Chapter 4

4 Sequences and series

Objectives

- To explore **sequences** of numbers and their **recurrence relations**.
- To use a CAS calculator to generate sequences and display their graphs.
- To recognise **arithmetic sequences**, and to find their terms, recurrence relations and numbers of terms.
- To calculate the sum of the terms in an **arithmetic series**.
- To recognise **geometric sequences**, and to find their terms, recurrence relations and numbers of terms.
- To calculate the sum of the terms in a **geometric series**.
- To calculate the sum of the terms in an **infinite geometric series**.
- To apply sequences and series to solving problems.

The following are examples of sequences of numbers:

a $1, 3, 5, 7, 9, \ldots$

b $0.1, 0.11, 0.111, 0.1111, 0.11111, \ldots$

c $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

d $10, 7, 4, 1, -2, \ldots$

e $0.6, 1.7, 2.8, \ldots, 9.4$

Each sequence is a list of numbers, with order being important. Sequence **e** is an example of a finite sequence, and the others are infinite sequences.

For some sequences of numbers, we can give a rule for getting from one number to the next:

- a rule for sequence **a** is: add 2
- a rule for sequence **c** is: multiply by $\frac{1}{3}$
- a rule for sequence **d** is: subtract 3
- a rule for sequence **e** is: add 1.1

In this chapter we study the theory required to solve problems involving sequences, and look at some applications of sequences.

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4A Introduction to sequences

The numbers of a sequence are called its **terms**. The nth term of a sequence is denoted by the symbol t_n. So the first term is t_1, the 12th term is t_{12}, and so on.

Recurrence relations

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation**, a **recursive formula** or an **iterative rule**. For example:

- The sequence $1, 3, 5, 7, 9, \dots$ may be defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ may be defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Example 1

Use the recurrence relation to find the first four terms of the sequence

$$t_1 = 3, \quad t_n = t_{n-1} + 5$$

Solution

$$\begin{aligned} t_1 &= 3 \\ t_2 &= t_1 + 5 = 8 \\ t_3 &= t_2 + 5 = 13 \\ t_4 &= t_3 + 5 = 18 \end{aligned}$$

The first four terms are 3, 8, 13, 18.

Example 2

Find the recurrence relation for the following sequence:

$$9, -3, 1, -\frac{1}{3}, \dots$$

Solution

$$-3 = -\frac{1}{3} \times 9 \qquad \text{i.e. } t_2 = -\frac{1}{3}t_1$$

$$1 = -\frac{1}{3} \times -3 \qquad \text{i.e. } t_3 = -\frac{1}{3}t_2$$

The sequence is defined by $t_1 = 9$ and $t_n = -\frac{1}{3}t_{n-1}$.

A sequence may also be defined explicitly by a rule that is stated in terms of n. For example:

- The rule $t_n = 2n$ defines the sequence $t_1 = 2, t_2 = 4, t_3 = 6, t_4 = 8, \dots$
- The rule $t_n = 2^{n-1}$ defines the sequence $t_1 = 1, t_2 = 2, t_3 = 4, t_4 = 8, \dots$
- The sequence $1, 3, 5, 7, 9, \dots$ can be defined by $t_n = 2n - 1$.
- The sequence $t_1 = \frac{1}{3}$, $t_n = \frac{1}{3}t_{n-1}$ can be defined by $t_n = \frac{1}{3^n}$.

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For an infinite sequence, there is a term t_n of the sequence for each natural number n. Therefore we can consider an infinite sequence to be a function whose domain is the natural numbers. For example, we can write $t\colon \mathbb{N} \to \mathbb{R}$, $t_n = 2n + 3$.

Example 3

Find the first four terms of the sequence defined by the rule $t_n = 2n + 3$.

Solution

$$t_1 = 2(1) + 3 = 5$$
$$t_2 = 2(2) + 3 = 7$$
$$t_3 = 2(3) + 3 = 9$$
$$t_4 = 2(4) + 3 = 11$$

The first four terms are 5, 7, 9, 11.

Example 4

Find a rule for the nth term of the sequence 1, 4, 9, 16 in terms of n.

Solution

$$t_1 = 1 = 1^2$$
$$t_2 = 4 = 2^2$$
$$t_3 = 9 = 3^2$$
$$t_4 = 16 = 4^2$$
$$\therefore \quad t_n = n^2$$

Example 5

At a particular school, the number of students studying Specialist Mathematics increases each year. There are presently 40 students studying Specialist Mathematics.

a Set up the recurrence relation if the number is increasing by five students each year.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the number of students expected to be studying Specialist Mathematics at the school in five years time.

Solution

a $t_n = t_{n-1} + 5$

b $t_1 = 40$

$$t_2 = t_1 + 5 = 45 = 40 + 1 \times 5$$
$$t_3 = t_2 + 5 = 50 = 40 + 2 \times 5$$

Therefore $t_n = 40 + (n - 1) \times 5$

$$= 35 + 5n$$

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c Five years from now implies $n = 6$:

$$t_6 = 40 + 5 \times 5 = 65$$

Sixty-five students will be studying Specialist Mathematics in five years.

Example 6

The height of a sand dune is increasing by 10% each year. It is currently 4 m high.

a Set up the recurrence relation that describes the height of the sand dune.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the height of the sand dune seven years from now.

Solution

a $t_n = t_{n-1} \times 1.1$

b $t_1 = 4$

$t_2 = 4 \times 1.1 = 4.4$

$t_3 = 4 \times (1.1)^2 = 4.84$

Therefore $t_n = 4 \times (1.1)^{n-1}$

c Seven years from now implies $n = 8$:

$$t_8 = 4 \times (1.1)^7 \approx 7.795$$

The sand dune will be 7.795 m high in seven years.

► Using a calculator with explicitly defined sequences

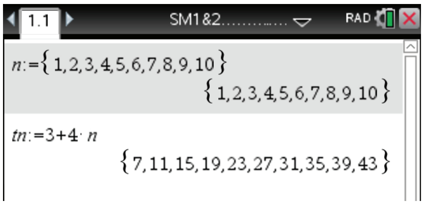
Example 7

Use a calculator to generate the first 10 terms of the sequence of numbers defined by the rule $t_n = 3 + 4n$.

Using the TI-Nspire

Sequences defined in terms of n can be investigated in a **Calculator** application.

- To generate the first 10 terms of the sequence defined by the rule $t_n = 3 + 4n$, complete as shown.



Note: Assigning (storing) the resulting list as *tn* enables the sequence to be graphed. If preferred, the variable *tn* can be entered as t_n using the subscript template $\square_{\square}$, which is accessed via .

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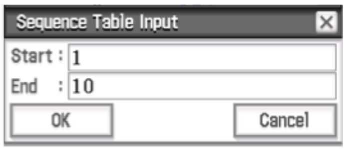
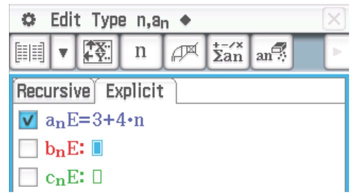
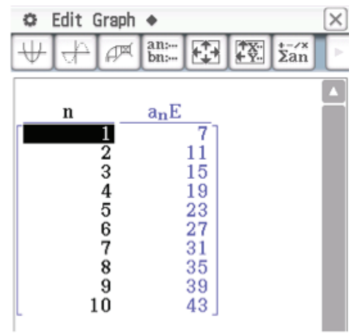
Using the Casio ClassPad

- Open the menu Menu; select **Sequence**.
- Ensure that the **Explicit** window is activated.
- Select the setting n.
- Tap the cursor next to $a_n E$ and enter $3 + 4n$.
- Tick the box or tap EXE.
- Tap to view the sequence values.
- Tap to open the Sequence Table Input window and complete as shown below; tap OK.



- Tap Resize to see the sequence of numbers.





► Using a calculator with recursively defined sequences

Example 8

Use a CAS calculator to generate the sequence defined by the recurrence relation

$$t_n = t_{n-1} + 3, \quad t_1 = 1$$

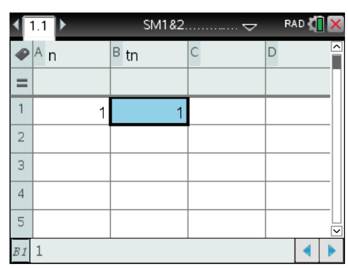
and plot the graph of the sequence against n.

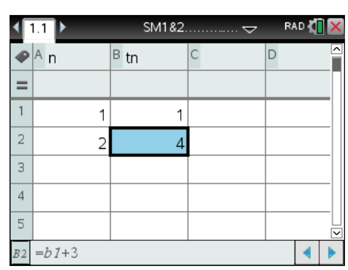
Using the TI-Nspire

- In a **Lists & Spreadsheet** page, name the first two lists n and tn respectively.
- Enter 1 in cell A1 and enter 1 in cell B1.

Note: If preferred, the variable tn can be entered as t_n using the subscript template, which is accessed via.

- Enter $= a1 + 1$ in cell A2 and enter $= b1 + 3$ in cell B2.





