Chapter 5

5 Algebra II

Objectives

- To understand equality of **polynomials**.
- To use **equating coefficients** to solve problems.
- To solve **quadratic equations** by various methods.
- To use quadratic equations to solve problems involving **rates**.
- To resolve rational algebraic expressions into **partial fractions**.
- To find the coordinates of the points of intersection of straight lines with parabolas, circles and rectangular hyperbolas.

In this chapter we first consider equating coefficients of polynomial functions, and then apply this technique to establish partial fractions.

In Chapter 1 we added and subtracted algebraic fractions such as

$$\frac{2}{x+3} + \frac{4}{x-3} = \frac{6(x+1)}{x^2-9}$$

In this chapter we learn how to go from right to left in similar equations. This process is sometimes called **partial fraction decomposition**. Another example is

$$\frac{4x^2+2x+6}{(x^2+3)(x-3)} = \frac{2}{x^2+3} + \frac{4}{x-3}$$

This is a useful tool in integral calculus, and partial fractions are applied this way in Specialist Mathematics Units 3 & 4.

This chapter also includes further study of quadratic functions: solving quadratic equations, using the discriminant, applying quadratic functions to problems involving rates and using quadratic equations to find the intersection of straight lines with parabolas, circles and rectangular hyperbolas.

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5A Polynomial identities

Polynomials are introduced in Mathematical Methods Units 1 & 2.

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Two polynomials are **equal** if they give the same value for all x. It can be proved that, if two polynomials are equal, then they have the same degree and corresponding coefficients are equal. For example:

- If $ax + b = cx^2 + dx + e$, then $c = 0$, $d = a$ and $e = b$.
- If $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

This process is called **equating coefficients**.

Example 1

If the expressions $(a + 2b)x^2 - (a - b)x + 8$ and $3x^2 - 6x + 8$ are equal for all x, find the values of a and b.

Solution

Assume that

$$(a + 2b)x^2 - (a - b)x + 8 = 3x^2 - 6x + 8 \quad \text{for all } x$$

Then by equating coefficients:

$$a + 2b = 3 \qquad (1)$$

$$-(a - b) = -6 \qquad (2)$$

Solve as simultaneous equations.

Add (1) and (2):

$$3b = -3$$

$$\therefore \quad b = -1$$

Substitute into (1):

$$a - 2 = 3$$

$$\therefore \quad a = 5$$

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Example 2

Express x^2 in the form $c(x-3)^2 + a(x-3) + d$.

Solution

Let $x^2 = c(x-3)^2 + a(x-3) + d$

$= c(x^2 - 6x + 9) + a(x-3) + d$

$= cx^2 + (a - 6c)x + 9c - 3a + d$

This implies that

$$c = 1 \qquad (1)$$

$$a - 6c = 0 \qquad (2)$$

$$9c - 3a + d = 0 \qquad (3)$$

From (2): $a = 6$

From (3): $9 - 18 + d = 0$

i.e. $d = 9$

Hence $x^2 = (x-3)^2 + 6(x-3) + 9$.

Example 3

Find the values of a, b, c and d such that

$$x^3 = a(x+2)^3 + b(x+1)^2 + cx + d \quad \text{for all } x$$

Solution

Expand the right-hand side and collect like terms:

$$x^3 = a(x^3 + 6x^2 + 12x + 8) + b(x^2 + 2x + 1) + cx + d$$

$$= ax^3 + (6a + b)x^2 + (12a + 2b + c)x + (8a + b + d)$$

Equate coefficients:

$$a = 1 \qquad (1)$$

$$6a + b = 0 \qquad (2)$$

$$12a + 2b + c = 0 \qquad (3)$$

$$8a + b + d = 0 \qquad (4)$$

Substituting $a = 1$ into (2) gives

$$6 + b = 0$$

$\therefore$ $b = -6$

Substituting $a = 1$ and $b = -6$ into (3) gives

$$12 - 12 + c = 0$$

$\therefore$ $c = 0$

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Substituting $a = 1$ and $b = -6$ into (4) gives

$$8 - 6 + d = 0$$

$$\therefore \quad d = -2$$

Hence $x^3 = (x + 2)^3 - 6(x + 1)^2 - 2$.

Example 4

Show that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Solution

Suppose that

$$2x^3 - 5x^2 + 4x + 1 = a(x + b)^3 + c$$

for some constants a, b and c.

Then expanding the right-hand side gives

$$\begin{aligned} 2x^3 - 5x^2 + 4x + 1 &= a(x^3 + 3bx^2 + 3b^2x + b^3) + c \\ &= ax^3 + 3abx^2 + 3ab^2x + ab^3 + c \end{aligned}$$

Equating coefficients:

$$\begin{aligned} a &= 2 \qquad &(1) \\ 3ab &= -5 \qquad &(2) \\ 3ab^2 &= 4 \qquad &(3) \\ ab^3 + c &= 1 \qquad &(4) \end{aligned}$$

From (2), we have $b = -\frac{5}{6}$. But from (3), we have $b = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$.

This is a contradiction, and therefore we have shown that $2x^3 - 5x^2 + 4x + 1$ cannot be expressed in the form $a(x + b)^3 + c$.

Section summary

- A **polynomial function** can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree** of a polynomial is the index n of the leading term.
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

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Exercise 5A

1 If $ax^2 + bx + c = 10x^2 - 7$, find the values of a, b and c.

Example 1 **2** If $(2a - b)x^2 + (a + 2b)x + 8 = 4x^2 - 3x + 8$, find the values of a and b.

3 If $(2a - 3b)x^2 + (3a + b)x + c = 7x^2 + 5x + 7$, find the values of a, b and c.

4 If $2x^2 + 4x + 5 = a(x + b)^2 + c$, find the values of a, b and c.

Example 2 **5** Express x^2 in the form $c(x + 2)^2 + a(x + 2) + d$.

6 Express x^3 in the form $(x + 1)^3 + a(x + 1)^2 + b(x + 1) + c$.

Example 3 **7** Find the values of a, b and c such that $x^2 = a(x + 1)^2 + bx + c$.

Example 4 **8** **a** Show that $3x^3 - 9x^2 + 8x + 2$ cannot be expressed in the form $a(x + b)^3 + c$.

b If $3x^3 - 9x^2 + 9x + 2$ can be expressed in the form $a(x + b)^3 + c$, then find the values of a, b and c.

9 Show that constants a, b, c and d can be found such that

$$n^3 = a(n + 1)(n + 2)(n + 3) + b(n + 1)(n + 2) + c(n + 1) + d$$

10 **a** Show that no constants a and b can be found such that

$$n^2 = a(n + 1)(n + 2) + b(n + 2)(n + 3)$$

b Express n^2 in the form $a(n + 1)(n + 2) + b(n + 1) + c$.

11 **a** Express $a(x + b)^2 + c$ in expanded form.

b Express $ax^2 + bx + c$ in completed-square form.

12 Prove that, if $ax^3 + bx^2 + cx + d = (x - 1)^2(px + q)$, then $b = d - 2a$ and $c = a - 2d$.

13 If $3x^2 + 10x + 3 = c(x - a)(x - b)$ for all values of x, find the values of a, b and c.

14 For any number n, show that n^2 can be expressed as $a(n - 1)^2 + b(n - 2)^2 + c(n - 3)^2$, and find the values of a, b and c.

15 If $x^3 + 3x^2 - 9x + c$ can be expressed in the form $(x - a)^2(x - b)$, show that either $c = 5$ or $c = -27$, and find a and b for each of these cases.

16 A polynomial P is said to be **even** if $P(-x) = P(x)$ for all x. A polynomial P is said to be **odd** if $P(-x) = -P(x)$ for all x.

a Show that, if $P(x) = ax^4 + bx^3 + cx^2 + dx + e$ is even, then $b = d = 0$.

b Show that, if $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is odd, then $b = d = f = 0$.

