definitions of **factor**, **prime**, **highest common factor** and **lowest common multiple**.
- To be able to solve **linear Diophantine equations**.
- To be able to solve problems with sets.

This chapter introduces set notation and discusses sets of numbers and their properties. Set notation is used widely in mathematics and in this book it is employed where appropriate. In this chapter we discuss natural numbers, integers and rational numbers, and then continue on to consider irrational numbers.

Irrational numbers such as $\sqrt{2}$ naturally arise when applying Pythagoras' theorem. When solving a quadratic equation, using the method of completing the square or the quadratic formula, we obtain answers such as $x = \frac{1}{2}(1 \pm \sqrt{5})$. These numbers involve surds.

Since these numbers are irrational, we cannot express them in exact form using decimals or fractions. Sometimes we may wish to approximate them using decimals, but mostly we prefer to leave them in exact form. Thus we need to be able to manipulate these types of numbers and to simplify combinations of them which arise when solving a problem.

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2A Set notation

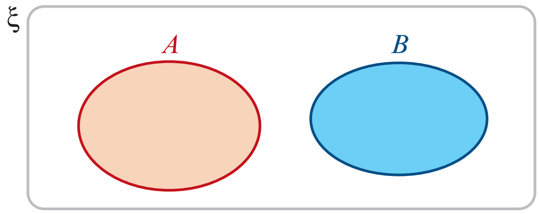
A **set** is a general name for any collection of things or numbers. There must be a way of deciding whether any particular object is a member of the set or not. This may be done by referring to a list of the members of the set or a statement describing them.

For example: $A = \{-3, 3\} = \{ x : x^2 = 9 \}$

Note: $\{ x : \dots \}$ is read as 'the set of all x such that ...'.

- The symbol $\in$ means 'is a member of' or 'is an element of'.
  For example: $3 \in$ {prime numbers} is read '3 is a member of the set of prime numbers'.
- The symbol $\notin$ means 'is not a member of' or 'is not an element of'.
  For example: $4 \notin$ {prime numbers} is read '4 is not a member of the set of prime numbers'.
- Two sets are **equal** if they contain exactly the same elements, not necessarily in the same order. For example: if $A =$ {prime numbers less than 10} and $B = \{2, 3, 5, 7\}$, then $A = B$.
- The set with no elements is called the **empty set** and is denoted by $\varnothing$.
- The **universal set** will be denoted by ξ. The universal set is the set of all elements which are being considered.
- If all the elements of a set B are also elements of a set A, then the set B is called a **subset** of A. This is written $B \subseteq A$. For example: $\{a, b, c\} \subseteq \{a, b, c, d, e, f, g\}$ and $\{3, 9, 27\} \subseteq$ {multiples of 3}. We note also that $A \subseteq A$ and $\varnothing \subseteq A$.

Venn diagrams are used to illustrate sets. For example, the diagram on the right shows two subsets A and B of a universal set ξ such that A and B have no elements in common. Two such sets are said to be **disjoint**.



The union of two sets

The set of all the elements that are members of set A or set B (or both) is called the **union** of A and B. The union of A and B is written $A \cup B$.

Example 1

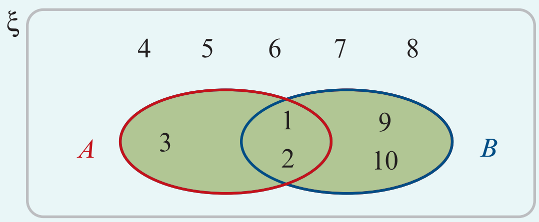
Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3\}$ and $B = \{1, 2, 9, 10\}$.

Find $A \cup B$ and illustrate on a Venn diagram.

Solution

$A \cup B = \{1, 2, 3, 9, 10\}$

The shaded area illustrates $A \cup B$.



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► The intersection of two sets

The set of all the elements that are members of both set A and set B is called the **intersection** of A and B. The intersection of A and B is written $A \cap B$.

Example 2

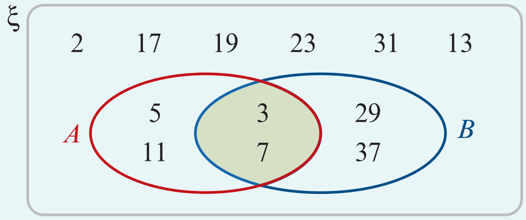
Let ξ = {prime numbers less than 40}, $A = \{3, 5, 7, 11\}$ and $B = \{3, 7, 29, 37\}$.

Find $A \cap B$ and illustrate on a Venn diagram.

Solution

$A \cap B = \{3, 7\}$

The shaded area illustrates $A \cap B$.



► The complement of a set

The **complement** of a set A is the set of all elements of ξ that are not members of A. The complement of A is denoted by A'.

If ξ = {students at Highland Secondary College} and A = {students with blue eyes}, then A' is the set of all students at Highland Secondary College who do not have blue eyes.

Similarly, if $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and $A = \{1, 3, 5, 7, 9\}$, then $A' = \{2, 4, 6, 8, 10\}$.

Example 3

Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

A = {odd numbers} $= \{1, 3, 5, 7, 9\}$

B = {multiples of 3} $= \{3, 6, 9\}$

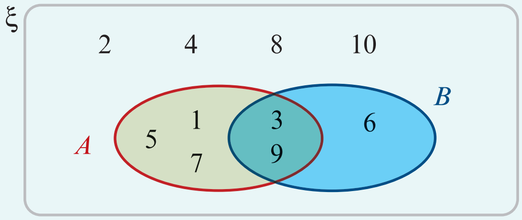
a Show these sets on a Venn diagram.

b Use the diagram to list the following sets:

i A' **ii** B' **iii** $A \cup B$ **iv** the complement of $A \cup B$, i.e. $(A \cup B)'$ **v** $A' \cap B'$

Solution

a



b From the diagram:

i $A' = \{2, 4, 6, 8, 10\}$

ii $B' = \{1, 2, 4, 5, 7, 8, 10\}$

iii $A \cup B = \{1, 3, 5, 6, 7, 9\}$

iv $(A \cup B)' = \{2, 4, 8, 10\}$

v $A' \cap B' = \{2, 4, 8, 10\}$

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Finite and infinite sets

When all the elements of a set may be counted, the set is called a **finite** set. For example, the set A = {months of the year} is finite. The number of elements of a set A will be denoted $|A|$. In this example, $|A| = 12$. If C = {letters of the alphabet}, then $|C| = 26$.

Sets which are not finite are called **infinite** sets. For example, the set of real numbers, $\mathbb{R}$, and the set of integers, $\mathbb{Z}$, are infinite sets.

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- The **empty set** is denoted by $\varnothing$ and the **universal set** by ξ.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The **complement** of A, denoted by A', is the set of all elements of ξ that are not in A.
- If two sets A and B have no elements in common, we say that they are **disjoint** and write $A \cap B = \varnothing$.

Exercise 2A

Example 1

1 Let $\xi = \{1, 2, 3, 4, 5\}$, $A = \{1, 2, 3, 5\}$ and $B = \{2, 4\}$.
Show these sets on a Venn diagram and use the diagram to find:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

Example 2

2 Let ξ = {natural numbers less than 17}, P = {multiples of 3} and Q = {even numbers}.
Show these sets on a Venn diagram and use it to find:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

Example 3

3 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, A = {multiples of 4} and B = {even numbers}.
Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cup B$ **d** $(A \cup B)'$ **e** $A' \cap B'$

4 Let ξ = {natural numbers from 10 to 25}, P = {multiples of 4} and Q = {multiples of 5}.
Show these sets on a Venn diagram and use this diagram to list the sets:

a P' **b** Q' **c** $P \cup Q$ **d** $(P \cup Q)'$ **e** $P' \cap Q'$

5 Let $\xi = \{p, q, r, s, t, u, v, w\}$, $X = \{r, s, t, w\}$ and $Y = \{q, s, t, u, v\}$.
Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cap Y'$ **d** $X' \cup Y'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

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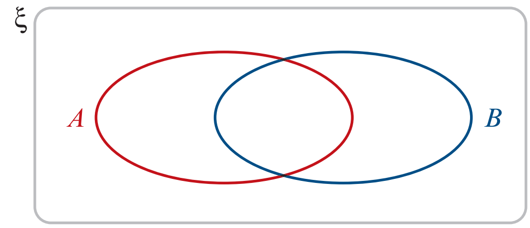
6 Let $\xi = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $X = \{\text{factors of } 12\}$ and $Y = \{\text{even numbers}\}$. Show ξ, X and Y on a Venn diagram, entering all members. Hence list the sets:

a X' **b** Y' **c** $X' \cup Y'$ **d** $(X \cap Y)'$ **e** $X \cup Y$ **f** $(X \cup Y)'$

Which two sets are equal?

7 Draw this diagram six times.
Use shading to illustrate each of the following sets:

a A' **b** B' **c** $A' \cap B'$
d $A' \cup B'$ **e** $A \cup B$ **f** $(A \cup B)'$



8 Let ξ = {different letters in the word *GENERAL*},
A = {different letters in the word *ANGEL*},
B = {different letters in the word *LEAN*}

Show these sets on a Venn diagram and use this diagram to list the sets:

a A' **b** B' **c** $A \cap B$ **d** $A \cup B$ **e** $(A \cup B)'$ **f** $A' \cup B'$

9 Let ξ = {different letters in the word *MATHEMATICS*}
A = {different letters in the word *ATTIC*}
B = {different letters in the word *TASTE*}

Show ξ, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cap B$ **d** $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

2B Sets of numbers

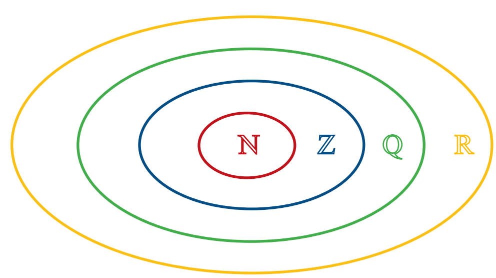
Recall that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

The numbers of the form $\dfrac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational**. Some examples of irrational numbers are $\sqrt{2}$, $\sqrt{3}$, π, $\pi + 2$ and $\sqrt{6} + \sqrt{7}$. These numbers cannot be written in the form $\dfrac{p}{q}$, for integers p, q; the decimal representations of these numbers do not terminate or repeat.

- The set of real numbers is denoted by $\mathbb{R}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of natural numbers is denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



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We can use set notation to describe subsets of the real numbers.

For example:

- $\{x : 0 < x < 1\}$ is the set of all real numbers strictly between 0 and 1
- $\{x : x > 0,\ x \in \mathbb{Q}\}$ is the set of all positive rational numbers
- $\{2n : n = 0, 1, 2, \dots\}$ is the set of all non-negative even numbers.

The set of all ordered pairs of real numbers is denoted by $\mathbb{R}^2$. That is,

$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$

This set is known as the **Cartesian product** of $\mathbb{R}$ with itself.

Rational numbers

Every rational number can be expressed as a terminating or recurring decimal.

To find the decimal representation of a rational number $\frac{m}{n}$, perform the division $m \div n$.

For example, to find the decimal representation of $\frac{3}{7}$, divide $3.0000000\dots$ by 7.

$$\begin{array}{r|l} & 0.\ 4\ \ 2\ \ 8\ \ 5\ \ 7\ \ 1\ \ 4\ \dots \\ \hline 7 & 3.\ {}^{3}0\ {}^{2}0\ {}^{6}0\ {}^{4}0\ {}^{5}0\ {}^{1}0\ {}^{3}0\ \dots \end{array}$$

Therefore $\frac{3}{7} = 0.\dot{4}2857\dot{1}$.

Theorem

Every rational number can be written as a terminating or recurring decimal.

Proof Consider any two natural numbers m and n. At each step in the division of m by n, there is a remainder. If the remainder is 0, then the division algorithm stops and the decimal is a terminating decimal.

If the remainder is never 0, then it must be one of the numbers $1, 2, 3, \dots, n - 1$. (In the above example, $n = 7$ and the remainders can only be 1, 2, 3, 4, 5 and 6.) Hence the remainder must repeat after at most $n - 1$ steps.

