Chapter 7

7 Principles of counting

Objectives

- ▶ To solve problems using the **addition** and **multiplication principles**.
- ▶ To solve problems involving **permutations**.
- ▶ To solve problems involving **combinations**.
- ▶ To establish and use identities associated with **Pascal's triangle**.
- ▶ To solve problems using the **pigeonhole principle**.
- ▶ To understand and apply the **inclusion–exclusion principle**.

Take a deck of 52 playing cards. This simple, familiar deck can be arranged in so many ways that if you and every other living human were to shuffle a deck once per second from the beginning of time, then by now only a tiny fraction of all possible arrangements would have been obtained. So, remarkably, every time you shuffle a deck you are likely to be the first person to have created that particular arrangement of cards!

To see this, note that we have 52 choices for the first card, and then 51 choices for the second card, and so on. This gives a total of

$$52 \times 51 \times \cdots \times 2 \times 1 \approx 8.1 \times 10^{67}$$

arrangements. This is quite an impressive number, especially in light of the fact that the universe is estimated to be merely 1.4×10^{10} years old.

Combinatorics is concerned with counting the number of ways of doing something. Our goal is to find clever ways of doing this without explicitly listing all the possibilities. This is particularly important in the study of probability. For instance, we can use combinatorics to explain why certain poker hands are more likely to occur than others without considering all 2 598 960 possible hands.

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7A Basic counting methods

► Tree diagrams

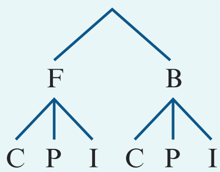
In most combinatorial problems, we are interested in the *number of solutions* to a given problem, rather than the solutions themselves. Nonetheless, for simple counting problems it is sometimes practical to list and then count all the solutions. Tree diagrams provide a systematic way of doing this, especially when the problem involves a small number of steps.

Example 1

A restaurant has a fixed menu, offering a choice of fish or beef for the main meal, and cake, pudding or ice-cream for dessert. How many different meals can be chosen?

Solution

We illustrate the possibilities on a tree diagram:



This gives six different meals, which we can write as

FC, FP, FI, BC, BP, BI

► The multiplication principle

In the above example, for each of the two ways of selecting the main meal, there were three ways of selecting the dessert. This gives a total of $2 \times 3 = 6$ ways of choosing a meal. This is an example of the **multiplication principle**, which will be used extensively throughout this chapter.

Multiplication principle

If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.

Example 2

Sandra has three different skirts, four different tops and five different pairs of shoes. How many choices does she have for a complete outfit?

Solution

$3 \times 4 \times 5 = 60$

Explanation

Using the multiplication principle, we multiply the number of ways of making each choice.

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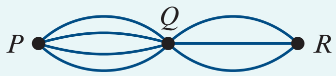
Example 3

How many paths are there from point P to point R travelling from left to right?



Solution

$4 \times 3 = 12$

Explanation

For each of the four paths from P to Q, there are three paths from Q to R.

The addition principle

In some instances, we have to count the number of ways of choosing between two alternative tasks. In this case, we use the **addition principle**.

Addition principle

Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, then there are $m + n$ ways to perform one of the tasks.

Example 4

To travel from Melbourne to Sydney tomorrow, Kara has a choice between three different flights and two different trains. How many choices does she have?

Solution

$3 + 2 = 5$

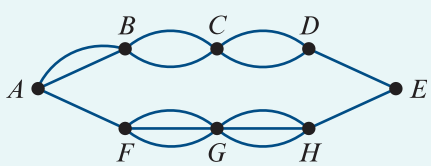
Explanation

The addition principle applies because Kara cannot travel by both plane and train. Therefore, we add the number of ways of making each choice.

Some problems will require use of both the multiplication and the addition principles.

Example 5

How many paths are there from point A to point E travelling from left to right?



Solution

We can take *either* an upper path *or* a lower path:

- Going from A to B to C to D to E there are $2 \times 2 \times 2 \times 1 = 8$ paths.
- Going from A to F to G to H to E there are $1 \times 3 \times 3 \times 1 = 9$ paths.

Using the addition principle, there is a total of $8 + 9 = 17$ paths from A to E.

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► Harder problems involving tree diagrams

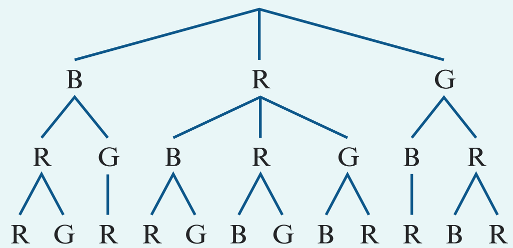
For some problems, a straightforward application of the multiplication and addition principles is not possible.

Example 6

A bag contains one blue token, two red tokens and one green token. Three tokens are removed from the bag and placed in a row. How many arrangements are possible?

Solution

The three tokens are selected without replacement. So once a blue or green token is taken, these cannot appear again. We use a tree diagram to systematically find every arrangement.



The complete set of possible arrangements can be read by tracing out each path from top to bottom of the diagram. This gives 12 different arrangements:

BRR, BRG, BGR, RBR, RBG, RRB, RRG, RGB, RGR, GBR, GRB, GRR

Section summary

Three useful approaches to solving simple counting problems:

- **Tree diagrams**
  These can be used to systematically list all solutions to a problem.
- **Multiplication principle**
  If there are m ways of performing one task and then there are n ways of performing another task, then there are $m \times n$ ways of performing *both* tasks.
- **Addition principle**
  Suppose there are m ways of performing one task and n ways of performing another task. If we cannot perform both tasks, there are $m + n$ ways to perform one of the tasks.

Some problems require use of both the addition and the multiplication principles.

Exercise 7A

Skillsheet
Example 2

1 Sam has five T-shirts, three pairs of pants and three pairs of shoes. How many different outfits can he assemble using these clothes?

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Example 4 **2** A restaurant offers five beef dishes and three chicken dishes. How many selections of one main meal does a customer have?

3 Each of the 10 boys at a party shakes hands with each of the 12 girls. How many handshakes take place?

4 Draw a tree diagram showing all the two-digit numbers that can be formed using the digits 7, 8 and 9 if each digit:

a cannot be repeated **b** can be repeated.

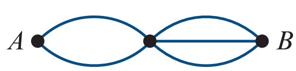
5 How many different three-digit numbers can be formed using the digits 2, 4 and 6 if each digit can be used:

a as many times as you would like **b** at most once?

6 Jack wants to travel from Sydney to Perth via Adelaide. There are four flights and two trains from Sydney to Adelaide. There are two flights and three trains from Adelaide to Perth. How many ways can Jack travel from Sydney to Perth?

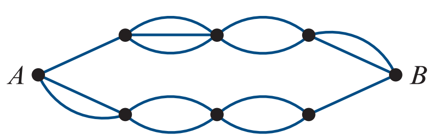
7 Travelling from left to right, how many paths are there from point A to point B in each of the following diagrams?

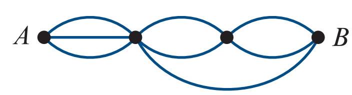
Example 3

a 

b 

Example 5

c 

d 

Example 6 **8** A bag contains two blue, one red and two green tokens. Two tokens are removed from the bag and placed in a row. With the help of a tree diagram, list all the different arrangements.

9 How many ways can you make change for 50 cents using 5, 10 and 20 cent pieces?

10 Four teachers decide to swap desks at work. How many ways can this be done if no teacher is to sit at their previous desk?

11 Three runners compete in a race. In how many ways can the runners complete the race assuming:

a there are no tied places **b** the runners can tie places?

12 A six-sided die has faces labelled with the numbers 0, 2, 3, 5, 7 and 11. If the die is rolled twice and the two results are multiplied, how many different answers can be obtained?

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7B Factorial notation and permutations

Factorial notation

Factorial notation provides a convenient way of expressing products of consecutive natural numbers. For each natural number n, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot 2 \cdot 1$$

where the notation $n!$ is read as 'n factorial'.

We also define $0! = 1$. Although it might seem strange at first, this definition will turn out to be very convenient, as it is compatible with formulas that we will establish shortly.

Another very useful identity is

$$n! = n \cdot (n-1)!$$

Example 7

Evaluate:

a $3!$ **b** $\dfrac{50!}{49!}$ **c** $\dfrac{10!}{2!\,8!}$

Solution

a $3! = 3 \cdot 2 \cdot 1$
$= 6$

b $\dfrac{50!}{49!} = \dfrac{50 \cdot 49!}{49!}$
$= 50$

c $\dfrac{10!}{2!\,8!} = \dfrac{10 \cdot 9 \cdot 8!}{2! \cdot 8!}$
$= \dfrac{10 \cdot 9}{2 \cdot 1}$
$= 45$

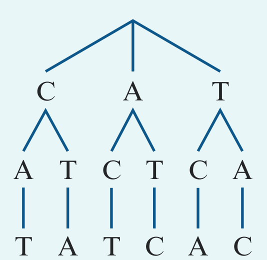
Permutations of n objects

A **permutation** is an ordered arrangement of a collection of objects.

Example 8

Using a tree diagram, list all the permutations of the letters in the word CAT.

Solution



There are six permutations:

CAT, CTA, ACT, ATC, TCA, TAC

Explanation

There are three choices for the first letter. This leaves only two choices for the second letter, and then one for the third.

