Chapter 13

13 Trigonometric ratios and applications

Objectives

- To solve practical problems using the trigonometric ratios.
- To use the **sine rule** and the **cosine rule** to solve problems.
- To find the **area of a triangle** given two sides and the included angle.
- To find the **length of an arc** and the **length of a chord** of a circle.
- To find the **area of a sector** and the **area of a segment** of a circle.
- To solve problems involving **angles of elevation** and **angles of depression**.
- To identify the **line of greatest slope of a plane**.
- To solve problems in **three dimensions**, including determining the angle between planes.

Trigonometry deals with the side lengths and angles of a triangle: the word *trigonometry* comes from the Greek words for triangle and measurement.

In Chapter 9, we used the four standard congruence tests for triangles. If you have the information about a triangle given in one of the congruence tests, then the triangle is uniquely determined (up to congruence). You can find the unknown side lengths and angles of the triangle using the **sine rule** or the **cosine rule**. In this chapter, we will apply these rules in two- and three-dimensional problems.

In Chapter 10, we studied the geometry of the circle, and the results involved chords, secants and arcs. In this chapter, we use trigonometry to determine the associated lengths and angles. We also find the areas of sectors and segments of circles.

Note: An introduction to sine, cosine and tangent as functions is given in Mathematical Methods Units 1 & 2 and also in an online chapter for this book, available in the Interactive Textbook.

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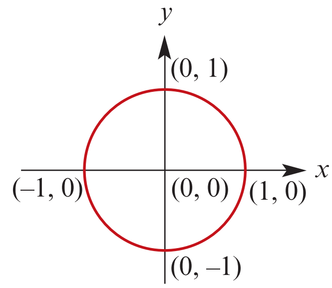
13A Reviewing trigonometry

In this section we review sine, cosine and tangent for angles between 0° and 180°.

Defining sine and cosine

The unit circle is a circle of radius 1 with centre at the origin.

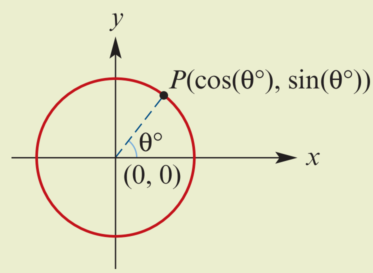
We can define the sine and cosine of any angle by using the unit circle.



Unit-circle definition of sine and cosine

For each angle θ°, there is a point P on the unit circle as shown. The angle is measured anticlockwise from the positive direction of the x-axis.

- $\cos(\theta°)$ is defined as the x-coordinate of the point P
- $\sin(\theta°)$ is defined as the y-coordinate of the point P



The trigonometric ratios

For acute angles, the unit-circle definition of sine and cosine given above is equivalent to the ratio definition.

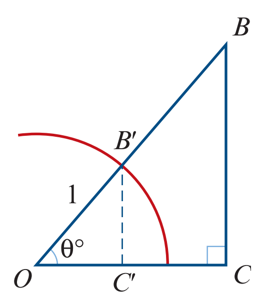
For a right-angled triangle OBC, we can construct a similar triangle $OB'C'$ that lies in the unit circle. From the diagram:

$$B'C' = \sin(\theta°) \qquad \text{and} \qquad OC' = \cos(\theta°)$$

The similarity factor is the length OB, giving

$$BC = OB\sin(\theta°) \quad \text{and} \quad OC = OB\cos(\theta°)$$

$$\therefore \quad \frac{BC}{OB} = \sin(\theta°) \quad \text{and} \quad \frac{OC}{OB} = \cos(\theta°)$$

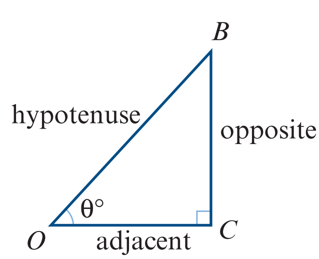


This gives the ratio definition of sine and cosine for a right-angled triangle. The naming of sides with respect to an angle θ° is as shown.

$$\sin(\theta°) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta°) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan(\theta°) = \frac{\text{opposite}}{\text{adjacent}}$$



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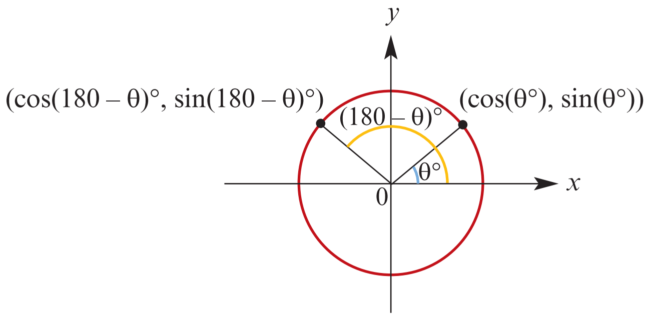
▶ Obtuse angles

From the unit circle, we see that

$$\sin(180 - \theta)^\circ = \sin(\theta^\circ)$$
$$\cos(180 - \theta)^\circ = -\cos(\theta^\circ)$$



For example:

$$\sin 135^\circ = \sin 45^\circ$$
$$\cos 135^\circ = -\cos 45^\circ$$

In this chapter, we will generally use the ratio definition of tangent for acute angles. But we can also find the tangent of an obtuse angle by defining

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

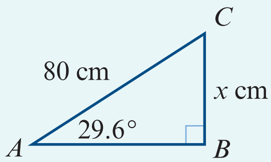
We will not consider angles greater than 180° or less than 0° in this chapter, since we are dealing with triangles.

▶ Solving right-angled triangles

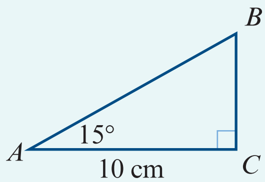
Here we provide some examples of using the trigonometric ratios.

Example 1

a Find the value of x correct to two decimal places.



b Find the length of the hypotenuse correct to two decimal places.



Solution

a $\dfrac{x}{80} = \sin 29.6^\circ$

$\therefore\ x = 80\sin 29.6^\circ$

$= 39.5153\ldots$

Hence $x = 39.52$, correct to two decimal places.

b $\dfrac{10}{AB} = \cos 15^\circ$

$10 = AB\cos 15^\circ$

$\therefore\ AB = \dfrac{10}{\cos 15^\circ}$

$= 10.3527\ldots$

The length of the hypotenuse is 10.35 cm, correct to two decimal places.

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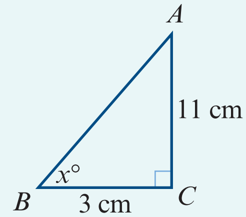
Example 2

Find the magnitude of $\angle ABC$.