Further examples:

$$\frac{1}{2} = 0.5, \quad \frac{1}{5} = 0.2, \quad \frac{1}{10} = 0.1, \quad \frac{1}{3} = 0.\dot{3}, \quad \frac{1}{7} = 0.\dot{1}4285\dot{7}$$

Theorem

A real number has a terminating decimal representation if and only if it can be written as

$$\frac{m}{2^{\alpha} \times 5^{\beta}}$$

for some $m \in \mathbb{Z}$ and some $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

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Proof Assume that $x = \dfrac{m}{2^\alpha \times 5^\beta}$ with $\alpha \geq \beta$. Multiply the numerator and denominator by $5^{\alpha-\beta}$. Then

$$x = \frac{m \times 5^{\alpha-\beta}}{2^\alpha \times 5^\alpha} = \frac{m \times 5^{\alpha-\beta}}{10^\alpha}$$

and so x can be written as a terminating decimal. The case $\alpha < \beta$ is similar.

Conversely, if x can be written as a terminating decimal, then there is $m \in \mathbb{Z}$ and $\alpha \in \mathbb{N} \cup \{0\}$ such that $x = \dfrac{m}{10^\alpha} = \dfrac{m}{2^\alpha \times 5^\alpha}$.

The method for finding a rational number $\dfrac{m}{n}$ from its decimal representation is demonstrated in the following example.

Example 4

Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a 0.05 **b** $0.\dot{4}2857\dot{1}$

Solution

a $0.05 = \dfrac{5}{100} = \dfrac{1}{20}$

b We can write

$$0.\dot{4}2857\dot{1} = 0.428571428571\ldots \qquad (1)$$

Multiply both sides by 10^6:

$$0.\dot{4}2857\dot{1} \times 10^6 = 428571.428571428571\ldots \qquad (2)$$

Subtract (1) from (2):

$$0.\dot{4}2857\dot{1} \times (10^6 - 1) = 428571$$

$$\therefore \quad 0.\dot{4}2857\dot{1} = \frac{428571}{10^6 - 1} = \frac{3}{7}$$

Real numbers

The set of real numbers is made up of two important subsets: the **algebraic numbers** and the **transcendental numbers**.

An algebraic number is a solution to a polynomial equation of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0, \quad \text{where } a_0, a_1, \ldots, a_n \text{ are integers}$$

Every rational number is algebraic. The irrational number $\sqrt{2}$ is algebraic, as it is a solution of the equation

$$x^2 - 2 = 0$$

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It can be shown that π is not an algebraic number; it is a transcendental number. The proof is too difficult to be given here.

The proof that $\sqrt{2}$ is irrational is presented in Chapter 8.

Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 5.

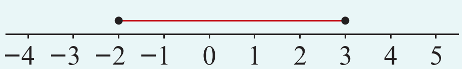
Example 5

Illustrate each of the following intervals of real numbers:

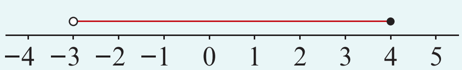
a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 2]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

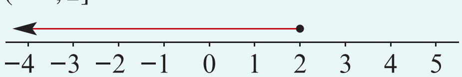
a $[-2, 3]$



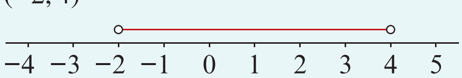
b $(-3, 4]$



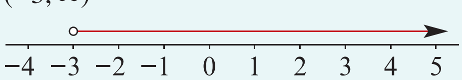
c $(-\infty, 2]$



d $(-2, 4)$



e $(-3, \infty)$



Explanation

The square brackets indicate that the endpoints are included; this is shown with closed circles.

The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as 'negative infinity'. The right endpoint is included.

Both brackets are round; the endpoints are not included.

The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as 'infinity'. The left endpoint is not included.

Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.

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The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Section summary

- Sets of numbers
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Exercise 2B

1 **a** Is the sum of two rational numbers also rational?
b Is the product of two rational numbers also rational?
c Is the quotient of two rational numbers also rational (if defined)?

2 **a** Is the sum of two irrational numbers always irrational?
b Is the product of two irrational numbers always irrational?
c Is the quotient of two irrational numbers always irrational?

Example 4

3 Write each of the following in the form $\dfrac{m}{n}$, where m and n are integers:

a 0.45 **b** $0.\dot{2}\dot{7}$ **c** 0.12
d $0.\dot{2}8571\dot{4}$ **e** $0.\dot{3}\dot{6}$ **f** $0.\dot{2}$

4 Give the decimal representation of each of the following rational numbers:

a $\dfrac{2}{7}$ **b** $\dfrac{5}{11}$ **c** $\dfrac{7}{20}$ **d** $\dfrac{4}{13}$ **e** $\dfrac{1}{17}$

Example 5

5 Illustrate each of the following intervals of real numbers:

a $[-1, 4]$ **b** $(-2, 2]$ **c** $(-\infty, 3]$ **d** $(-1, 5)$ **e** $(-2, \infty)$

6 Write each of the following sets using interval notation:

a $\{ x : x < 3 \}$ **b** $\{ x : x \ge -3 \}$ **c** $\{ x : x \le -3 \}$
d $\{ x : x > 5 \}$ **e** $\{ x : -2 \le x < 3 \}$ **f** $\{ x : -2 \le x \le 3 \}$
g $\{ x : -2 < x \le 3 \}$ **h** $\{ x : -5 < x < 3 \}$

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2C The modulus function

The **modulus** or **absolute value** of a real number x is denoted by $|x|$ and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

It may also be defined as $|x| = \sqrt{x^2}$. For example: $|5| = 5$ and $|-5| = 5$.

Example 6

Evaluate each of the following:

a **i** $|-3 \times 2|$ **ii** $|-3| \times |2|$

b **i** $\left|\dfrac{-4}{2}\right|$ **ii** $\dfrac{|-4|}{|2|}$

c **i** $|-6 + 2|$ **ii** $|-6| + |2|$

Solution

a **i** $|-3 \times 2| = |-6| = 6$ **ii** $|-3| \times |2| = 3 \times 2 = 6$ Note: $|-3 \times 2| = |-3| \times |2|$

b **i** $\left|\dfrac{-4}{2}\right| = |-2| = 2$ **ii** $\dfrac{|-4|}{|2|} = \dfrac{4}{2} = 2$ Note: $\left|\dfrac{-4}{2}\right| = \dfrac{|-4|}{|2|}$

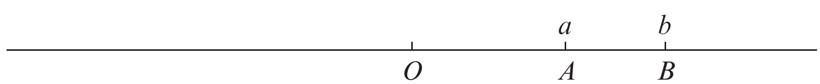
c **i** $|-6 + 2| = |-4| = 4$ **ii** $|-6| + |2| = 6 + 2 = 8$ Note: $|-6 + 2| \neq |-6| + |2|$

Properties of the modulus function

- $|ab| = |a|\,|b|$ and $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
- $|x| = a$ implies $x = a$ or $x = -a$
- $|a + b| \leq |a| + |b|$
- If a and b are both positive or both negative, then $|a + b| = |a| + |b|$.
- If $a \geq 0$, then $|x| \leq a$ is equivalent to $-a \leq x \leq a$.
- If $a \geq 0$, then $|x - k| \leq a$ is equivalent to $k - a \leq x \leq k + a$.

► The modulus function as a measure of distance

Consider two points A and B on a number line:



On a number line, the distance between points A and B is $|a - b| = |b - a|$.

Thus $|x - 2| \leq 3$ can be read as 'on the number line, the distance of x from 2 is less than or equal to 3', and $|x| \leq 3$ can be read as 'on the number line, the distance of x from the origin is less than or equal to 3'. Note that $|x| \leq 3$ is equivalent to $-3 \leq x \leq 3$ or $x \in [-3, 3]$.

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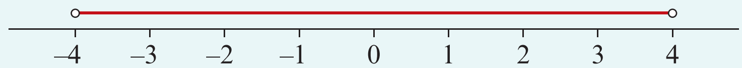
Example 7

Illustrate each of the following sets on a number line and represent the sets using interval notation:

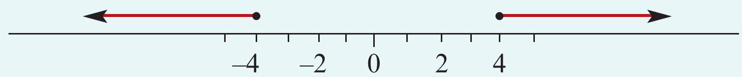
a $\{ x : |x| < 4 \}$ **b** $\{ x : |x| \geq 4 \}$ **c** $\{ x : |x - 1| \leq 4 \}$

Solution

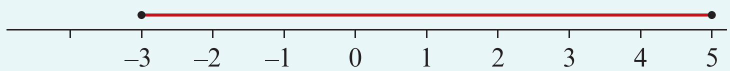
a $(-4, 4)$



b $(-\infty, -4] \cup [4, \infty)$



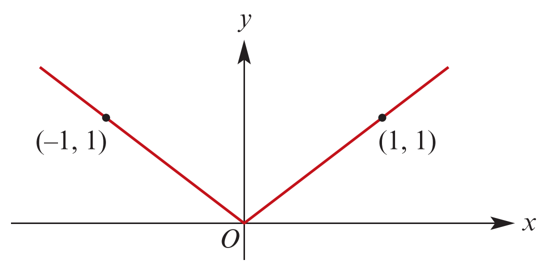
c $[-3, 5]$



The graph of $y = |x|$

The graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ is shown here.

This graph is symmetric about the y-axis, since $|x| = |-x|$.



Example 8

For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 3| + 1$ **b** $f(x) = -|x - 3| + 1$

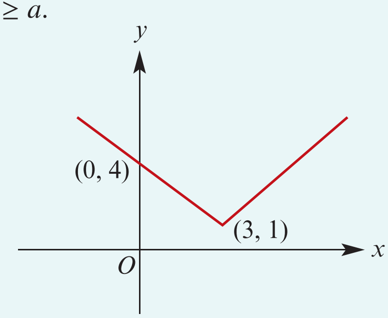
Solution

Note that $|a - b| = a - b$ if $a \geq b$, and $|a - b| = b - a$ if $b \geq a$.

a $$f(x) = |x - 3| + 1 = \begin{cases} x - 3 + 1 & \text{if } x \geq 3 \\ 3 - x + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} x - 2 & \text{if } x \geq 3 \\ 4 - x & \text{if } x < 3 \end{cases}$$

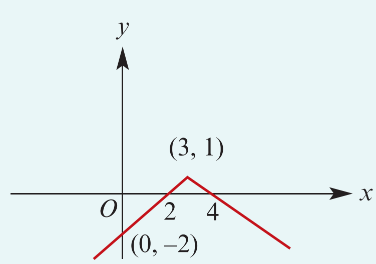
Range = $[1, \infty)$



b $$f(x) = -|x - 3| + 1 = \begin{cases} -(x - 3) + 1 & \text{if } x \geq 3 \\ -(3 - x) + 1 & \text{if } x < 3 \end{cases}$$

$$= \begin{cases} -x + 4 & \text{if } x \geq 3 \\ -2 + x & \text{if } x < 3 \end{cases}$$

Range = $(-\infty, 1]$



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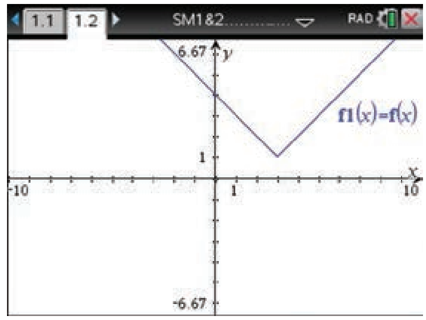
Using the TI-Nspire

- Use menu > **Actions** > **Define** to define the function $f(x) = \text{abs}(x - 3) + 1$.

Note: The absolute value function can be obtained by typing **abs( )** or using the 2D-template palette.

- Open a **Graphs** application (ctrl I > **Graphs**) and let $f1(x) = f(x)$.
- Press enter to obtain the graph.