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Another way to find the number of permutations for the previous example is to draw three boxes, corresponding to the three positions. In each box, we write the number of choices we have for that position.

- We have 3 choices for the first letter (C, A or T).
- We have 2 choices for the second letter (because we have already used one letter).
- We have 1 choice for the third letter (because we have already used two letters).

3	2	1

By the multiplication principle, the total number of arrangements is

$$3 \times 2 \times 1 = 3!$$

So three objects can be arranged in 3! ways. More generally:

The number of permutations of n objects is $n!$.

Proof The reason for this is simple:

- The first item can be chosen in n ways.
- The second item can be chosen in $n - 1$ ways, since only $n - 1$ objects remain.
- The third item can be chosen in $n - 2$ ways, since only $n - 2$ objects remain.

⋮

- The last item can be chosen in 1 way, since only 1 object remains.

Therefore, by the multiplication principle, there are

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1 = n!$$

permutations of n objects.

Example 9

How many ways can six different books be arranged on a shelf?

Solution

$$\begin{aligned} 6! &= 6 \times 5 \times 4 \times 3 \times 2 \times 1 \\ &= 720 \end{aligned}$$

Explanation

Six books can be arranged in 6! ways.

Example 10

Using your calculator, find how many ways 12 students can be lined up in a row.

Using the TI-Nspire

Evaluate 12! as shown.

Note: The factorial symbol (!) can be accessed using ?!▸, the Symbols palette (ctrl 📖) or menu > **Probability** > **Factorial**.

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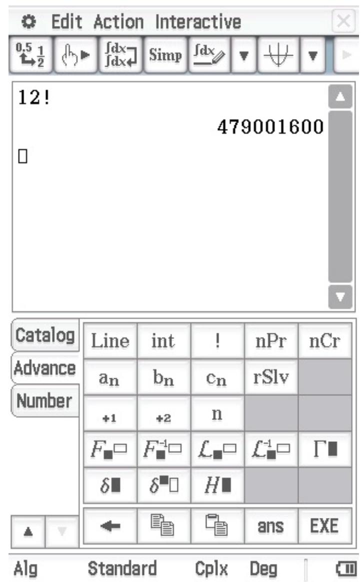
Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, open the keyboard.
- Enter the number 12, followed by the factorial symbol. Tap EXE.

Note: The factorial symbol (!) is found in the Advance keyboard; you need to scroll down to see this.



Example 11

How many four-digit numbers can be formed using the digits 1, 2, 3 and 4 if:

a they cannot be repeated

b they can be repeated?

Solution

a $4! = 4 \times 3 \times 2 \times 1 = 24$

b $4^4 = 4 \times 4 \times 4 \times 4 = 256$

Explanation

Four numbers can be arranged in 4! ways.

Using the multiplication principle, there are 4 choices for each of the 4 digits.

► Permutations of n objects taken r at a time

Imagine a very small country with very few cars. Licence plates consist of a sequence of four digits, and repetitions of the digits are not allowed. How many such licence plates are there?

Here, we are asking for the number of permutations of 10 digits taken four at a time. We will denote this number by ${}^{10}P_4$.

To solve this problem, we draw four boxes. In each box, we write the number of choices we have for that position. For the first digit, we have a choice of 10 digits. Once chosen, we have only 9 choices for the second digit, then 8 choices for the third and 7 choices for the fourth.

10	9	8	7

By the multiplication principle, the total number of licence plates is

$$10 \times 9 \times 8 \times 7$$

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There is a clever way of writing this product as a fraction involving factorials:

$$
\begin{aligned}
{}^{10}P_4 &= 10 \cdot 9 \cdot 8 \cdot 7 \\
&= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \\
&= \frac{10!}{6!} \\
&= \frac{10!}{(10-4)!}
\end{aligned}
$$

More generally:

Number of permutations

The number of permutations of n objects taken r at a time is denoted by ${}^nP_r$ and is given by the formula

$$
{}^nP_r = \frac{n!}{(n-r)!}
$$

Proof To establish this formula we note that:

- The 1st item can be chosen in n ways.
- The 2nd item can be chosen in $n - 1$ ways.

⋮

- The rth item can be chosen in $n - r + 1$ ways.

Therefore, by the multiplication principle, the number of permutations of n objects taken r at a time is

$$
\begin{aligned}
{}^nP_r &= n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \\
&= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-r+1) \cdot (n-r) \cdot \cdots \cdot 2 \cdot 1}{(n-r) \cdot \cdots \cdot 2 \cdot 1} \\
&= \frac{n!}{(n-r)!}
\end{aligned}
$$

Notes:

- If $r = n$, then we have ${}^nP_n$, which is simply the number of permutations of n objects and so must equal $n!$. The formula still works in this instance, since

$$
\begin{aligned}
{}^nP_n &= \frac{n!}{(n-n)!} \\
&= \frac{n!}{0!} \\
&= n!
\end{aligned}
$$

  Note that this calculation depends crucially on our decision to define $0! = 1$.
- If $r = 1$, then we obtain ${}^nP_1 = n$. Given n objects, there are n choices of one object, and each of these can be arranged in just one way.

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Example 12

a Using the letters A, B, C, D and E without repetition, how many different two-letter arrangements are there?

b Six runners compete in a race. In how many ways can the gold, silver and bronze medals be awarded?

Solution

a There are five letters to arrange in two positions:

$$\begin{aligned} {}^5P_2 &= \frac{5!}{(5-2)!} \\ &= \frac{5!}{3!} \\ &= \frac{5 \cdot 4 \cdot 3!}{3!} \\ &= 20 \end{aligned}$$

b There are six runners to arrange in three positions:

$$\begin{aligned} {}^6P_3 &= \frac{6!}{(6-3)!} \\ &= \frac{6!}{3!} \\ &= \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} \\ &= 120 \end{aligned}$$

Although the formula developed for ${}^nP_r$ will have an important application later in this chapter, you do not actually have to use it when solving problems. It is often more convenient to simply draw boxes corresponding to the positions, and to write in each box the number of choices for that position.

Example 13

How many ways can seven friends sit along a park bench with space for only four people?

Solution

7	6	5	4

By the multiplication principle, the total number of arrangements is

$$7 \times 6 \times 5 \times 4 = 840$$

Explanation

We draw four boxes, representing the positions to be filled. In each box we write the number of ways we can fill that position.

Using the TI-Nspire

- To evaluate ${}^7P_4$, use (menu) > **Probability** > **Permutations** as shown.

Note: Alternatively, you can simply type npr(7, 4). The command is not case sensitive.

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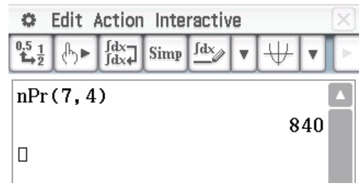
Using the Casio ClassPad

To evaluate ${}^7P_4$:

- In Main $\sqrt{\alpha}$, select nPr from the Advance keyboard. (You need to scroll down to find this keyboard.)
- In the brackets, enter the numbers 7 and 4, separated by a comma. Then tap EXE.



Section summary

- $n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 2 \cdot 1$ and $0! = 1$
- $n! = n \cdot (n-1)!$
- A **permutation** is an ordered arrangement of objects.
- The number of permutations of n objects is $n!$.
- The number of permutations of n objects taken r at a time is given by

$$ {}^nP_r = \frac{n!}{(n-r)!} $$

Exercise 7B

1 Evaluate $n!$ for $n = 0, 1, 2, \dots, 10$.

Example 7 **2** Evaluate each of the following:

a $\dfrac{5!}{4!}$ **b** $\dfrac{10!}{8!}$ **c** $\dfrac{12!}{10!\,2!}$ **d** $\dfrac{100!}{97!\,3!}$

3 Simplify the following expressions:

a $\dfrac{(n+1)!}{n!}$ **b** $\dfrac{(n+2)!}{(n+1)!}$ **c** $\dfrac{n!}{(n-2)!}$ **d** $\dfrac{1}{n!} + \dfrac{1}{(n+1)!}$

4 Evaluate ${}^4P_r$ for $r = 0, 1, 2, 3, 4$.

Example 8 **5** Use a tree diagram to find all the permutations of the letters in the word DOG.

Example 9 **6** How many ways can five books on a bookshelf be arranged?

7 How many ways can the letters in the word HYPERBOLA be arranged?

Example 12 **8** Write down all the two-letter permutations of the letters in the word FROG.

Example 13 **9** How many ways can six students be arranged along a park bench if the bench has:

a six seats **b** five seats **c** four seats?

10 Using the digits 1, 2, 5, 7 and 9 without repetition, how many numbers can you form that have:

a five digits **b** four digits **c** three digits?

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11 How many ways can six students be allocated to eight vacant desks?

12 How many ways can three letters be posted in five mailboxes if each mailbox can receive:

a more than one letter **b** at most one letter?

13 Using six differently coloured flags without repetition, how many signals can you make using:

a three flags in a row **b** four flags in a row **c** five flags in a row?

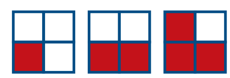
14 You are in possession of four flags, each coloured differently. How many signals can you make using at least two flags arranged in a row?

15 Many Australian car licence plates consist of a sequence of three letters followed by a sequence of three digits.

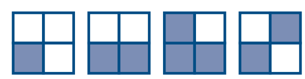
a How many different car licence plates have letters and numbers arranged this way?

b How many of these have no repeated letters or numbers?