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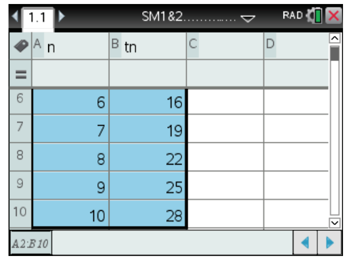
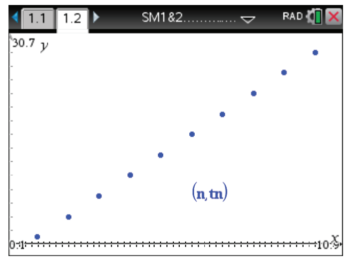
- Highlight the cells A2 and B2 using (shift) and the arrows.
- Use (menu) > **Data** > **Fill** and arrow down to row 10.
- Press (enter) to populate the lists.

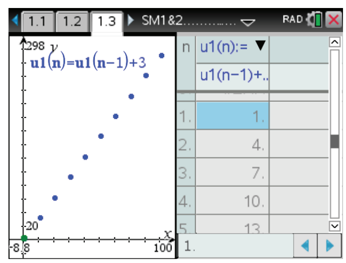


- To graph the sequence, open a **Graphs** application ((ctrl)(I) > **Add Graphs**).
- Graph the sequence as a scatter plot using (menu) > **Graph Entry/Edit** > **Scatter Plot**. Enter the list variables as n and tn in their respective fields.
- Set an appropriate window using (menu) > **Window/Zoom** > **Zoom – Data**.



Note: It is possible to see the coordinates of the points: (menu) > **Trace** > **Graph Trace**. The scatter plot can also be graphed in a **Data & Statistics** page.

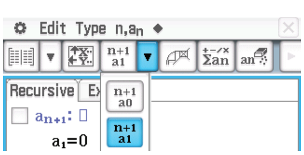
- Alternatively, the sequence can be graphed directly in the sequence plotter ((menu) > **Graph Entry/Edit** > **Sequence** > **Sequence**).
- Enter the rule $u1(n) = u1(n-1) + 3$ and the initial value 1. Change **nStep** to 10.
- Set an appropriate window using (menu) > **Window/Zoom** > **Zoom – Fit**.
- Use (ctrl)(T) to show a table of values.

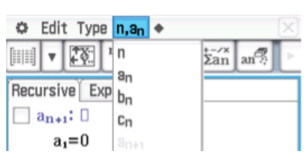


Using the Casio ClassPad

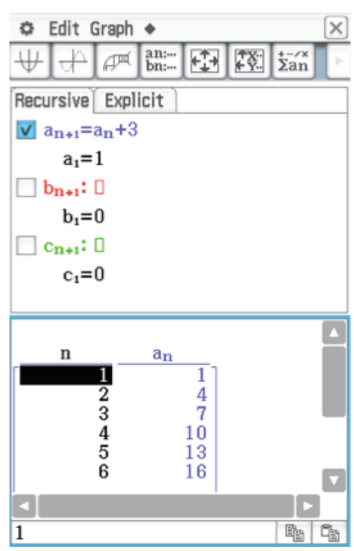
- Open the menu (Menu); select **Sequence** (Sequence).
- Ensure that the **Recursive** window is activated.
- Select the setting $\frac{n+1}{a1}$ as shown below.
- Tap the cursor next to a_{n+1} and enter $a_n + 3$.

Note: The symbol a_n can be found in the dropdown menu in the toolbar as shown below.





- Enter 1 for the value of the first term, a_1.
- Tick the box. Tap (table icon) to view the sequence values.



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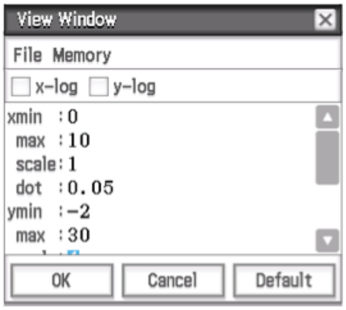
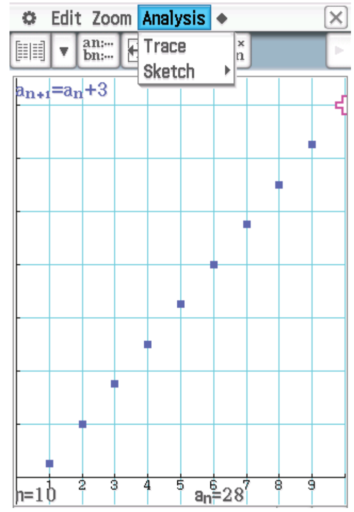
- Tap to view the graph.
- Tap Resize and then . Set the View Window as shown below.



- Select **Analysis** > **Trace** and use the cursor ► to view each value in the sequence.



Section summary

A sequence may be defined by a rule which enables each subsequent term to be found from the previous term. This type of rule is called a **recurrence relation** and we say that the sequence has been defined **recursively**. For example:

- The sequence $1, 3, 5, 7, 9, \dots$ is defined by $t_1 = 1$ and $t_n = t_{n-1} + 2$.
- The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ is defined by $t_1 = \frac{1}{3}$ and $t_n = \frac{1}{3}t_{n-1}$.

Exercise 4A

Example 1 **1** In each of the following, a recursive definition for a sequence is given. List the first five terms.

a $t_1 = 3,\ t_n = t_{n-1} + 4$ **b** $t_1 = 5,\ t_n = 3t_{n-1} + 4$ **c** $t_1 = 1,\ t_n = 5t_{n-1}$

d $t_1 = -1,\ t_n = t_{n-1} + 2$ **e** $t_{n+1} = 2t_n + t_{n-1},\ t_1 = 1,\ t_2 = 3$

Example 2 **2** For each of the following sequences, find the recurrence relation:

a $3, 6, 9, 12, \dots$ **b** $1, 2, 4, 8, \dots$ **c** $3, -6, 12, -24, \dots$

d $4, 7, 10, 13, \dots$ **e** $4, 9, 14, 19, \dots$

Example 3 **3** Each of the following is a rule for a sequence. In each case, find t_1, t_2, t_3, t_4.

a $t_n = \frac{1}{n}$ **b** $t_n = n^2 + 1$ **c** $t_n = 2n$

d $t_n = 2^n$ **e** $t_n = 3n + 2$ **f** $t_n = (-1)^n n^3$

g $t_n = 2n + 1$ **h** $t_n = 2 \times 3^{n-1}$

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Example 4 **4** For each of the following sequences, find a possible rule for t_n in terms of n:

a $3, 6, 9, 12, \ldots$ **b** $1, 2, 4, 8, \ldots$ **c** $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

d $3, -6, 12, -24, \ldots$ **e** $4, 7, 10, 13, \ldots$ **f** $4, 9, 14, 19, \ldots$

5 Consider a sequence for which $t_n = 3n + 1$. Find t_{n+1} and t_{2n}.

Example 5 **6** Hamish collects football cards. He currently has 15 and he adds three to his collection every week.

a Set up the recurrence relation that will generate the number of cards Hamish has in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the number of cards Hamish should have after another 12 weeks.

Example 6 **7** Isobel can swim 100 m in 94.3 s. She aims to reduce her time by 4% each week.

a Set up the recurrence relation that generates Isobel's time for the 100 m in any given week.

b Write down an expression for t_n in terms of n for the recurrence relation found in **a**.

c Find the time in which Isobel expects to be able to complete the 100 m after another 8 weeks.

8 Stephen is a sheep farmer with a flock of 100 sheep. He wishes to increase the size of his flock by both breeding and buying new stock. He estimates that 80% of his sheep will produce one lamb each year and he intends to buy 20 sheep to add to the flock each year. Assuming that no sheep die:

a Write the recurrence relation for the expected number of sheep at the end of each year. (Let $t_0 = 100$.)

b Calculate the number of sheep at the end of each of the first five years.

9 Alison invests \$2000 at the beginning of the year. At the beginning of each of the following years, she puts a further \$400 into the account. Compound interest of 6% p.a. is paid on the investment at the end of each year.

a Write down the amount of money in the account at the end of each of the first three years.

b Set up a recurrence relation to generate the sequence for the investment. (Let t_1 be the amount in the account at the end of the first year.)

c With a calculator or spreadsheet, use the recurrence relation to find the amount in the account after 10 years.

Example 7 **10** For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = 3n - 2$ **b** $t_n = 5 - 2n$ **c** $t_n = 2^{n-2}$ **d** $t_n = 2^{6-n}$

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Example 8 **11** For each of the following, use a CAS calculator to find the first six terms of the sequence and plot the graph of these terms against n:

a $t_n = (t_{n-1})^2,\ t_1 = 1.1$ **b** $t_n = \frac{2}{3}t_{n-1},\ t_1 = 27$

c $t_n = 2t_{n-1} + 5,\ t_1 = -1$ **d** $t_n = 4 - t_{n-1},\ t_1 = -3$

12 **a** For a sequence for which $t_n = 2^{n-1}$, find t_1, t_2 and t_3.

b For a sequence for which $u_n = \frac{1}{2}(n^2 - n) + 1$, find u_1, u_2 and u_3.

c What do you notice?

d Find t_4 and u_4.

13 If $S_n = an^2 + bn$, for constants $a, b \in \mathbb{R}$, find S_1, S_2, S_3 and $S_{n+1} - S_n$.

14 For the sequence defined by $t_1 = 1$ and $t_{n+1} = \frac{1}{2}\left(t_n + \frac{2}{t_n}\right)$, find t_2, t_3 and t_4.