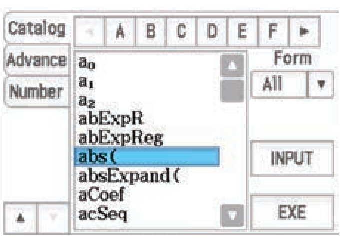


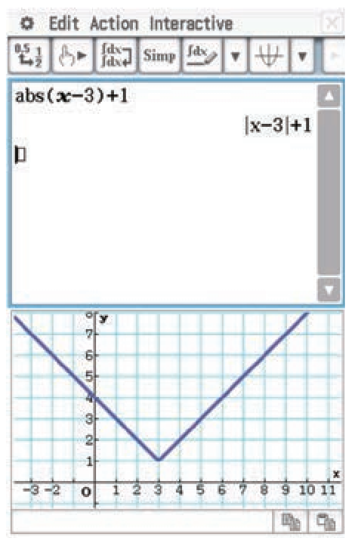
Note: The expression $\text{abs}(x - 3) + 1$ could have been entered directly for $f1(x)$.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $|x - 3| + 1$.

Note: To obtain the absolute value function, either choose **abs(** from the catalog (as shown below) or select from the Math1 keyboard.





- Tap to open the graph window.
- Highlight $|x - 3| + 1$ and drag into the graph window.
- Select **Zoom** > **Initialize** or use to adjust the window manually.

Note: Alternatively, the function can be graphed using the **Graph & Table** application. Enter the expression in $y1$, tick the box, and tap .

Exercise 2C

Skillsheet

Example 6

1 Evaluate each of the following:

a $|-5| + 3$ **b** $|-5| + |-3|$ **c** $|-5| - |-3|$

d $|-5| - |-3| - 4$ **e** $|-5| - |-3| - |-4|$ **f** $|-5| + |-3| - |-4|$

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2 Solve each of the following equations for x:

a $|x - 1| = 2$ **b** $|2x - 3| = 4$ **c** $|5x - 3| = 9$ **d** $|x - 3| - 9 = 0$

e $|3 - x| = 4$ **f** $|3x + 4| = 8$ **g** $|5x + 11| = 9$

Example 7 **3** Illustrate each of the following sets on a number line and represent the sets using interval notation:

a $\{ x : |x| < 3 \}$ **b** $\{ x : |x| \geq 5 \}$ **c** $\{ x : |x - 2| \leq 1 \}$

d $\{ x : |x - 2| < 3 \}$ **e** $\{ x : |x + 3| \geq 5 \}$ **f** $\{ x : |x + 2| \leq 1 \}$

Example 8 **4** For each of the following functions, sketch the graph and state the range:

a $f(x) = |x - 4| + 1$ **b** $f(x) = -|x + 3| + 2$

c $f(x) = |x + 4| - 1$ **d** $f(x) = 2 - |x - 1|$

5 Solve each of the following inequalities, giving your answer using set notation:

a $\{ x : |x| \leq 5 \}$ **b** $\{ x : |x| \geq 2 \}$ **c** $\{ x : |2x - 3| \leq 1 \}$

d $\{ x : |5x - 2| < 3 \}$ **e** $\{ x : |-x + 3| \geq 7 \}$ **f** $\{ x : |-x + 2| \leq 1 \}$

6 Solve each of the following for x:

a $|x - 4| - |x + 2| = 6$ **b** $|2x - 5| - |4 - x| = 10$ **c** $|2x - 1| + |4 - 2x| = 10$

7 If $f(x) = |x - a| + b$ with $f(3) = 3$ and $f(-1) = 3$, find the values of a and b.

8 Prove that $|x - y| \leq |x| + |y|$.

9 Prove that $|x| - |y| \leq |x - y|$.

10 Prove that $|x + y + z| \leq |x| + |y| + |z|$.

2D Surds

A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.

Note: $\sqrt{a}$ is taken to mean the positive square root.

In general, a **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.

For example:

- $\sqrt{7}, \sqrt{24}, \sqrt{\frac{9}{7}}, \sqrt{\frac{1}{2}}$ are quadratic surds
- $\sqrt{9}, \sqrt{16}, \sqrt{\frac{9}{4}}$ are *not* surds
- $\sqrt[3]{7}, \sqrt[3]{15}$ are surds of order 3
- $\sqrt[4]{100}, \sqrt[4]{26}$ are surds of order 4

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Quadratic surds hold a prominent position in school mathematics. For example, the solutions of quadratic equations often involve surds:

$$x = \frac{1+\sqrt{5}}{2} \text{ is a solution of the quadratic equation } x^2 - x - 1 = 0.$$

Some well-known values of trigonometric functions involve surds. For example:

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \qquad \sin 15^\circ = \frac{\sqrt{6}-\sqrt{2}}{4}$$

Exact solutions are often required in Mathematical Methods Units 3 & 4 and Specialist Mathematics Units 3 & 4.

Properties of square roots

The following properties of square roots are often used.

For positive numbers a and b:

- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ e.g. $\sqrt{50} = \sqrt{25} \times \sqrt{2} = 5\sqrt{2}$
- $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$ e.g. $\sqrt{\frac{7}{9}} = \frac{\sqrt{7}}{\sqrt{9}} = \frac{\sqrt{7}}{3}$

Properties of surds

Simplest form

If possible, a factor which is the square of a rational number is 'taken out' of a square root. When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.

Example 9

Write each of the following in simplest form:

a $\sqrt{72}$ **b** $\sqrt{28}$ **c** $\sqrt{\frac{700}{117}}$ **d** $\sqrt{\frac{99}{64}}$

Solution

a $\sqrt{72} = \sqrt{36 \times 2} = 6\sqrt{2}$

b $\sqrt{28} = \sqrt{4 \times 7} = 2\sqrt{7}$

c $$\sqrt{\frac{700}{117}} = \frac{\sqrt{700}}{\sqrt{117}} = \frac{\sqrt{7 \times 100}}{\sqrt{9 \times 13}} = \frac{10}{3}\sqrt{\frac{7}{13}}$$

d $$\sqrt{\frac{99}{64}} = \frac{\sqrt{99}}{\sqrt{64}} = \frac{\sqrt{9 \times 11}}{8} = \frac{3\sqrt{11}}{8}$$

Like surds

Surds which have the same 'irrational factor' are called **like surds**.

For example: $3\sqrt{7}$, $2\sqrt{7}$ and $\sqrt{7}$ are like surds.

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The sum or difference of two like surds can be simplified:

- $m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p}$
- $m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$

Example 10

Express each of the following as a single surd in simplest form:

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

c $\sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}}$

d $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

Solution

a $\sqrt{147} + \sqrt{108} - \sqrt{363}$

$= \sqrt{7^2 \times 3} + \sqrt{6^2 \times 3} - \sqrt{11^2 \times 3}$

$= 7\sqrt{3} + 6\sqrt{3} - 11\sqrt{3}$

$= 2\sqrt{3}$

b $\sqrt{3} + \sqrt{5} + \sqrt{20} + \sqrt{27} - \sqrt{45} - \sqrt{48}$

$= \sqrt{3} + \sqrt{5} + 2\sqrt{5} + 3\sqrt{3} - 3\sqrt{5} - 4\sqrt{3}$

$= 0\sqrt{3} + 0\sqrt{5}$

$= 0$

c $\sqrt{\frac{1}{8}} - \sqrt{\frac{1}{18}} - 5\sqrt{\frac{1}{72}}$

$= \sqrt{\frac{1}{4 \times 2}} - \sqrt{\frac{1}{9 \times 2}} - 5\sqrt{\frac{1}{36 \times 2}}$

$= \frac{1}{2}\sqrt{\frac{1}{2}} - \frac{1}{3}\sqrt{\frac{1}{2}} - \frac{5}{6}\sqrt{\frac{1}{2}}$

$= \frac{3}{6}\sqrt{\frac{1}{2}} - \frac{2}{6}\sqrt{\frac{1}{2}} - \frac{5}{6}\sqrt{\frac{1}{2}}$

$= \frac{-4}{6}\sqrt{\frac{1}{2}}$

$= \frac{-2}{3}\sqrt{\frac{1}{2}}$

d $\sqrt{50} + \sqrt{2} - 2\sqrt{18} + \sqrt{8}$

$= 5\sqrt{2} + \sqrt{2} - 2 \times 3\sqrt{2} + 2\sqrt{2}$

$= 8\sqrt{2} - 6\sqrt{2}$

$= 2\sqrt{2}$

► Rationalising the denominator

In the past, a labour-saving procedure with surds was to **rationalise** any surds in the denominator of an expression. This is still considered to be a neat way of expressing final answers.

For $\sqrt{5}$, a rationalising factor is $\sqrt{5}$, as $\sqrt{5} \times \sqrt{5} = 5$.

For $1 + \sqrt{2}$, a rationalising factor is $1 - \sqrt{2}$, as $(1 + \sqrt{2})(1 - \sqrt{2}) = 1 - 2 = -1$.

For $\sqrt{3} + \sqrt{6}$, a rationalising factor is $\sqrt{3} - \sqrt{6}$, as $(\sqrt{3} + \sqrt{6})(\sqrt{3} - \sqrt{6}) = 3 - 6 = -3$.

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Example 11

Rationalise the denominator of each of the following:

a $\dfrac{1}{2\sqrt{7}}$ **b** $\dfrac{1}{2-\sqrt{3}}$ **c** $\dfrac{1}{\sqrt{3}-\sqrt{6}}$ **d** $\dfrac{3+\sqrt{8}}{3-\sqrt{8}}$

Solution

a $\dfrac{1}{2\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}} = \dfrac{\sqrt{7}}{14}$

b $$\begin{aligned}\frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} &= \frac{2+\sqrt{3}}{4-3}\\ &= 2+\sqrt{3}\end{aligned}$$

c $$\begin{aligned}\frac{1}{\sqrt{3}-\sqrt{6}} \times \frac{\sqrt{3}+\sqrt{6}}{\sqrt{3}+\sqrt{6}} &= \frac{\sqrt{3}+\sqrt{6}}{3-6}\\ &= \frac{-1}{3}(\sqrt{3}+\sqrt{6})\end{aligned}$$

d $$\begin{aligned}\frac{3+\sqrt{8}}{3-\sqrt{8}} &= \frac{3+2\sqrt{2}}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}\\ &= \frac{9+12\sqrt{2}+8}{9-8}\\ &= 17+12\sqrt{2}\end{aligned}$$

Example 12

Expand the brackets in each of the following and collect like terms, expressing surds in simplest form:

a $(3-\sqrt{2})^2$ **b** $(3-\sqrt{2})(1+\sqrt{2})$

Solution

a $$\begin{aligned}&(3-\sqrt{2})^2\\ &= (3-\sqrt{2})(3-\sqrt{2})\\ &= 3(3-\sqrt{2}) - \sqrt{2}(3-\sqrt{2})\\ &= 9-3\sqrt{2}-3\sqrt{2}+2\\ &= 11-6\sqrt{2}\end{aligned}$$

b $$\begin{aligned}&(3-\sqrt{2})(1+\sqrt{2})\\ &= 3(1+\sqrt{2}) - \sqrt{2}(1+\sqrt{2})\\ &= 3+3\sqrt{2}-\sqrt{2}-2\\ &= 1+2\sqrt{2}\end{aligned}$$

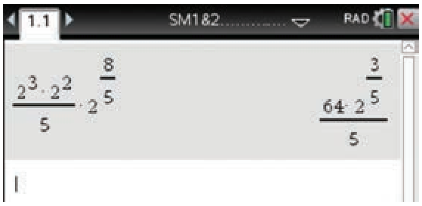
Using the TI-Nspire

A CAS calculator can be used to work with irrational numbers.

Expressions on the screen can be selected using the up arrow ▲. This returns the expression to the entry line and modifications can be made.

For example:

- Evaluate $\dfrac{2^3 \cdot 2^2}{5} \cdot 2^{\frac{8}{5}}$ as shown.