16 **a** The three tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



b The four tiles shown are to be arranged in a row, and can be rotated. How many different ways can this be done?



17 Find all possible values of m and n if $m! \cdot n! = 720$ and $m > n$.

18 Show that $n! = (n^2 - n) \cdot (n - 2)!$ for $n \geq 2$.

19 Given six different colours, how many ways can you paint a cube so that all the faces have different colours? Two colourings are considered to be the same when one can be obtained from the other by rotating the cube.

7C Permutations with restrictions

Suppose we want to know how many three-digit numbers have no repeated digits. The answer is *not* simply ${}^{10}P_3$, the number of permutations of 10 digits taken three at a time. This is because the digit 0 cannot be used in the hundreds place.

- There are 9 choices for the first digit (1, 2, 3, . . . , 9).
- There are 9 choices for the second digit (0 and the eight remaining non-zero digits).
- This leaves 8 choices for the third digit.

100s	10s	units
9	9	8

By the multiplication principle, there are $9 \times 9 \times 8 = 648$ different three-digit numbers.

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When considering permutations with restrictions, we deal with the restrictions first.

Example 14

a How many arrangements of the word DARWIN begin and end with a vowel?

b Using the digits 0, 1, 2, 3, 4 and 5 without repetition, how many odd four-digit numbers can you form?

Solution

a We draw six boxes. In each box, we write the number of choices we have for that position. We first consider restrictions. There are two choices of vowel (A or I) for the first letter, leaving only one choice for the last letter.

2					1

This leaves four choices for the second letter, three for the next, and so on.

2	4	3	2	1	1

By the multiplication principle, the number of arrangements is

$$2 \times 4 \times 3 \times 2 \times 1 \times 1 = 48$$

b We draw four boxes. Again, we first consider restrictions. The last digit must be odd (1, 3 or 5), giving three choices. We cannot use 0 in the first position, so this leaves four choices for that position.

4			3

Once these two digits have been chosen, this leaves four choices and then three choices for the remaining two positions.

4	4	3	3

Thus the number of arrangements is

$$4 \times 4 \times 3 \times 3 = 144$$

▶ Permutations with items grouped together

For some arrangements, we may want certain items to be grouped together. In this case, the trick is to initially treat each group of items as a single object. We then multiply by the numbers of arrangements within each group.

Example 15

a How many arrangements of the word EQUALS are there if the vowels are kept together?

b How many ways can two chemistry, four physics and five biology books be arranged on a shelf if the books of each subject are kept together?

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Solution	Explanation
a $4! \times 3! = 144$	We group the three vowels together so that we have four items to arrange: (E, U, A), Q, L, S. They can be arranged in 4! ways. Then the three vowels can be arranged among themselves in 3! ways. We use the multiplication principle.
b $3! \times 2! \times 4! \times 5! = 34\,560$	There are three groups and so they can be arranged in 3! ways. The two chemistry books can be arranged among themselves in 2! ways, the four physics books in 4! ways and the five biology books in 5! ways. We use the multiplication principle.

Section summary

- To count permutations that are subject to restrictions, we draw a series of boxes. In each box, we write the number of choices we have for that position. We always consider the restrictions first.
- When items are to be grouped together, we initially treat each group as a single object. We find the number of arrangements of the groups, and then multiply by the numbers of arrangements within each group.

Exercise 7C

Skillsheet

Example 14

1 Using the digits 1, 2, 3, 4 and 5 without repetition, how many five-digit numbers can you form:

a without restriction
b that are odd
c that begin with 5
d that do not begin with 5?

Example 15

2 In how many ways can three girls and two boys be arranged in a row:

a without restriction
b if the two boys sit together
c if the two boys do not sit together
d if girls and boys alternate?

3 How many permutations of the word QUEASY:

a begin with a vowel
b begin and end with a vowel
c keep the vowels together
d keep the vowels and consonants together?

4 How many ways can four boys and four girls be arranged in a row if:

a boys and girls sit in alternate positions
b boys sit together and girls sit together?

5 The digits 0, 1, 2, 3, 4 and 5 can be combined without repetition to form new numbers. In how many ways can you form:

a a six-digit number
b a four-digit number divisible by 5
c a number less than 6000
d an even three-digit number?

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6 Two parents and four children are seated in a cinema along six consecutive seats. How many ways can this be done:

a without restriction
b if the two parents sit at either end
c if the children sit together
d if the parents sit together and the children sit together
e if the youngest child must sit between and next to both parents?

7 12321 is a **palindromic number** because it reads the same backwards as forwards. How many palindromic numbers have:

a five digits
b six digits?

8 How many arrangements of the letters in VALUE do not begin and end with a vowel?

9 Using each of the digits 1, 2, 3 and 4 at most once, how many even numbers can you form?

10 How many ways can six girls be arranged in a row so that two of the girls, A and B:

a do not sit together
b have one person between them?

11 How many ways can three girls and three boys be arranged in a row if no two girls sit next to each other?

7D Permutations of like objects

The name for the Sydney suburb of WOOLLOOMOOLOO has the unusual distinction of having 13 letters in total, of which only four are different. Finding the number of permutations of the letters in this word is not as simple as evaluating 13!. This is because switching like letters does not result in a new permutation.

Our aim is to find an expression for P, where P is the number of permutations of the letters in the word WOOLLOOMOOLOO. First notice that the word has

1 letter W, 1 letter M, 3 letter Ls, 8 letter Os

Replace the three identical Ls with L_1, L_2 and L_3. These three letters can be arranged in 3! different ways. Therefore, by the multiplication principle, there are now

$$P \cdot 3!$$

permutations. Likewise, replace the eight identical Os with $O_1, O_2, \ldots, O_8$. These eight letters can be arranged in 8! different ways. Therefore there are now

$$P \cdot 3! \cdot 8!$$

permutations.

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On the other hand, notice that the 13 letters are now distinct, so there are 13! permutations of these letters. Therefore

$$P \cdot 3! \cdot 8! = 13! \quad \text{and so} \quad P = \frac{13!}{3!\,8!}$$

We can easily generalise this procedure to give the following result.

Permutations of like objects

The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2! \cdots n_r!}$$

Example 16

a Find the number of permutations of the letters in the word RIFFRAFF.

b There are four identical knives, three identical forks and two identical spoons in a drawer. They are taken out of the drawer and lined up in a row. How many ways can this be done?

Solution

a $\dfrac{8!}{4!\,2!} = 840$

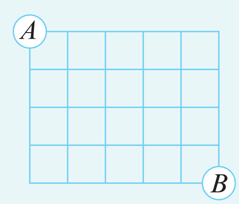
Explanation: There are 8 letters of which 4 are alike and 2 are alike.

b $\dfrac{9!}{4!\,3!\,2!} = 1260$

Explanation: There are 9 items of which 4 are alike, 3 are alike and 2 are alike.

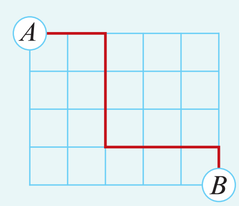
Example 17

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of four Ds and five Rs in some order. For example, the path shown can be described by the sequence RRDDDRRRD.



There are

$$\frac{9!}{4!\,5!} = 126$$

permutations of these letters, since there are 9 letters of which 4 are alike and 5 are alike.

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Section summary

- Switching like objects does not give a new arrangement.
- The number of permutations of n objects of which n_1 are alike, n_2 are alike, ... and n_r are alike is given by

$$\frac{n!}{n_1!\,n_2!\cdots n_r!}$$

Exercise 7D

Skillsheet
Example 16

1 Ying has four identical 20 cent pieces and three identical 10 cent pieces. How many ways can she arrange these coins in a row?

2 How many ways can the letters in the word MISSISSIPPI be arranged?

3 Find the number of permutations of the letters in the word WARRNAMBOOL.

4 Using five 9s and three 7s, how many eight-digit numbers can be made?

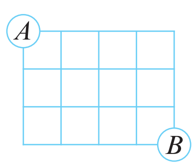
5 Using three As, four Bs and five Cs, how many sequences of 12 letters can be made?

6 How many ways can two red, two black and four blue flags be arranged in a row:

a without restriction
b if the first flag is red
c if the first and last flags are blue
d if every alternate flag is blue
e if the two red flags are adjacent?

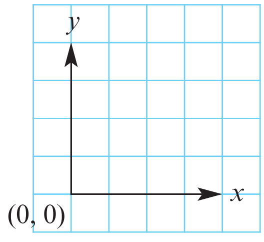
Example 17

7 The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



8 The grid shown consists of unit squares. By travelling only along the grid lines, how many paths are there:

a of length 6 from $(0, 0)$ to the point $(2, 4)$
b of length $m + n$ from $(0, 0)$ to the point (m, n), where m and n are natural numbers?



9 Consider a deck of 52 playing cards.

a How many ways can the deck be arranged? Express your answer in the form $a!$.
b If two identical decks are combined, how many ways can the cards be arranged? Express your answer in the form $\dfrac{a!}{(b!)^c}$.
c If n identical decks are combined, find an expression for the number of ways that the cards can be arranged.

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10 An ant starts at position $(0, 0)$ and walks north, east, south or west, one unit at a time. How many different paths of length 8 units finish at $(0, 0)$?

11 Jessica is about to walk up a flight of 10 stairs. She can take either one or two stairs at a time. How many different ways can she walk up the flight of stairs?