The terms of this sequence are successive rational approximations of a real number. Can you recognise the number?

15 The Fibonacci sequence is defined by $t_1 = 1$, $t_2 = 1$ and $t_{n+2} = t_{n+1} + t_n$ for $n \in \mathbb{N}$. Use the rule to find t_3, t_4 and t_5. Show that $t_{n+2} = 2t_n + t_{n-1}$ for all $n \in \mathbb{N} \setminus \{1\}$.

4B Arithmetic sequences

A sequence in which each successive term is found by adding a fixed amount to the previous term is called an **arithmetic sequence**. That is, an arithmetic sequence has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant.

For example: $2, 5, 8, 11, 14, 17, \ldots$ is an arithmetic sequence.

The nth term of an arithmetic sequence is given by

$$t_n = a + (n-1)d$$

where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Example 9

Find the 10th term of the arithmetic sequence $-4, -1, 2, 5, \ldots$.

Solution

$a = -4$, $d = 3$

$$\begin{aligned} t_n &= a + (n-1)d \\ \therefore\quad t_{10} &= -4 + (10-1) \times 3 \\ &= 23 \end{aligned}$$

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Example 10

If 41 is the nth term in the arithmetic sequence $-4, -1, 2, 5, \ldots$, find the value of n.

Solution

$a = -4$, $d = 3$

$$\begin{aligned} t_n &= a + (n-1)d = 41 \\ \therefore \quad -4 + (n-1) \times 3 &= 41 \\ 3(n-1) &= 45 \\ n - 1 &= 15 \\ n &= 16 \end{aligned}$$

Hence 41 is the 16th term of the sequence.

Example 11

The 12th term of an arithmetic sequence is 9 and the 25th term is 100. Find a and d, and hence find the 8th term.

Solution

An arithmetic sequence has rule

$$t_n = a + (n-1)d$$

Since the 12th term is 9, we have

$$9 = a + 11d \qquad (1)$$

Since the 25th term is 100, we have

$$100 = a + 24d \qquad (2)$$

To find a and d, we solve the two equations simultaneously.

Subtract (1) from (2):

$$\begin{aligned} 91 &= 13d \\ \therefore \quad d &= 7 \end{aligned}$$

From (1), we have

$$\begin{aligned} 9 &= a + 11(7) \\ \therefore \quad a &= -68 \end{aligned}$$

Therefore

$$\begin{aligned} t_8 &= a + 7d \\ &= -68 + 7 \times 7 \\ &= -19 \end{aligned}$$

The 8th term of the sequence is -19.

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Example 12

A national park has a series of huts along one of its mountain trails. The first hut is 5 km from the start of the trail, the second is 8 km from the start, the third 11 km and so on.

a How far from the start of the trail is the sixth hut?

b How far is it from the sixth hut to the twelfth hut?

Solution

The distances of the huts from the start of the trail form an arithmetic sequence with $a = 5$ and $d = 3$.

a For the sixth hut:

$$\begin{aligned} t_6 &= a + 5d \\ &= 5 + 5 \times 3 = 20 \end{aligned}$$

The sixth hut is 20 km from the start of the trail.

b For the twelfth hut:

$$\begin{aligned} t_{12} &= a + 11d \\ &= 5 + 11 \times 3 = 38 \end{aligned}$$

The distance from the sixth hut to the twelfth hut is $t_{12} - t_6 = 38 - 20 = 18$ km.

▶ Arithmetic mean

The **arithmetic mean** of two numbers a and b is defined as $\dfrac{a+b}{2}$.

If the numbers a, c, b are consecutive terms of an arithmetic sequence, then

$$\begin{aligned} c - a &= b - c \\ 2c &= a + b \\ \therefore \quad c &= \frac{a+b}{2} \end{aligned}$$

That is, the middle term c is the arithmetic mean of a and b.

Section summary

- An **arithmetic sequence** has a recurrence relation of the form $t_n = t_{n-1} + d$, where d is a constant. Each successive term is found by adding a fixed amount to the previous term. For example: $2, 5, 8, 11, \dots$
- The nth term of an arithmetic sequence is given by

  $$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** between successive terms, that is, $d = t_k - t_{k-1}$, for all $k > 1$.

Exercise 4B

1 For the arithmetic sequence where $t_n = a + (n-1)d$, find the first four terms given that:

a $a = 0,\ d = 2$

b $a = -3,\ d = 5$

c $a = d = -\sqrt{5}$

d $a = 11,\ d = -2$

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Example 9 **2** **a** If an arithmetic sequence has a first term of 5 and a common difference of -3, find the 13th term.

b If an arithmetic sequence has a first term of -12 and a common difference of 4, find the 10th term.

c For the arithmetic sequence with $a = 25$ and $d = -2.5$, find the ninth term.

d For the arithmetic sequence with $a = 2\sqrt{3}$ and $d = \sqrt{3}$, find the fifth term.

3 Find the rule of the arithmetic sequence whose first few terms are:

a $3, 7, 11$ **b** $3, -1, -5$ **c** $-\frac{1}{2}, \frac{3}{2}, \frac{7}{2}, \frac{11}{2}$ **d** $5 - \sqrt{5},\ 5,\ 5 + \sqrt{5}$

Example 10 **4** In each of the following, t_n is the nth term of an arithmetic sequence:

a If 54 is the nth term in the sequence $6, 10, 14, 18, \dots$, find the value of n.

b If -16 is the nth term in the sequence $5, 2, -1, -4, \dots$, find the value of n.

c Find n if $t_1 = 16$, $t_2 = 13$ and $t_n = -41$.

d Find n if $t_1 = 7$, $t_2 = 11$ and $t_n = 227$.

Example 11 **5** For an arithmetic sequence with fourth term 7 and thirtieth term 85, find the values of a and d, and hence find the seventh term.

6 If an arithmetic sequence has $t_3 = 18$ and $t_6 = 486$, find the rule for the sequence, i.e. find t_n.

7 For the arithmetic sequence with $t_7 = 0.6$ and $t_{12} = -0.4$, find t_{20}.

8 The number of laps that a swimmer swims each week follows an arithmetic sequence. In the 5th week she swims 24 laps and in the 10th week she swims 39 laps. How many laps does she swim in the 15th week?

9 For an arithmetic sequence, find t_6 if $t_{15} = 3 + 9\sqrt{3}$ and $t_{20} = 38 - \sqrt{3}$.

Example 12 **10** A small company producing wallets plans an increase in output. In the first week it produces 280 wallets. The number of wallets produced each week is to be increased by 8 per week until the weekly number produced reaches 1000.

a How many wallets are produced in the 50th week?

b In which week does the production reach 1000?

11 An amphitheatre has 25 seats in row A, 28 seats in row B, 31 seats in row C, and so on.

a How many seats are there in row P?

b How many seats are there in row X?

c Which row has 40 seats?

12 The number of people who go to see a movie over a period of a week follows an arithmetic sequence. On the first day only three people go to the movie, but on the sixth day 98 people go. Find the rule for the sequence and hence determine how many attend on the seventh day.

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13 An arithmetic sequence contains 10 terms. If the first is 4 and the tenth is 30, what is the eighth term?

14 The number of goals kicked by a team in the first six games of a season follows an arithmetic sequence. If the team kicked 5 goals in the first game and 15 in the sixth, how many did they kick in each of the other four games?

15 The first term of an arithmetic sequence is a and the mth term is 0. Find the rule for t_n for this sequence.

16 Find the arithmetic mean of:

a 8 and 15

b $\dfrac{1}{2\sqrt{2}-1}$ and $\dfrac{1}{2\sqrt{2}+1}$

17 Find x if $3x - 2$ is the arithmetic mean of $5x + 1$ and 11.

18 If a, $4a - 4$ and $8a - 13$ are successive terms of an arithmetic sequence, find a.

19 If $t_m = n$ and $t_n = m$, prove that $t_{m+n} = 0$. (Here t_m and t_n are the mth and nth terms of an arithmetic sequence).

20 If a, $2a$ and a^2 are consecutive terms of an arithmetic sequence, find a ($a \neq 0$).

21 Show that there is no infinite arithmetic sequence whose terms are all prime numbers.

4C Arithmetic series

The sum of the terms in a sequence is called a **series**. If the sequence is arithmetic, then the series is called an **arithmetic series**.

The symbol S_n is used to denote the sum of the first n terms of a sequence. That is,

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

Writing this sum in reverse order, we have

$$S_n = (a + (n - 1)d) + (a + (n - 2)d) + \cdots + (a + d) + a$$

Adding these two expressions together gives

$$2S_n = n(2a + (n - 1)d)$$

Therefore

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big)$$

Since the last term $\ell = t_n = a + (n - 1)d$, we can also write

$$S_n = \frac{n}{2}(a + \ell)$$

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Example 13

For the arithmetic sequence 2, 5, 8, 11, . . . , calculate the sum of the first 14 terms.

Solution

$a = 2,\ d = 3,\ n = 14$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\therefore \quad S_{14} = \frac{14}{2}(2 \times 2 + 13 \times 3) = 301$$

Example 14

For the arithmetic sequence 27, 23, 19, 15, . . . , −33, find:

a the number of terms

b the sum of the terms.