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5B Quadratic equations

A polynomial function of degree 2 is called a **quadratic function**. The general quadratic function can be written as $P(x) = ax^2 + bx + c$, where $a \neq 0$.

Quadratic functions are studied extensively in Mathematical Methods Units 1 & 2. In this section we provide further practice exercises.

A quadratic equation $ax^2 + bx + c = 0$ may be solved by factorising, by completing the square or by using the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The following example demonstrates each method.

Example 5

Solve the following quadratic equations for x:

a $2x^2 + 5x = 12$ **b** $3x^2 + 4x = 2$ **c** $9x^2 + 6x + 1 = 0$

Solution

a

$$2x^2 + 5x - 12 = 0$$

$$(2x - 3)(x + 4) = 0$$

$$2x - 3 = 0 \text{ or } x + 4 = 0$$

Therefore $x = \frac{3}{2}$ or $x = -4$.

Explanation

Rearrange the quadratic equation.

Factorise.

Use the null factor theorem.

b

$$3x^2 + 4x - 2 = 0$$

$$x^2 + \frac{4}{3}x - \frac{2}{3} = 0$$

$$x^2 + \frac{4}{3}x + \left(\frac{2}{3}\right)^2 - \left(\frac{2}{3}\right)^2 - \frac{2}{3} = 0$$

$$\left(x + \frac{2}{3}\right)^2 - \frac{4}{9} - \frac{2}{3} = 0$$

$$\left(x + \frac{2}{3}\right)^2 = \frac{10}{9}$$

$$x + \frac{2}{3} = \pm\frac{\sqrt{10}}{3}$$

$$x = -\frac{2}{3} \pm \frac{\sqrt{10}}{3}$$

Therefore $x = \frac{-2 + \sqrt{10}}{3}$ or $x = \frac{-2 - \sqrt{10}}{3}$.

Explanation

Rearrange the quadratic equation.

Divide both sides by 3.

Add and subtract $\left(\frac{b}{2}\right)^2$ to 'complete the square'.

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c If $9x^2 + 6x + 1 = 0$, then

$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 9 \times 1}}{2 \times 9}$$

$$= \frac{-6 \pm \sqrt{0}}{18}$$

$$= -\frac{1}{3}$$

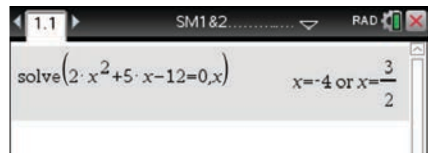
Use the general quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Alternatively, the equation can be solved by noting that $9x^2 + 6x + 1 = (3x + 1)^2$.

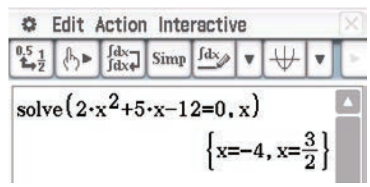
Using the TI-Nspire

Use (menu) > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $2x^2 + 5x - 12 = 0$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x. Tap OK.



The discriminant: real solutions

The number of solutions to a quadratic equation $ax^2 + bx + c = 0$ can be determined by the **discriminant** Δ, where $\Delta = b^2 - 4ac$.

- If $\Delta > 0$, the equation has two real solutions.
- If $\Delta = 0$, the equation has one real solution.
- If $\Delta < 0$, the equation has no real solutions.

Note: In parts **a** and **b** of Example 5, we have $\Delta > 0$ and so there are two real solutions. In part **c**, we have $\Delta = 6^2 - 4 \times 9 \times 1 = 0$ and so there is only one real solution.

The discriminant: rational solutions

For a quadratic equation $ax^2 + bx + c = 0$ such that a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Note: In part **a** of Example 5, we have $\Delta = 121$, which is a perfect square.

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Example 6

Consider the quadratic equation $x^2 - 4x = t$. Make x the subject and give the values of t for which real solution(s) to the equation can be found.

Solution

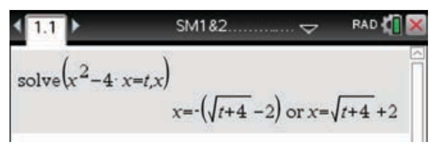
$$\begin{aligned} x^2 - 4x &= t \\ x^2 - 4x + 4 &= t + 4 \qquad \text{(completing the square)} \\ (x-2)^2 &= t + 4 \\ x - 2 &= \pm\sqrt{t+4} \\ x &= 2 \pm \sqrt{t+4} \end{aligned}$$

For real solutions to exist, we must have $t + 4 \geq 0$, i.e. $t \geq -4$.

Note: In this case the discriminant is $\Delta = 16 + 4t$. There are real solutions when $\Delta \geq 0$.

Using the TI-Nspire

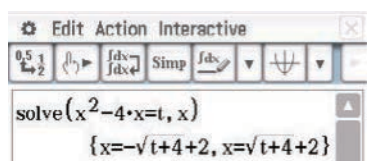
Use (menu) > **Algebra** > **Solve** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $x^2 - 4x = t$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.

Note: The variable t is found in the (Var) keyboard.



Example 7

a Find the discriminant of the quadratic $x^2 + px - \dfrac{25}{4}$ in terms of p.

b Solve the quadratic equation $x^2 + px - \dfrac{25}{4} = 0$ in terms of p.

c Prove that there are two solutions for all values of p.

d Find the values of p, where p is a non-negative integer, for which the quadratic equation has rational solutions.

Solution

Here we have $a = 1$, $b = p$ and $c = -\dfrac{25}{4}$.

a $\Delta = b^2 - 4ac = p^2 + 25$

b The quadratic formula gives $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-p \pm \sqrt{p^2 + 25}}{2}$.

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c We have $\Delta = p^2 + 25 > 0$, for all values of p. Thus there are always two solutions.

d If there are rational solutions, then $\Delta = p^2 + 25$ is a perfect square. Since p is an integer, we can write $p^2 + 25 = k^2$, where k is an integer with $k \geq 0$.

Rearranging, we have

$$k^2 - p^2 = 25$$

$$\therefore \quad (k - p)(k + p) = 25$$

We can factorise 25 as 5×5 or 1×25.

Note: We do not need to consider negative factors of 25, as p and k are non-negative, and so $k + p \geq 0$. Since p is non-negative, we also know that $k - p \leq k + p$.

The table on the right shows the values of k and p in each of the two cases.

Hence $p = 0$ and $p = 12$ are the only values for which the solutions are rational.

$k - p$	$k + p$	k	p
5	5	5	0
1	25	13	12

Example 8

A rectangle has an area of 288 cm^2. If the width is decreased by 1 cm and the length increased by 1 cm, the area would be decreased by 3 cm^2. Find the original dimensions of the rectangle.

Solution

Let w and ℓ be the width and length, in centimetres, of the original rectangle.

Then $\quad w\ell = 288 \quad (1)$

The dimensions of the new rectangle are $w - 1$ and $\ell + 1$, and the area is 285 cm^2.

Thus $\quad (w - 1)(\ell + 1) = 285 \quad (2)$

Rearranging (1) to make ℓ the subject and substituting in (2) gives

$$(w - 1)\left(\frac{288}{w} + 1\right) = 285$$

$$288 - \frac{288}{w} + w - 1 = 285$$

$$w - \frac{288}{w} + 2 = 0$$

$$w^2 + 2w - 288 = 0$$

Using the general quadratic formula gives

$$w = \frac{-2 \pm \sqrt{2^2 - 4 \times (-288)}}{2}$$

$$= -18 \text{ or } 16$$

But $w > 0$, and so $w = 16$. The original dimensions of the rectangle are 16 cm by 18 cm.

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Example 9

Solve the equation $x - 4\sqrt{x} - 12 = 0$ for x.

Solution

For $\sqrt{x}$ to be defined, we must have $x \geq 0$.

Let $x = a^2$, where $a \geq 0$.