7E Combinations

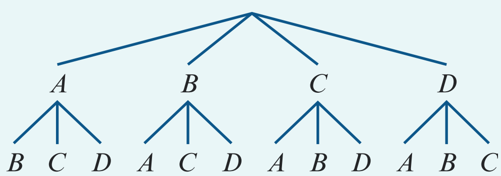
We have seen that a permutation is an ordered arrangement of objects. In contrast, a **combination** is a selection made regardless of order. We use the notation ${}^nP_r$ to denote the number of permutations of n distinct objects taken r at a time. Similarly, we use the notation ${}^nC_r$ to denote the number of combinations of n distinct objects taken r at a time.

Example 18

How many ways can two letters be chosen from the set $\{A, B, C, D\}$?

Solution

The tree diagram below shows the ways that the first and second choices can be made.



This gives 12 arrangements. But there are only six selections, since

$\{A, B\}$ is the same as $\{B, A\}$, $\{A, C\}$ is the same as $\{C, A\}$, $\{A, D\}$ is the same as $\{D, A\}$,
$\{B, C\}$ is the same as $\{C, B\}$, $\{B, D\}$ is the same as $\{D, B\}$ $\{C, D\}$ is the same as $\{D, C\}$

Suppose we want to count the number of ways that three students can be chosen from a group of seven. Let's label the students with the letters $\{A, B, C, D, E, F, G\}$. One such combination might be BDE. Note that this combination corresponds to 3! permutations:

$$BDE, \quad BED, \quad DBE, \quad DEB, \quad EBD, \quad EDB$$

In fact, each combination of three items corresponds to 3! permutations, and so there are 3! times as many permutations as combinations. Therefore

$${}^7P_3 = 3! \times {}^7C_3 \quad \text{and so} \quad {}^7C_3 = \frac{{}^7P_3}{3!}$$

Since we have already established that ${}^7P_3 = \dfrac{7!}{(7-3)!}$, we obtain

$${}^7C_3 = \frac{7!}{3!\,(7-3)!}$$

This argument generalises easily so that we can establish a formula for ${}^nC_r$.

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Number of combinations

The number of combinations of n objects taken r at a time is given by the formula

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

Example 19

a A pizza can have three toppings chosen from nine options. How many different pizzas can be made?

b How many subsets of $\{1, 2, 3, \ldots, 20\}$ have exactly two elements?

Solution

a Three objects are to be chosen from nine options. This can be done in $^{9}C_3$ ways, and

$$\begin{aligned} ^{9}C_3 &= \frac{9!}{3!\,(9-3)!} \\ &= \frac{9!}{3!\,6!} \\ &= \frac{9 \cdot 8 \cdot 7 \cdot 6!}{3! \cdot 6!} \\ &= \frac{9 \cdot 8 \cdot 7}{3 \cdot 2} \\ &= 84 \end{aligned}$$

b Two objects are to be chosen from 20 options. This can be done in $^{20}C_2$ ways, and

$$\begin{aligned} ^{20}C_2 &= \frac{20!}{2!\,(20-2)!} \\ &= \frac{20!}{2!\,18!} \\ &= \frac{20 \cdot 19 \cdot 18!}{2! \cdot 18!} \\ &= \frac{20 \cdot 19}{2 \cdot 1} \\ &= 190 \end{aligned}$$

Example 20

Using your calculator, find how many ways 10 students can be selected from a class of 20 students.

Using the TI-Nspire

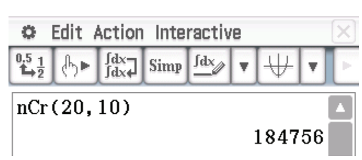
- To evaluate $^{20}C_{10}$, use menu > **Probability** > **Combinations** as shown.

Note: Alternatively, you can simply type ncr(20, 10). The command is not case sensitive.

Using the Casio ClassPad

To evaluate $^{20}C_{10}$:

- In Main $\sqrt{\alpha}$, select nCr from the Advance keyboard.
- In the brackets, enter the numbers 20 and 10, separated by a comma. Then tap EXE.



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Example 21

Consider a group of six students. In how many ways can a group of:

a two students be selected

b four students be selected?

Solution

a
$$\begin{aligned} {}^6C_2 &= \frac{6!}{2!\,(6-2)!} \\ &= \frac{6!}{2!\,4!} \\ &= \frac{6\cdot 5\cdot 4!}{2!\cdot 4!} \\ &= \frac{6\cdot 5}{2\cdot 1} \\ &= 15 \end{aligned}$$

b
$$\begin{aligned} {}^6C_4 &= \frac{6!}{4!\,(6-4)!} \\ &= \frac{6!}{4!\,2!} \\ &= \frac{6\cdot 5\cdot 4!}{4!\cdot 2!} \\ &= \frac{6\cdot 5}{2\cdot 1} \\ &= 15 \end{aligned}$$

The fact that parts **a** and **b** of the previous example have the same answer is not a coincidence. Choosing two students out of six is the same as *not choosing* the other four students out of six. Therefore ${}^6C_2 = {}^6C_4$.

More generally:

$${}^nC_r = {}^nC_{n-r}$$

Quick calculations

In some instances, you can avoid unnecessary calculations by noting that:

- ${}^nC_0 = 1$, since there is only one way to select no objects from n objects
- ${}^nC_n = 1$, since there is only one way to select n objects from n objects
- ${}^nC_1 = n$, since there are n ways to select one object from n objects
- ${}^nC_{n-1} = n$, since this corresponds to the number of ways of not selecting one object from n objects.

Example 22

a Six points lie on a circle. How many triangles can you make using these points as the vertices?

b Each of the 20 people at a party shakes hands with every other person. How many handshakes take place?

Solution	*Explanation*
a ${}^6C_3 = 20$	This is the same as asking how many ways three vertices can be chosen out of six.
b ${}^{20}C_2 = 190$	This is the same as asking how many ways two people can be chosen to shake hands out of 20 people.

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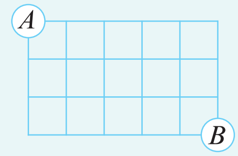
Example 23

The grid shown consists of unit squares. By travelling only right (R) or down (D) along the grid lines, how many paths are there from point A to point B?



Solution

Each path from A to B can be described by a sequence of three Ds and five Rs in some order. Therefore, the number of paths is equal to the number of ways of selecting three of the eight boxes below to be filled with the three Ds. (The rest will be Rs.) This can be done in ${}^{8}C_3 = 56$ ways.

Alternative notation

We will consistently use the notation ${}^{n}C_r$ to denote the number of ways of selecting r objects from n objects, regardless of order. However, it is also common to denote this number by $\binom{n}{r}$.

For example:

$$\binom{6}{4} = \frac{6!}{4!\,2!} = 15$$

Section summary

- A **combination** is a selection made regardless of order.
- The number of combinations of n objects taken r at a time is given by

$$ {}^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

Exercise 7E

1 Evaluate ${}^{5}C_r$ for $r = 0, 1, 2, 3, 4, 5$.

2 Evaluate each of the following without the use of your calculator:

a ${}^{7}C_1$ **b** ${}^{6}C_5$ **c** ${}^{12}C_{10}$
d ${}^{8}C_5$ **e** ${}^{100}C_{99}$ **f** ${}^{1000}C_{998}$

3 Simplify each of the following:

a ${}^{n}C_1$ **b** ${}^{n}C_2$ **c** ${}^{n}C_{n-1}$
d ${}^{n+1}C_1$ **e** ${}^{n+2}C_n$ **f** ${}^{n+1}C_{n-1}$

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Example 19 **4** A playlist contains ten of Nandi's favourite songs. How many ways can he:

a arrange three songs in a list **b** select three songs for a list?

5 How many ways can five cards be selected from a deck of 52 playing cards?

6 How many subsets of $\{1, 2, 3, \dots, 10\}$ contain exactly:

a 1 element **b** 2 elements

c 8 elements **d** 9 elements?

7 A lottery consists of drawing seven balls out of a barrel of balls numbered from 1 to 45. How many ways can this be done if their order does not matter?

Example 22 **8** Eight points lie on a circle. How many triangles can you make using these points as the vertices?

9 **a** In a hockey tournament, each of the 10 teams plays every other team once. How many games take place?

b In another tournament, each team plays every other team once and 120 games take place. How many teams competed?

10 At a party, every person shakes hands with every other person. Altogether there are 105 handshakes. How many people are at the party?

11 Prove that ${}^nC_r = {}^nC_{n-r}$.

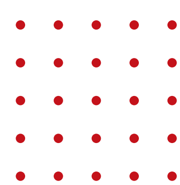
12 Explain why the number of diagonals in a regular polygon with n sides is ${}^nC_2 - n$.

13 Ten students are divided into two teams of five. Explain why the number of ways of doing this is $\dfrac{{}^{10}C_5}{2}$.

14 Twelve students are to be divided into two teams of six. In how many ways can this be done? (Hint: First complete the previous question.)

15 Using the formula for ${}^nC_r$, prove that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$, where $1 \le r < n$.

16 Consider the 5×5 grid shown.



a How many ways can three dots be chosen?

b How many ways can three dots be chosen so that they lie on a straight line?

c How many ways can three dots be chosen so that they are the vertices of a triangle? (Hint: Use parts **a** and **b**.)

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7F Combinations with restrictions

▶ Combinations including specific items

In some problems, we want to find the number of combinations that include specific items. This reduces both the number of items we have to select and the number of items from which we are selecting.