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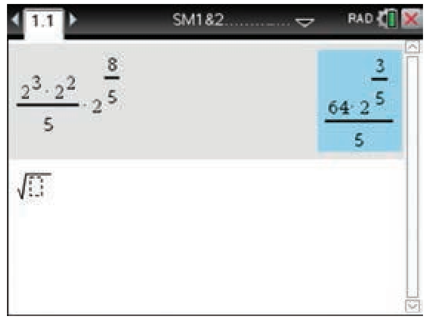
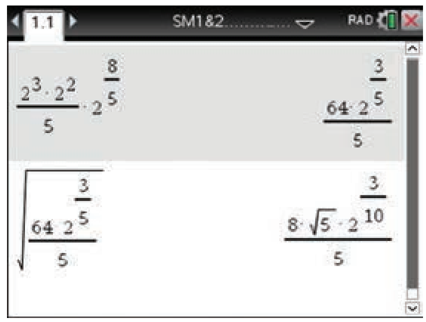
- To find the square root of this expression, first type (ctrl)(x^2). Then move upwards by pressing the up arrow ▲, so that the expression is highlighted.



- Press (enter) to paste this expression into the square root sign.
- Press (enter) once more to evaluate the square root of this expression.

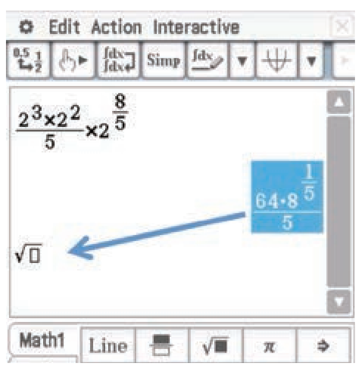


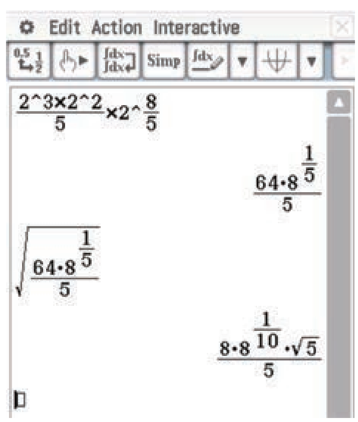
Using the Casio ClassPad

Expressions on the screen can be selected using the stylus. Highlight and drag the expression to the next entry line, where modifications can be made.

For example:

- Evaluate $\dfrac{2^3 \cdot 2^2}{5} \cdot 2^{\frac{8}{5}}$ as shown.
- In the next entry line, tap [√■] from the (Math1) keyboard.
- Highlight the expression and drag to the square root sign.
- Tap (EXE) to evaluate.
- Alternatively, highlight the expression and select **Edit** > **Copy**. Then tap the cursor in the desired position and select **Edit** > **Paste**.





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Exercise 2D

Skillsheet

Example 9

1 Express each of the following in terms of the simplest possible surds:

a $\sqrt{8}$ **b** $\sqrt{12}$ **c** $\sqrt{27}$ **d** $\sqrt{50}$
e $\sqrt{45}$ **f** $\sqrt{1210}$ **g** $\sqrt{98}$ **h** $\sqrt{108}$
i $\sqrt{25}$ **j** $\sqrt{75}$ **k** $\sqrt{512}$

Example 10

2 Simplify each of the following:

a $\sqrt{8} + \sqrt{18} - 2\sqrt{2}$ **b** $\sqrt{75} + 2\sqrt{12} - \sqrt{27}$
c $\sqrt{28} + \sqrt{175} - \sqrt{63}$ **d** $\sqrt{1000} - \sqrt{40} - \sqrt{90}$
e $\sqrt{512} + \sqrt{128} + \sqrt{32}$ **f** $\sqrt{24} - 3\sqrt{6} - \sqrt{216} + \sqrt{294}$

3 Simplify each of the following:

a $\sqrt{75} + \sqrt{108} + \sqrt{14}$ **b** $\sqrt{847} - \sqrt{567} + \sqrt{63}$
c $\sqrt{720} - \sqrt{245} - \sqrt{125}$ **d** $\sqrt{338} - \sqrt{288} + \sqrt{363} - \sqrt{300}$
e $\sqrt{12} + \sqrt{8} + \sqrt{18} + \sqrt{27} + \sqrt{300}$ **f** $2\sqrt{18} + 3\sqrt{5} - \sqrt{50} + \sqrt{20} - \sqrt{80}$

Example 11

4 Express each of the following with rational denominators:

a $\dfrac{1}{\sqrt{5}}$ **b** $\dfrac{1}{\sqrt{7}}$ **c** $-\dfrac{1}{\sqrt{2}}$ **d** $\dfrac{2}{\sqrt{3}}$ **e** $\dfrac{3}{\sqrt{6}}$

f $\dfrac{1}{2\sqrt{2}}$ **g** $\dfrac{1}{\sqrt{2}+1}$ **h** $\dfrac{1}{2-\sqrt{3}}$ **i** $\dfrac{1}{4-\sqrt{10}}$ **j** $\dfrac{2}{\sqrt{6}+2}$

k $\dfrac{1}{\sqrt{5}-\sqrt{3}}$ **l** $\dfrac{3}{\sqrt{6}-\sqrt{5}}$ **m** $\dfrac{1}{3-2\sqrt{2}}$

Example 12

5 Express each of the following in the form $a + b\sqrt{c}$:

a $\dfrac{2}{3-2\sqrt{2}}$ **b** $(\sqrt{5}+2)^2$ **c** $(1+\sqrt{2})(3-2\sqrt{2})$ **d** $(\sqrt{3}-1)^2$

e $\sqrt{\dfrac{1}{3}} - \dfrac{1}{\sqrt{27}}$ **f** $\dfrac{\sqrt{3}+2}{2\sqrt{3}-1}$ **g** $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ **h** $\dfrac{\sqrt{8}+3}{\sqrt{18}+2}$

6 Expand and simplify each of the following:

a $(2\sqrt{a}-1)^2$ **b** $(\sqrt{x+1}+\sqrt{x+2})^2$

7 For real numbers a and b, we have $a > b$ if and only if $a - b > 0$. Use this to state the larger of:

a $5 - 3\sqrt{2}$ and $6\sqrt{2} - 8$ **b** $2\sqrt{6} - 3$ and $7 - 2\sqrt{6}$

8 For positive real numbers a and b, we have $a > b$ if and only if $a^2 - b^2 > 0$. Use this to state the larger of:

a $\dfrac{2}{\sqrt{3}}$ and $\dfrac{3}{\sqrt{2}}$ **b** $\dfrac{\sqrt{7}}{3}$ and $\dfrac{\sqrt{5}}{2}$ **c** $\dfrac{\sqrt{3}}{7}$ and $\dfrac{\sqrt{5}}{5}$ **d** $\dfrac{\sqrt{10}}{2}$ and $\dfrac{8}{\sqrt{3}}$

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9 Find the values of b and c for a quadratic function $f(x) = x^2 + bx + c$ such that the solutions of the equation $f(x) = 0$ are:

a $\sqrt{3},\ -\sqrt{3}$ **b** $2\sqrt{3},\ -2\sqrt{3}$ **c** $1 - \sqrt{2},\ 1 + \sqrt{2}$

d $2 - \sqrt{3},\ 2 + \sqrt{3}$ **e** $3 - 2\sqrt{2},\ 3 + 2\sqrt{2}$ **f** $4 - 7\sqrt{5},\ 3 + 2\sqrt{5}$

10 Express $\dfrac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$ with a rational denominator.

11 **a** Show that $a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$.

b Express $\dfrac{1}{1 - 2^{\frac{1}{3}}}$ with a rational denominator.

2E Natural numbers

▶ Factors and composites

The factors of 8 are 1, 2, 4 and 8.

The factors of 24 are 1, 2, 3, 4, 6, 8, 12 and 24.

The factors of 5 are 1 and 5.

A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.

A natural number greater than 1 is called a **prime number** if its only factors are itself and 1.

The prime numbers less than 100 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97.

A natural number m is called a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

▶ Prime decomposition

Expressing a composite number as a product of powers of prime numbers is called **prime decomposition**. For example:

$$3000 = 3 \times 2^3 \times 5^3$$

$$2294 = 2 \times 31 \times 37$$

This is useful for finding the factors of a number. For example, the prime decomposition of 12 is given by $12 = 2^2 \times 3$. The factors of 12 are

$$1, \quad 2, \quad 2^2 = 4, \quad 3, \quad 2 \times 3 = 6 \quad \text{and} \quad 2^2 \times 3 = 12$$

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This property of natural numbers is described formally by the following theorem.

Fundamental theorem of arithmetic

Every natural number greater than 1 either is a prime number or can be represented as a product of prime numbers. Furthermore, this representation is unique apart from rearrangement of the order of the prime factors.

Example 13

Give the prime decomposition of 17 248 and hence list the factors of this number.

Solution

The prime decomposition can be found by repeated division, as shown on the right.

2	17248
2	8624
2	4312
2	2156
2	1078
7	539
7	77
11	11
	1

The prime decomposition of 17 248 is

$$17\,248 = 2^5 \times 7^2 \times 11$$

Therefore each factor must be of the form

$$2^\alpha \times 7^\beta \times 11^\gamma$$

where $\alpha = 0, 1, 2, 3, 4, 5$, $\beta = 0, 1, 2$ and $\gamma = 0, 1$.

The factors of 17 248 can be systematically listed as follows:

1	2	2^2	2^3	2^4	2^5
7	2×7	$2^2 \times 7$	$2^3 \times 7$	$2^4 \times 7$	$2^5 \times 7$
7^2	2×7^2	$2^2 \times 7^2$	$2^3 \times 7^2$	$2^4 \times 7^2$	$2^5 \times 7^2$
11	2×11	$2^2 \times 11$	$2^3 \times 11$	$2^4 \times 11$	$2^5 \times 11$
7×11	$2 \times 7 \times 11$	$2^2 \times 7 \times 11$	$2^3 \times 7 \times 11$	$2^4 \times 7 \times 11$	$2^5 \times 7 \times 11$
$7^2 \times 11$	$2 \times 7^2 \times 11$	$2^2 \times 7^2 \times 11$	$2^3 \times 7^2 \times 11$	$2^4 \times 7^2 \times 11$	$2^5 \times 7^2 \times 11$

▶ Highest common factor

The **highest common factor** of two natural numbers a and b is the largest natural number that is a factor of both a and b. It is denoted by HCF(a, b).

For example, the highest common factor of 15 and 24 is 3. We write HCF$(15, 24) = 3$.

Note: The highest common factor is also called the **greatest common divisor**.

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Using prime decomposition to find HCF

Prime decomposition can be used to find the highest common factor of two numbers.

For example, consider the numbers 140 and 110. Their prime factorisations are

$$140 = 2^2 \times 5 \times 7 \quad \text{and} \quad 110 = 2 \times 5 \times 11$$

A number which is a factor of both 140 and 110 must have prime factors which occur in both these factorisations. The highest common factor of 140 and 110 is $2 \times 5 = 10$.

Next consider the numbers

$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the highest common factor, we take the lower power of each prime factor:

$$\text{HCF}(396\,000, 1\,960\,200) = 2^3 \cdot 3^2 \cdot 5^2 \cdot 11$$

Example 14

a Find the highest common factor of 528 and 3168.

b Find the highest common factor of 3696 and 3744.

Solution

a

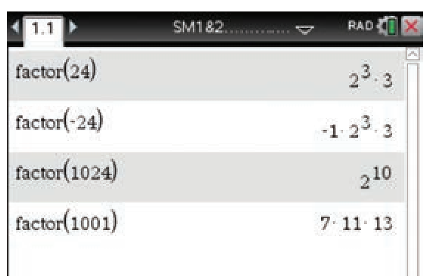
$$528 = 2^4 \times 3 \times 11$$
$$3168 = 2^5 \times 3^2 \times 11$$
$$\therefore \text{HCF}(528, 3168) = 2^4 \times 3 \times 11 = 528$$

b

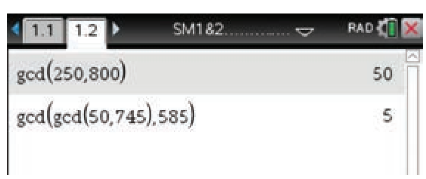
$$3696 = 2^4 \times 3 \times 7 \times 11$$
$$3744 = 2^5 \times 3^2 \times 13$$
$$\therefore \text{HCF}(3696, 3744) = 2^4 \times 3 = 48$$

Using the TI-Nspire

- The prime decomposition of a natural number can be obtained using (menu) > **Algebra** > **Factor** as shown.