Solution

$$\tan x = \frac{11}{3}$$

$$\therefore \quad x = \tan^{-1}\left(\frac{11}{3}\right)$$

$$= (74.7448\ldots)^\circ$$

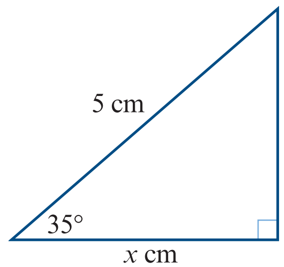
Hence $x = 74.74^\circ$, correct to two decimal places.

Exercise 13A

Example 1, 2

1 Find the value of x in each of the following:

a



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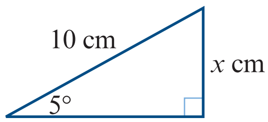

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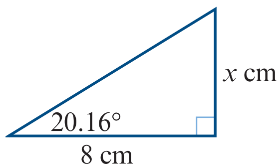

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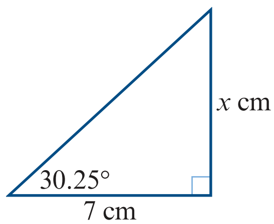
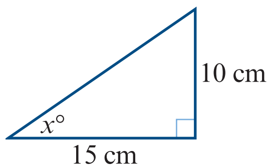
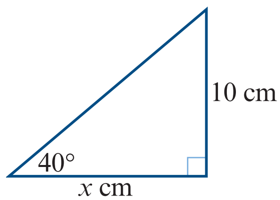


e



f



2 An equilateral triangle has altitudes of length 20 cm. Find the length of one side.

3 The base of an isosceles triangle is 12 cm long and the equal sides are 15 cm long. Find the magnitude of each of the three angles of the triangle.

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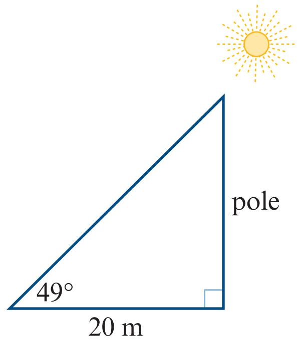
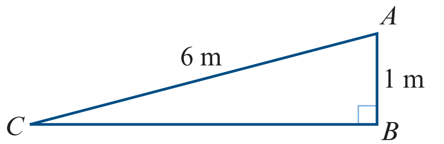
4 A pole casts a shadow 20 m long when the altitude of the sun is 49°. Calculate the height of the pole.



5 This figure represents a ramp.

a Find the magnitude of angle ACB.

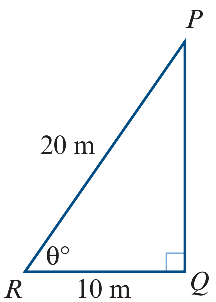
b Find the distance BC.



6 This figure shows a vertical mast PQ, which stands on horizontal ground. A straight wire 20 m long runs from P at the top of the mast to a point R on the ground, which is 10 m from the foot of the mast.

a Calculate the angle of inclination, $\theta°$, of the wire to the ground.

b Calculate the height of the mast.



7 A ladder leaning against a vertical wall makes an angle of 26° with the wall. If the foot of the ladder is 3 m from the wall, calculate:

a the length of the ladder

b the height it reaches above the ground.

8 An engineer is designing a straight concrete entry ramp, 60 m long, for a car park that is 13 m above street level. Calculate the angle of the ramp to the horizontal.

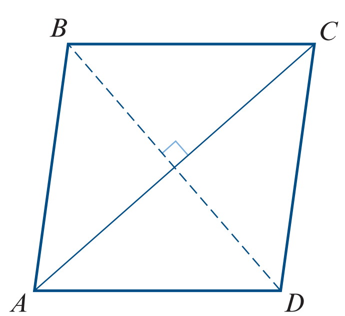
9 A vertical mast is secured from its top by straight cables 200 m long fixed at the ground. The cables make angles of 66° with the ground. What is the height of the mast?

10 A mountain railway rises 400 m at a uniform slope of 16° with the horizontal. What is the distance travelled by a train for this rise?

11 The diagonals of a rhombus bisect each other at right angles. If $BD = AC = 10$ cm, find:

a the length of the sides of the rhombus

b the magnitude of angle ABC.



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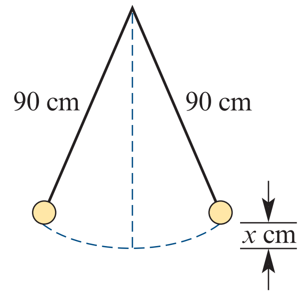
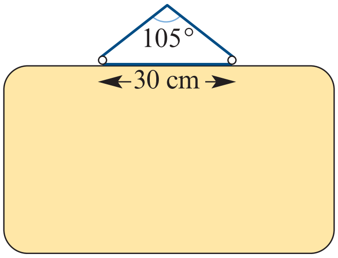
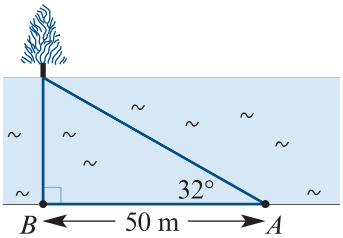
12 A pendulum swings from the vertical through an angle of 15° on each side of the vertical. If the pendulum is 90 cm long, what is the distance, x cm, between its highest and lowest points?



13 A picture is hung symmetrically by means of a string passing over a nail, with the ends of the string attached to two rings on the upper edge of the picture. The distance between the rings is 30 cm, and the string makes an angle of 105° at the nail. Find the length of the string.

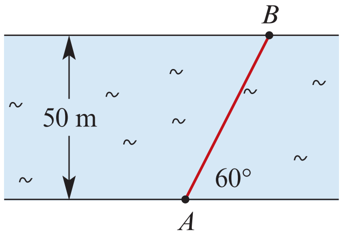


14 The distance AB is 50 m. If the line of sight to the tree of a person standing at A makes an angle of 32° with the bank, how wide is the river?



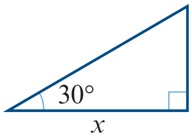
15 A ladder 4.7 m long is placed against a wall. The foot of the ladder must not be placed in a flower bed, which extends a distance of 1.7 m out from the base of the wall. How high up the wall can the ladder reach?

16 A river is known to be 50 m wide. A swimmer sets off from A to cross the river, and the path of the swimmer AB is as shown. How far does the person swim?



17 A rope is tied to the top of a flagpole. When it hangs straight down, it is 2 m longer than the pole. When the rope is pulled tight with the lower end on the ground, it makes an angle of 60° to the horizontal. How tall is the flagpole?

18 The triangle shown has perimeter 10. Find the value of x.



19 Consider the circle with equation $x^2 + y^2 - 4y = 0$ and the point $P(5, 2)$. Draw a diagram to show the circle and the two lines from P that are tangent to the circle. Find the angle between the two tangent lines, $\angle APB$, where A and B are the two points of contact.