Example 24

a Grace belongs to a group of eight workers. How many ways can a team of four workers be selected if Grace must be on the team?

b A hand of cards consists of five cards drawn from a deck of 52 playing cards. How many hands contain both the queen and the king of hearts?

Solution	*Explanation*
a ${}^{7}C_3 = 35$	Grace must be in the selection. Therefore three more workers are to be selected from the remaining seven workers.
b ${}^{50}C_3 = 19\,600$	The queen and king of hearts must be in the selection. So three more cards are to be selected from the remaining 50 cards.

In some other problems, it can be more efficient to count the selections that we don't want.

Example 25

Four students are to be chosen from a group of eight students for the school tennis team. Two members of the group, Sam and Tess, do not get along and cannot both be on the team. How many ways can the team be selected?

Solution

There are ${}^{8}C_4$ ways of selecting four students from eight. We then subtract the number of combinations that include both Sam and Tess. If Sam and Tess are on the team, then we can select two more students from the six that remain in ${}^{6}C_2$ ways. This gives

$$ {}^{8}C_4 - {}^{6}C_2 = 55 $$

▶ Combinations from multiple groups

Sometimes we are required to make multiple selections from separate groups. In this case, the multiplication principle dictates that we simply multiply the number of ways of performing each task.

Example 26

From seven women and four men in a workplace, how many groups of five can be chosen:

a without restriction

b containing three women and two men

c containing at least one man

d containing at most one man?

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Solution

a There are 11 people in total, from which we must select five. This gives

$$^{11}C_5 = 462$$

b There are $^7C_3$ ways of selecting three women from seven. There are $^4C_2$ ways of selecting two men from four. We then use the multiplication principle to give

$$^7C_3 \cdot {}^4C_2 = 210$$

c Method 1

If you select at least one man, then you select 1, 2, 3 or 4 men and fill the remaining positions with women. We use the multiplication and addition principles to give

$$^4C_1 \cdot {}^7C_4 + {}^4C_2 \cdot {}^7C_3 + {}^4C_3 \cdot {}^7C_2 + {}^4C_4 \cdot {}^7C_1 = 441$$

Method 2

It is more efficient to consider all selections of 5 people from 11 and then subtract the number of combinations containing all women. This gives

$$^{11}C_5 - {}^7C_5 = 441$$

d If there is at most one man, then either there are no men or there is one man. If there are no men, then there are $^7C_5$ ways of selecting all women. If there is one man, then there are $^4C_1$ ways of selecting one man and $^7C_4$ ways of selecting four women. This gives

$$^7C_5 + {}^4C_1 \cdot {}^7C_4 = 161$$

Permutations and combinations combined

In the following example, we first select the items and then arrange them.

Example 27

a How many arrangements of the letters in the word DUPLICATE can be made that have two vowels and three consonants?

b A president, vice-president, secretary and treasurer are to be chosen from a group containing seven women and six men. How many ways can this be done if exactly two women are chosen?

Solution	*Explanation*
a $^4C_2 \cdot {}^5C_3 \cdot 5! = 7200$	There are $^4C_2$ ways of selecting 2 of 4 vowels and $^5C_3$ ways of selecting 3 of 5 consonants. Once chosen, the 5 letters can be arranged in 5! ways.
b $^7C_2 \cdot {}^6C_2 \cdot 4! = 7560$	There are $^7C_2$ ways of selecting 2 of 7 women and $^6C_2$ ways of selecting 2 of 6 men. Once chosen, the 4 people can be arranged into the positions in 4! ways.

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Section summary

- If a selection must include specific items, then this reduces both the number of items that we have to select and the number of items that we select from.
- If we are required to make multiple selections from separate groups, then we multiply the number of ways of performing each task.
- Some problems will require us to select and then arrange objects.

Exercise 7F

Skillsheet
Example 24

1 Jane and Jenny belong to a class of 20 students. How many ways can you select a group of four students from the class if both Jane and Jenny are to be included?

2 How many subsets of $\{1, 2, 3, \ldots, 10\}$ have exactly five elements and contain the number 5?

3 Five cards are dealt from a deck of 52 playing cards. How many hands contain the jack, queen and king of hearts?

Example 25

4 Six students are to be chosen from a group of 10 students for the school basketball team. Two members of the group, Rachel and Nethra, do not get along and cannot both be on the team. How many ways can the team be selected?

Example 26

5 From eight girls and five boys, a team of seven is selected for a mixed netball team. How many ways can this be done if:

a there are no restrictions
b there are four girls and three boys on the team
c there must be at least three boys and three girls on the team
d there are at least two boys on the team?

6 There are 10 student leaders at a secondary school. Four are needed for a fundraising committee and three are needed for a social committee. How many ways can the students be selected if they can serve on:

a both committees
b at most one committee?

7 There are 18 students in a class. Seven are required for a basketball team and eight are required for a netball team. How many ways can the teams be selected if students can play in:

a both teams
b at most one team?

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8 From 10 Labor senators and 10 Liberal senators, a committee of five is formed. How many ways can this be done if:

a there are no restrictions

b there are at least two senators from each political party

c there is at least one Labor senator?

9 Consider the set of numbers $\{1, 2, 3, 4, 5, 6, 7\}$.

a How many subsets have exactly five elements?

b How many five-element subsets contain the numbers 2 and 3?

c How many five-element subsets do not contain both 2 and 3?

10 Four letters are selected from the English alphabet. How many of these selections will contain exactly two vowels?

11 A seven-card hand is dealt from a deck of 52 playing cards. How many distinct hands contain:

a four hearts and three spades

b exactly two hearts and three spades?

12 A committee of five people is chosen from four doctors, four dentists and three physiotherapists. How many ways can this be done if the committee contains:

a exactly three doctors and one dentist

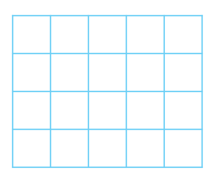
b exactly two doctors?

Example 27 **13** There are four girls and five boys. Two of each are chosen and then arranged on a bench. How many ways can this be done?

14 A president, vice-president, secretary and treasurer are to be chosen from a group containing six women and five men. How many ways can this be done if exactly two women must be chosen?

15 Using five letters from the word TRAMPOLINE, how many arrangements contain two vowels and three consonants?

16 How many rectangles are there in the grid shown on the right?
Hint: Every rectangle is determined by a choice of two vertical and two horizontal lines.



17 Five cards are dealt from a deck of 52 playing cards. A full house is a hand that contains 3 cards of one rank and 2 cards of another rank (for example, 3 kings and 2 sevens). How many ways can a full house be dealt?

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7G Pascal's triangle

The diagram below consists of the binomial coefficients ${}^nC_r$ for $0 \leq n \leq 5$. They form the first 6 rows of **Pascal's triangle**, named after the seventeenth century French mathematician Blaise Pascal, one of the founders of probability theory.

Interestingly, the triangle was well known to Chinese and Indian mathematicians many centuries earlier.

$n = 0$:	${}^0C_0$	1
$n = 1$:	${}^1C_0 \quad {}^1C_1$	1 1
$n = 2$:	${}^2C_0 \quad {}^2C_1 \quad {}^2C_2$	1 2 1
$n = 3$:	${}^3C_0 \quad {}^3C_1 \quad {}^3C_2 \quad {}^3C_3$	1 3 3 1
$n = 4$:	${}^4C_0 \quad {}^4C_1 \quad {}^4C_2 \quad {}^4C_3 \quad {}^4C_4$	1 4 6 4 1
$n = 5$:	${}^5C_0 \quad {}^5C_1 \quad {}^5C_2 \quad {}^5C_3 \quad {}^5C_4 \quad {}^5C_5$	1 5 10 10 5 1

Pascal's rule

Pascal's triangle has many remarkable properties. Most importantly:

> Each entry in Pascal's triangle is the sum of the two entries immediately above.

Pascal's triangle has this property because of the following identity.

> **Pascal's rule**
>
> $${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r \qquad \text{where } 1 \leq r < n$$

Proof In Question 15 of Exercise 7E, you are asked to prove Pascal's rule using the formula for ${}^nC_r$. However, there is a much nicer argument.

The number of subsets of $\{1, 2, \ldots, n\}$ containing exactly r elements is ${}^nC_r$. Each of these subsets can be put into one of two groups:

1. those that contain n
2. those that do not contain n.

If the subset contains n, then each of the remaining $r - 1$ elements must be chosen from $\{1, 2, \ldots, n - 1\}$. Therefore the first group contains ${}^{n-1}C_{r-1}$ subsets.

If the subset does not contain n, then we still have to choose r elements from $\{1, 2, \ldots, n - 1\}$. Therefore the second group contains ${}^{n-1}C_r$ subsets.

The two groups together contain all ${}^nC_r$ subsets and so

$${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

which establishes Pascal's rule.

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Example 28

Given that $^{17}C_2 = 136$ and $^{17}C_3 = 680$, evaluate $^{18}C_3$.

Solution

$$\begin{aligned} {}^{18}C_3 &= {}^{17}C_2 + {}^{17}C_3 \\ &= 136 + 680 \\ &= 816 \end{aligned}$$

Explanation

We let $n = 18$ and $r = 3$ in Pascal's rule:

$$^{n}C_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$$

Example 29

Write down the $n = 6$ row of Pascal's triangle and then write down the value of $^{6}C_3$.