Solution

a $a = 27,\ d = -4,\ \ell = t_n = -33$

$$\begin{aligned} t_n &= a + (n-1)d \\ -33 &= 27 + (n-1)(-4) \\ -60 &= (n-1)(-4) \\ 15 &= n - 1 \\ n &= 16 \end{aligned}$$

There are 16 terms in the sequence.

b $a = 27,\ \ell = t_n = -33,\ n = 16$

$$S_n = \frac{n}{2}(a + \ell)$$

$$\begin{aligned} \therefore \quad S_{16} &= \frac{16}{2}(27 - 33) \\ &= -48 \end{aligned}$$

The sum of the terms is −48.

Example 15

For the arithmetic sequence 3, 6, 9, 12, . . . , calculate:

a the sum of the first 25 terms

b the number of terms in the series if $S_n = 1395$.

Solution

a $a = 3,\ d = 3,\ n = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{25} &= \frac{25}{2}\big(2(3) + (24)(3)\big) \\ &= 975 \end{aligned}$$

b $a = 3,\ d = 3,\ S_n = 1395$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1395 \\ \frac{n}{2}\big(2(3) + (n-1)(3)\big) &= 1395 \\ n(6 + 3n - 3) &= 2790 \\ 3n + 3n^2 &= 2790 \\ 3n^2 + 3n - 2790 &= 0 \\ n^2 + n - 930 &= 0 \\ (n - 30)(n + 31) &= 0 \end{aligned}$$

Therefore $n = 30$, since $n > 0$.

Hence there are 30 terms in the series.

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Example 16

A hardware store sells nails in a range of packet sizes. Packet A contains 50 nails, packet B contains 75 nails, packet C contains 100 nails, and so on.

a Find the number of nails in packet J.

b Lachlan buys one each of packets A to J. How many nails in total does Lachlan have?

c Assuming he buys one of each packet starting at A, how many packets does he need to buy to have a total of 1100 nails?

Solution

a $a = 50$, $d = 25$

$$t_n = a + (n-1)d$$

For packet J, we take $n = 10$:

$$\begin{aligned} t_{10} &= 50 + 9 \times 25 \\ &= 275 \end{aligned}$$

Packet J contains 275 nails.

b $a = 50$, $d = 25$

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big)$$

$$\begin{aligned} \therefore \quad S_{10} &= \frac{10}{2}\big(2 \times 50 + 9 \times 25\big) \\ &= 1625 \end{aligned}$$

Packets A to J contain 1625 nails.

c $a = 50$, $d = 25$, $S_n = 1100$

$$\begin{aligned} S_n = \frac{n}{2}\big(2a + (n-1)d\big) &= 1100 \\ \frac{n}{2}\big(2(50) + (n-1)(25)\big) &= 1100 \\ n(100 + 25n - 25) &= 2200 \\ 25n^2 + 75n - 2200 &= 0 \\ n^2 + 3n - 88 &= 0 \\ (n+11)(n-8) &= 0 \end{aligned}$$

Thus $n = 8$, since $n > 0$. If Lachlan buys one each of the first eight packets (A to H), he will have exactly 1100 nails.

Example 17

The sum of the first 10 terms of an arithmetic sequence is $48\frac{3}{4}$. If the fourth term is $3\frac{3}{4}$, find the first term and the common difference.

Solution

$$t_4 = a + 3d = 3\tfrac{3}{4}$$

$$\therefore \quad a + 3d = \frac{15}{4} \qquad (1)$$

$$S_{10} = \frac{10}{2}(2a + 9d) = 48\tfrac{3}{4}$$

$$\therefore \quad 10a + 45d = \frac{195}{4} \qquad (2)$$

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Solve equations (1) and (2) simultaneously:

$$(1)\times 40: \quad 40a + 120d = 150$$
$$(2)\times 4: \quad 40a + 180d = 195$$
$$60d = 45$$
$$\therefore \quad d = \frac{3}{4}$$

Substitute in (1) to obtain $a + 3\left(\frac{3}{4}\right) = \frac{15}{4}$ and therefore $a = \frac{3}{2}$.

The first term is $1\frac{1}{2}$ and the common difference is $\frac{3}{4}$.

Section summary

The sum of the first n terms of an arithmetic sequence

$$S_n = a + (a + d) + (a + 2d) + \cdots + (a + (n - 1)d)$$

is given by

$$S_n = \frac{n}{2}\big(2a + (n - 1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Exercise 4C

Skillsheet

Example 13

1 For each arithmetic sequence, find the specified sum:

a $8, 13, 18, \ldots$ find S_{12}

b $-3.5, -1.5, 0.5, \ldots$ find S_{10}

c $\frac{1}{\sqrt{2}}, \sqrt{2}, \frac{3}{\sqrt{2}}, \ldots$ find S_{15}

d $-4, 1, 6, \ldots$ find S_8

2 Greg goes fishing every day for a week. On the first day he catches seven fish and each day he catches three more than the previous day. How many fish did he catch in total?

3 Find the sum of the first 16 multiples of 5.

4 Find the sum of all the even numbers between 1 and 99.

Example 14

5 For the arithmetic sequence $-3, 1, 5, 9, \ldots, 49$, find:

a the number of terms

b the sum of the terms.

6 For the arithmetic sequence $24, 20, 16, 12, \ldots, -52$, find:

a the number of terms

b the sum of the terms.

7 For the arithmetic sequence $\frac{1}{2}, 2, \frac{7}{2}, 5, \ldots, 17$, find:

a the number of terms

b the sum of the terms.

Example 15

8 For the sequence $4, 8, 12, \ldots$, find:

a the sum of the first 9 terms

b $\{\, n : S_n = 180 \,\}$.

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9 There are 110 logs to be put in a pile, with 15 logs in the bottom layer, 14 in the next, 13 in the next, and so on. How many layers will there be?

10 The sum of the first m terms of an arithmetic sequence with first term -5 and common difference 4 is 660. Find m.

11 Evaluate $54 + 48 + 42 + \cdots + (-54)$.

Example 16 **12** Dora's walking club plans 15 walks for the summer. The first walk is a distance of 6 km, the last walk is a distance of 27 km, and the distances of the walks form an arithmetic sequence.

a How far is the 8th walk?

b How far does the club plan to walk in the first five walks?

c Dora's husband, Alan, can only complete the first n walks. If he walks a total of 73.5 km, how many walks does he complete?

d Dora goes away on holiday and misses the 9th, 10th and 11th walks, but completes all other walks. How far does Dora walk in total?

13 Liz has to proofread 500 pages of a new novel. She plans to read 30 pages on the first day and to increase the number of pages she reads by five each day.

a How many days will it take her to complete the proofreading?

She has only five days to complete the task. She therefore decides to read 50 pages on the first day and to increase the number she reads by a constant amount each day.

b By how many should she increase the number of pages she reads each day if she is to meet her deadline?

14 An assembly hall has 50 seats in row A, 54 seats in row B, 58 seats in row C, and so on. That is, there are four more seats in each row.

a How many seats are there in row J?

b How many seats are there altogether if the back row is row Z?

On a particular day, the front four rows are reserved for parents (and there is no other seating for parents).

c How many parents can be seated?

d How many students can be seated?

The hall is extended by adding more rows following the same pattern.

e If the final capacity of the hall is 3410, how many rows were added?

15 A new golf club is formed with 40 members in its first year. Each following year, the number of new members exceeds the number of retirements by 15. Each member pays \$120 p.a. in membership fees. Calculate the amount received from fees in the first 12 years of the club's existence.

Example 17 **16** For the arithmetic sequence with $t_2 = -12$ and $S_{12} = 18$, find a, d, t_6 and S_6.

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17 The sum of the first 10 terms of an arithmetic sequence is 120, and the sum of the first 20 terms is 840. Find the sum of the first 30 terms.

18 If $t_6 = 16$ and $t_{12} = 28$, find S_{14}.

19 For an arithmetic sequence, find t_n if:

a $t_3 = 6.5$ and $S_8 = 67$

b $t_4 = \dfrac{6}{\sqrt{5}}$ and $S_5 = 16\sqrt{5}$

20 For the sequence with $t_n = bn$, where $b \in \mathbb{R}$, find:

a $t_{n+1} - t_n$

b $t_1 + t_2 + \cdots + t_n$

21 For a sequence where $t_n = 15 - 5n$, find t_5 and find the sum of the first 25 terms.

22 An arithmetic sequence has a common difference of d and the sum of the first 20 terms is 25 times the first term. Find the sum of the first 30 terms in terms of d.

23 The sum of the first n terms of a particular sequence is given by $S_n = 17n - 3n^2$.

a Find an expression for the sum of the first $(n - 1)$ terms.

b Find an expression for the nth term of the sequence.

c Show that the sequence is arithmetic and find a and d.

24 Three consecutive terms of an arithmetic sequence have a sum of 36 and a product of 1428. Find the three terms.

25 Show that the sum of the first $2n$ terms of an arithmetic sequence is n times the sum of the two middle terms.

26 Find the sum of numbers between 1 and 120 inclusive that are multiples of 2 or 3.

27 Find all arithmetic sequences consisting of four positive integers whose sum is 100.

28 How many triangles have three angles that are positive integers in an arithmetic sequence?

4D Geometric sequences

A sequence in which each successive term is found by multiplying the previous term by a fixed amount is called a **geometric sequence**. That is, a geometric sequence has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant.