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13B The sine rule

In the previous section, we focused on right-angled triangles. In this section and the next, we consider non-right-angled triangles.

The **sine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 one side and two angles are given

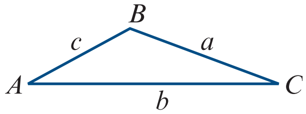
2 two sides and a non-included angle are given (that is, the given angle is not 'between' the two given sides).

In the first case, the triangle is uniquely defined up to congruence. In the second case, there may be two triangles.

Labelling triangles

The following convention is used in the remainder of this chapter:

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

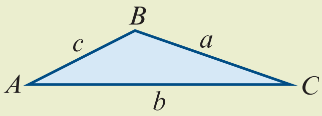


For example, the magnitude of angle BAC is denoted by A, and the length of side BC is denoted by a.

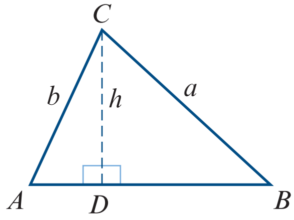
Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.



In triangle ACD:

$$\sin A = \frac{h}{b}$$

$$\therefore \quad h = b \sin A$$

In triangle BCD:

$$\sin B = \frac{h}{a}$$

$$\therefore \quad a \sin B = b \sin A$$

i.e. $$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Similarly, starting with a perpendicular from A to BC would give

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

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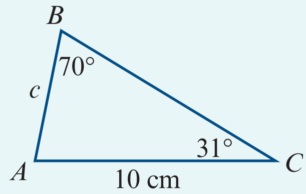
One side and two angles

When one side and two angles are given, this corresponds to the AAS congruence test. The triangle is uniquely defined up to congruence.

Example 3

Use the sine rule to find the length of AB.



Solution

$$\frac{c}{\sin 31^\circ} = \frac{10}{\sin 70^\circ}$$

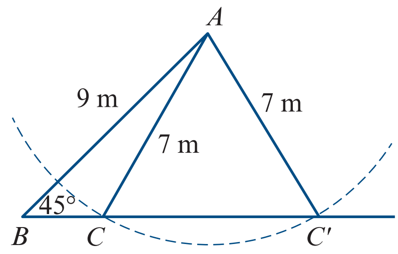
$$\therefore \quad c = \frac{10 \sin 31^\circ}{\sin 70^\circ}$$

$$= 5.4809\ldots$$

The length of AB is 5.48 cm, correct to two decimal places.

Two sides and a non-included angle

Suppose that we are given the two side lengths 7 m and 9 m and a non-included angle of 45°. There are two triangles that satisfy these conditions, as shown in the diagram.



Warning

- When you are given two sides and a non-included angle, you must consider the possibility that there are two such triangles.
- An angle found using the sine rule is possible if the sum of the given angle and the found angle is less than 180°.

Note: If the given angle is obtuse or a right angle, then there is only one such triangle.

The following example illustrates the case where there are two possible triangles.

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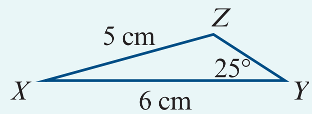
Example 4

Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



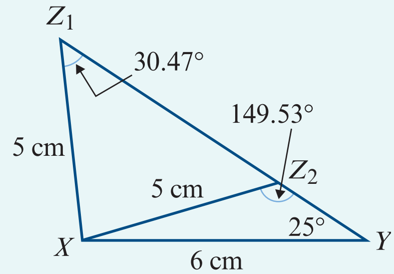
Solution

$$\frac{5}{\sin 25°} = \frac{6}{\sin Z}$$

$$\frac{\sin Z}{6} = \frac{\sin 25°}{5}$$

$$\sin Z = \frac{6 \sin 25°}{5}$$

$$= 0.5071\ldots$$



$\therefore \quad Z = (30.473\ldots)°$ or $Z = (180 - 30.473\ldots)°$

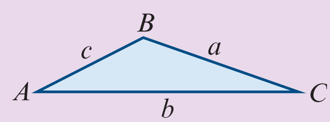
Hence $Z = 30.47°$ or $Z = 149.53°$, correct to two decimal places.

Note: Remember that $\sin(180 - \theta)° = \sin(\theta°)$.

Section summary

- **Sine rule** For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



- When to use the sine rule:
  - one side and two angles are given (AAS)
  - two sides and a non-included angle are given.

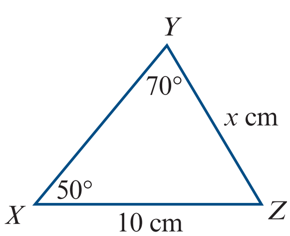
Exercise 13B

Skillsheet

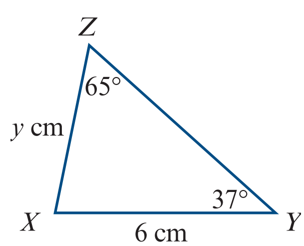
Example 3

1 Find the value of the pronumeral for each of the following triangles:

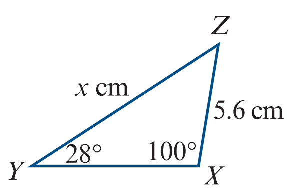
a



b



c



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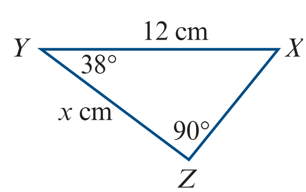

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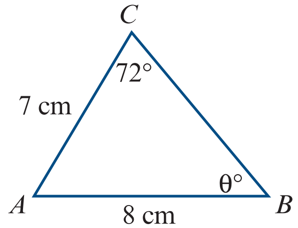
Example 4

2 Find the value of θ for each of the following triangles:

a



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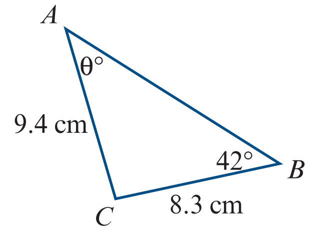

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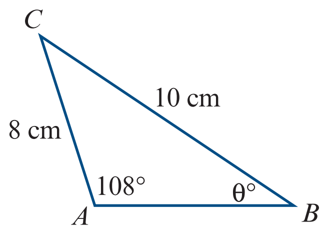

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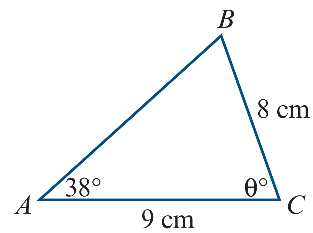