Solution

$n = 6$: $\quad 1 \quad 6 \quad 15 \quad \boxed{20} \quad 15 \quad 6 \quad 1$

$^{6}C_3 = 20$

Explanation

Each entry in the $n = 6$ row is the sum of the two entries immediately above.

Note that $^{6}C_3$ is the fourth entry in the row, since the first entry corresponds to $^{6}C_0$.

Subsets of a set

Suppose your friend says to you: 'I have five books that I no longer need, take any that you want.' How many different selections are possible?

We will look at two solutions to this problem.

Solution 1

You could select none of the books ($^{5}C_0$ ways), or one out of five ($^{5}C_1$ ways), or two out of five ($^{5}C_2$ ways), and so on. This gives the answer

$$^{5}C_0 + {}^{5}C_1 + {}^{5}C_2 + {}^{5}C_3 + {}^{5}C_4 + {}^{5}C_5 = 32$$

Note that this is simply the sum of the entries in the $n = 5$ row of Pascal's triangle.

Solution 2

For each of the five books we have two options: either accept or reject the book. Using the multiplication principle, we obtain the answer

$$2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$$

There are two important conclusions that we can draw from this example.

1. The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

2. A set of size n has 2^n subsets, including the empty set and the set itself.

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Example 30

a Your friend offers you any of six books that she no longer wants. How many selections are possible assuming that you take at least one book?

b How many subsets of $\{1, 2, 3, \dots, 10\}$ have at least two elements?

Solution	Explanation
a $2^6 - 1 = 63$	There are 2^6 subsets of a set of size 6. We subtract 1 because we discard the empty set of no books.
b $2^{10} - {}^{10}C_1 - {}^{10}C_0$ $= 2^{10} - 10 - 1$ $= 1013$	There are 2^{10} subsets of a set of size 10. There are ${}^{10}C_1$ subsets containing 1 element and ${}^{10}C_0$ subsets containing 0 elements.

Section summary

- The values of ${}^nC_r$ can be arranged to give Pascal's triangle.
- Each entry in Pascal's triangle is the sum of the two entries immediately above.
- Pascal's rule: ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$ {}^nC_0 + {}^nC_1 + \cdots + {}^nC_{n-1} + {}^nC_n = 2^n $$

- A set of size n has 2^n subsets, including the empty set and the set itself.

Exercise 7G

Example 28 **1** Evaluate ${}^7C_2$, ${}^6C_2$ and ${}^6C_1$, and verify that the first is the sum of the other two.

Example 29 **2** Write down the $n = 7$ row of Pascal's triangle. Use your answer to write down the values of ${}^7C_2$ and ${}^7C_4$.

3 Write down the $n = 8$ row of Pascal's triangle. Use your answer to write down the values of ${}^8C_4$ and ${}^8C_6$.

Example 30 **4** Your friend offers you any of six different DVDs that he no longer wants. How many different selections are possible?

5 How many subsets does the set $\{A, B, C, D, E\}$ have?

6 How many subsets does the set $\{1, 2, 3, \dots, 10\}$ have?

7 How many subsets of $\{1, 2, 3, 4, 5, 6\}$ have at least one element?

8 How many subsets of $\{1, 2, 3, \dots, 8\}$ have at least two elements?

9 How many subsets of $\{1, 2, 3, \dots, 10\}$ contain the numbers 9 and 10?

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10 You have one 5 cent, one 10 cent, one 20 cent and one 50 cent piece. How many different sums of money can you make assuming that at least one coin is used?

11 Let's call a set **selfish** if it contains its size as an element. For example, the set $\{1, 2, 3\}$ is selfish because the set has size 3 and the number 3 belongs to the set.

a How many subsets of $\{1, 2, 3, \dots, 8\}$ are selfish?

b How many subsets of $\{1, 2, 3, \dots, 8\}$ have the property that both the subset and its complement are selfish?

7H The pigeonhole principle

The pigeonhole principle is an intuitively obvious counting technique which can be used to prove some remarkably counterintuitive results. It gets its name from the following simple observation: If $n + 1$ pigeons are placed into n holes, then some hole contains at least two pigeons. Obviously, in most instances we will not be working with pigeons, so we will recast the principle as follows.

Pigeonhole principle

If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.

Proof Suppose that each of the n holes contains at most one object. Then the total number of objects is at most n, which is a contradiction.

We are now in a position to prove a remarkable fact: There are at least two people in Australia with the same number of hairs on their head. The explanation is simple. No one has more than 1 million hairs on their head, so let's make 1 million holes labelled with the numbers from 1 to 1 million. We now put each of the 24 million Australians into the hole corresponding to the number of hairs on their head. Clearly, some hole contains at least two people, and all the people in that hole will have the same number of hairs on their head.

Example 31

You have thirteen red, ten blue and eight green socks. How many socks need to be selected at random to ensure that you have a matching pair?

Solution

Label three holes with the colours red, blue and green.

$\boxed{R}$ $\boxed{B}$ $\boxed{G}$

Selecting just three socks is clearly not sufficient, as you might pick one sock of each colour. Select four socks and place each sock into the hole corresponding to the colour of the sock. As there are four socks and three holes, the pigeonhole principle guarantees that some hole contains at least two socks. This is the required pair.

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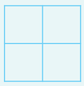
Example 32

a Show that for any five points chosen inside a 2×2 square, at least two of them will be no more than $\sqrt{2}$ units apart.

b Seven football teams play 22 games of football. Show that some pair of teams play each other at least twice.

Solution

a Split the 2×2 square into four unit squares.



Now we have four squares and five points. By the pigeonhole principle, some square contains at least two points. The distance between any two of these points cannot exceed the length of the square's diagonal, $\sqrt{1^2 + 1^2} = \sqrt{2}$.

b There are ${}^7C_2 = 21$ ways that two teams can be chosen to compete from seven. There are 22 games of football, and so some pair of teams play each other at least twice.

► The generalised pigeonhole principle

Suppose that 13 pigeons are placed into four holes. By the pigeonhole principle, there is some hole with at least two pigeons. In fact, some hole must contain at least four pigeons. The reason is simple: If each of the four holes contained no more than three pigeons, then there would be no more than 12 pigeons.

This observation generalises as follows.

Generalised pigeonhole principle

If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Proof Again, let's suppose that the statement is false. Then each of the n holes contains no more than m objects. However, this means that there are no more than mn objects, which is a contradiction.

Example 33

Sixteen natural numbers are written on a whiteboard. Prove that at least four numbers will leave the same remainder when divided by 5.

Solution

We label five holes with each of the possible remainders on division by 5.

$\boxed{0}$ $\boxed{1}$ $\boxed{2}$ $\boxed{3}$ $\boxed{4}$

There are 16 numbers to be placed into five holes. Since $16 = 3 \times 5 + 1$, there is some hole with at least four numbers, each of which leaves the same remainder when divided by 5.

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► Pigeons in multiple holes

In some instances, objects can be placed into more than one hole.

Example 34

Seven people sit at a round table with 10 chairs. Show that there are three consecutive chairs that are occupied.

Solution

Number the chairs from 1 to 10. There are 10 groups of three consecutive chairs:

$$\{1, 2, 3\},\quad \{2, 3, 4\},\quad \{3, 4, 5\},\quad \{4, 5, 6\},\quad \{5, 6, 7\},$$
$$\{6, 7, 8\},\quad \{7, 8, 9\},\quad \{8, 9, 10\},\quad \{9, 10, 1\},\quad \{10, 1, 2\}$$

Each of the seven people will belong to three of these groups, and so 21 people have to be allocated to 10 groups. Since $21 = 2 \times 10 + 1$, the generalised pigeonhole principle guarantees that some group must contain three people.

Section summary

- **Pigeonhole principle**
  If $n + 1$ or more objects are placed into n holes, then some hole contains at least two objects.
- **Generalised pigeonhole principle**
  If at least $mn + 1$ objects are placed into n holes, then some hole contains at least $m + 1$ objects.

Exercise 7H

Example 31

1 You have twelve red, eight blue and seven green socks. How many socks need to be selected at random to ensure that you have a matching pair?

2 A sentence contains 27 English words. Show that there are at least two words that begin with the same letter.

3 Show that in any collection of five natural numbers, at least two will leave the same remainder when divided by 4.

4 How many cards need to be dealt from a deck of 52 playing cards to be certain that you will obtain at least two cards of the same:

a colour **b** suit **c** rank?

5 Eleven points on the number line are located somewhere between 0 and 1. Show that there are at least two points no more than 0.1 apart.

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Example 32 **6** An equilateral triangle has side length 2 units. Choose any five points inside the triangle. Prove that there are at least two points that are no more than 1 unit apart.

7 Thirteen points are located inside a rectangle of length 6 and width 8. Show that there are at least two points that are no more than $2\sqrt{2}$ units apart.

8 The **digital sum** of a natural number is defined to be the sum of its digits. For example, the digital sum of 123 is $1 + 2 + 3 = 6$.

a Nineteen two-digit numbers are selected. Prove that at least two of them have the same digital sum.

b Suppose that 82 three-digit numbers are selected. Prove that at least four of them have the same digital sum.

Example 33 **9** Whenever Eva writes down 13 integers, she notices that at least four of them leave the same remainder when divided by 4. Explain why this is always the case.

10 Twenty-nine games of football are played among eight teams. Prove that there is some pair of teams who play each other more than once.