The equation becomes

$$a^2 - 4\sqrt{a^2} - 12 = 0$$
$$a^2 - 4a - 12 = 0$$
$$(a - 6)(a + 2) = 0$$
$$\therefore \quad a = 6 \text{ or } a = -2$$

But $a \geq 0$. Hence $a = 6$ and so $x = 36$.

Section summary

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though there may be no solution for some quadratic equations.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

  For the equation $ax^2 + bx + c = 0$:
  - If $\Delta > 0$, there are two solutions.
  - If $\Delta = 0$, there is one solution.
  - If $\Delta < 0$, there are no real solutions.
- For the equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers:
  - If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
  - If $\Delta = 0$, there is one rational solution.
  - If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

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Exercise 5B

Skillsheet
Example 5

1 Solve the following quadratic equations for x:

a $x^2 - 2x = -1$ **b** $x^2 - 6x + 9 = 0$ **c** $5x^2 - 10x = 1$
d $-2x^2 + 4x = 1$ **e** $2x^2 + 4x = 7$ **f** $6x^2 + 13x + 1 = 0$

2 The following equations have the number of solutions shown in brackets. Find the possible values of m.

a $x^2 + 3x + m = 0$ (0) **b** $x^2 - 5x + m = 0$ (2) **c** $mx^2 + 5x - 8 = 0$ (1)
d $x^2 + mx + 9 = 0$ (2) **e** $x^2 - mx + 4 = 0$ (0) **f** $4x^2 - mx - m = 0$ (1)

Example 6

3 Make x the subject in each of the following and give the values of t for which real solution(s) to the equation can be found:

a $2x^2 - 4t = x$ **b** $4x^2 + 4x - 4 = t - 2$
c $5x^2 + 4x + 10 = t$ **d** $tx^2 + 4tx + 10 = t$

Example 7

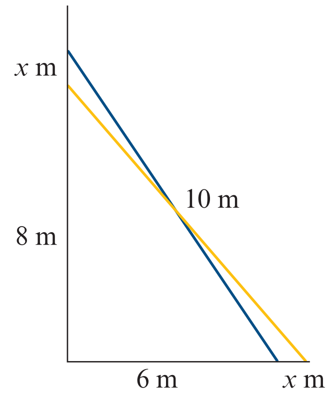
4 **a** Solve the quadratic equation $x^2 + px - 16 = 0$ in terms of p.
b Find the values of p, where p is an integer with $0 \le p \le 10$, for which the quadratic equation in **a** has rational solutions.

5 **a** Show that the solutions of the equation $2x^2 - 3px + (3p - 2) = 0$ are rational for all integer values of p.
b Find the value of p for which there is only one solution.
c Solve the equation when:
i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

6 **a** Show that the solutions of the equation $4(4p - 3)x^2 - 8px + 3 = 0$ are rational for all integer values of p.
b Find the value of p for which there is only one solution.
c Solve the equation when:
i $p = 1$ **ii** $p = 2$ **iii** $p = -1$

Example 8

7 A pole 10 m long leans against a wall. The bottom of the pole is 6 m from the wall. If the bottom of the pole is pulled away x m so that the top slides down by the same amount, find x.



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8 A wire of length 200 cm is cut into two parts and each part is bent to form a square. If the area of the larger square is 9 times the area of the smaller square, find the length of the sides of the larger square.

Example 9

9 Solve each of the following equations for x:

a $x - 8\sqrt{x} + 12 = 0$

b $x - 8 = 2\sqrt{x}$

c $x - 5\sqrt{x} - 14 = 0$

d $\sqrt[3]{x^2} - 9\sqrt[3]{x} + 8 = 0$

e $\sqrt[3]{x^2} - \sqrt[3]{x} - 6 = 0$

f $x - 29\sqrt{x} + 100 = 0$

10 Find constants a, b and c such that

$$3x^2 - 5x + 1 = a(x + b)^2 + c$$

for all values of x. Hence find the minimum value of $3x^2 - 5x + 1$.

11 Show that the graphs of $y = 2 - 4x - x^2$ and $y = 24 + 8x + x^2$ do not intersect.

12 Solve the quadratic equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ for x.

13 Given that the two solutions of the equation $2x^2 - 6x - m = 0$ differ by 5, find the value of m.

14 For the equation $(b^2 - 2ac)x^2 + 4(a + c)x = 8$:

a Prove that there are always (real) solutions of the equation.

b Find the conditions that there is only one solution.

15 The equation $\dfrac{1}{2} + \dfrac{1}{x + k} = \dfrac{1}{x}$ has no solutions. Find the possible values of k.

16 Find the smallest positive integer p for which the equation $3x^2 + px + 7 = 0$ has solutions.

5C Applying quadratic equations to rate problems

A **rate** describes how a certain quantity changes with respect to the change in another quantity (often time). An example of a rate is ‘speed’. A speed of 60 km/h gives us a measure of how fast an object is travelling. A further example is ‘flow’, where a rate of 20 L/min is going to fill an empty swimming pool faster than a rate of 6 L/min.

Many problems are solved using rates, which can be expressed as fractions. For example, a speed of 60 km/h can be expressed in fraction form as

$$\frac{\text{distance (km)}}{\text{time taken (h)}} = \frac{60}{1}$$

When solving rate problems, it is often necessary to add two or more fractions with different denominators, as shown in the following examples.

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Example 10

a Express $\frac{6}{x} + \frac{6}{x+8}$ as a single fraction. **b** Solve the equation $\frac{6}{x} + \frac{6}{x+8} = 2$ for x.

Solution

a
$$\frac{6}{x} + \frac{6}{x+8} = \frac{6(x+8)}{x(x+8)} + \frac{6x}{x(x+8)}$$
$$= \frac{6x + 48 + 6x}{x(x+8)}$$
$$= \frac{12(x+4)}{x(x+8)}$$

b Since $\frac{6}{x} + \frac{6}{x+8} = \frac{12(x+4)}{x(x+8)}$, we have

$$\frac{12(x+4)}{x(x+8)} = 2$$
$$12(x+4) = 2x(x+8)$$
$$6(x+4) = x(x+8)$$
$$6x + 24 = x^2 + 8x$$
$$x^2 + 2x - 24 = 0$$
$$(x+6)(x-4) = 0$$

Therefore $x = -6$ or $x = 4$.

Example 11

A car travels 500 km at a constant speed. If it had travelled at a speed of 10 km/h less, it would have taken 1 hour more to travel the distance. Find the speed of the car.

Solution

Let x km/h be the speed of the car.

It takes $\frac{500}{x}$ hours for the journey.

If the speed is 10 km/h less, then the new speed is $(x - 10)$ km/h.

The time taken is $\frac{500}{x} + 1$ hours.

We can now write:

$$500 = (x - 10) \times \left(\frac{500}{x} + 1\right)$$
$$500x = (x - 10)(500 + x)$$
$$500x = 490x - 5000 + x^2$$

Thus

$$x^2 - 10x - 5000 = 0$$

and so

$$x = \frac{10 \pm \sqrt{100 + 4 \times 5000}}{2}$$
$$= 5(1 \pm \sqrt{201})$$

The speed is $5(1 + \sqrt{201}) \approx 75.887$ km/h.

Explanation

For an object travelling at a constant speed in one direction:

$$\text{speed} = \frac{\text{distance travelled}}{\text{time taken}}$$

and so

$$\text{time taken} = \frac{\text{distance travelled}}{\text{speed}}$$

and

$$\text{distance travelled} = \text{speed} \times \text{time taken}$$

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Example 12

A tank is filled by two pipes. The smaller pipe alone will take 24 minutes longer than the larger pipe alone, and 32 minutes longer than when both pipes are used. How long will each pipe take to fill the tank alone? How long will it take for both pipes used together to fill the tank?

Solution

Let C cubic units be the capacity of the tank, and let x minutes be the time it takes for the larger pipe alone to fill the tank.

Then the average rate of flow for the larger pipe is $\frac{C}{x}$ cubic units per minute.