- The highest common factor of two numbers (also called their *greatest common divisor*) can be found by using the command **gcd( )** from (menu) > **Number** > **Greatest Common Divisor**, or by just typing it in, as shown.



Note: Nested **gcd( )** commands may be used to find the greatest common divisor of several numbers.

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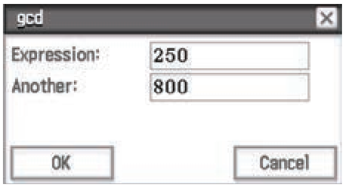
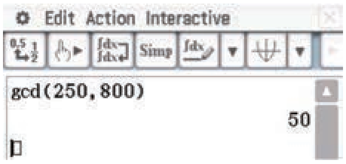
Using the Casio ClassPad

- To find the highest common factor of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **gcd**.
- Enter the required numbers in the two lines provided, and tap OK.





Lowest common multiple

A natural number a is a **multiple** of a natural number b if there exists a natural number k such that $a = kb$.

The **lowest common multiple** of two natural numbers a and b is the smallest natural number that is a multiple of both a and b. It is denoted by $\mathrm{LCM}(a, b)$.

For example: $\mathrm{LCM}(24, 36) = 72$ and $\mathrm{LCM}(256, 100) = 6400$.

Using prime decomposition to find LCM

Consider again the numbers

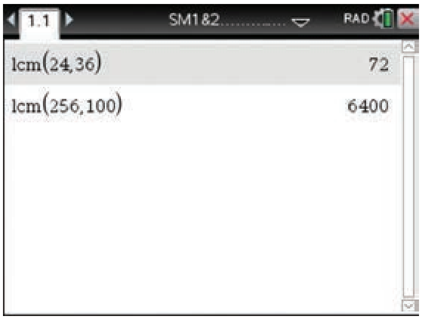
$$396\,000 = 2^5 \cdot 3^2 \cdot 5^3 \cdot 11 \quad \text{and} \quad 1\,960\,200 = 2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$$

To obtain the lowest common multiple, we take the higher power of each prime factor:

$$\mathrm{LCM}(396\,000, 1\,960\,200) = 2^5 \cdot 3^4 \cdot 5^3 \cdot 11^2 = 39\,204\,000$$

Using the TI-Nspire

The lowest common multiple of two numbers (also called their *least common multiple*) can be found by using the command **lcm( )** from menu > **Number** > **Least Common Multiple**, or by just typing it in, as shown.



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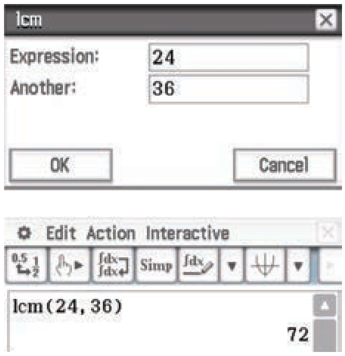
Using the Casio ClassPad

- To find the lowest common multiple of two numbers, go to **Interactive** > **Calculation** > **gcd/lcm** > **lcm**.
- Enter the required numbers in the two lines provided, and tap OK.



Section summary

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.
- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by LCM(a, b), is the smallest natural number that is a multiple of both a and b.

Exercise 2E

Example 13

1 Give the prime decomposition of each of the following numbers:

a 60 **b** 676 **c** 228 **d** 900 **e** 252
f 6300 **g** 68 640 **h** 96 096 **i** 32 032 **j** 544 544

Example 14

2 Find the highest common factor of each of the following pairs of numbers:

a 4361, 9281 **b** 999, 2160 **c** 5255, 716 845 **d** 1271, 3875 **e** 804, 2358

3 **a** List all the factors of 18 and all the factors of 36.
b Why does 18 have an even number of factors and 36 an odd number of factors?
c Find the smallest number greater than 100 with exactly three factors.

4 A woman has three children and two of them are teenagers, aged between 13 and 19. The product of their three ages is 1050. How old is each child?

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5 By using prime decomposition, find a natural number n such that $22^2 \times 55^2 = 10^2 \times n^2$.

6 The natural number n has exactly eight different factors. Two of these factors are 15 and 21. What is the value of n?

7 Let n be the smallest of three natural numbers whose product is 720. What is the largest possible value of n?

8 When all eight factors of 30 are multiplied together, the product is 30^k. What is the value of k?

9 A bell rings every 36 minutes and a buzzer rings every 42 minutes. If they sound together at 9 a.m., when will they next sound together?

10 The LCM of two numbers is $2^5 \times 3^3 \times 5^3$ and the HCF is $2^3 \times 3 \times 5^2$. Find all the possible numbers.

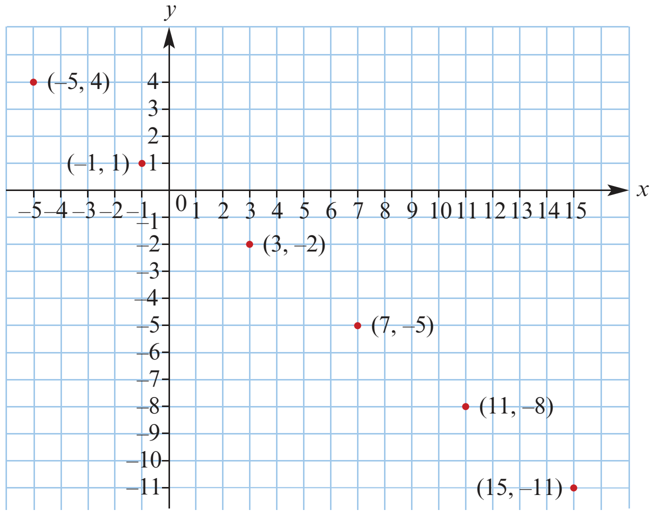
2F Linear Diophantine equations

A **Diophantine equation** is an equation in which only integer solutions are allowed.

The equation $x^2 + y^2 = z^2$ can be considered as a Diophantine equation. There are infinitely many integer solutions. For example: $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^2$. Of course, these solutions correspond to right-angled triangles with these side lengths.

An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.

For example, consider the equation $3x + 4y = 1$. This equation defines a straight line. A family of integer solutions to this equation is illustrated on the following graph.



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From the graph, we can see that as the integer solutions for x increase by 4, the corresponding integer solutions for y decrease by 3.

The solutions may be built up in the following way using $(-1, 1)$ as the starting point.

x	y
$-1 + 1 \times 4$	$1 - 1 \times 3$
$-1 + 2 \times 4$	$1 - 2 \times 3$
$-1 + 3 \times 4$	$1 - 3 \times 3$

i.e.

x	y
3	-2
7	-5
11	-8

The family of solutions may be described as

$$x = -1 + 4t, \quad y = 1 - 3t \qquad \text{for } t \in \mathbb{Z}$$

The solution set is

$$\{(x, y) : x = -1 + 4t,\ y = 1 - 3t,\ t \in \mathbb{Z}\}$$

If a linear Diophantine equation has one solution, then it has infinitely many:

Theorem

If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by

$$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$

where d is the highest common factor of a and b.

Proof Assume that (x_1, y_1) is another solution to the equation. Then

$$ax_1 + by_1 = c \qquad (1)$$
$$ax_0 + by_0 = c \qquad (2)$$

Subtract (2) from (1):

$$a(x_1 - x_0) = b(y_0 - y_1) \qquad (3)$$

Divide both sides by d:

$$\frac{a}{d}(x_1 - x_0) = \frac{b}{d}(y_0 - y_1)$$

The integers $\frac{a}{d}$ and $\frac{b}{d}$ have no common factor. Hence $x_1 - x_0$ must be divisible by $\frac{b}{d}$ and so

$$x_1 - x_0 = \frac{b}{d}t, \quad \text{for some } t \in \mathbb{Z}$$

Therefore $x_1 = x_0 + \frac{b}{d}t$ and from (3) it follows that $y_1 = y_0 - \frac{a}{d}t$.

It can be checked by substitution that $x = x_0 + \frac{b}{d}t$ and $y = y_0 - \frac{a}{d}t$ is a solution of the equation for any $t \in \mathbb{Z}$.

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Example 15

A man has \$200 in his wallet, made up of \$50 and \$20 notes. What are the possible numbers of each of these types of notes?

Solution

Let x and y be the numbers of \$50 and \$20 notes respectively.

The linear Diophantine equation is

$$50x + 20y = 200$$
$$\therefore \quad 5x + 2y = 20$$

By inspection, a solution is

$$x = 4, \; y = 0$$

Therefore the general solution is

$$x = 4 + 2t, \; y = 0 - 5t, \quad \text{for } t \in \mathbb{Z}$$

We are only interested in the case where $x \geq 0$ and $y \geq 0$, that is, $4 + 2t \geq 0$ and $-5t \geq 0$.

Hence $-2 \leq t \leq 0$.

For $t = -2$: $\quad x = 0, \quad y = 10$

For $t = -1$: $\quad x = 2, \quad y = 5$

For $t = 0$: $\quad x = 4, \quad y = 0$

The man could have ten \$20 notes, or two \$50 notes and five \$20 notes, or four \$50 notes.

Theorem

A linear Diophantine equation $ax + by = c$ has integer solutions if and only if HCF(a, b) divides c.

This means, for example, that there are no integer solutions to $2x + 6y = 3$. In the next section we will see a method for finding a solution to $ax + by = c$ when HCF(a, b) divides c.

Section summary

- A **Diophantine equation** is an equation in which only integer solutions are allowed.
- An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.
- If a linear Diophantine equation has one solution, then it has infinitely many:
  If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by
  $$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$
  where d is the highest common factor of a and b.

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Exercise 2F

1 Find all solutions to each of the following Diophantine equations:

a $11x + 3y = 1$ **b** $2x + 7y = 2$ **c** $24x + 63y = 99$

d $22x + 6y = 2$ **e** $2x + 7y = 22$ **f** $10x + 35y = 110$

2 For **e** in Question 1, find the solution(s) with both x and y positive.

3 Prove that, if $ax + by = c$ and the highest common factor of a and b does not divide c, then there is no solution to the Diophantine equation.

Example 15

4 A student puts a number of spiders (with eight legs each) and a number of beetles (with six legs each) in a box. She counted 54 legs in all.

a Form a Diophantine equation.

b Find the number of spiders and the number of beetles in the box.

5 Helena has a number of coins in her purse. They are all either 20c or 50c coins. The total value of the coins is $5. What are the possible numbers of each type of coin?

6 One of the solutions of the equation $19x + 83y = 1983$ in positive integers x and y is obviously $x = 100$, $y = 1$. Show that there is only one other pair of positive integers which satisfies this equation and find it. Consider the equation $19x + 98y = 1998$.

7 A man has $500 in his wallet, made up of $50 and $10 notes. Find the possible combinations of notes that he could have.

8 There are seven coconuts and 63 heaps of pineapples, where each heap has exactly the same number of pineapples. The fruit is to be divided equally between 23 people. Let x be the number of pineapples in each heap and let y be the number of pieces of fruit that each person receives. Form a Diophantine equation and find the possible values for x and y.

9 A dealer spent $10 000 buying cattle, some at $410 each and the rest at $530 each. How many of each sort did she buy?

10 Find the smallest positive number which when divided by 7 leaves a remainder of 6, and when divided by 11 leaves a remainder of 9. Also find the general form of such numbers.

11 Given a 3 litre jug and a 5 litre jug, can I measure exactly 7 litres of water? If it is possible, explain how this may be done as efficiently as possible.

12 The Guadeloupe Post Office has only 3c and 5c stamps. What amounts of postage can the post office sell?

13 A man spent $29.60 buying party hats. There were two types of party hat: type A cost $1.70, while type B cost $1.00. How many of each type did he buy?

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14 Why has the equation $6x - 9y = 10$ no integer solutions?