3 Solve the following triangles (i.e. find all sides and angles):

a $a = 12$, $B = 59°$, $C = 73°$

b $A = 75.3°$, $b = 5.6$, $B = 48.25°$

c $A = 123.2°$, $a = 11.5$, $C = 37°$

d $A = 23°$, $a = 15$, $B = 40°$

e $B = 140°$, $b = 20$, $A = 10°$

4 Solve the following triangles (i.e. find all sides and angles):

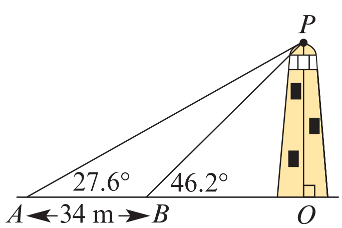
a $b = 17.6$, $C = 48.25°$, $c = 15.3$

b $B = 129°$, $b = 7.89$, $c = 4.56$

c $A = 28.35°$, $a = 8.5$, $b = 14.8$

5 A landmark A is observed from two points B and C, which are 400 m apart. The magnitude of angle ABC is measured as 68° and the magnitude of angle ACB as 70°. Find the distance of A from C.

6 P is a point at the top of a lighthouse. Measurements of the length AB and angles PBO and PAO are as shown in the diagram. Find the height of the lighthouse.

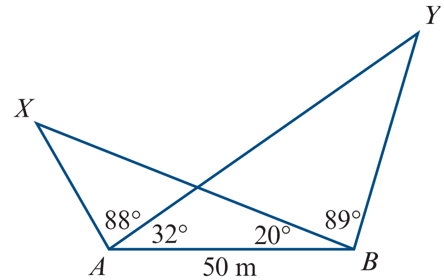


7 A and B are two points on a coastline, and C is a point at sea. The points A and B are 1070 m apart. The angles CAB and CBA have magnitudes of 74° and 69° respectively. Find the distance of C from A.

8 Find:

a AX

b AY



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9 Use the sine rule to establish the following identities for triangles:

a $\dfrac{a+b}{c} = \dfrac{\sin A + \sin B}{\sin C}$ **b** $\dfrac{a-b}{c} = \dfrac{\sin A - \sin B}{\sin C}$

13C The cosine rule

The **cosine rule** is used to find unknown side lengths or angles of a triangle in the following two situations:

1 two sides and the included angle are given

2 three sides are given.

In each case, the triangle is uniquely defined up to congruence.

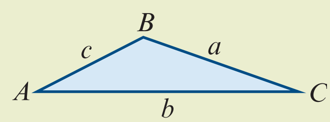
Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

or equivalently

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

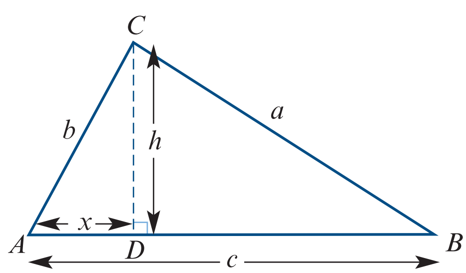


Proof We will give a proof for acute-angled triangles. The proof for obtuse-angled triangles is similar.

In triangle ACD:

$$\cos A = \frac{x}{b}$$

$$\therefore \quad x = b\cos A$$



Using Pythagoras' theorem in triangles ACD and BCD:

$$b^2 = x^2 + h^2$$

$$a^2 = (c-x)^2 + h^2$$

Expanding gives

$$a^2 = c^2 - 2cx + x^2 + h^2$$

$$= c^2 - 2cx + b^2 \qquad (\text{as } b^2 = x^2 + h^2)$$

$$\therefore \quad a^2 = b^2 + c^2 - 2bc\cos A \qquad (\text{as } x = b\cos A)$$

▶ Two sides and the included angle

When two sides and the included angle are given, this corresponds to the SAS congruence test. The triangle is uniquely defined up to congruence.

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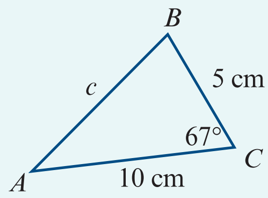
Example 5

For triangle ABC, find the length of AB in centimetres correct to two decimal places.



Solution

$$c^2 = 5^2 + 10^2 - 2 \times 5 \times 10 \cos 67^\circ$$
$$= 85.9268\ldots$$
$$\therefore \quad c = 9.2696\ldots$$

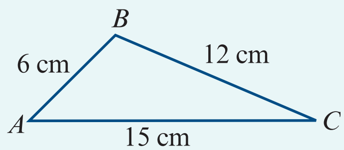
The length of AB is 9.27 cm, correct to two decimal places.

▶ Three sides

When three sides are given, this corresponds to the SSS congruence test. The triangle is uniquely defined up to congruence.

Example 6

Find the magnitude of angle ABC.



Solution

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{12^2 + 6^2 - 15^2}{2 \times 12 \times 6}$$
$$= -0.3125$$
$$\therefore \quad B = (108.2099\ldots)^\circ$$

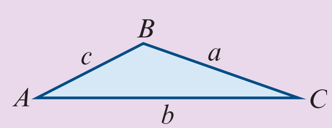
The magnitude of angle ABC is 108.21°, correct to two decimal places.

Section summary

- **Cosine rule** For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc \cos A \quad \text{or} \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

- When to use the cosine rule:
  - two sides and the included angle are given (SAS)
  - three sides are given (SSS).



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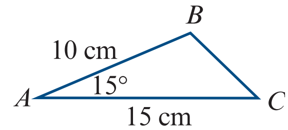
Exercise 13C

Skillsheet

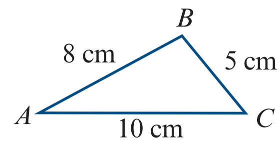
Example 5

1 Find the length of BC.



Example 6

2 Find the magnitudes of angles ABC and ACB.



3 For triangle ABC with:

a $A = 60°$ $b = 16$ $c = 30$, find a

b $a = 14$ $B = 53°$ $c = 12$, find b

c $a = 27$ $b = 35$ $c = 46$, find the magnitude of angle ABC

d $a = 17$ $B = 120°$ $c = 63$, find b

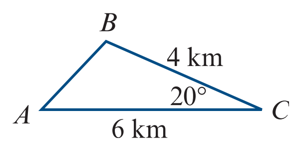
e $a = 31$ $b = 42$ $C = 140°$, find c

f $a = 10$ $b = 12$ $c = 9$, find the magnitude of angle BCA

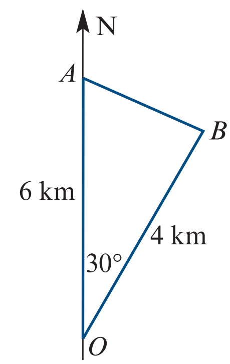
g $a = 11$ $b = 9$ $C = 43.2°$, find c

h $a = 8$ $b = 10$ $c = 15$, find the magnitude of angle CBA.