For example: $2, 6, 18, 54, \dots$ is a geometric sequence.

> The nth term of a geometric sequence is given by
>
> $$t_n = ar^{n-1}$$
>
> where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

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Example 18

Find the 10th term of the sequence $2, 6, 18, \ldots$.

Solution

$a = 2$, $r = 3$

$$\begin{aligned} t_n &= ar^{n-1} \\ \therefore \quad t_{10} &= 2 \times 3^{10-1} \\ &= 39\,366 \end{aligned}$$

Example 19

For a geometric sequence, the first term is 18 and the fourth term is 144. Find the common ratio.

Solution

$a = 18$, $t_4 = 144$

$$\begin{aligned} t_4 = 18 \times r^{4-1} &= 144 \\ 18r^3 &= 144 \\ r^3 &= 8 \\ \therefore \quad r &= 2 \end{aligned}$$

The common ratio is 2.

Example 20

For a geometric sequence $36, 18, 9, \ldots$, the nth term is $\frac{9}{16}$. Find the value of n.

Solution

$a = 36$, $r = \frac{1}{2}$

$$\begin{aligned} t_n = 36 \times \left(\frac{1}{2}\right)^{n-1} &= \frac{9}{16} \\ \left(\frac{1}{2}\right)^{n-1} &= \frac{9}{576} \\ \left(\frac{1}{2}\right)^{n-1} &= \frac{1}{64} \\ \left(\frac{1}{2}\right)^{n-1} &= \left(\frac{1}{2}\right)^{6} \\ n - 1 &= 6 \\ \therefore \quad n &= 7 \end{aligned}$$

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Example 21

The third term of a geometric sequence is 10 and the sixth term is 80. Find the common ratio and the first term.

Solution

$$t_3 = ar^2 = 10 \qquad (1)$$
$$t_6 = ar^5 = 80 \qquad (2)$$

Divide (2) by (1):

$$\frac{ar^5}{ar^2} = \frac{80}{10}$$
$$r^3 = 8$$
$$\therefore \quad r = 2$$

Substitute in (1):

$$a \times 4 = 10$$
$$\therefore \quad a = \frac{5}{2}$$

The common ratio is 2 and the first term is $\frac{5}{2}$.

Example 22

Georgina draws a pattern consisting of a number of equilateral triangles. The first triangle has sides of length 4 cm and the side length of each successive triangle is one and a half times the side length of the previous one.

a What is the side length of the fifth triangle?

b Which triangle has a side length of $45\frac{9}{16}$ cm?

Solution

a $a = 4,\ r = \frac{3}{2}$

$$t_n = ar^{n-1}$$
$$\therefore \quad t_5 = ar^4 = 4 \times \left(\frac{3}{2}\right)^4$$
$$= 20\tfrac{1}{4}$$

The fifth triangle has a side length of $20\frac{1}{4}$ cm.

b $a = 4,\ r = \frac{3}{2},\ t_n = 45\frac{9}{16}$

$$t_n = ar^{n-1} = 45\tfrac{9}{16}$$
$$4 \times \left(\frac{3}{2}\right)^{n-1} = \frac{729}{16}$$
$$\left(\frac{3}{2}\right)^{n-1} = \frac{729}{64} = \left(\frac{3}{2}\right)^6$$

Therefore $n - 1 = 6$ and so $n = 7$.

The seventh triangle has a side length of $45\frac{9}{16}$ cm.

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Compound interest

One application of geometric sequences is **compound interest**. Compound interest is interest calculated at regular intervals on the total of the amount originally invested and the amount accumulated in the previous years.

Assume that \$1000 is invested at 10% per annum. At the end of the first year, the amount will have grown to

$$1000 + 10\%(1000) = \$1100$$

At the end of the second year, it will have grown to

$$(1000 + 10\%(1000)) + 10\%(1000 + 10\%(1000)) = \$1210$$

The value of the investment at the end of each year forms a geometric sequence. For this example, we have

$$a = 1000 \text{ and } r = 1.1, \quad \text{i.e. } r = 100\% + 10\%$$

Example 23

Marta invests \$2500 at 7% p.a. compounded annually. Find:

a the value of her investment after 5 years

b how long it takes until her investment is worth \$10 000.

Solution

Let t_n be the value at the end of the $(n-1)$st year. Then $a = 2500$ and $r = 1.07$.

a

$$\begin{aligned} t_6 &= ar^5 \\ &= 2500(1.07)^5 \\ &= 3506.38 \end{aligned}$$

The value of the investment after 5 years is \$3506.38.

b

$$\begin{aligned} t_n = ar^{n-1} &= 10\,000 \\ 2500(1.07)^{n-1} &= 10\,000 \\ 1.07^{n-1} &= 4 \\ \log_{10}(1.07^{n-1}) &= \log_{10} 4 \\ (n-1)\log_{10}(1.07) &= \log_{10} 4 \\ n - 1 &= \frac{\log_{10} 4}{\log_{10}(1.07)} \\ \therefore \quad n &\approx 21.49 \end{aligned}$$

By the end of the 21st year, the investment will be worth over \$10 000.

Note: For part **b**, the number of years can also be found by trial and error or by using a CAS calculator.

Geometric mean

The **geometric mean** of two numbers a and b is defined as $\sqrt{ab}$.

If positive numbers a, c, b are consecutive terms of a geometric sequence, then

$$\frac{c}{a} = \frac{b}{c} \qquad \therefore \quad c = \sqrt{ab}$$

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Section summary

- A **geometric sequence** has a recurrence relation of the form $t_n = rt_{n-1}$, where r is a constant. Each successive term is found by multiplying the previous term by a fixed amount. For example: $2, 6, 18, 54, \ldots$
- The nth term of a geometric sequence is given by

  $$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** of successive terms, that is, $r = \dfrac{t_k}{t_{k-1}}$, for all $k > 1$.

Exercise 4D

1 For a geometric sequence $t_n = ar^{n-1}$, find the first four terms given that:

a $a = 3, \ r = 2$ **b** $a = 3, \ r = -2$

c $a = 10\,000, \ r = 0.1$ **d** $a = r = 3$

Example 18

2 Find the specified term in each of the following geometric sequences:

a $\frac{15}{7}, \frac{5}{7}, \frac{5}{21}, \ldots$ find t_6 **b** $1, -\frac{1}{4}, \frac{1}{16}, \ldots$ find t_5

c $\sqrt{2}, 2, 2\sqrt{2}, \ldots$ find t_{10} **d** $a^x, a^{x+1}, a^{x+2}, \ldots$ find t_6

3 Find the rule for the geometric sequence whose first few terms are:

a $3, 2, \frac{4}{3}$ **b** $2, -4, 8, -16$ **c** $2, 2\sqrt{5}, 10$

Example 19

4 Find the common ratio for the following geometric sequences:

a the first term is 2 and the sixth term is 486

b the first term is 25 and the fifth term is $\frac{16}{25}$

Example 20

5 A geometric sequence has first term $\frac{1}{4}$ and common ratio 2. Which term of the sequence is 64?

6 If t_n is the nth term of the following geometric sequences, find n in each case:

a $2, 6, 18, \ldots$ $t_n = 486$ **b** $5, 10, 20, \ldots$ $t_n = 1280$

c $768, 384, 192, \ldots$ $t_n = 3$ **d** $\frac{8}{9}, \frac{4}{3}, 2, \ldots$ $t_n = \frac{27}{4}$

e $-\frac{4}{3}, \frac{2}{3}, -\frac{1}{3}, \ldots$ $t_n = \frac{1}{96}$

Example 21

7 The 12th term of a geometric sequence is 2 and the 15th term is 54. Find the 7th term.

8 A geometric sequence has $t_2 = \dfrac{1}{2\sqrt{2}}$ and $t_4 = \sqrt{2}$. Find t_8.

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9 The number of fish in the breeding tanks of a fish farm follow a geometric sequence. The third tank contains 96 fish and the sixth tank contains 768 fish.

a How many fish are in the first tank? b How many fish are in the 10th tank?

Example 22 10 An algal bloom is growing in a lake. The area it covers triples each day.

a If it initially covers an area of 10 m^2, how many square metres will it cover after one week?

b If the lake has a total area of 200 000 m^2, how long before the entire lake is covered?

11 A ball is dropped from a height of 2 m and continues to bounce so that it rebounds to three-quarters of the height from which it previously falls. Find the height it rises to on the fifth bounce.

12 An art collector has a painting that is increasing in value by 8% each year. If the painting is currently valued at $2500:

a How much will it be worth in 10 years?

b How many years before its value exceeds $100 000?

13 The Tour de Moravia is a cycling event which lasts for 15 days. On the first day the cyclists must ride 120 km, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride on the 8th day?

b On which day do they ride 30.5 km?

14 A child negotiates a new pocket-money deal with her unsuspecting father in which she receives 1 cent on the first day of the month, 2 cents on the second, 4 cents on the third, 8 cents on the fourth, and so on, until the end of the month. How much would the child receive on the 30th day of the month? (Give your answer to the nearest thousand dollars.)