Since the smaller pipe alone takes $(x + 24)$ minutes to fill the tank, the average rate of flow for the smaller pipe is $\frac{C}{x+24}$ cubic units per minute.

The average rate of flow when both pipes are used together is the sum of these two rates:

$$\frac{C}{x} + \frac{C}{x+24} \text{ cubic units per minute}$$

Expressed as a single fraction:

$$\begin{aligned}\frac{C}{x} + \frac{C}{x+24} &= \frac{C(x+24) + Cx}{x(x+24)} \\ &= \frac{2C(x+12)}{x(x+24)}\end{aligned}$$

The time taken to fill the tank using both pipes is

$$\begin{aligned}C \div \frac{2C(x+12)}{x(x+24)} &= C \times \frac{x(x+24)}{2C(x+12)} \\ &= \frac{x(x+24)}{2(x+12)}\end{aligned}$$

Therefore the time taken for the smaller pipe alone to fill the tank can be also be expressed as $\frac{x(x+24)}{2(x+12)} + 32$ minutes.

Thus

$$\begin{aligned}\frac{x(x+24)}{2(x+12)} + 32 &= x + 24 \\ \frac{x(x+24)}{2(x+12)} &= x - 8 \\ x(x+24) &= 2(x+12)(x-8) \\ x^2 + 24x &= 2x^2 + 8x - 192 \\ x^2 - 16x - 192 &= 0 \\ (x-24)(x+8) &= 0\end{aligned}$$

But $x > 0$, and hence $x = 24$.

It takes 24 minutes for the larger pipe alone to fill the tank, 48 minutes for the smaller pipe alone to fill the tank, and 16 minutes for both pipes together to fill the tank.

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Exercise 5C

Skillsheet

Example 10

1 **a** Express $\frac{6}{x} - \frac{6}{x+3}$ as a single fraction.

b Solve the equation $\frac{6}{x} - \frac{6}{x+3} = 1$ for x.

2 Solve the equation $\frac{300}{x+5} = \frac{300}{x} - 2$ for x.

3 The sum of the reciprocals of two consecutive odd numbers is $\frac{36}{323}$. Form a quadratic equation and hence determine the two numbers.

Example 11

4 A cyclist travels 40 km at a speed of x km/h.

a Find the time taken in terms of x.

b Find the time taken when his speed is reduced by 2 km/h.

c If the difference between the times is 1 hour, find his original speed.

5 A car travels from town A to town B, a distance of 600 km, in x hours. A plane, travelling 220 km/h faster than the car, takes $5\frac{1}{2}$ hours less to cover the same distance.

a Express, in terms of x, the average speed of the car and the average speed of the plane.

b Find the actual average speed of each of them.

6 A car covers a distance of 200 km at a speed of x km/h. A train covers the same distance at a speed of $(x + 5)$ km/h. If the time taken by the car is 2 hours more than that taken by the train, find x.

7 A man travels 108 km, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled at an average speed 2 km/h faster. What was the man's average speed when he made the trip?

8 A bus is due to reach its destination 75 km away at a certain time. The bus usually travels with an average speed of x km/h. Its start is delayed by 18 minutes but, by increasing its average speed by 12.5 km/h, the driver arrives on time.

a Find x. **b** How long did the journey actually take?

9 Ten minutes after the departure of an express train, a slow train starts, travelling at an average speed of 20 km/h less. The slow train reaches a station 250 km away 3.5 hours after the arrival of the express. Find the average speed of each of the trains.

10 When the average speed of a car is increased by 10 km/h, the time taken for the car to make a journey of 105 km is reduced by 15 minutes. Find the original average speed.

11 A tank can be filled with water by two pipes running together in $11\frac{1}{9}$ minutes. If the larger pipe alone takes 5 minutes less to fill the tank than the smaller pipe, find the time that each pipe will take to fill the tank.

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Example 12 **12** At first two different pipes running together will fill a tank in $\frac{20}{3}$ minutes. The rate that water runs through each of the pipes is then adjusted. If one pipe, running alone, takes 1 minute less to fill the tank at its new rate, and the other pipe, running alone, takes 2 minutes more to fill the tank at its new rate, then the two running together will fill the tank in 7 minutes. Find in what time the tank will be filled by each pipe running alone at the new rates.

13 The journey between two towns by one route consists of 233 km by rail followed by 126 km by sea. By a second route the journey consists of 405 km by rail followed by 39 km by sea. If the time taken for the first route is 50 minutes longer than for the second route, and travelling by rail is 25 km/h faster than travelling by sea, find the average speed by rail and the average speed by sea.

14 A sea freighter travelling due north at 12 km/h sights a cruiser straight ahead at an unknown distance and travelling due east at unknown speed. After 15 minutes the vessels are 10 km apart and then, 15 minutes later, they are 13 km apart. (Assume that both travel at constant speeds.) How far apart are the vessels when the cruiser is due east of the freighter?

15 Cask A, which has a capacity of 20 litres, is filled with wine. A certain quantity of wine from cask A is poured into cask B, which also has a capacity of 20 litres. Cask B is then filled with water. After this, cask A is filled with some of the mixture from cask B. A further $\frac{20}{3}$ litres of the mixture now in A is poured back into B, and the two casks now have the same amount of wine. How much wine was first taken out of cask A?

16 Two trains travel between two stations 80 km apart. If train A travels at an average speed of 5 km/h faster than train B and completes the journey 20 minutes faster, find the average speeds of the two trains, giving your answers correct to two decimal places.

17 A tank is filled by two pipes. The smaller pipe running alone will take 24 minutes longer than the larger pipe alone, and a minutes longer than when both pipes are running together.

a Find, in terms of a, how long each pipe takes to fill the tank.

b Find how long each pipe takes to fill the tank when:

i $a = 49$ **ii** $a = 32$ **iii** $a = 27$ **iv** $a = 25$

18 Train A leaves Armadale and travels at constant speed to Bundong, which is a town 300 km from Armadale. At the same time, train B leaves Bundong and travels at constant speed to Armadale. They meet at a town Yunga, which is between the two towns. Nine hours after leaving Yunga, train A reaches Bundong, and four hours after leaving Yunga, train B reaches Armadale.

a Find the distance of Yunga from Armadale.

b Find the speed of each of the trains.

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5D Partial fractions

A **rational function** is the quotient of two polynomials. If $g(x)$ and $h(x)$ are polynomials, then $f(x) = \dfrac{g(x)}{h(x)}$ is a rational function; e.g. $f(x) = \dfrac{4x+2}{x^2-1}$.

- If the degree of $g(x)$ is less than the degree of $h(x)$, then $f(x)$ is a **proper fraction**.
- If the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then $f(x)$ is an **improper fraction**.

By convention, we consider a rational function for its maximal domain. For example, the function $f(x) = \dfrac{4x+2}{x^2-1}$ is only considered for $x \in \mathbb{R} \setminus \{-1, 1\}$.

A rational function may be expressed as a sum of simpler functions by resolving it into what are called **partial fractions**. For example:

$$\frac{4x+2}{x^2-1} = \frac{3}{x-1} + \frac{1}{x+1}$$

This technique can help when sketching the graphs of rational functions or when performing other mathematical procedures such as integration.

Proper fractions

For proper fractions, the technique used for obtaining partial fractions depends on the type of factors in the denominator of the original algebraic fraction. We only consider examples where the denominators have factors that are either degree 1 (linear) or degree 2 (quadratic).

- For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax+b}$.
- For every repeated linear factor $(cx+d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx+d}$ and $\dfrac{C}{(cx+d)^2}$.
- For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx+E}{ax^2+bx+c}$.

Note: A quadratic expression is said to be **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, both $x^2 + 1$ and $x^2 + 4x + 10$ are irreducible.

To resolve an algebraic fraction into its partial fractions:

Step 1 Write a statement of identity between the original fraction and a sum of the appropriate number of partial fractions.

Step 2 Express the sum of the partial fractions as a single fraction, and note that the numerators of both sides are equivalent.