11 A teacher instructs each member of her class to write down a different whole number between 1 and 49. She says that there will be at least one pair of students such that the sum of their two numbers is 50. How many students must be in her class?

Example 34 **12** There are 10 students seated at a round table with 14 chairs. Show that there are three consecutive chairs that are occupied.

13 There are four points on a circle. Show that three of these points lie on a half-circle.
Hint: Pick any one of the four points and draw a diameter through that point.

14 There are 35 players on a football team and each player has a different number chosen from 1 to 99. Prove that there are at least four pairs of players whose numbers have the same sum.

15 Seven boys and five girls sit evenly spaced at a round table. Prove that some pair of boys are sitting opposite each other.

16 There are n guests at a party and some of these guests shake hands when they meet. Use the pigeonhole principle to show that there is a pair of guests who shake hands with the same number of people.
Hint: Place the n guests into holes labelled from 0 to $n - 1$, corresponding to the number of hands that they shake. Why must either the first or the last hole be empty?

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7I The inclusion–exclusion principle

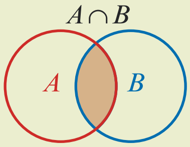
▶ Basic set theory

A set is any collection of objects where order is not important. The set with no elements is called the **empty set** and is denoted by $\varnothing$. We say that set B is a **subset** of set A if each element of B is also in A. In this case, we can write $B \subseteq A$. Note that $\varnothing \subseteq A$ and $A \subseteq A$.

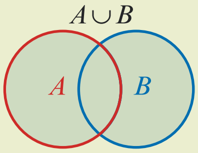
If A is a finite set, then the number of elements in A will be denoted by $|A|$.

Given any two sets A and B we define two important sets:

1 The **intersection** of sets A and B is denoted by $A \cap B$ and consists of elements belonging to A *and* B.



2 The **union** of sets A and B is denoted by $A \cup B$ and consists of elements belonging to A *or* B.



Note: It is important to realise that $A \cup B$ includes elements belonging to A and B.

Example 35

Consider the three sets of numbers $A = \{2, 3\}$, $B = \{1, 2, 3, 4\}$ and $C = \{3, 4, 5\}$.

a Find $B \cap C$.
b Find $A \cup C$.
c Find $A \cap B \cap C$.
d Find $A \cup B \cup C$.
e Find $|A|$.
f List all the subsets of C.

Solution

a $B \cap C = \{3, 4\}$
b $A \cup C = \{2, 3, 4, 5\}$
c $A \cap B \cap C = \{3\}$
d $A \cup B \cup C = \{1, 2, 3, 4, 5\}$
e $|A| = 2$
f $\varnothing, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}$

Earlier in the chapter we encountered the addition principle. This principle can be concisely expressed using set notation.

Addition principle

If A and B are two finite sets of objects such that $A \cap B = \varnothing$, then

$$|A \cup B| = |A| + |B|$$

Our aim is to extend this rule for instances where $A \cap B \neq \varnothing$.

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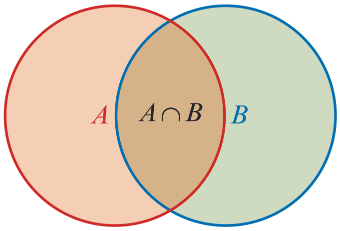
Two sets

To count the number of elements in the set $A \cup B$, we first add (include) $|A|$ and $|B|$. However, this counts the elements in $A \cap B$ twice, and so we subtract (exclude) $|A \cap B|$.



Inclusion–exclusion principle for two sets

If A and B are two finite sets of objects, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Example 36

Each of the 25 students in a Year 11 class studies Physics or Chemistry. Of these students, 15 study Physics and 18 study Chemistry. How many students study both subjects?

Solution

$$\begin{aligned} |P \cup C| &= |P| + |C| - |P \cap C| \\ 25 &= 15 + 18 - |P \cap C| \\ 25 &= 33 - |P \cap C| \\ \therefore \quad |P \cap C| &= 8 \end{aligned}$$

Explanation

Let P and C be the sets of students who study Physics and Chemistry respectively.

Since each student studies Physics or Chemistry, we know that $|P \cup C| = 25$.

Example 37

A bag contains 100 balls labelled with the numbers from 1 to 100. How many ways can a ball be chosen that is a multiple of 2 or 5?

Solution

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ &= 50 + 20 - 10 \\ &= 60 \end{aligned}$$

Explanation

Within the set of numbers $\{1, 2, 3, \dots, 100\}$, let A be the set of multiples of 2 and let B be the set of multiples of 5.

Then $A \cap B$ consists of numbers that are multiples of both 2 and 5, that is, multiples of 10.

Therefore $|A| = 50$, $|B| = 20$ and $|A \cap B| = 10$. We then use the inclusion–exclusion principle.

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Example 38

A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly:

a two clubs

b three spades

c two clubs and three spades

d two clubs or three spades?

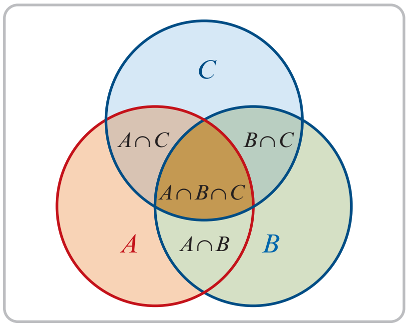
Solution	*Explanation*
a $^{13}C_2 \cdot {}^{39}C_3 = 712\,842$	There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{39}C_3$ ways of choosing 3 more cards from the 39 non-clubs.
b $^{13}C_3 \cdot {}^{39}C_2 = 211\,926$	There are $^{13}C_3$ ways of choosing 3 spades from 13 and $^{39}C_2$ ways of choosing 2 more cards from the 39 non-spades.
c $^{13}C_2 \cdot {}^{13}C_3 = 22\,308$	There are $^{13}C_2$ ways of choosing 2 clubs from 13 and $^{13}C_3$ ways of choosing 3 spades from 13.
d $\begin{aligned} &\lvert A \cup B \rvert \\ &= \lvert A \rvert + \lvert B \rvert - \lvert A \cap B \rvert \\ &= 712\,842 + 211\,926 - 22\,308 \\ &= 902\,460 \end{aligned}$	We let A be the set of all hands with 2 clubs and let B be the set of all hands with 3 spades. Then $A \cap B$ is the set of all hands with 2 clubs and 3 spades. We use the inclusion–exclusion principle to find $\lvert A \cup B \rvert$.

Three sets

For three sets A, B and C, the formula for $|A \cup B \cup C|$ is slightly harder to establish.

We first add $|A|$, $|B|$ and $|C|$. However, we have counted the elements in $A \cap B$, $A \cap C$ and $B \cap C$ twice, and the elements in $A \cap B \cap C$ three times.

Therefore we subtract $|A \cap B|$, $|A \cap C|$ and $|B \cap C|$ to compensate. But then the elements in $A \cap B \cap C$ will have been excluded once too often, and so we add $|A \cap B \cap C|$.



Inclusion–exclusion principle for three sets

If A, B and C are three finite sets of objects, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

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Example 39

How many integers from 1 to 140 inclusive are not divisible by 2, 5 or 7?

Solution

Let A, B and C be the sets of all integers from 1 to 140 that are divisible by 2, 5 and 7 respectively. We then have

A	multiples of 2	$\lvert A\rvert = 140 \div 2 = 70$
B	multiples of 5	$\lvert B\rvert = 140 \div 5 = 28$
C	multiples of 7	$\lvert C\rvert = 140 \div 7 = 20$
$A \cap B$	multiples of 10	$\lvert A \cap B\rvert = 140 \div 10 = 14$
$A \cap C$	multiples of 14	$\lvert A \cap C\rvert = 140 \div 14 = 10$
$B \cap C$	multiples of 35	$\lvert B \cap C\rvert = 140 \div 35 = 4$
$A \cap B \cap C$	multiples of 70	$\lvert A \cap B \cap C\rvert = 140 \div 70 = 2$

We use the inclusion–exclusion principle to give

$$\begin{aligned}|A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|\\ &= 70 + 28 + 20 - 14 - 10 - 4 + 2\\ &= 92\end{aligned}$$

Therefore the number of integers not divisible by 2, 5 or 7 is $140 - 92 = 48$.

Section summary

- The inclusion–exclusion principle extends the addition principle to instances where the two sets have objects in common.
- The principle works by ensuring that objects belonging to multiple sets are not counted more than once.
- The inclusion–exclusion principles for two sets and three sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Exercise 7I

Example 35

1 Consider the three sets of numbers $A = \{4, 5, 6\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 3, 4, 6\}$.

a Find $B \cap C$. **b** Find $A \cup C$. **c** Find $A \cap B \cap C$.

d Find $A \cup B \cup C$. **e** Find $|A|$. **f** List all the subsets of A.

Example 36

2 In an athletics team, each athlete competes in track or field events. There are 25 athletes who compete in track events, 23 who compete in field events and 12 who compete in both track and field events. How many athletes are in the team?

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3 Fifty patients at a medical clinic are being treated for a disease using two types of medication, A and B. There are 25 patients using medication A and 29 patients using medication B. How many patients are using both types of medication?

Example 37

4 How many integers from 1 to 630 inclusive are multiples of 7 or 9?

5 Consider the integers from 1 to 96 inclusive. How many of these are:

a divisible by 2 or 3
b not divisible by 2 or 3?