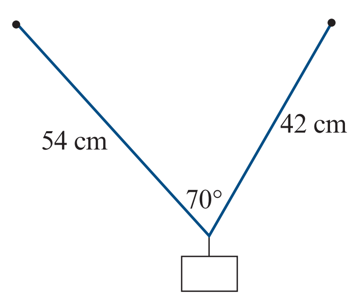
4 A section of an orienteering course is as shown. Find the length of leg AB.



5 Two ships sail in different directions from a point O. At a particular time, their positions A and B are as shown. Find the distance between the ships at this time.



6 A weight is hung from two hooks in a ceiling by strings of length 54 cm and 42 cm, which are inclined at 70° to each other. Find the distance between the hooks.



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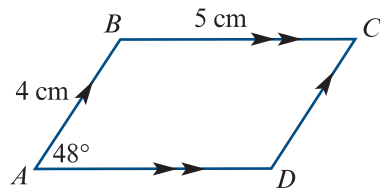
7 $ABCD$ is a parallelogram. Find the lengths of the diagonals:

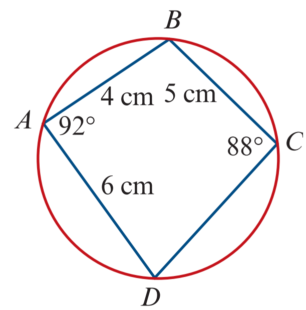
a AC

b BD



8 **a** Find the length of diagonal BD.

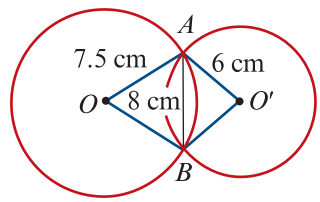
b Use the sine rule to find the length of CD.



9 Two circles of radius 7.5 cm and 6 cm have a common chord of length 8 cm.

a Find the magnitude of angle $AO'B$.

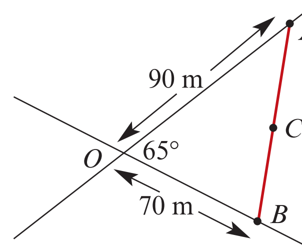
b Find the magnitude of angle AOB.



10 Two straight roads intersect at an angle of 65°. A point A on one road is 90 m from the intersection and a point B on the other road is 70 m from the intersection, as shown.

a Find the distance of A from B.

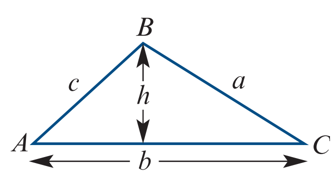
b If C is the midpoint of AB, find the distance of C from the intersection.



13D The area of a triangle

The area of a triangle is given by

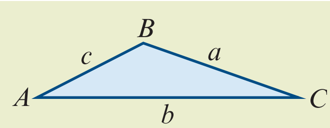
$$\text{Area} = \frac{1}{2} \times \text{base length} \times \text{height}$$
$$= \frac{1}{2}bh$$



By observing that $h = c \sin A$, we obtain the following formula.

For triangle ABC:

$$\text{Area} = \frac{1}{2}bc \sin A$$



That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

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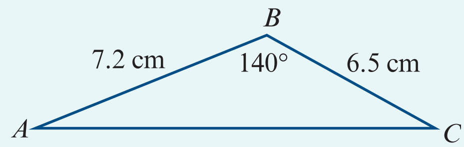
Example 7

Find the area of triangle ABC shown in the diagram.



Solution

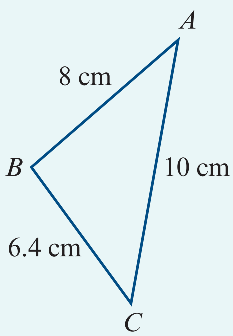
$$\text{Area} = \frac{1}{2} \times 7.2 \times 6.5 \sin 140^\circ$$

$$= 15.04 \text{ cm}^2 \qquad \text{(correct to two decimal places)}$$

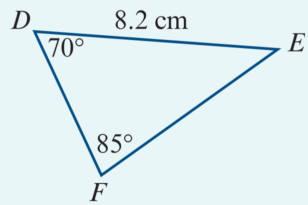
Example 8

Find the area of each of the following triangles, correct to three decimal places:

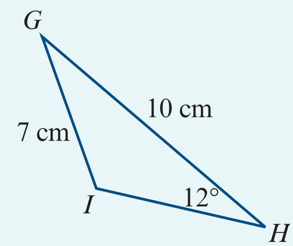
a



b



c



Solution

a Using the cosine rule:

$$8^2 = 6.4^2 + 10^2 - 2 \times 6.4 \times 10 \cos C$$

$$64 = 140.96 - 128 \cos C$$

$$\cos C = 0.60125$$

$$\therefore \quad C^\circ = (53.0405\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle ABC = \frac{1}{2} \times 6.4 \times 10 \times \sin C$$

$$= 25.570 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

b Note that $E^\circ = (180 - (70 + 85))^\circ = 25^\circ$.

Using the sine rule:

$$DF = \sin 25^\circ \times \frac{8.2}{\sin 85^\circ}$$

$$= 3.4787\ldots \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle DEF = \frac{1}{2} \times 8.2 \times DF \times \sin 70^\circ$$

$$= 13.403 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

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c Using the sine rule:

$$\sin I = 10 \times \frac{\sin 12^\circ}{7}$$
$$= 0.2970\ldots$$

$$\therefore \quad I^\circ = (180 - 17.27\ldots)^\circ \qquad \text{(since } I \text{ is an obtuse angle)}$$
$$= (162.72\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

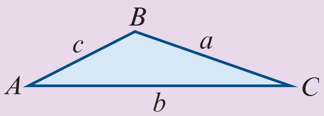
$$\therefore \quad G^\circ = (180 - (12 + I))^\circ$$
$$= (5.27\ldots)^\circ \qquad \text{(store exact value on your calculator)}$$

$$\text{Area } \triangle GHI = \frac{1}{2} \times 10 \times 7 \times \sin G$$
$$= 3.220 \text{ cm}^2 \qquad \text{(correct to three decimal places)}$$

Section summary

For triangle ABC:

$$\text{Area} = \frac{1}{2}bc\sin A$$



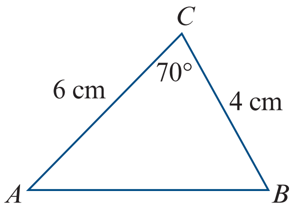
That is, the area is half the product of the lengths of two sides and the sine of the angle included between them.