Step 3 Find the values of the introduced constants $A, B, C, \dots$ by substituting appropriate values for x or by equating coefficients.

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Example 13

Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution

Method 1

Let

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A}{x-1} + \frac{B}{x+3} \qquad (1)$$

for all $x \in \mathbb{R} \setminus \{1, -3\}$.

Express the right-hand side as a single fraction:

$$\frac{3x+5}{(x-1)(x+3)} = \frac{A(x+3) + B(x-1)}{(x-1)(x+3)}$$

$$\therefore \quad \frac{3x+5}{(x-1)(x+3)} = \frac{(A+B)x + 3A - B}{(x-1)(x+3)}$$

$$\therefore \quad 3x + 5 = (A+B)x + 3A - B$$

Equate coefficients:

$$A + B = 3$$

$$3A - B = 5$$

Solving these equations simultaneously gives

$$4A = 8$$

and so $A = 2$ and $B = 1$.

Therefore

$$\frac{3x+5}{(x-1)(x+3)} = \frac{2}{x-1} + \frac{1}{x+3}$$

Explanation

Since the denominator has two linear factors, there will be two partial fractions of the form $\dfrac{A}{x-1}$ and $\dfrac{B}{x+3}$.

Method 2

From equation (1) we can write:

$$3x + 5 = A(x+3) + B(x-1) \qquad (2)$$

Substitute $x = 1$ in equation (2):

$$8 = 4A$$

$$\therefore \quad A = 2$$

Substitute $x = -3$ in equation (2):

$$-4 = -4B$$

$$\therefore \quad B = 1$$

Explanation

We know that equation (2) is true for all $x \in \mathbb{R} \setminus \{1, -3\}$.

But if this is the case, then it also has to be true for $x = 1$ and $x = -3$.

Note: You could substitute any values of x to find A and B in this way, but these values simplify the calculations.

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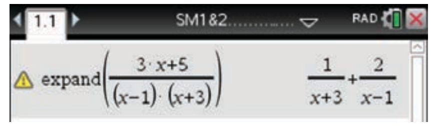
Using the TI-Nspire

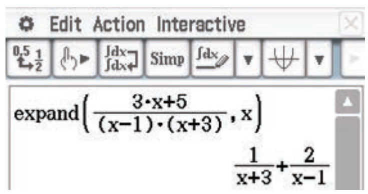
Use (menu) > **Algebra** > **Expand** as shown.

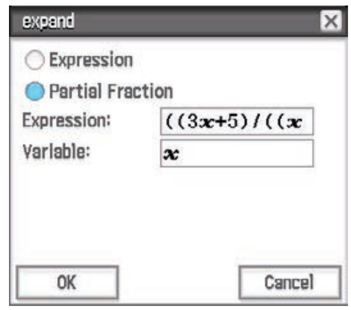
Note: You can access the fraction template using (ctrl) (÷).



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight $\dfrac{3x+5}{(x-1)(x+3)}$.
- Go to **Interactive** > **Transformation** > **expand** and select the **Partial Fraction** option.
- Enter the variable and tap OK.





Example 14

Resolve $\dfrac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution

Since the denominator has a repeated linear factor and a single linear factor, there are three partial fractions:

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

$$\therefore \quad \frac{2x+10}{(x+1)(x-1)^2} = \frac{A(x-1)^2 + B(x+1)(x-1) + C(x+1)}{(x+1)(x-1)^2}$$

This gives the equation

$$2x + 10 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$$

We will use a combination of methods to find A, B and C.

Let $x = 1$: $\quad 2(1) + 10 = C(1+1)$

$$12 = 2C$$

$$\therefore \quad C = 6$$

Let $x = -1$: $\quad 2(-1) + 10 = A(-1-1)^2$

$$8 = 4A$$

$$\therefore \quad A = 2$$

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Now substitute these values for A and C:

$$\begin{aligned} 2x + 10 &= 2(x-1)^2 + B(x+1)(x-1) + 6(x+1) \qquad (1) \\ &= 2(x^2 - 2x + 1) + B(x^2 - 1) + 6(x+1) \\ &= (2+B)x^2 + 2x + 8 - B \end{aligned}$$

Equate coefficients:

$$2 + B = 0$$
$$8 - B = 10$$

Therefore $B = -2$ and hence

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{2}{x+1} - \frac{2}{x-1} + \frac{6}{(x-1)^2}$$

Alternatively, the value of B could be found by substituting $x = 0$ into equation (1).

Note: In Exercise 5D, you will show that it is impossible to find A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

Example 15

Resolve $\dfrac{x^2 + 6x + 5}{(x-2)(x^2+x+1)}$ into partial fractions.

Solution

Since the denominator has a single linear factor and an irreducible quadratic factor (i.e. cannot be reduced to linear factors), there are two partial fractions:

$$\frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+x+1}$$

$$\therefore \quad \frac{x^2+6x+5}{(x-2)(x^2+x+1)} = \frac{A(x^2+x+1) + (Bx+C)(x-2)}{(x-2)(x^2+x+1)}$$

This gives the equation

$$x^2 + 6x + 5 = A(x^2+x+1) + (Bx+C)(x-2) \qquad (1)$$

Subsituting $x = 2$:

$$\begin{aligned} 2^2 + 6(2) + 5 &= A(2^2 + 2 + 1) \\ 21 &= 7A \\ \therefore \quad A &= 3 \end{aligned}$$

We can rewrite equation (1) as

$$\begin{aligned} x^2 + 6x + 5 &= A(x^2+x+1) + (Bx+C)(x-2) \\ &= A(x^2+x+1) + Bx^2 - 2Bx + Cx - 2C \\ &= (A+B)x^2 + (A - 2B + C)x + A - 2C \end{aligned}$$

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Since $A = 3$, this gives

$$x^2 + 6x + 5 = (3 + B)x^2 + (3 - 2B + C)x + 3 - 2C$$

Equate coefficients:

$$3 + B = 1 \quad \text{and} \quad 3 - 2C = 5$$

$$\therefore \quad B = -2 \qquad \therefore \quad C = -1$$

Check: $3 - 2B + C = 3 - 2(-2) + (-1) = 6$

Therefore

$$\frac{x^2 + 6x + 5}{(x - 2)(x^2 + x + 1)} = \frac{3}{x - 2} + \frac{-2x - 1}{x^2 + x + 1}$$

$$= \frac{3}{x - 2} - \frac{2x + 1}{x^2 + x + 1}$$

Note: The values of B and C could also be found by substituting $x = 0$ and $x = 1$ in equation (1).

Improper fractions

Improper algebraic fractions can be expressed as a sum of partial fractions by first dividing the denominator into the numerator to produce a quotient and a proper fraction. This proper fraction can then be resolved into its partial fractions using the techniques just introduced.

Example 16

Express $\dfrac{x^5 + 2}{x^2 - 1}$ as partial fractions.

Solution

Dividing through:

$$\begin{array}{r|l} & x^3 + x \\ \hline x^2 - 1 & x^5 + 2 \\ & x^5 - x^3 \\ \hline & \quad x^3 + 2 \\ & \quad x^3 - x \\ \hline & \qquad x + 2 \end{array}$$

Therefore

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x + \frac{x + 2}{x^2 - 1}$$

By expressing $\dfrac{x + 2}{x^2 - 1} = \dfrac{x + 2}{(x - 1)(x + 1)}$ as partial fractions, we obtain

$$\frac{x^5 + 2}{x^2 - 1} = x^3 + x - \frac{1}{2(x + 1)} + \frac{3}{2(x - 1)}$$

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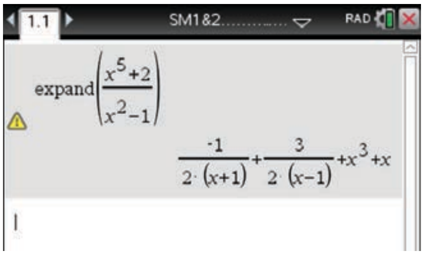
Using the TI-Nspire

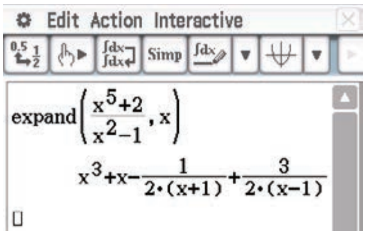
Use (menu) > **Algebra** > **Expand** as shown.