15 Find the smallest multiple of 13 which when divided by 18 leaves 5 as a remainder.

16 A two-digit number exceeds five times the sum of its digits by 17. Find two such numbers.

2G The Euclidean algorithm

The Euclidean algorithm provides a method for finding the highest common factor of two numbers and also a method for solving linear Diophantine equations.

From your earlier work on arithmetic, you will know that dividing one natural number by another gives a result such as:

- $65 \div 7$ gives $65 = 7 \times 9 + 2$
- $92 \div 3$ gives $92 = 3 \times 30 + 2$

We can formalise this observation as follows.

Division algorithm

If a and b are integers with $a > 0$, then there are unique integers q and r such that

$$b = aq + r \quad \text{and} \quad 0 \le r < a$$

Proof Suppose there is another pair of integers q_1 and r_1 with $b = aq_1 + r_1$ and $0 \le r_1 < a$.

Then $\quad aq_1 + r_1 = aq + r$

$\therefore \quad a(q_1 - q) = r - r_1 \qquad (1)$

First suppose that $r > r_1$. The left-hand side of equation (1) is an integer which is a multiple of a. Therefore a divides the right-hand side. But the right-hand side is an integer which is greater than 0 and less than a. This is a contradiction, and so the assumption that $r > r_1$ must be false.

Next suppose that $r < r_1$. Multiplying both sides of equation (1) by -1, we obtain

$$a(q - q_1) = r_1 - r$$

and a similar contradiction will arise.

Thus we have shown that $r = r_1$, and so equation (1) gives $a(q_1 - q) = 0$. Hence $q_1 = q$ and $r_1 = r$, and the uniqueness of the integers q and r has been proved.

Example 16

Express -45 in the form $6q + r$, where $0 \le r < 6$.

Solution

$-45 = 6(-8) + 3$

Note: The answer $-45 = 6(-7) - 3$ is not correct, since the remainder -3 is less than zero.

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The following theorem is useful for determining the highest common factor of any two given integers. We use HCF(a, b) to denote the highest common factor of two integers a and b.

Theorem

Let a and b be two integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then HCF(a, b) = HCF(a, r).

Proof If d is a common factor of a and r, then d divides the right-hand side of the equation $b = aq + r$, and so d divides b.

This proves that all common factors of a and r are also common factors of a and b.

But HCF(a, r) is a common factor of a and r, and therefore HCF(a, r) must divide a and b. It follows that HCF(a, r) must divide HCF(a, b). That is,

$$\text{HCF}(a, b) = m \cdot \text{HCF}(a, r) \qquad \text{for some integer } m \qquad (1)$$

Now rewrite the equation $b = aq + r$ as $r = b - aq$.

If d is a common factor of a and b, then d divides the right-hand side of the equation $r = b - aq$, and so d divides r.

This proves that all common factors of a and b are also common factors of a and r. It follows that HCF(a, b) must divide HCF(a, r). That is,

$$\text{HCF}(a, r) = n \cdot \text{HCF}(a, b) \qquad \text{for some integer } n \qquad (2)$$

From equations (1) and (2), we obtain

$$\text{HCF}(a, r) = mn \cdot \text{HCF}(a, r)$$

$$\therefore \qquad 1 = mn$$

This equation in integers m, n is possible only if $m = n = 1$ or $m = n = -1$.

Hence HCF(a, b) = HCF(a, r), since both must be positive.

Example 17

Find HCF(1271, 3875).

Solution

At each step, we use the division algorithm and the previous theorem:

$$\underline{3875} = 3 \times \underline{1271} + \underline{62} \qquad \text{and so} \qquad \text{HCF}(1271, 3875) = \text{HCF}(1271, 62)$$

$$\underline{1271} = 20 \times \underline{62} + \underline{31} \qquad \text{and so} \qquad \text{HCF}(62, 1271) = \text{HCF}(62, 31)$$

$$62 = 2 \times 31 + 0 \qquad \text{and so} \qquad \text{HCF}(31, 62) = \text{HCF}(31, 0) = 31$$

Hence it follows that HCF(1271, 3875) = 31.

Note: Keep using the division algorithm until you get remainder zero. Then the HCF is the last non-zero remainder.

This procedure is called the **Euclidean algorithm**.

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▶ Method for finding a solution of a linear Diophantine equation

The method presented here uses the Euclidean algorithm.

Example 18

Find $a, b \in \mathbb{Z}$ such that $22a + 6b = 2$.

Solution

Apply the Euclidean algorithm to 22 and 6:

$$\underline{22} = 3 \times \underline{6} + \underline{4} \qquad (1)$$
$$\underline{6} = 1 \times \underline{4} + \underline{2} \qquad (2)$$
$$4 = 2 \times 2 + 0 \qquad (3)$$

Hence $\text{HCF}(22, 6) = 2$.

Now use these equations backwards:

$$\begin{aligned} \underline{2} &= \underline{6} - 1 \times \underline{4} && \text{from (2)} \\ &= \underline{6} - 1 \times (\underline{22} - 3 \times \underline{6}) && \text{from (1)} \\ &= \underline{6} - 1 \times \underline{22} + 3 \times \underline{6} \\ &= 4 \times \underline{6} - 1 \times \underline{22} \end{aligned}$$

Therefore

$$-1 \times 22 + 4 \times 6 = 2$$

and so one solution is $a = -1$ and $b = 4$.

The general solution is $a = -1 + 3t$ and $b = 4 - 11t$, where $t \in \mathbb{Z}$.

Example 19

Find $a, b \in \mathbb{Z}$ such that $125a + 90b = 5$.

Solution

First divide by 5:

$$25a + 18b = 1$$

Apply the Euclidean algorithm to 25 and 18:

$$\underline{25} = 1 \times \underline{18} + \underline{7} \qquad (1)$$
$$\underline{18} = 2 \times \underline{7} + \underline{4} \qquad (2)$$
$$\underline{7} = 1 \times \underline{4} + \underline{3} \qquad (3)$$
$$\underline{4} = 1 \times \underline{3} + \underline{1} \qquad (4)$$
$$3 = 3 \times 1 + 0 \qquad (5)$$

Hence $\text{HCF}(25, 18) = 1$.

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Now use the equations backwards:

$$\begin{aligned}
\underline{1} &= \underline{4} - 1 \times \underline{3} && \text{from (4)}\\
&= \underline{4} - 1 \times (\underline{7} - 1 \times \underline{4}) && \text{from (3)}\\
&= 2 \times \underline{4} - 1 \times \underline{7} &&\\
&= 2 \times (\underline{18} - 2 \times \underline{7}) - 1 \times \underline{7} && \text{from (2)}\\
&= 2 \times \underline{18} - 5 \times \underline{7} &&\\
&= 2 \times \underline{18} - 5 \times (\underline{25} - 1 \times \underline{18}) && \text{from (1)}\\
&= 7 \times \underline{18} - 5 \times \underline{25} &&
\end{aligned}$$

Therefore

$$-5 \times 25 + 7 \times 18 = 1$$

and so one solution is $a = -5$ and $b = 7$.

The general solution is $a = -5 + 18t$ and $b = 7 - 25t$, where $t \in \mathbb{Z}$.

Section summary

- **Division algorithm**
  If a and b are integers with $a > 0$, then there are unique integers q and r such that $b = aq + r$ and $0 \le r < a$.
- **Euclidean algorithm**
  Let a and b be integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then $\text{HCF}(a, b) = \text{HCF}(a, r)$.
  The repeated application of this result can be used to find the highest common factor of two natural numbers and to solve linear Diophantine equations.

Exercise 2G

Example 16

1 For each of the following, express b in the form $b = aq + r$ with $0 \le r < a$, and show that $\text{HCF}(a, b) = \text{HCF}(a, r)$:

a $a = 5,\ b = 43$
b $a = 13,\ b = 39$
c $a = 17,\ b = 37$
d $a = 16,\ b = 128$

2 If d is a common factor of a and b, prove that d is also a common factor of $a + b$ and $a - b$.

Example 17

3 Use the Euclidean algorithm to find:

a $\text{HCF}(4361, 9284)$
b $\text{HCF}(999, 2160)$
c $\text{HCF}(-372, 762)$
d $\text{HCF}(5255, 716\,485)$

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Example 18, 19

4 Solve each equation in the integers:

a $804x + 2358y = 6$ **b** $18x + 24y = 6$ **c** $3x + 4y = 478$

d $3x - 5y = 38$ **e** $804x + 2688y = 12$ **f** $1816x + 2688y = 8$

2H Problems involving sets

Sets can be used to help sort information, as each of the following examples demonstrates. Recall that, if A is a finite set, then the number of elements in A is denoted by $|A|$.

Example 20

Two hundred and eighty students each travel to school by either train or tram or both. Of these students, 150 travel by train, and 60 travel by both train and tram.

a Show this information on a Venn diagram.

b Hence find the number of students who travel by:

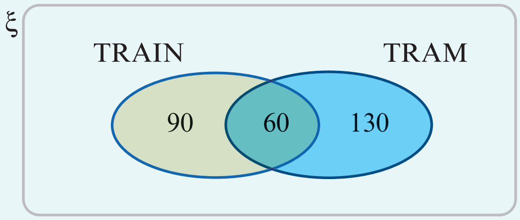
i tram

ii train but not tram

iii just one of these modes of transport.

Solution

a



$|\xi| = 280$

b **i** $|\text{TRAM}| = 130 + 60 = 190$

ii $|\text{TRAIN} \cap (\text{TRAM})'| = 90$

iii $|\text{TRAIN} \cap (\text{TRAM})'| + |(\text{TRAIN})' \cap \text{TRAM}| = 90 + 130 = 220$

Example 21

An athletics team has 18 members. Each member competes in at least one of three events: sprints (S), jumps (J) and hurdles (H). Every hurdler also jumps or sprints. The following additional information is available:

$$|S| = 11, \quad |J| = 10, \quad |J \cap H' \cap S'| = 5, \quad |J' \cap H' \cap S| = 5 \quad \text{and} \quad |J \cap H'| = 7$$

a Draw a Venn diagram.

b Find:

i $|H|$ **ii** $|S \cap H \cap J|$ **iii** $|S \cup J|$ **iv** $|S \cap J \cap H'|$

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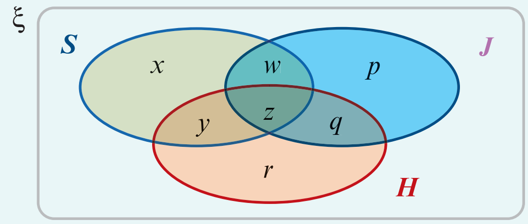
Solution

a Assign a variable to the number of members in each region of the Venn diagram.



The information in the question can be summarised in terms of these variables:

$$x + y + z + w = 11 \quad \text{as } |S| = 11 \quad (1)$$
$$p + q + z + w = 10 \quad \text{as } |J| = 10 \quad (2)$$
$$x + y + z + w + p + q + r = 18 \quad \text{as all members compete} \quad (3)$$
$$p = 5 \quad \text{as } |J \cap H' \cap S'| = 5 \quad (4)$$
$$x = 5 \quad \text{as } |J' \cap H' \cap S| = 5 \quad (5)$$
$$r = 0 \quad \text{as every hurdler also jumps or sprints} \quad (6)$$
$$w + p = 7 \quad \text{as } |J \cap H'| = 7 \quad (7)$$

From (4) and (7): $w = 2$.
Equation (3) now becomes

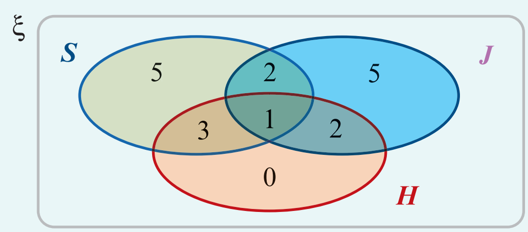
$$5 + y + z + 2 + 5 + q = 18$$
$$\therefore \quad y + z + q = 6 \quad (8)$$

Equation (1) becomes

$$y + z = 4$$

Therefore from (8): $q = 2$.
Equation (2) becomes

$$5 + 2 + z + 2 = 10$$
$$\therefore \quad z = 1$$
$$\therefore \quad y = 3$$



The Venn diagram can now be completed as shown.

b **i** $|H| = 6$ **ii** $|S \cap H \cap J| = 1$ **iii** $|S \cup J| = 18$ **iv** $|S \cap J \cap H'| = 2$

Exercise 2H

Skillsheet

Example 20

1 There are 28 students in a class, all of whom take either History or Economics or both. Of the 14 students who take History, five also take Economics.

a Show this information on a Venn diagram.

b Hence find the number of students who take:

i Economics **ii** History but not Economics **iii** just one of these subjects.