Exercise 13D

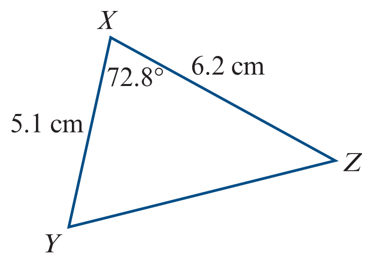
Skillsheet
Example 7

1 Find the area of each of the following triangles:

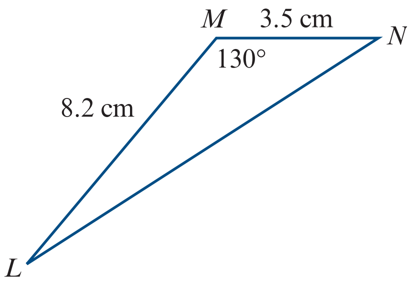
a



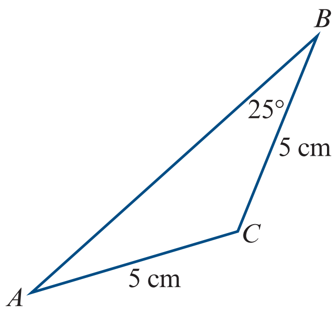
b



c



d



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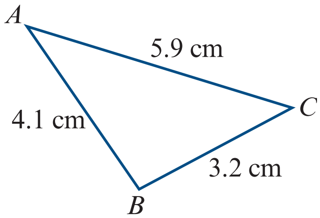
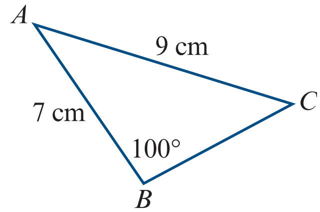
Example 8

2 Find the area of each of the following triangles, correct to three decimal places:

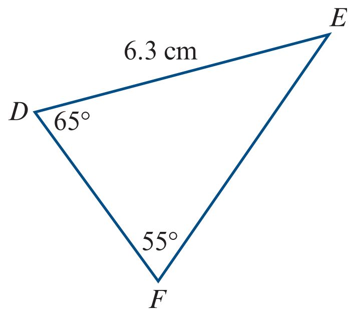
a



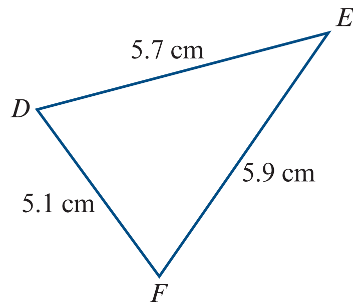
b



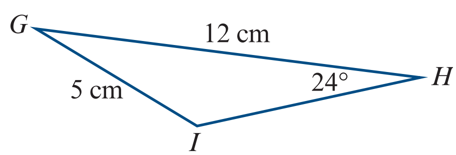
c



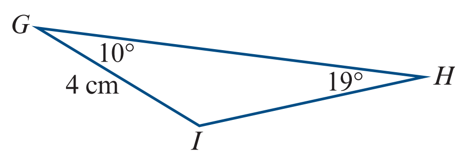
d



e

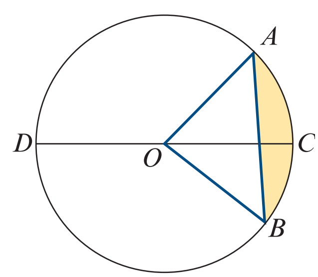


f



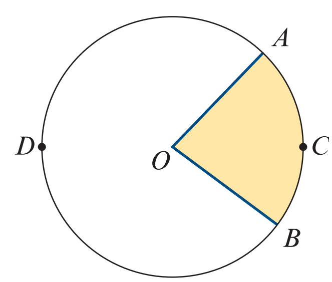
13E Circle mensuration

▶ Terminology



In the diagram, the circle has centre O.

- Chords A **chord** of a circle is a line segment with endpoints on the circle; e.g. line segment AB in the diagram. A chord passing through the centre of the circle is called a **diameter**; e.g. line segment CD in the diagram.
- Arcs Any two points on a circle divide the circle into arcs. The shorter arc is called the **minor arc** and the longer is the **major arc**. In the diagram, arc ACB is a minor arc and arc ADB is a major arc. The arcs DAC and DBC are called **semicircular arcs**.
- Segments Every chord divides the interior of a circle into two regions called segments. The smaller is called the **minor segment** and the larger is the **major segment**. In the above diagram, the minor segment has been shaded.
- Sectors Two radii and an arc define a region called a sector. In this diagram, with circle centre O, the shaded region is a **minor sector** and the unshaded region is a **major sector**.



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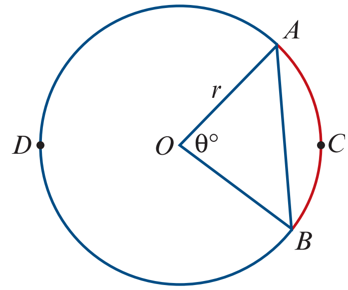
Arc length

The circle in the diagram has centre O and radius r. The arc ACB and the corresponding chord AB are said to **subtend** the angle $\angle AOB$ at the centre of the circle.



The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The length ℓ of arc ACB will be the same fraction of the circumference of the circle, $2\pi r$.

Length of an arc using degrees

$$\ell = \frac{\theta}{360} \times 2\pi r$$

$$= \frac{\pi r \theta}{180} \qquad \text{(where } \theta \text{ is measured in degrees)}$$

Radian measure of angles is introduced in Mathematical Methods Units 1 & 2.

We recall that, in the unit circle, an arc of length θ units subtends an angle of θ radians at the centre. A circle of radius r is similar to the unit circle, with similarity factor r, and therefore an arc of length $r\theta$ units subtends an angle of θ radians at the centre.

Length of an arc using radians

$$\ell = r\theta \qquad \text{(where } \theta \text{ is measured in radians)}$$

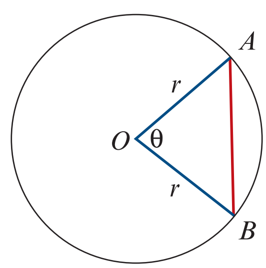
Note: As there are 2π radians in a circle, the arc length is $\ell = \dfrac{\theta}{2\pi} \times 2\pi r = r\theta$.

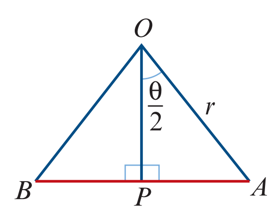
Chord length

In triangle OAP:

$$AP = r\sin\left(\frac{\theta}{2}\right)$$

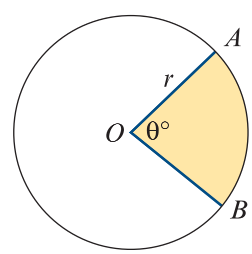
$$\therefore \quad \boxed{AB = 2r\sin\left(\frac{\theta}{2}\right)}$$





Area of a sector

The magnitude $\theta°$ of angle $\angle AOB$ is a fraction of $360°$. The area of the sector will be the same fraction of the area of the circle, πr^2.