Note: You can access the fraction template using (ctrl) (÷).



Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight $\dfrac{x^5+2}{x^2-1}$.
- Go to **Interactive** > **Transformation** > **expand** and choose the **Partial Fraction** option.
- Enter the variable and tap OK.



Section summary

- A rational function may be expressed as a sum of simpler functions by resolving it into **partial fractions**. For example:

$$\frac{4x+2}{x^2-1}=\frac{3}{x-1}+\frac{1}{x+1}$$

- Examples of resolving a proper fraction into partial fractions:
  - Single linear factors

$$\frac{3x-4}{(2x-3)(x+5)}=\frac{A}{2x-3}+\frac{B}{x+5}$$

  - Repeated linear factor

$$\frac{3x-4}{(2x-3)(x+5)^2}=\frac{A}{2x-3}+\frac{B}{x+5}+\frac{C}{(x+5)^2}$$

  - Irreducible quadratic factor

$$\frac{3x-4}{(2x-3)(x^2+5)}=\frac{A}{2x-3}+\frac{Bx+C}{x^2+5}$$

- A quadratic polynomial is **irreducible** if it cannot be factorised over $\mathbb{R}$. For example, the quadratics x^2+5 and $x^2+4x+10$ are irreducible.
- If $f(x)=\dfrac{g(x)}{h(x)}$ is an improper fraction, i.e. if the degree of $g(x)$ is greater than or equal to the degree of $h(x)$, then the division must be performed first.

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Exercise 5D

Example 13

1 Resolve the following rational expressions into partial fractions:

a $\dfrac{5x+1}{(x-1)(x+2)}$ **b** $\dfrac{-1}{(x+1)(2x+1)}$ **c** $\dfrac{3x-2}{x^2-4}$

d $\dfrac{4x+7}{x^2+x-6}$ **e** $\dfrac{7-x}{(x-4)(x+1)}$

Example 14

2 Resolve the following rational expressions into partial fractions:

a $\dfrac{2x+3}{(x-3)^2}$ **b** $\dfrac{9}{(1+2x)(1-x)^2}$ **c** $\dfrac{2x-2}{(x+1)(x-2)^2}$

Example 15

3 Resolve the following rational expressions into partial fractions:

a $\dfrac{3x+1}{(x+1)(x^2+x+1)}$ **b** $\dfrac{3x^2+2x+5}{(x^2+2)(x+1)}$ **c** $\dfrac{x^2+2x-13}{2x^3+6x^2+2x+6}$

Example 16

4 Resolve $\dfrac{3x^2-4x-2}{(x-1)(x-2)}$ into partial fractions.

5 Show that it is not possible to find values of A and C such that

$$\frac{2x+10}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{C}{(x-1)^2}$$

6 Express each of the following as partial fractions:

a $\dfrac{1}{(x-1)(x+1)}$ **b** $\dfrac{x}{(x-2)(x+3)}$ **c** $\dfrac{3x+1}{(x-2)(x+5)}$

d $\dfrac{1}{(2x-1)(x+2)}$ **e** $\dfrac{3x+5}{(3x-2)(2x+1)}$ **f** $\dfrac{2}{x^2-x}$

g $\dfrac{3x+1}{x^3+x}$ **h** $\dfrac{3x^2+8}{x(x^2+4)}$ **i** $\dfrac{1}{x^2-4x}$

j $\dfrac{x+3}{x^2-4x}$ **k** $\dfrac{x^3-x^2-1}{x^2-x}$ **l** $\dfrac{x^3-x^2-6}{2x-x^2}$

m $\dfrac{x^2-x}{(x+1)(x^2+2)}$ **n** $\dfrac{x^2+2}{x^3-3x-2}$ **o** $\dfrac{2x^2+x+8}{x(x^2+4)}$

p $\dfrac{1-2x}{2x^2+7x+6}$ **q** $\dfrac{3x^2-6x+2}{(x-1)^2(x+2)}$ **r** $\dfrac{4}{(x-1)^2(2x+1)}$

s $\dfrac{x^3-2x^2-3x+9}{x^2-4}$ **t** $\dfrac{x^3+3}{(x+1)(x-1)}$ **u** $\dfrac{2x-1}{(x+1)(3x+2)}$

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5E Simultaneous equations

In this section, we look at methods for finding the coordinates of the points of intersection of a linear graph with different non-linear graphs: parabolas, circles and rectangular hyperbolas. We also consider the intersections of two parabolas. These types of graphs are studied further in Mathematical Methods Units 1 & 2.

Example 17

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

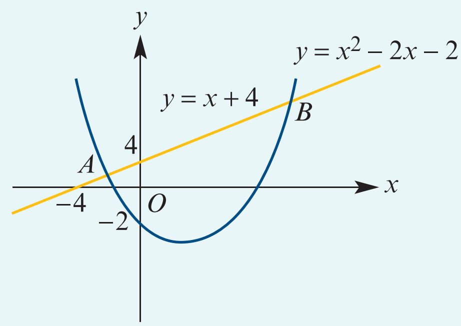
Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$

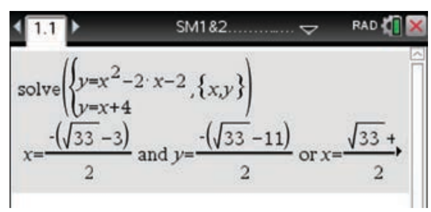
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\frac{3 - \sqrt{33}}{2}, \frac{11 - \sqrt{33}}{2}\right)$ and $B\left(\frac{3 + \sqrt{33}}{2}, \frac{11 + \sqrt{33}}{2}\right)$.

Using the TI-Nspire

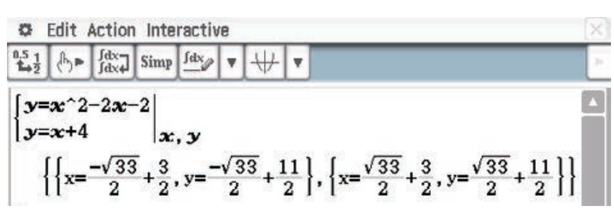
- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.
- Use the touchpad to move the cursor up to the solution and see all the solutions.



Using the Casio ClassPad

The exact coordinates of the points of intersection can be obtained in the Main $\sqrt{\alpha}$ application.

- To select the simultaneous equations template, tap [icon] from the Math1 keyboard.
- Enter the two equations and the variables x, y in the spaces provided. Then tap EXE.
- Tap Rotate [icon] from the icon panel and ► on the touch screen to view the entire solution.



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Example 18

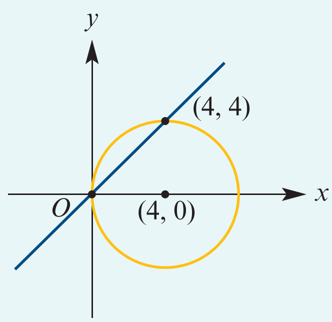
Find the points of intersection of the circle with equation $(x-4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.

Substitute $y = x$ into the equation of the circle:

$$\begin{aligned}(x-4)^2 + x^2 &= 16\\ x^2 - 8x + 16 + x^2 &= 16\\ 2x^2 - 8x &= 0\\ 2x(x-4) &= 0\\ \therefore \quad x = 0 \text{ or } x &= 4\end{aligned}$$



The points of intersection are $(0, 0)$ and $(4, 4)$.