Example 23 15 $5000 is invested at 6% p.a. compounded annually.

a Find the value of the investment after 6 years.

b Find how long it will take for the original investment to double in value.

16 How much would need to be invested at 8.5% p.a. compounded annually to yield a return of $8000 after 12 years?

17 What annual compound interest rate would be required to triple the value of an investment of $200 in 10 years?

18 The first three terms of a geometric sequence are $4, 8, 16$. Find the first term in the sequence which exceeds 2000.

19 The first three terms of a geometric sequence are $3, 9, 27$. Find the first term in the sequence which exceeds 500.

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20 The number of 'type A' apple bugs present in an orchard is estimated to be 40 960, and the number is reducing by 50% each week. At the same time it is estimated that there are 40 'type B' apple bugs, whose number is doubling each week. After how many weeks will there be the same number of each type of bug?

21 Find the geometric mean of:

a 5 and 720 **b** 1 and 6.25 **c** $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ **d** x^2y^3 and x^6y^{11}

22 The fourth, seventh and sixteenth terms of an arithmetic sequence also form consecutive terms of a geometric sequence. Find the common ratio of the geometric sequence.

23 Consider the geometric sequence $1, a, a^2, a^3, \dots$. Suppose that the sum of two consecutive terms in the sequence gives the next term in the sequence. Find a.

24 A bottle contains 1000 mL of pure ethanol. Then 300 mL is removed and the bottle is topped up with pure water. The mixture is stirred.

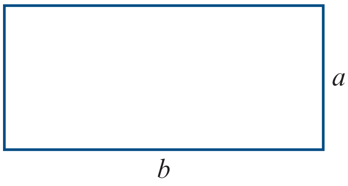
a What is the volume of ethanol in the bottle if this process is repeated five times in total?

b How many times should the process be repeated for there to be less than 1 mL of ethanol in the bottle?

25 The rectangle shown has side lengths a and b.

a Find the side length of a square with the same perimeter. Comment.

b Find the side length of a square with the same area. Comment.



4E Geometric series

The sum of the terms in a geometric sequence is called a **geometric series**. An expression for S_n, the sum of the first n terms of a geometric sequence, can be found using a similar method to that used for arithmetic series.

Let $\qquad S_n = a + ar + ar^2 + \cdots + ar^{n-1} \qquad (1)$

Then $\qquad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n \qquad (2)$

Subtract (1) from (2):

$$rS_n - S_n = ar^n - a$$

$$S_n(r-1) = a(r^n - 1)$$

Therefore

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

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For values of r such that $-1 < r < 1$, it is often more convenient to use the equivalent formula

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

which is obtained by multiplying both the numerator and the denominator by -1.

Example 24

Find the sum of the first nine terms of the sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$.

Solution

$a = \frac{1}{3}$, $r = \frac{1}{3}$, $n = 9$

$$\begin{aligned} \therefore \quad S_9 &= \frac{\frac{1}{3}\left(1 - \left(\frac{1}{3}\right)^9\right)}{1 - \frac{1}{3}} \\ &= \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^9\right) \\ &\approx 0.499975 \end{aligned}$$

Example 25

For the geometric sequence $1, 3, 9, \ldots$, find how many terms must be added together to obtain a sum of 1093.

Solution

$a = 1$, $r = 3$, $S_n = 1093$

$$\begin{aligned} S_n = \frac{a(r^n - 1)}{r - 1} &= 1093 \\ \frac{1(3^n - 1)}{3 - 1} &= 1093 \\ 3^n - 1 &= 1093 \times 2 \\ \therefore \quad 3^n &= 2187 \end{aligned}$$

A CAS calculator can be used to find $n = 7$.

Seven terms are required to give a sum of 1093.

Example 26

In the 15-day Tour de Moravia, the cyclists must ride 120 km on the first day, and each successive day they ride 90% of the distance of the previous day.

a How far do they ride in total to the nearest kilometre?

b After how many days will they have ridden half that distance?

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Solution

a $a = 120$, $r = 0.9$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore \quad S_{15} = \frac{120(1-(0.9)^{15})}{1-0.9}$$

$$= 952.93$$

They ride 953 km.

b $a = 120$, $r = 0.9$, $S_n = 476.5$

$$S_n = \frac{a(1-r^n)}{1-r} = 476.5$$

$$\frac{120(1-(0.9)^n)}{1-0.9} = 476.5$$

$$1 - 0.9^n = \frac{476.5 \times 0.1}{120}$$

$$1 - 0.9^n = 0.3971$$

$$0.9^n = 1 - 0.3971$$

$$\therefore \quad 0.9^n = 0.6029$$

A CAS calculator can be used to find $n \approx 4.8$. Thus they pass the halfway mark on the fifth day.

Section summary

The sum of the first n terms of a geometric sequence

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

is given by

$$S_n = \frac{a(r^n - 1)}{r-1} \quad \text{or} \quad S_n = \frac{a(1-r^n)}{1-r}$$

Exercise 4E

Example 24

1 Find the specified sum for each of the following geometric series:

a $5 + 10 + 20 + \cdots$ find S_{10}

b $1 - 3 + 9 - \cdots$ find S_6

c $-\frac{4}{3} + \frac{2}{3} - \frac{1}{3} + \cdots$ find S_9

2 Find:

a $2 - 6 + 18 - \cdots + 1458$

b $-4 + 8 - 16 + \cdots - 1024$

c $6250 + 1250 + 250 + \cdots + 2$

Example 25

3 For the geometric sequence $3, 6, 12, \ldots$, find how many terms must be added together to obtain a sum of 3069.

4 For the geometric sequence $24, -12, 6, \ldots$, find how many terms must be added together to obtain a sum of $16\frac{1}{8}$.

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Example 26

5 Gerry owns a milking cow. On the first day he milks the cow, it produces 600 mL of milk. On each successive day, the amount of milk increases by 10%.

a How much milk does the cow produce on the seventh day?

b How much milk does it produce in the first week?

c After how many days will it have produced a total in excess of 10 000 mL?

6 On Monday, William spends 20 minutes playing the piano. On Tuesday, he spends 25 minutes playing, and on each successive day he increases the time he spends playing in the same ratio.

a For how many minutes does he play on Friday?

b How many minutes in total does he play from Monday to Friday?

c On which day of the following week will his total time playing pass 15 hours?

7 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound, it rises to two-thirds of the height of the previous rebound. What total distance will it have travelled when it strikes the ground for the 10th time?

8 An insurance broker makes \$15 000 commission on sales in her first year. Each year, she increases her sales by 5%.

a How much commission would she make in her fifth year?

b How much commission would she make in total over 5 years?

9 Andrew invests \$1000 at 20% simple interest for 10 years. Bianca invests her \$1000 at 12.5% compound interest for 10 years. At the end of 10 years, whose investment is worth more?

10 For a geometric sequence with nth term t_n:

a if $t_3 = 20$ and $t_6 = 160$, find S_5

b if $t_3 = \sqrt{2}$ and $t_8 = 8$, find S_8.

11 **a** How many terms of the geometric sequence where $t_1 = 1, t_2 = 2, t_3 = 4, \dots$ must be taken for $S_n = 255$?

b Let $S_n = 1 + 2 + 4 + \cdots + 2^{n-1}$. Find $\{ n : S_n > 1\,000\,000 \}$.

12 Find $1 - x^2 + x^4 - x^6 + x^8 - \cdots + x^{2m}$, where m is even.

13 A sheet of A4 paper is about 0.05 mm thick. The paper is torn in half, and each half is again torn in half, and this process is repeated for a total of 40 times.

a How high will the stack of paper be if the pieces are placed one on top of the other?

b How many times would the process have to be repeated for the stack to first reach the moon, 384 400 km away?

14 Which would you prefer: \$1 million given to you every week for a year, or 1c in the first week, 2c in the second, 4c in the third, and so on, until the end of the year?

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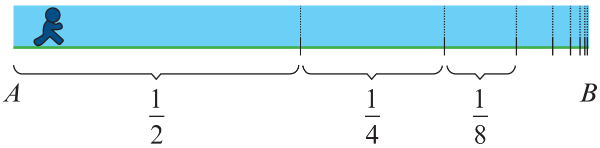
4F Zeno's paradox and infinite geometric series

A runner wants to go from point A to point B. To do this, he would first have to run half the distance, then half the remaining distance, then half the remaining distance, and so on.



The Greek philosopher Zeno of Elea, who lived about 450 BC, argued that since the runner has to complete an infinite number of stages to get from A to B, he cannot do this in a finite amount of time, and so he cannot reach B. In this section we see how to resolve this paradox.

▶ Infinite geometric series

If a geometric sequence has a common ratio with magnitude less than 1, that is, if $-1 < r < 1$, then each successive term is closer to zero. For example, consider the sequence

$$\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$$

In Example 24 we found that the sum of the first 9 terms is $S_9 \approx 0.499975$. The sum of the first 20 terms is $S_{20} \approx 0.49999999986$. We might conjecture that, as we add more and more terms of the sequence, the sum will get closer and closer to 0.5, that is, $S_n \to 0.5$ as $n \to \infty$.