Using degrees: Area of sector $= \dfrac{\pi r^2 \theta}{360}$

Using radians: Area of sector $= \dfrac{1}{2} r^2 \theta$

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Example 9

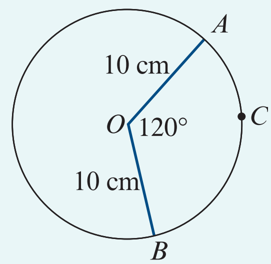
The circle shown has centre O and radius length 10 cm. The angle subtended at O by arc ACB has magnitude $120°$. Find:

a **i** the exact length of the chord AB

ii the exact length of the arc ACB

b the exact area of the minor sector AOB

c the magnitude of angle AOC, in degrees, if the minor arc AC has length 4 cm.



Solution

a **i** Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$

$= 20\sin 60°$ since $r = 10$ and $\theta = 120°$

$= 20 \times \frac{\sqrt{3}}{2}$

$= 10\sqrt{3}$

Length of chord is $10\sqrt{3}$ cm.

ii Arc length $\ell = r\theta$ using radians

$= 10 \times \frac{2\pi}{3}$ since $r = 10$ and $\theta = \frac{2\pi}{3}$

$= \frac{20\pi}{3}$

Length of arc is $\frac{20\pi}{3}$ cm.

Check: Verify that length of arc is greater than length of chord.

b Area of sector $= \frac{1}{2}r^2\theta$ using radians

$= \frac{1}{2} \times 10^2 \times \frac{2\pi}{3}$ since $r = 10$ and $\theta = \frac{2\pi}{3}$

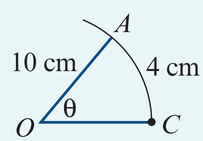
$= \frac{100\pi}{3}$

Area of minor sector AOB is $\frac{100\pi}{3}$ cm^2.

c Using radians: $\ell = r\theta$

$4 = 10\theta$

$\therefore \quad \theta = \frac{4}{10}$



Convert to degrees: $\angle AOC = 0.4 \times \frac{180}{\pi}$

$= (22.9183\ldots)°$

$= 22.92°$ (correct to two decimal places)

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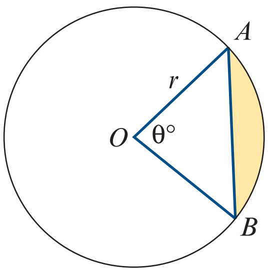
Area of a segment

The area of the shaded segment is found by subtracting the area of $\triangle AOB$ from the area of the minor sector OAB.



Using degrees: Area of segment $= \dfrac{\pi r^2 \theta}{360} - \dfrac{1}{2} r^2 \sin\theta$

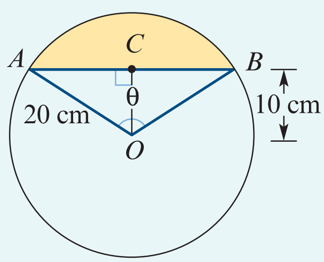
Using radians: Area of segment $= \dfrac{1}{2} r^2 \theta - \dfrac{1}{2} r^2 \sin\theta$

Example 10

A circle, with centre O and radius length 20 cm, has a chord AB that is 10 cm from the centre of the circle. Calculate the area of the minor segment formed by this chord.

Solution

The area of the segment is $\frac{1}{2} r^2(\theta - \sin\theta)$. We know $r = 20$, but we need to find θ.



In $\triangle OCB$:

$$\cos\left(\frac{\theta}{2}\right) = \frac{10}{20}$$

$$\frac{\theta}{2} = \frac{\pi}{3}$$

$$\therefore \quad \theta = \frac{2\pi}{3}$$

$$\text{Area of segment} = \frac{1}{2} \times 20^2 \left(\frac{2\pi}{3} - \sin\left(\frac{2\pi}{3}\right)\right)$$

$$= 200\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right)$$

$$= 200\left(\frac{4\pi - 3\sqrt{3}}{6}\right)$$

$$= \frac{100(4\pi - 3\sqrt{3})}{3} \text{ cm}^2$$

Section summary

- Circle mensuration formulas with θ in radians
  - Arc length $= r\theta$
  - Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$
  - Area of sector $= \frac{1}{2} r^2 \theta$
  - Area of segment $= \frac{1}{2} r^2 (\theta - \sin\theta)$
- Circle mensuration formulas with θ in degrees
  - Arc length $= \dfrac{\pi r \theta}{180}$
  - Chord length $= 2r\sin\left(\frac{\theta}{2}\right)$
  - Area of sector $= \dfrac{\pi r^2 \theta}{360}$
  - Area of segment $= \dfrac{\pi r^2 \theta}{360} - \dfrac{1}{2} r^2 \sin\theta$

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Exercise 13E

Skillsheet
Example 9

1 Find the length of an arc which subtends an angle of magnitude $105°$ at the centre of a circle of radius length 25 cm.

2 Find the magnitude, in degrees, of the angle subtended at the centre of a circle of radius length 30 cm by:

a an arc of length 50 cm

b a chord of length 50 cm.

Example 10

3 A chord of length 6 cm is drawn in a circle of radius 7 cm. Find:

a the length of the minor arc cut off by the chord

b the area of the smaller region inside the circle cut off by the chord.

4 Sketch, on the same set of axes, the graphs of $A = \{ (x, y) : x^2 + y^2 \leq 16 \}$ and $B = \{ (x, y) : y \geq 2 \}$. Find the area measure of the region $A \cap B$.

5 Find the area of the region between an equilateral triangle of side length 10 cm and the circumcircle of the triangle (the circle that passes through the three vertices of the triangle).

6 A person stands on level ground 60 m from the nearest point of a cylindrical tank of radius length 20 m. Calculate:

a the circumference of the tank

b the percentage of the circumference that is visible to the person.

7 The minute hand of a large clock is 4 m long.

a How far does the tip of the minute hand move between 12:10 p.m. and 12:35 p.m.?

b What is the area covered by the minute hand between 12:10 p.m. and 12:35 p.m.?

8 Two circles of radii 3 cm and 4 cm have their centres 5 cm apart. Calculate the area of the region common to both circles.

9 A sector of a circle has perimeter 32 cm and area 63 cm^2. Find the radius length and the magnitude of the angle subtended at the centre of the two possible sectors.

10 Two wheels (pulleys) have radii of length 15 cm and 25 cm and have their centres 60 cm apart. What is the length of the belt required to pass tightly around the pulleys without crossing?