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2 **a** Draw a Venn diagram to show three sets A, B and C in a universal set ξ. Enter numbers in the correct parts of the diagram using the following information:

$$|A \cap B \cap C| = 2, \quad |A \cap B| = 7, \quad |B \cap C| = 6,$$
$$|A \cap C| = 8, \quad |A| = 16, \quad |B| = 20, \quad |C| = 19, \quad |\xi| = 50$$

b Use the diagram to find:

i $|A' \cap C'|$ **ii** $|A \cup B'|$ **iii** $|A' \cap B \cap C'|$

3 In a border town in the Balkans, 60% of people speak Bulgarian, 40% speak Greek and 20% speak neither. What percentage of the town speak both Bulgarian and Greek?

4 A survey of a class of 40 students showed that 16 own at least one dog and 25 at least one cat. Six students have neither. How many students own both?

5 At an international conference there were 105 delegates. Seventy spoke English, 50 spoke French and 50 spoke Japanese. Twenty-five spoke English and French, 15 spoke French and Japanese and 30 spoke Japanese and English.

a How many delegates spoke all three languages?

b How many spoke Japanese only?

6 A restaurant serves lunch to 350 people. It offers three desserts: profiteroles, gelati and fruit. Forty people have all three desserts, 70 have gelati only, 50 have profiteroles only and 60 have fruit only. Forty-five people have fruit and gelati only, 30 people have gelati and profiteroles only and 10 people have fruit and profiteroles only. How many people do not have a dessert?

Example 21

7 Forty travellers were questioned about the various methods of transport they had used the previous day. Every traveller used at least one of the following methods: car (C), bus (B), train (T). Of these travellers:

- eight had used all three methods of transport
- four had travelled by bus and car only
- two had travelled by car and train only
- the number (x) who had travelled by train only was equal to the number who had travelled by bus and train only.

If 20 travellers had used a train and 33 had used a bus, find:

a the value of x

b the number who travelled by bus only

c the number who travelled by car only.

8 Let ξ be the set of all integers and let

$$X = \{ x : 21 < x < 37 \}, \quad Y = \{ 3y : 0 < y \le 13 \}, \quad Z = \{ z^2 : 0 < z < 8 \}$$

a Draw a Venn diagram representing these sets.

b **i** Find $X \cap Y \cap Z$. **ii** Find $|X \cap Y|$.

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9 A number of students bought red, green and black pens. Three bought one of each colour. Of the students who bought two colours, three did not buy red, five not green and two not black. The same number of students bought red only as bought red with other colours. The same number bought black only as bought green only. More students bought red and black but not green than bought black only. More bought only green than bought green and black but not red. How many students were there and how many pens of each colour were sold?

10 For three subsets B, M and F of a universal set ξ,

$$|B \cap M| = 12, \quad |M \cap F \cap B| = |F'|, \quad |F \cap B| > |M \cap F|,$$
$$|B \cap F' \cap M'| = 5, \quad |M \cap B' \cap F'| = 5, \quad |F \cap M' \cap B'| = 5, \quad |\xi| = 28$$

Find $|M \cap F|$.

11 A group of 80 students were interviewed about which sports they play. It was found that 23 do athletics, 22 swim and 18 play football. If 10 students do athletics and swim only, 11 students do athletics and play football only, six students swim and play football only and 46 students do none of these activities on a regular basis, how many students do all three?

12 At a certain secondary college, students have to be proficient in at least one of the languages Italian, French and German. In a particular group of 33 students, two are proficient in all three languages, three in Italian and French only, four in French and German only and five in German and Italian only. The number of students proficient in Italian only is x, in French only is x and in German only is $x + 1$. Find x and then find the total number of students proficient in Italian.

13 At a certain school, 201 students study one or more of Mathematics, Physics and Chemistry. Of these students: 35 take Chemistry only, 50% more students study Mathematics only than study Physics only, four study all three subjects, 25 study both Mathematics and Physics but not Chemistry, seven study both Mathematics and Chemistry but not Physics, and 20 study both Physics and Chemistry but not Mathematics. Find the number of students studying Mathematics.

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Review

AS

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Chapter summary

Sets

- Set notation

$x \in A$	x is an element of A
$x \notin A$	x is not an element of A
ξ	the universal set
$\varnothing$	the empty set
$A \subseteq B$	A is a subset of B
$A \cup B$	the union of A and B consists of all elements that are in either A or B or both
$A \cap B$	the intersection of A and B consists of all elements that are in both A and B
A'	the complement of A consists of all elements of ξ that are not in A

- Sets of numbers

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

The modulus function

- The **modulus** or **absolute value** of a real number x is

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example: $|5| = 5$ and $|-5| = 5$.

- On the number line, the distance between two numbers a and b is given by $|a - b| = |b - a|$. For example: $|x - 2| < 5$ can be read as 'the distance of x from 2 is less than 5'.

Surds

- A **quadratic surd** is a number of the form $\sqrt{a}$, where a is a rational number which is not the square of another rational number.
- A **surd of order n** is a number of the form $\sqrt[n]{a}$, where a is a rational number which is not a perfect nth power.
- When the number under the square root has no factors which are squares of a rational number, the surd is said to be in **simplest form**.
- Surds which have the same 'irrational factor' are called **like surds**. The sum or difference of two like surds can be simplified:

$$m\sqrt{p} + n\sqrt{p} = (m + n)\sqrt{p} \quad \text{and} \quad m\sqrt{p} - n\sqrt{p} = (m - n)\sqrt{p}$$

Natural numbers

- A natural number a is a **factor** of a natural number b if there exists a natural number k such that $b = ak$.
- A natural number greater than 1 is a **prime number** if its only factors are itself and 1.
- A natural number m is a **composite number** if it can be written as a product $m = a \times b$, where a and b are natural numbers greater than 1 and less than m.

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- Every composite number can be expressed as a product of powers of prime numbers; this is called **prime decomposition**. For example: $1300 = 2^2 \times 5^2 \times 13$
- The **highest common factor** of two natural numbers a and b, denoted by HCF(a, b), is the largest natural number that is a factor of both a and b.
- The **lowest common multiple** of two natural numbers a and b, denoted by LCM(a, b), is the smallest natural number that is a multiple of both a and b.

Diophantine equations and the Euclidean algorithm

- A **Diophantine equation** is an equation in which only integer solutions are allowed.
- An equation of the form $ax + by = c$, where the coefficients a, b, c are integers, is called a **linear Diophantine equation** when the intention is to find integer solutions for x, y.
- If $ax + by = c$ is a linear Diophantine equation and (x_0, y_0) is found to be one solution, then the general solution is given by

$$x = x_0 + \frac{b}{d}t, \quad y = y_0 - \frac{a}{d}t \qquad \text{for } t \in \mathbb{Z}$$

  where d is the highest common factor of a and b.
- A linear Diophantine equation $ax + by = c$ has integer solutions if and only if HCF(a, b) divides c.
- Division algorithm If a and b are integers with $a > 0$, then there are unique integers q and r such that $b = aq + r$ and $0 \le r < a$.
- Euclidean algorithm Let a and b be integers with $a \neq 0$. If $b = aq + r$, where q and r are integers, then HCF(a, b) = HCF(a, r).

  The repeated application of this result can be used to find the highest common factor of two natural numbers and to solve linear Diophantine equations.

Technology-free questions

1 Express the following as fractions in their simplest form:

a $0.0\dot{7}$ **b** $0.\dot{4}\dot{5}$ **c** 0.005 **d** 0.405 **e** $0.2\dot{6}$ **f** $0.1\dot{7}1428\dot{5}$

2 Express 504 as a product of powers of prime numbers.

3 **a** Find the four integer values of n such that $|n^2 - 9|$ is a prime number.

b Solve each equation for x:

i $x^2 + 5|x| - 6 = 0$ **ii** $x + |x| = 0$

c Solve the inequality $5 - |x| < 4$ for x.

4 Express each of the following with a rational denominator:

a $\dfrac{2\sqrt{3} - 1}{\sqrt{2}}$ **b** $\dfrac{\sqrt{5} + 2}{\sqrt{5} - 2}$ **c** $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$

5 Express $\dfrac{3 + 2\sqrt{75}}{3 - \sqrt{12}}$ in the form $a + b\sqrt{3}$, where $a, b \in \mathbb{Q} \setminus \{0\}$.

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6 Express each of the following with a rational denominator:

a $\dfrac{6\sqrt{2}}{3\sqrt{2}-2\sqrt{3}}$ **b** $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$

7 In a class of 100 students, 55 are girls, 45 have blue eyes, 40 are blond, 25 are blond girls, 15 are blue-eyed blonds, 20 are blue-eyed girls, and five are blue-eyed blond girls. Find:

a the number of blond boys

b the number of boys who are neither blond nor blue-eyed.

8 A group of 30 students received prizes in at least one of the subjects of English, Mathematics and French. Two students received prizes in all three subjects. Fourteen received prizes in English and Mathematics but not French. Two received prizes in English alone, two in French alone and five in Mathematics alone. Four received prizes in English and French but not Mathematics.

a How many received prizes in Mathematics and French but not English?

b How many received prizes in Mathematics?

c How many received prizes in English?

9 Fifty people are interviewed. Twenty-three people like Brand X, 25 like Brand Y and 19 like Brand Z. Eleven like X and Z. Eight like Y and Z. Five like X and Y. Two like all three. How many like none of them?

10 Three rectangles A, B and C overlap (intersect). Their areas are 20 cm^2, 10 cm^2 and 16 cm^2 respectively. The area common to A and B is 3 cm^2, that common to A and C is 6 cm^2 and that common to B and C is 4 cm^2. How much of the area is common to all three if the total area covered is 35 cm^2?

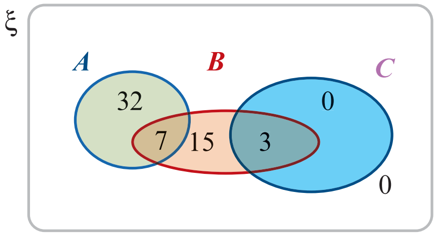
11 Express $\sqrt{112}-\sqrt{63}-\dfrac{224}{\sqrt{28}}$ in simplest form.

12 If $\dfrac{\sqrt{7}-\sqrt{3}}{x}=\dfrac{x}{\sqrt{7}+\sqrt{3}}$, find the values of x.

13 Express $\dfrac{1+\sqrt{2}}{\sqrt{5}+\sqrt{3}}+\dfrac{1-\sqrt{2}}{\sqrt{5}-\sqrt{3}}$ in the form $a\sqrt{5}+b\sqrt{6}$.

14 Simplify $\sqrt{27}-\sqrt{12}+2\sqrt{75}-\sqrt{\dfrac{48}{25}}$.

15 A, B and C are three sets and $\xi = A \cup B \cup C$. The number of elements in the regions of the Venn diagram are as shown. Find:



a the number of elements in $A \cup B$

b the number of elements in C

c the number of elements in $B' \cap A$.

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16 Using the result that $(\sqrt{a}+\sqrt{b})^2 = a+b+2\sqrt{ab}$, determine the square root of $17+6\sqrt{8}$.

17 Find the highest common factor of 1885 and 365 using the Euclidean algorithm.

18 Consider $9x + 43y = 7$. Solve for x and y where:

a $x \in \mathbb{Z}, y \in \mathbb{Z}$ **b** $x \in \mathbb{N}, y \in \mathbb{N}$

19 Prove that the product of two odd integers is odd. (You may assume that the sum and product of any two integers is an integer.)