An infinite series $t_1 + t_2 + t_3 + \cdots$ is said to be **convergent** if the sum of the first n terms, S_n, approaches a limiting value as $n \to \infty$. This limit is called the **sum to infinity** of the series.

If $-1 < r < 1$, then the infinite geometric series $a + ar + ar^2 + \cdots$ is convergent and the sum to infinity is given by

$$S_\infty = \frac{a}{1-r}$$

Proof We know that

$$S_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

As $n \to \infty$, we have $r^n \to 0$ and so $\dfrac{ar^n}{1-r} \to 0$. Hence $S_n \to \dfrac{a}{1-r}$ as $n \to \infty$.

Resolution of Zeno's paradox Assume that the runner is travelling at a constant speed and that he takes 1 minute to run half the distance from A to B. Then he takes $\frac{1}{2}$ minute to run half the remaining distance, and so on. The total time taken is

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

This is an infinite geometric series, and the formula gives $S_\infty = \dfrac{a}{1-r} = \dfrac{1}{1-\frac{1}{2}} = 2$.

This fits with our common sense: If the runner takes 1 minute to cover half the distance, then he will take 2 minutes to cover the whole distance.

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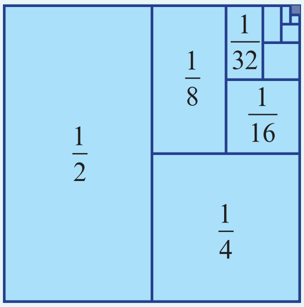
Example 27

Find the sum to infinity of the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

Solution

$a = \frac{1}{2}$, $r = \frac{1}{2}$ and so $S_\infty = \dfrac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$



Note: This result is illustrated by the unit square shown. Divide the square in two, then divide one of the resulting rectangles in two, and so on. The sum of the areas of the rectangles equals the area of the square.

Example 28

A square has a side length of 40 cm. A copy of the square is made so that the area of the copy is 80% of the original. The process is repeated so that each time the area of the new square is 80% of the previous one. If this process is repeated indefinitely, find the total area of all the squares.

Solution

The area of the first square is $40^2 = 1600\text{ cm}^2$.

We have $a = 1600$ and $r = 0.8$, giving

$$S_\infty = \frac{1600}{1 - 0.8} = 8000\text{ cm}^2$$

Example 29

Express the recurring decimal $0.\dot{3}\dot{2}$ as the ratio of two integers.

Solution

$$0.\dot{3}\dot{2} = 0.32 + 0.0032 + 0.000032 + \cdots$$

We have $a = 0.32$ and $r = 0.01$, giving

$$S_\infty = \frac{0.32}{0.99} = \frac{32}{99}$$

i.e. $0.\dot{3}\dot{2} = \dfrac{32}{99}$

Exercise 4F

Skillsheet

Example 27

1 Find:

a $1 + \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \cdots$

b $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots$

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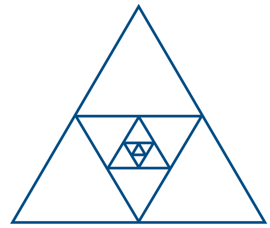
Example 28

2 An equilateral triangle has perimeter p cm. The midpoints of the sides are joined to form another triangle, and this process is repeated. Find the perimeter and area of the nth triangle, and find the limits as $n \to \infty$ of the sum of the perimeters and the sum of the areas of the first n triangles.



3 A rocket is launched into the air so that it reaches a height of 200 m in the first second. Each subsequent second it gains 6% less height. Find how high the rocket will climb.

4 A patient has an infection that, if it exceeds a certain level, will kill him. He is given a drug to inhibit the spread of the infection. The drug acts such that each day the level of infection only increases by 65% of the previous day's level. The level of infection on the first day is 450, and the critical level is 1280. Will the infection kill him?

5 A man can walk 3 km in the first hour of a journey, but in each succeeding hour walks half the distance covered in the preceding hour. Can he complete a journey of 6 km? Where does this problem cease to be realistic?

6 A frog standing 10 m from the edge of a pond sets out to jump towards it. Its first jump is 2 m, its second jump is $1\frac{1}{2}$ m, its third jump is $1\frac{1}{8}$ m, and so on. Show that the frog will never reach the edge of the pond.

7 A computer virus acts in such a way that on the first day it blocks out one-third of the area of the screen of an infected computer. Each successive day it blocks out more of the screen: an area one-third of that it blocked the previous day. If this continues indefinitely, what percentage of the screen will eventually be blocked out?

8 A stone is thrown so that it skips across the surface of a lake. If each skip is 30% less that the previous skip, how long should the first skip be so that the total distance travelled by the stone is 40 m?

9 A ball dropped from a height of 15 m rebounds from the ground to a height of 10 m. With each successive rebound it rises two-thirds of the height of the previous rebound. If it continues to bounce indefinitely, what is the total distance it will travel?

Example 29

10 Express each of the following periodic decimals as the ratio of a pair of integers:

a $0.\dot{4}$ **b** $0.0\dot{3}$ **c** $10.\dot{3}$ **d** $0.0\dot{3}\dot{5}$ **e** $0.\dot{9}$ **f** $4.\dot{1}$

11 The sum of the first four terms of a geometric series is 30 and the sum to infinity is 32. Find the first two terms.

12 Find the third term of a geometric sequence that has a common ratio of $-\dfrac{1}{4}$ and a sum to infinity of 8.

13 Find the common ratio of a geometric sequence with first term 5 and sum to infinity 15.

14 For any number $x > 2$, show that there is an infinite geometric series such that $a = 2$ and the sum to infinity is x.

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Review

Chapter summary

AS

Nrich

- The nth term of a sequence is denoted by t_n.
- A **recurrence relation** enables each subsequent term to be found from previous terms. A sequence specified in this way is said to be defined **recursively**.

  e.g. $t_1 = 1, \quad t_n = t_{n-1} + 2$
- A sequence may also be defined by a rule that is stated in terms of n.

  e.g. $t_n = 2n$

Arithmetic sequences and series

- An **arithmetic sequence** has a rule of the form

$$t_n = a + (n-1)d$$

  where a is the first term and d is the **common difference** (i.e. $d = t_k - t_{k-1}$ for all $k > 1$).
- The **arithmetic mean** of two numbers a and b is $\dfrac{a+b}{2}$.
- The sum of the terms in an arithmetic sequence is called an **arithmetic series**.
- The sum of the first n terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}\big(2a + (n-1)d\big) \quad \text{or} \quad S_n = \frac{n}{2}(a + \ell), \text{ where } \ell = t_n$$

Geometric sequences and series

- A **geometric sequence** has a rule of the form

$$t_n = ar^{n-1}$$

  where a is the first term and r is the **common ratio** (i.e. $r = \dfrac{t_k}{t_{k-1}}$ for all $k > 1$).
- The **geometric mean** of two numbers a and b is $\sqrt{ab}$.
- The sum of the terms in a geometric sequence is called a **geometric series**.
- For $r \neq 1$, the sum of the first n terms of a geometric sequence is given by

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

- For $-1 < r < 1$, the sum S_n approaches a limiting value as $n \to \infty$, and the series is said to be **convergent**. This limit is called the **sum to infinity** and is given by $S_\infty = \dfrac{a}{1-r}$.

Technology-free questions

1 Find the first six terms of the following sequences:

a $t_1 = 3, \; t_n = t_{n-1} - 4$ **b** $t_1 = 5, \; t_n = 2t_{n-1} + 2$

2 Find the first six terms of the following sequences:

a $t_n = 2n$ **b** $t_n = -3n + 2$

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3 Nick invests \$5000 at 5% p.a. compound interest at the beginning of the year. At the beginning of each of the following years, he puts a further \$500 into the account.

- **a** Write down the amount of money in the account at the end of each of the first two years.
- **b** Set up a recurrence relation to generate the sequence for the investment.

4 The 4th term of an arithmetic sequence is 19 and the 7th term is 43. Find the 20th term.

5 For an arithmetic sequence with $t_5 = 0.35$ and $t_9 = 0.15$, find t_{14}.

6 For an arithmetic sequence with $t_6 = -24$ and $t_{14} = 6$, find S_{10}.

7 For the arithmetic sequence $-5, 2, 9, \ldots$, find $\{\, n : S_n = 402 \,\}$.

8 The 6th term of a geometric sequence is 9 and the 10th term is 729. Find the 4th term.

9 One thousand dollars is invested at 3.5% p.a. compounded annually. Find the value of the investment after n years.

10 The first term of a geometric sequence is 9 and the third term is 4. Find the possible values for the second and fourth terms.

11 The sum of three consecutive terms of a geometric sequence is 24 and the sum of the next three terms is also 24. Find the sum of the first 12 terms.