Example 19

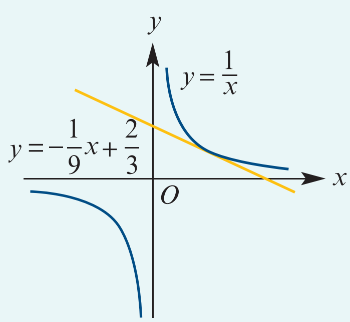
Find the point of contact of the straight line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.

Equate the expressions for y:

$$\begin{aligned}-\frac{1}{9}x + \frac{2}{3} &= \frac{1}{x}\\ -x^2 + 6x &= 9\\ x^2 - 6x + 9 &= 0\\ (x-3)^2 &= 0\\ \therefore \quad x &= 3\end{aligned}$$

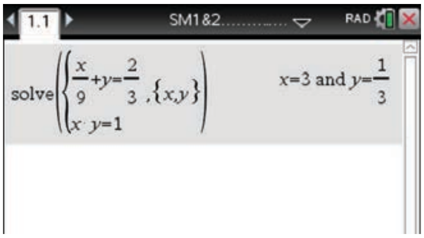


The point of intersection is $(3, \frac{1}{3})$.

Using the TI-Nspire

Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations** as shown.

Note: The multiplication sign between x and y is required, as the calculator will consider xy to be a single variable.



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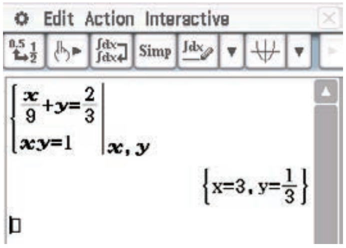
Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, select the simultaneous equations template by tapping [{■] from the [Math1] keyboard.
- Enter the two equations and the variables x, y in the spaces provided; tap [EXE].



Example 20

Find the coordinates of the points of intersection of the graphs of $y = -3x^2 - 4x + 1$ and $y = 2x^2 - x - 1$.

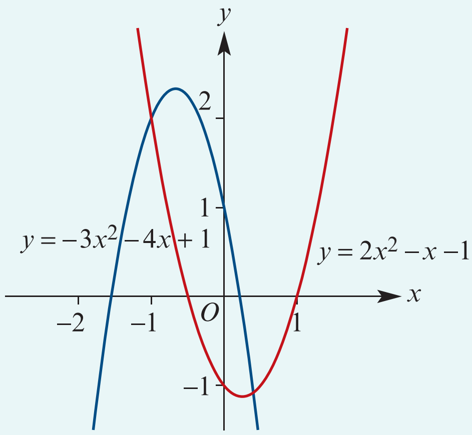
Solution

$$-3x^2 - 4x + 1 = 2x^2 - x - 1$$
$$-5x^2 - 3x + 2 = 0$$
$$5x^2 + 3x - 2 = 0$$
$$(5x - 2)(x + 1) = 0$$
$$\therefore \quad x = \frac{2}{5} \text{ or } x = -1$$

Substitute in $y = 2x^2 - x - 1$:

When $x = -1$, $y = 2$.

When $x = \frac{2}{5}$, $y = 2 \times \frac{4}{25} - \frac{2}{5} - 1 = -\frac{27}{25}$.



The points of intersection are $(-1, 2)$ and $\left(\frac{2}{5}, -\frac{27}{25}\right)$.

Exercise 5E

Skillsheet
Example 17

1 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$ **b** $y - 2x^2 = 0$, $y - x = 0$ **c** $y = x^2 - x$, $y = 2x + 1$

Example 18

2 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$ **b** $x^2 + y^2 = 125$, $x + y = 15$ **c** $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$ **e** $x^2 + y^2 = 106$, $x - y = 4$

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Example 19

3 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$
$xy = 187$

b $x + y = 51$
$xy = 518$

c $x - y = 5$
$xy = 126$

4 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

5 Find the coordinates of the points of intersection of the curves with equations $y = \dfrac{1}{x-2} + 3$ and $y = x$.

6 Find the coordinates of the points A and B where the line with equation $x - 3y = 0$ meets the circle with equation $x^2 + y^2 - 10x - 5y + 25 = 0$.

7 Find the coordinates of the points of intersection of the line with equation $\dfrac{y}{4} - \dfrac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

8 Find the coordinates of the points of intersection of the curve with equation $y = \dfrac{1}{x+2} - 3$ and the line with equation $y = -x$.

9 Find the point where the line $4y = 9x + 4$ touches the parabola $y^2 = 9x$.

10 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle with equation $x^2 + y^2 = 9$.

11 Find the coordinates of the point where the straight line with equation $y = \dfrac{1}{4}x + 1$ touches the curve with equation $y = -\dfrac{1}{x}$.

12 Find points of intersection of the curve $y = \dfrac{2}{x-2}$ and the line $y = x - 1$.

Example 20

13 Find the coordinates of the points of intersection of the graphs of the following pairs of quadratic functions:

a $y = 2x^2 - 4x + 1, \; y = 2x^2 - x - 1$

b $y = -2x^2 + x + 1, \; y = 2x^2 - x - 1$

c $y = x^2 + x + 1, \; y = x^2 - x - 2$

d $y = 3x^2 + x + 2, \; y = x^2 - x + 2$

14 In each of the following, use the discriminant of the resulting quadratic equation:

a Find the possible values of k for which the straight line $y = k(1 - 2x)$ touches but does not cross the parabola $y = x^2 + 2$.

b Find the possible values of c for which the line $y = 2x + c$ intersects the circle $x^2 + y^2 = 20$ in two distinct points.

c Find the value of p for which the line $y = 6$ meets the parabola $y = x^2 + (1 - p)x + 2p$ at only one point.

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Chapter summary

AS

Nrich

Polynomials

- A **polynomial function** can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.

- The **degree** of a polynomial is the index n of the leading term (the term of highest index among those terms with a non-zero coefficient).
- Equating coefficients

  Two polynomials are equal if they give the same value for all x. If two polynomials are equal, then they have the same degree and corresponding coefficients are equal.

  For example: if $x^2 - x - 12 = x^2 + (a + b)x + ab$, then $a + b = -1$ and $ab = -12$.

Quadratics

- A quadratic function can be written in the form $y = ax^2 + bx + c$, where $a \neq 0$.
- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the **general quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two real solutions.
  - If $\Delta = 0$, the quadratic equation has one real solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.

Partial fractions

- A **rational function** has the form $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are polynomials.

  For example: $f(x) = \dfrac{2x + 10}{x^3 - x^2 - x + 1}$

- Some rational functions may be expressed as a sum of **partial fractions**:
  - For every linear factor $ax + b$ in the denominator, there will be a partial fraction of the form $\dfrac{A}{ax + b}$.
  - For every repeated linear factor $(cx + d)^2$ in the denominator, there will be partial fractions of the form $\dfrac{B}{cx + d}$ and $\dfrac{C}{(cx + d)^2}$.
  - For every irreducible quadratic factor $ax^2 + bx + c$ in the denominator, there will be a partial fraction of the form $\dfrac{Dx + E}{ax^2 + bx + c}$.