20 Using the Euclidean algorithm, find the highest common factor of 10 659 and 12 121.

21 **a** Solve the Diophantine equation $5x + 7y = 1$.

b Hence solve the Diophantine equation $5x + 7y = 100$.

c Find $\{(x, y) : 5x + 7y = 1,\ y \geq x \text{ and } x, y \in \mathbb{Z}\}$.

22 The sum of the ages of Tom and Fred is 63. Tom is twice as old as Fred was when Tom was as old as Fred is now. What are the ages of Tom and Fred?

Multiple-choice questions

1 $\dfrac{4}{3+2\sqrt{2}}$ expressed in the form $a + b\sqrt{2}$ is

A $12 - 8\sqrt{2}$ **B** $3 + 2\sqrt{2}$ **C** $\dfrac{3}{17} - \dfrac{8}{17}\sqrt{2}$

D $\dfrac{3}{17} + \dfrac{8}{17}\sqrt{2}$ **E** $12 + 8\sqrt{2}$

2 The prime decomposition of 86 400 is

A $2^5 \times 3^2 \times 5$ **B** $2^6 \times 3^3 \times 5^2$ **C** $2^7 \times 3^3 \times 5$

D $2^7 \times 3^3 \times 5^2$ **E** $2^6 \times 3^3 \times 5^3$

3 $(\sqrt{6}+3)(\sqrt{6}-3)$ is equal to

A $3 - 12\sqrt{6}$ **B** $-3 - 6\sqrt{6}$ **C** $-3 + 6\sqrt{6}$ **D** -3 **E** 3

4 For the Venn diagram shown, ξ is the set of natural numbers less than 20, A is the set of natural numbers less than 10, and B is the set of natural numbers less than 20 that are divisible by 3. The set $B' \cap A$ is

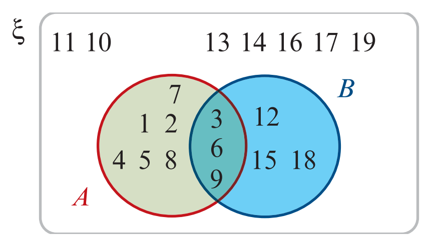
A $\{3, 6, 9\}$

B $\{12, 15, 18\}$

C $\{10, 11, 13, 14, 16, 17, 19\}$

D $\{1, 2, 4, 5, 7, 8\}$

E $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 15, 18\}$



5 $(3, \infty) \cap (-\infty, 5] =$

A $(-\infty, 3)$ **B** $(-\infty, 5]$ **C** $(3, 5]$ **D** $\mathbb{R}$ **E** $[3, 5]$

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Review

6 A bell is rung every 6 minutes and a gong is sounded every 14 minutes. If these occur together at a particular time, then the smallest number of minutes until the bell and the gong are again heard simultaneously is

A 10 **B** 20 **C** 72 **D** 42 **E** 84

7 If X is the set of multiples of 2, Y the set of multiples of 7, and Z the set of multiples of 5, then $X \cap Y \cap Z$ can be described as

A the set of multiples of 2
B the set of multiples of 70
C the set of multiples of 35
D the set of multiples of 14
E the set of multiples of 10

8 In a class of students, 50% play football, 40% play tennis and 30% play neither. The percentage that plays both is

A 10 **B** 20 **C** 30 **D** 50 **E** 40

9 $\dfrac{\sqrt{7}-\sqrt{6}}{\sqrt{7}+\sqrt{6}} =$

A $5+2\sqrt{7}$ **B** $13+2\sqrt{6}$ **C** $13-2\sqrt{42}$ **D** $1+2\sqrt{42}$ **E** $13-2\sqrt{13}$

10 There are 40 students in a class, all of whom take either Literature or Economics or both. Twenty take Literature and five of these also take Economics. The number of students who take only Economics is

A 20 **B** 5 **C** 10 **D** 15 **E** 25

11 The number of solutions of the Diophantine equation $3x + 5y = 1008$, where x and y are positive integers, is

A 1 **B** 134 **C** 68 **D** 67 **E** infinite

12 The number of factors that the integer $2^p 3^q 5^r$ has is

A $\dfrac{(p+q+r)!}{p!\,q!\,r!}$ **B** pqr **C** $p+q+r$

D $(p+1)(q+1)(r+1)$ **E** $p+q+r+1$

13 The number of pairs of integers (m, n) which satisfy the equation $m + n = mn$ is

A 1 **B** 2 **C** 3 **D** 4 **E** more than 4

Extended-response questions

1 **a** Show that $(\sqrt{x}+\sqrt{y})^2 = x + y + 2\sqrt{xy}$.

b Substitute $x = 3$ and $y = 5$ in the identity from part **a** to show that

$$\sqrt{3}+\sqrt{5} = \sqrt{8+2\sqrt{15}}$$

c Use this technique to find the square root of:

i $14+2\sqrt{33}$ (Hint: Use $x = 11$ and $y = 3$.) **ii** $15-2\sqrt{56}$ **iii** $51-36\sqrt{2}$

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2 In this question, we consider the set $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$. In Chapter 16, the set $\mathbb{C}$ of complex numbers is introduced, where $\mathbb{C} = \{ a + b\sqrt{-1} : a, b \in \mathbb{R} \}$.

a If $(2 + 3\sqrt{3}) + (4 + 2\sqrt{3}) = a + b\sqrt{3}$, find a and b.

b If $(2 + 3\sqrt{3})(4 + 2\sqrt{3}) = p + q\sqrt{3}$, find p and q.

c If $\dfrac{1}{3 + 2\sqrt{3}} = a + b\sqrt{3}$, find a and b.

d Solve each of the following equations for x:

i $(2 + 5\sqrt{3})x = 2 - \sqrt{3}$ **ii** $(x - 3)^2 - 3 = 0$ **iii** $(2x - 1)^2 - 3 = 0$

e Explain why every rational number is a member of $\{ a + b\sqrt{3} : a, b \in \mathbb{Q} \}$.

3 **a** Show that $\dfrac{1}{2 + \sqrt{3}} = 2 - \sqrt{3}$.

b Use the substitution $t = \left(\sqrt{2 + \sqrt{3}}\right)^x$ and part **a** to show that the equation

$$\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$$

can be written as $t + \dfrac{1}{t} = 4$.

c Show that the solutions of the equation are $t = 2 - \sqrt{3}$ and $t = 2 + \sqrt{3}$.

d Use this result to solve the equation $\left(\sqrt{2 + \sqrt{3}}\right)^x + \left(\sqrt{2 - \sqrt{3}}\right)^x = 4$.

4 Use Venn diagrams to illustrate:

a $|A \cup B| = |A| + |B| - |A \cap B|$

b $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$

5 A quadratic equation with integer coefficients $x^2 + bx + c = 0$ has a solution $x = 2 - \sqrt{3}$.

a Find the values of b and c.

Hint: Use the result that, for m, n rational, if $m + n\sqrt{3} = 0$, then $m = 0$ and $n = 0$.

b Find the other solution to this quadratic equation.

c Now consider a quadratic equation with integer coefficients $x^2 + bx + c = 0$ that has a solution $x = m - n\sqrt{q}$, where q is not a perfect square. Show that:

i $b = -2m$ **ii** $c = m^2 - n^2 q$

Hence show that:

iii $x^2 + bx + c = (x - (m - n\sqrt{q}))(x - (m + n\sqrt{q}))$

6 A **Pythagorean triple** (x, y, z) consists of three natural numbers x, y, z such that $x^2 + y^2 = z^2$. For example: $(3, 4, 5)$ and $(5, 12, 13)$ are Pythagorean triples.
A Pythagorean triple is in simplest form if x, y, z have no common factor. Up to swapping x and y, all Pythagorean triples in simplest form may be generated by:

$$x = 2mn, \quad y = m^2 - n^2, \quad z = m^2 + n^2 \qquad \text{where } m, n \in \mathbb{N}$$

For example, if $m = 2$ and $n = 1$, then $x = 4$, $y = 3$ and $z = 5$.

a Find the Pythagorean triple for $m = 5$ and $n = 2$.

b Verify that, if $x = 2mn$, $y = m^2 - n^2$ and $z = m^2 + n^2$, where $m, n \in \mathbb{N}$, then $x^2 + y^2 = z^2$.

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7 The factors of 12 are 1, 2, 3, 4, 6, 12.

a How many factors does each of the following numbers have?

i 2^3 **ii** 3^7

b How many factors does 2^n have?

c How many factors does each of the following numbers have?

i $2^3 \cdot 3^7$ **ii** $2^n \cdot 3^m$

d Every natural number greater than 1 may be expressed as a product of powers of primes; this is called prime decomposition. For example: $1080 = 2^3 \times 3^3 \times 5$.
Let x be a natural number greater than 1 and let

$$x = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \dots p_n^{\alpha_n}$$

be its prime decomposition, where each $\alpha_i \in \mathbb{N}$ and each p_i is a prime number.
How many factors does x have? (Answer to be given in terms of α_i.)

e Find the smallest number which has eight factors.

8 **a** Give the prime decompositions of 1080 and 25 200.

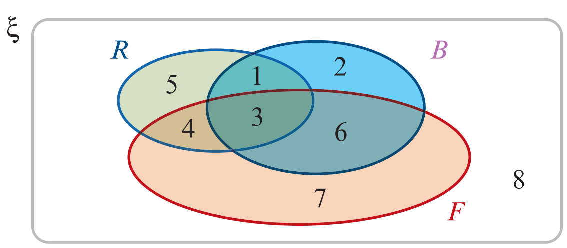
b Use your answer to part **a** to find the lowest common multiple of 1080 and 25 200.

c Carefully explain why, if m and n are integers, then $mn = \text{LCM}(m, n) \times \text{HCF}(m, n)$.

d **i** Find four consecutive even numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

ii Find four consecutive natural numbers such that the smallest is a multiple of 5, the second a multiple of 7, the third a multiple of 9 and the largest a multiple of 11.

9 **a** The Venn diagram shows the set ξ of all students enrolled at Argos Secondary College. Set R is the set of all students with red hair. Set B is the set of all students with blue eyes. Set F is the set of all female students.



The numbers on the diagram are to label the eight different regions.

i Identify the region in the Venn diagram which represents male students who have neither red hair nor blue eyes.

ii Describe the gender, hair colour and eye colour of students represented in region 1 of the diagram.

iii Describe the gender, hair colour and eye colour of students represented in region 2 of the diagram.

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b It is known that, at Argos Secondary College, 250 students study French (F), Greek (G) or Japanese (J). Forty-one students do not study French. Twelve students study French and Japanese but not Greek. Thirteen students study Japanese and Greek but not French. Thirteen students study only Greek. Twice as many students study French and Greek but not Japanese as study all three. The number studying only Japanese is the same as the number studying both French and Greek.

i How many students study all three languages?

ii How many students study only French?

10 Consider the universal set ξ as the set of all students enrolled at Sounion Secondary College. Let B denote the set of students taller than 180 cm and let A denote the set of female students.

a Give a brief description of each of the following sets:

i B' **ii** $A \cup B$ **iii** $A' \cap B'$

b Use a Venn diagram to show $(A \cup B)' = A' \cap B'$.

c Hence show that $A \cup B \cup C = (A' \cap B' \cap C')'$, where C is the set of students who play sport.

11 In a certain city, three Sunday newspapers (A, B and C) are available. In a sample of 500 people from this city, it was found that:

- nobody regularly reads both A and C
- a total of 100 people regularly read A
- 205 people regularly read only B
- of those who regularly read C, exactly half of them also regularly read B
- 35 people regularly read A and B but not C
- 35 people don't read any of the papers at all.

a Draw a Venn diagram showing the number of regular readers for each possible combination of A, B and C.

b How many people in the sample were regular readers of C?

c How many people in the sample regularly read A only?

d How many people are regular readers of A, B and C?

12 You have an inexhaustible supply of 5c and 8c stamps.

a List all possible ways of obtaining a total value of 38c with these stamps.

b List all possible ways of obtaining a total of $1.20 with these stamps.

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