12 Find the sum of the first eight terms of the geometric sequence with first term 6 and common ratio -3.

13 Find the sum to infinity of $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.

14 The numbers x, $x + 4$, $2x + 2$ are three successive terms of a geometric sequence. Find the value of x.

Multiple-choice questions

1 The first three terms of the sequence defined by the rule $t_n = 3n + 2$ are

A $1, 2, 3$ **B** $2, 4, 6$ **C** $5, 7, 9$ **D** $5, 8, 11$ **E** $5, 8, 10$

2 If $t_1 = 3$ and $t_{n+1} = t_n + 3$, then t_4 is

A 4 **B** 12 **C** 9 **D** 15 **E** 14

3 For the arithmetic sequence $10, 8, 6, \ldots$, we have $t_{10} =$

A -8 **B** -10 **C** -12 **D** 10 **E** 8

4 For the arithmetic sequence $10, 8, 6, \ldots$, we have $S_{10} =$

A 10 **B** 0 **C** -10 **D** 20 **E** -20

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5 If 58 is the nth term of the arithmetic sequence 8, 13, 18, . . . , then $n =$

A 12 **B** 11 **C** 10 **D** 5 **E** 3

6 The sixth term of the geometric sequence $12, 8, \frac{16}{3}, \ldots$ is

A $\frac{16}{3}$ **B** $\frac{128}{27}$ **C** $\frac{64}{81}$ **D** $\frac{128}{81}$ **E** $\frac{256}{81}$

7 For the sequence $8, 4, 2, \ldots$, we have $S_6 =$

A $\frac{1}{4}$ **B** $15\frac{1}{2}$ **C** $15\frac{7}{8}$ **D** 15 **E** $15\frac{3}{4}$

8 For the sequence $8, 4, 2, \ldots$, we have $S_\infty =$

A $\frac{1}{2}$ **B** 0 **C** 16 **D** 4 **E** ∞

9 If \$2000 is invested at 5.5% p.a. compounded annually, the value of the investment after 6 years is

A \$13 766.10 **B** \$11 162.18 **C** \$2550.00 **D** \$2613.92 **E** \$2757.69

10 If $S_\infty = 37.5$ and $r = \frac{1}{3}$, then a equals

A $\frac{2}{3}$ **B** 12.5 **C** $16\frac{2}{3}$ **D** 25 **E** 56.25

Extended-response questions

1 A do-it-yourself picture-framing kit is available in various sizes. Size 1 contains 0.8 m of moulding, size 2 contains 1.5 m, size 3 contains 2.2 m, and so on.

a Form the sequence of lengths of moulding.

b Is the sequence of lengths of moulding an arithmetic sequence?

c Find the length of moulding contained in the largest kit, size 12.

2 A firm proposes to sell coated seeds in packs containing the following number of seeds: 50, 75, 100, 125, ...

a Is this an arithmetic sequence?

b Find a formula for the nth term.

c Find the number of seeds in the 25th size packet.

3 A number of telegraph poles are to be placed in a straight line between two towns, A and B, which are 32 km apart. The first is placed 5 km from town A, and the last is placed 3 km from town B. The poles are placed so that the intervals starting from town A and finishing at town B are

$$5,\ 5-d,\ 5-2d,\ 5-3d,\ \ldots,\ 5-6d,\ 3$$

There are seven poles. How far is the fifth pole from town A, and how far is it from town B?

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4 A new electronic desktop telephone exchange, for use in large organisations, is available in various sizes.

Size 1 can handle 20 internal lines.
Size 2 can handle 36 internal lines.
Size 3 can handle 52 internal lines.
Size 4 can handle 68 internal lines, and so on.
Size n can handle T_n internal lines.

a Continue the sequence up to T_8.
b Write down a formula for T_n in terms of n.
c A customer needs an exchange to handle 196 lines. Is there a version of the desktop exchange which will just do this? If so, which size is it? If not, which is the next largest size?

5 A firm makes nylon thread in the following deniers (thicknesses):

$$2, 9, 16, 23, 30, \ldots$$

a Find the denier number, D_n, of the firm's nth thread in order of increasing thickness.

A request came in for some very heavy 191 denier thread, but this turned out to be one stage beyond the thickest thread made by the firm.

b How many different thicknesses does the firm make?

6 A new house appears to be slipping down a hillside. The first year it slipped 4 mm, the second year 16 mm, and the third year 28 mm. If it goes on like this, how far will it slip during the 40th year?

7 Anna sends 16 Christmas cards the first year, 24 the second year, 32 the next year, and so on. How many Christmas cards will she have sent altogether after 10 years if she keeps increasing the number sent each year in the same way?

8 Each time Lee rinses her hair after washing it, the result is to remove a quantity of shampoo from her hair. With each rinse, the quantity of shampoo removed is one-tenth of that removed by the previous rinse.

a If Lee washes out 90 mg of shampoo with the first rinse, how much will she have washed out altogether after six rinses?
b How much shampoo do you think was present in her hair at the beginning?

9 A prisoner is trapped in an underground cell, which is inundated by a sudden rush of water. The water comes up to a height of 1 m, which is one-third of the height of the ceiling (3 m). After an hour another inundation occurs, and the water level in the cell rises by $\frac{1}{3}$ m. After a second hour another inundation raises the water level by $\frac{1}{9}$ m. If this process continues for 6 hours, write down:

a the amount the water level will rise at the end of the sixth hour
b the total height of the water level then.

If this process continues, do you think the prisoner, who cannot swim, will drown? Why?

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Review

10 After an undetected leak in a storage tank, the staff at an experimental station were subjected to 500 curie hours of radiation the first day, 400 the second day, 320 the third day, and so on. Find the number of curie hours they were subjected to:

a on the 14th day

b during the first five days of the leak.

11 A rubber ball is dropped from a height of 81 m. Each time it strikes the ground, it rebounds two-thirds of the distance through which it has fallen.

a Find the height that the ball reaches after the sixth bounce.

b Assuming that the ball continues to bounce indefinitely, find the total distance travelled by the ball.

12 In payment for loyal service to the king, a wise peasant asked to be given one grain of rice for the first square of a chessboard, two grains for the second square, four for the third square, and so on for all 64 squares of the board. The king thought that this seemed fair and readily agreed, but was horrified when the court mathematician informed him how many grains of rice he would have to pay the peasant. How many grains of rice did the king have to pay? (Leave your answer in index form.)

13 **a** In its first month of operation, a cement factory, A, produces 4000 tonnes of cement. In each successive month, production rises by 250 tonnes per month. This growth in production is illustrated for the first five months in the table shown.

Month number (n)	1	2	3	4	5
Cement produced (tonnes)	4000	4250	4500	4750	5000

i Find an expression, in terms of n, for the amount of cement produced in the nth month.

ii Find an expression, in terms of n, for the total amount of cement produced in the first n months.

iii In which month is the amount of cement produced 9250 tonnes?

iv In month m, the amount of cement produced is T tonnes. Find m in terms of T.

v The total amount of cement produced in the first p months is 522 750 tonnes. Find the value of p.

b A second factory, B, commences production at exactly the same time as the first. In its first month it produces 3000 tonnes of cement. In each successive month, production increases by 8%.

i Find an expression for the total amount of cement produced by this factory after n months.

ii Let Q_A be the total amount of cement produced by factory A in the first n months and Q_B the total amount of cement produced by factory B in the first n months. Find an expression in terms of n for $Q_B - Q_A$ and find the smallest value of n for which $Q_B - Q_A \geq 0$.

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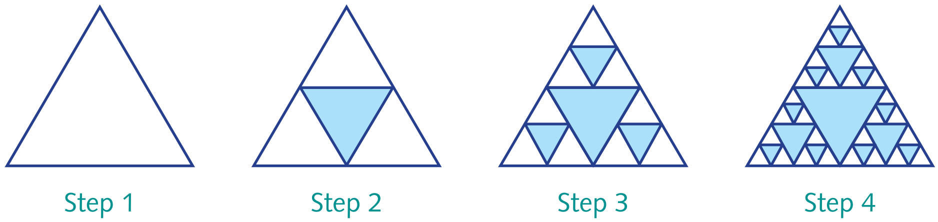
14 The following diagrams show the first four steps in forming the Sierpinski triangle.



The diagrams are produced in the following way:

Step 1 Start with an equilateral triangle of side length 1 unit.

Step 2 Subdivide it into four smaller congruent equilateral triangles and colour the central one blue.

Step 3 Repeat Step 2 with each of the smaller white triangles.

Step 4 Repeat again.

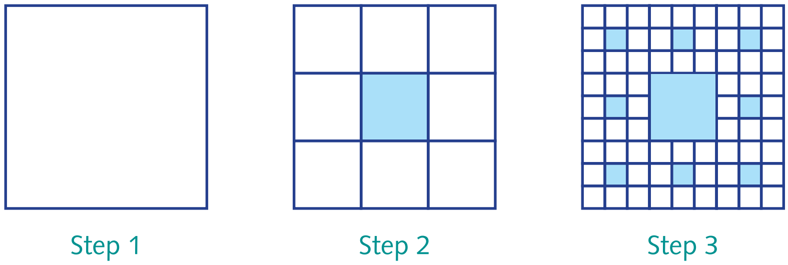
a How many white triangles are there in the nth diagram (that is, after Step n)?

b What is the side length of a white triangle in the nth diagram?

c What fraction of the area of the original triangle is still white in the nth diagram?

d Consider what happens as n approaches infinity.

15 The Sierpinski carpet is formed from a unit square in a way similar to the Sierpinski triangle. The following diagrams show the first three steps.



a How many white squares are there in the nth diagram (that is, after Step n)?

b What is the length of a side of a white square in the nth diagram?

c What is the fraction of the area of square which is white in the nth diagram?

d Consider what happens as n approaches infinity.

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