  For example: $\dfrac{2x + 10}{(x + 1)(x - 1)^2} = \dfrac{A}{x + 1} + \dfrac{B}{x - 1} + \dfrac{C}{(x - 1)^2}$, where $A = 2$, $B = -2$ and $C = 6$

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Technology-free questions

1 If $(3a + b)x^2 + (a - 2b)x + b + 2c = 11x^2 - x + 4$, find the values of a, b and c.

2 Express x^3 in the form $(x - 1)^3 + a(x - 1)^2 + b(x - 1) + c$.

3 Prove that, if $ax^3 + bx^2 + cx + d = (x + 1)^2(px + q)$, then $b = 2a + d$ and $c = a + 2d$.

4 Prove that, if $ax^3 + bx^2 + cx + d = (x - 2)^2(px + q)$, then $b = -4a + \frac{1}{4}d$ and $c = 4a - d$.

5 Solve the following quadratic equations for x:

a $x^2 + x = 12$ **b** $x^2 - 2 = x$ **c** $-x^2 + 3x + 11 = 1$

d $2x^2 - 4x + 1 = 0$ **e** $3x^2 - 2x + 5 = t$ **f** $tx^2 + 4 = tx$

6 Solve the equation $\dfrac{2}{x - 1} - \dfrac{3}{x + 2} = \dfrac{1}{2}$ for x.

7 Express each of the following as partial fractions:

a $\dfrac{-3x + 4}{(x - 3)(x + 2)}$ **b** $\dfrac{7x + 2}{x^2 - 4}$ **c** $\dfrac{7 - x}{x^2 + 2x - 15}$

d $\dfrac{3x - 9}{x^2 - 4x - 5}$ **e** $\dfrac{3x - 4}{(x + 3)(x + 2)^2}$ **f** $\dfrac{6x^2 - 5x - 16}{(x - 1)^2(x + 4)}$

g $\dfrac{x^2 - 6x - 4}{(x^2 + 2)(x + 1)}$ **h** $\dfrac{-x + 4}{(x - 1)(x^2 + x + 1)}$ **i** $\dfrac{-4x + 5}{(x + 4)(x - 3)}$

j $\dfrac{-2x + 8}{(x + 4)(x - 3)}$

8 Express each of the following as partial fractions:

a $\dfrac{14(x - 2)}{(x - 3)(x^2 + x + 2)}$ **b** $\dfrac{1}{(x + 1)(x^2 - x + 2)}$ **c** $\dfrac{3x^3}{x^2 - 5x + 4}$

9 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = -x$ **b** $x^2 + y^2 = 16$, $x + y = 4$ **c** $x + y = 5$, $xy = 4$

10 Find the coordinates of the points of intersection of the line with equation $3y - x = 1$ and the circle with equation $x^2 + 2x + y^2 = 9$.

11 A motorist makes a journey of 135 km at an average speed of x km/h.

a Write an expression for the number of hours taken for the journey.

b Owing to road works, on a certain day his average speed for the journey is reduced by 15 km/h. Write an expression for the number of hours taken on that day.

c If the second journey takes 45 minutes longer than the first, form an equation in x and solve it.

d Find his average speed for each journey.

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Multiple-choice questions

1 If x^2 is written in the form $(x+1)^2 + b(x+1) + c$, then the values of b and c are

A $b = 0,\ c = 0$ **B** $b = -2,\ c = 0$ **C** $b = -2,\ c = 1$
D $b = 1,\ c = 2$ **E** $b = 1,\ c = -2$

2 If $x^3 = a(x+2)^3 + b(x+2)^2 + c(x+2) + d$, then the values of a, b, c and d are

A $a = 0,\ b = -8,\ c = 10,\ d = -6$ **B** $a = 0,\ b = -6,\ c = 10,\ d = -8$
C $a = 1,\ b = -8,\ c = 10,\ d = -6$ **D** $a = 1,\ b = -6,\ c = 12,\ d = -8$
E $a = 1,\ b = -8,\ c = 12,\ d = -6$

3 The quadratic equation $3x^2 - 6x + 3 = 0$ has

A two real solutions, $x = \pm 1$ **B** one real solution, $x = -1$
C no real solutions **D** one real solution, $x = 1$
E two real solutions, $x = 1$ and $x = 2$

4 The quadratic equation whose solutions are 4 and −6 is

A $(x+4)(x-6) = 0$ **B** $x^2 - 2x - 24 = 0$ **C** $2x^2 + 4x = 48$
D $-x^2 + 2x - 24 = 0$ **E** $x^2 + 2x + 24 = 0$

5 $\dfrac{3}{x+4} - \dfrac{5}{x-2}$ is equal to

A $\dfrac{-2}{(x+4)(x-2)}$ **B** $\dfrac{2(x+1)}{(x+4)(x-2)}$ **C** $\dfrac{-2(x-7)}{(x+4)(x-2)}$
D $\dfrac{2(4x+13)}{(x+4)(x-2)}$ **E** $\dfrac{-2(x+13)}{(x+4)(x-2)}$

6 $\dfrac{4}{(x+3)^2} + \dfrac{2x}{x+1}$ is equal to

A $\dfrac{8x}{(x+3)^2(x+1)}$ **B** $\dfrac{2(3x^2+x+18)}{(x+3)^2(x+1)}$ **C** $\dfrac{3x^2+13x+18}{(x+3)^2(x+1)}$
D $\dfrac{2(3x^2+13x+18)}{(x+3)^2(x+1)}$ **E** $\dfrac{2(x^3+6x^2+11x+2)}{(x+3)^2(x+1)}$

7 If $\dfrac{7x^2+13}{(x-1)(x^2+x+2)}$ is expressed in the form $\dfrac{a}{x-1} + \dfrac{bx+c}{x^2+x+2}$, then

A $a = 5,\ b = 0,\ c = -13$ **B** $a = 5,\ b = 0,\ c = -10$ **C** $a = 5,\ b = 2,\ c = -3$
D $a = 7,\ b = 2,\ c = 3$ **E** $a = 7,\ b = 3,\ c = 13$

8 $\dfrac{4x-3}{(x-3)^2}$ is equal to

A $\dfrac{3}{x-3} + \dfrac{1}{x-3}$ **B** $\dfrac{4x}{x-3} - \dfrac{3}{x-3}$ **C** $\dfrac{9}{x-3} + \dfrac{4}{(x-3)^2}$
D $\dfrac{4}{x-3} + \dfrac{9}{(x-3)^2}$ **E** $\dfrac{4}{x-3} - \dfrac{15}{(x-3)^2}$

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Review

9 $\dfrac{8x+7}{2x^2+5x+2}$ is equal to

A $\dfrac{2}{2x+1}-\dfrac{3}{x+2}$ **B** $\dfrac{2}{2x+1}+\dfrac{3}{x+2}$ **C** $\dfrac{-4}{2x+2}-\dfrac{1}{x+1}$

D $\dfrac{-4}{2x+2}+\dfrac{1}{x+1}$ **E** $\dfrac{4}{2x+2}-\dfrac{1}{x+1}$

10 $\dfrac{-3x^2+2x-1}{(x^2+1)(x+1)}$ is equal to

A $\dfrac{2}{x^2+1}+\dfrac{3}{x+1}$ **B** $\dfrac{2}{x^2+1}-\dfrac{3}{x+1}$ **C** $\dfrac{5}{x^2+1}+\dfrac{2}{x+1}$

D $\dfrac{3}{x^2+1}-\dfrac{2}{x+1}$ **E** $\dfrac{3}{x^2+1}+\dfrac{2}{x+1}$

Extended-response questions

1 A train completes a journey of 240 km at a constant speed.

a If the train had travelled 4 km/h slower, it would have taken two hours more for the journey. Find the actual speed of the train.

b If the train had travelled a km/h slower and still taken two hours more for the journey of 240 km, what would have been the actual speed? (Answer in terms of a.) Discuss the practical possible values of a and also the possible values for the speed of the train.

c If the train had travelled a km/h slower and taken a hours more for the journey of 240 km, and if a is an integer and the speed is an integer, find the possible values for a and the speed of the train.

2 Two trains are travelling at constant speeds. The slower train takes a hours longer to cover b km. It travels 1 km less than the faster train in c hours.

a What is the speed of the faster train, in terms of a, b and c?

b If a, b, c and the speeds of the trains are all rational numbers, find five sets of values for a, b and c. Choose and discuss two sensible sets of values.

3 A tank can be filled using two pipes. The smaller pipe alone will take a minutes longer than the larger pipe alone to fill the tank. Also, the smaller pipe will take b minutes longer to fill the tank than when both pipes are used.

a In terms of a and b, how long will each of the pipes take to fill the tank?

b If $a = 24$ and $b = 32$, how long will each of the pipes take to fill the tank?

c If a and b are consecutive positive integers, find five pairs of values of a and b such that $b^2 - ab$ is a perfect square. Interpret these results in the context of this problem.

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