Example 38

6 How many five-letter arrangements of the word COMET:

a begin with a vowel
b end with a vowel
c begin and end with a vowel
d begin or end with a vowel?

7 **a** How many of the integers from 1 to 100 inclusive are perfect squares or perfect cubes?

b How many of the integers from 1 to 1000 inclusive are perfect squares or perfect cubes?

Example 39

8 How many of the integers from 1 to 120 inclusive are multiples of 2, 3 or 5?

9 How many of the integers from 1 to 220 inclusive are not divisible by 2, 5 or 11?

10 There are 98 Year 11 students at a secondary school and each of them must study at least one of Biology, Physics or Chemistry. There are 36 students who study Biology, 42 who study Physics and 40 who study Chemistry. Moreover, 9 study Biology and Physics, 8 study Biology and Chemistry and 7 study Physics and Chemistry. How many students study all three subjects?

11 A group of six students is selected from four students in Year 10, five in Year 11 and four in Year 12. How many selections have exactly:

a two Year 10 students
b two Year 11 students
c two Year 10 and two Year 11 students
d two Year 10 or two Year 11 students?

12 A hand of five cards is dealt from a deck of 52 cards. How many hands contain exactly one heart or exactly two diamonds?

13 Find the sum of all the integers from 1 to 100 inclusive that are divisible by 2 or 3.

14 There are eighty Year 11 students at a school and each of them must study at least one of three languages. Thirty are studying French, forty-five are studying Chinese, thirty are studying German and fifteen are studying all three languages. How many students are studying exactly two languages?

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Review

Chapter summary

AS

Nrich

- The addition and multiplication principles provide efficient methods for counting the number of ways of performing multiple tasks.
- The number of **permutations** (or arrangements) of n objects taken r at a time is given by

$$^{n}P_r = \frac{n!}{(n-r)!}$$

- The number of **combinations** (or selections) of n objects taken r at a time is given by

$$^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

- When permutations or combinations involve restrictions, we deal with them first.
- The values of $^{n}C_r$ can be arranged to give **Pascal's triangle**, where each entry is the sum of the two entries immediately above.
- The sum of the entries in row n of Pascal's triangle is 2^n. That is,

$$^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_{n-1} + {}^{n}C_n = 2^n$$

- A set of size n has 2^n subsets.
- The **pigeonhole principle** is used to show that some pair or group of objects have the same property.
- The **inclusion–exclusion principle** allows us to count the number of elements in a union of sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Technology-free questions

1 Evaluate:

a $^{6}C_3$ **b** $^{20}C_2$ **c** $^{300}C_1$ **d** $^{100}C_{98}$

2 Find the value of n if $^{n}C_2 = 55$.

3 How many three-digit numbers can be formed using the digits 1, 2 and 3 if the digits:

a can be repeated **b** cannot be repeated?

4 How many ways can six students be arranged on a bench seat with space for three?

5 How many ways can three students be allocated to five vacant desks?

6 There are four Year 11 and three Year 12 students in a school debating club. How many ways can a team of four be selected if two are chosen from each year level?

7 There are three boys and four girls in a group. How many ways can three children be selected if at least one of them is a boy?

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8 On a ship's mast are two identical red and three identical black flags that can be arranged to send messages to nearby ships. How many different arrangements using all five flags are possible?

9 There are 53 English words written on a page. How many are guaranteed to share the same first letter?

10 Each of the twenty students in a class plays netball or basketball. Twelve play basketball and four play both sports. How many students play netball?

11 Six people are to be seated in a row. Calculate the number of ways this can be done so that two particular people, A and B, always have exactly one person between them.

Multiple-choice questions

1 Bao plans to study six subjects in Year 12. He has already chosen three subjects and for the remaining three he plans to choose one of four languages, one of three mathematics subjects and one of four science subjects. How many ways can he select his remaining subjects?

A 6 **B** 11 **C** 48 **D** 165 **E** 990

2 There are three flights directly from Melbourne to Brisbane. There are also two flights from Melbourne to Sydney and then four choices of connecting flight from Sydney to Brisbane. How many different paths are there from Melbourne to Brisbane?

A 9 **B** 11 **C** 18 **D** 20 **E** 24

3 In how many ways can 10 people be arranged in a queue at the bank?

A $10!$ **B** 10^{10} **C** 2^{10} **D** ${}^{10}C_2$ **E** ${}^{10}C_1$

4 How many three-digit numbers can be formed using the digits 1, 2, 3, 4, 5 and 6 at most once?

A ${}^{6}C_3$ **B** $3!$ **C** $6!$ **D** $6 \times 5 \times 4$ **E** $6 + 5 + 4$

5 How many permutations of the word UTOPIA begin and end with a vowel?

A 90 **B** 288 **C** 384 **D** 720 **E** 4320

6 How many ways can four identical red flags and three identical blue flags be arranged in a row?

A 4×3 **B** $\dfrac{7!}{4! \times 3!}$ **C** $7! \times 3! \times 4!$ **D** $4! \times 3!$ **E** $2 \times 3! \times 4!$

7 How many ways can three DVDs be chosen from a collection of nine different DVDs?

A $3!$ **B** $9 \times 8 \times 7$ **C** ${}^{9}C_3$ **D** $\dfrac{9!}{3!}$ **E** 3×9

8 The number of subsets of $\{A, B, C, D, E, F\}$ with at least one element is

A ${}^{6}C_2$ **B** ${}^{6}C_2 - 1$ **C** $2^5 - 1$ **D** $2^6 - 1$ **E** 2^6

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9 A class consists of nine girls and eight boys. How many ways can a group of two boys and two girls be chosen?

A $\dfrac{17!}{2!\,2!}$ **B** ${}^{17}C_4$ **C** ${}^{9}C_2 \cdot {}^{8}C_2$ **D** $\dfrac{17!}{9!\,8!}$ **E** $9 \times 8 \times 8 \times 7$

10 There are six blue balls and five red balls in a bag. How many balls need to be selected at random before you are certain that three will have the same colour?

A 3 **B** 4 **C** 5 **D** 7 **E** 11

11 Each of the 30 students in a class studies French, German or Chinese. Of these students, 15 study French, 17 study German and 15 study Chinese. There are 15 students that study more than one subject. How many students study all three subjects?

A 2 **B** 3 **C** 4 **D** 5 **E** 6

Extended-response questions

1 A six-digit number is formed using the digits 1, 2, 3, 4, 5 and 6 without repetition. How many ways can this be done if:

a the first digit is 5
b the first digit is even
c even and odd digits alternate
d the even digits are kept together?

2 Three letters from the word AUNTIE are arranged in a row. How many ways can this be done if:

a the first letter is E
b the first letter is a vowel
c the letter E is used?

3 A student leadership team consists of four boys and six girls. A group of four students is required to organise a social function. How many ways can the group be selected:

a without restriction
b if the school captain is included
c if there are two boys
d if there is at least one boy?

4 Consider the eight letters N, N, J, J, T, T, T, T. How many ways can all eight letters be arranged if:

a there is no restriction
b the first and last letters are both N
c the two Js are adjacent
d no two Ts are adjacent?

5 A pizza restaurant offers the following toppings: onion, capsicum, mushroom, olives, ham and pineapple.

a How many different kinds of pizza can be ordered with:
  i three different toppings
  ii three different toppings including ham
  iii any number of toppings (between none and all six)?

b Another pizza restaurant boasts that they can make more than 200 varieties of pizza. What is the smallest number of toppings that they could use?

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6 In how many ways can a group of four people be chosen from five married couples if:

a there is no restriction

b any two women and two men are chosen

c any two married couples are chosen

d a husband and wife cannot both be selected?

7 The name David Smith has initials DS.

a How many different two-letter initials are possible?

b How many different two-letter initials contain at least one vowel?

c Given 50 000 people, how many of them can be guaranteed to share the same two-letter initials?

8 Consider the integers from 1 to 96 inclusive. Let sets A and B consist of those integers that are multiples of 6 and 8 respectively.

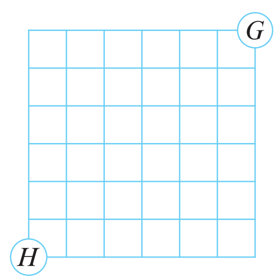
a What is the lowest common multiple of 6 and 8?

b How many integers belong to $A \cap B$?

c How many integers from 1 to 96 are divisible by 6 or 8?

d An integer from 1 to 96 is chosen at random. What is the probability that it is not divisible by 6 or 8?

9 Every morning, Milly walks from her home $H(0, 0)$ to the gym $G(6, 6)$ along city streets that are laid out in a square grid as shown. She always takes a path of shortest distance.



a How many paths are there from H to G?

b Show that there is some path that she takes at least twice in the course of three years.

c On her way to the gym, she often purchases a coffee at a cafe located at point $C(2, 2)$. How many paths are there from:

i H to C **ii** C to G **iii** H to C to G?

d A new cafe opens up at point $B(4, 4)$. How many paths can Milly take, assuming that she buys coffee at either cafe?

Hint: You will need to use the inclusion–exclusion principle here.

10 A box contains 400 balls, each of which is blue, red, green, yellow or orange. The ratio of blue to red to green balls is 1 : 4 : 2. The ratio of green to yellow to orange balls is 1 : 3 : 6. What is the smallest number of balls that must be drawn to ensure that at least 50 balls of one colour are selected?

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