11 A frame in the shape of an equilateral triangle encloses three circular discs of radius length 5 cm so that the discs touch each other. Find:

a the perimeter of the smallest frame which can enclose the discs

b the area enclosed between the discs.

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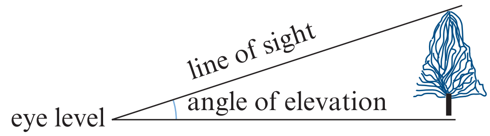
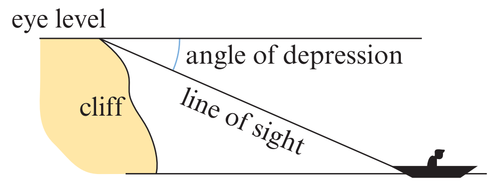
13F Angles of elevation, angles of depression and bearings

The **angle of elevation** is the angle between the horizontal and a direction above the horizontal.



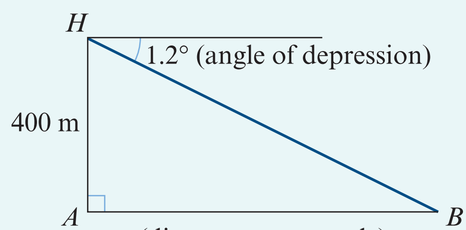
The **angle of depression** is the angle between the horizontal and a direction below the horizontal.



Example 11

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution



(diagram not to scale)

$$\frac{AH}{AB} = \tan 1.2^\circ$$

$$\frac{400}{AB} = \tan 1.2^\circ$$

$$\therefore \quad AB = \frac{400}{\tan 1.2^\circ}$$

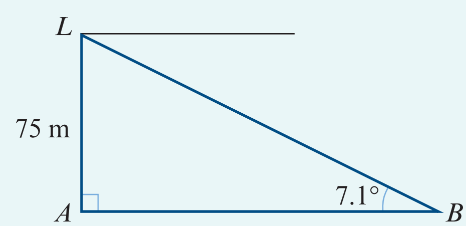
$$= 19\,095.800\ldots$$

The horizontal distance is 19 100 m, correct to the nearest 10 m.

Example 12

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the distance of the boat from the lighthouse.

Solution



$$\frac{75}{AB} = \tan 7.1^\circ$$

$$\therefore \quad AB = \frac{75}{\tan 7.1^\circ}$$

$$= 602.135\ldots$$

The distance of the boat from the lighthouse is 602 m, correct to the nearest metre.

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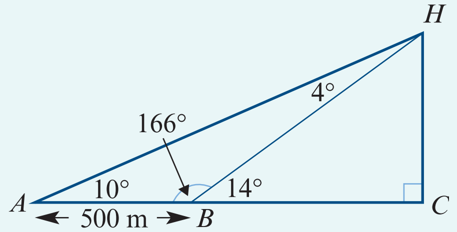
Example 13

From the point A, a man observes that the angle of elevation of the summit of a hill is $10°$. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of $14°$. Find the height of the hill above the level of A.

Solution

Magnitude of $\angle HBA = (180 - 14)° = 166°$

Magnitude of $\angle AHB = (180 - (166 + 10))° = 4°$



Using the sine rule in triangle ABH:

$$\frac{500}{\sin 4°} = \frac{HB}{\sin 10°}$$

$$\therefore \quad HB = \frac{500 \sin 10°}{\sin 4°} = 1244.67\ldots$$

In triangle BCH:

$$\frac{HC}{HB} = \sin 14°$$

$$\therefore \quad HC = HB \sin 14° = 301.11\ldots$$

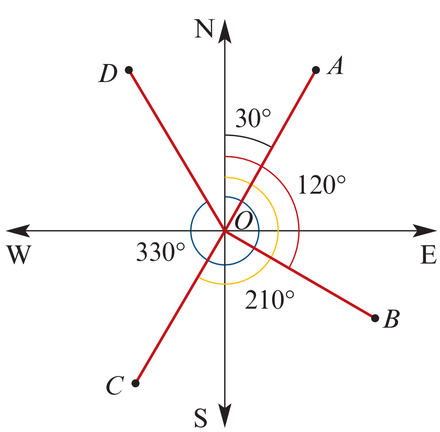
The height of the hill is 301 m, correct to the nearest metre.

Bearings

The **bearing** (or compass bearing) is the direction measured from north clockwise.

For example:

- The bearing of A from O is 030°.
- The bearing of B from O is 120°.
- The bearing of C from O is 210°.
- The bearing of D from O is 330°.



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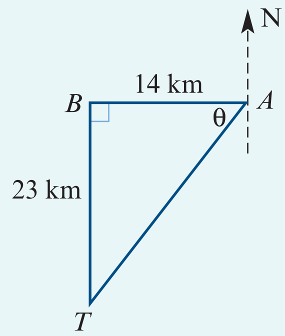
Example 14

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution



$$\tan\theta = \frac{23}{14}$$

$\therefore \quad \theta = 58.67^\circ$ (to two decimal places)

Thus the bearing is

$$180^\circ + (90 - 58.67)^\circ = 211.33^\circ$$

To find the distance, use Pythagoras' theorem:

$$\begin{aligned} AT^2 &= AB^2 + BT^2 \\ &= 14^2 + 23^2 \\ &= 725 \end{aligned}$$

$\therefore \quad AT = 26.925\ldots$

The mast is 27 km from the centre of town A (to the nearest kilometre) and on a bearing of 211.33°.

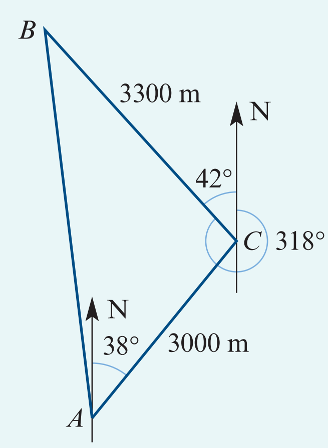
Example 15

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Find:

a the distance AB

b the bearing of B from A.

Solution



a The magnitude of angle ACB needs to be found so that the cosine rule can be applied in triangle ABC:

$$\angle ACB = (180 - (38 + 42))^\circ = 100^\circ$$

In triangle ABC:

$$\begin{aligned} AB^2 &= 3000^2 + 3300^2 - 2 \times 3000 \times 3300 \cos 100^\circ \\ &= 23\,328\,233.917\ldots \end{aligned}$$

$\therefore \quad AB = 4829.931\ldots$

The distance of B from A is 4830 m (to the nearest metre).

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b To find the bearing of B from A, the magnitude of angle BAC must first be found. Using the sine rule:

$$\frac{3300}{\sin A} = \frac{AB}{\sin 100^\circ}$$

$$\therefore \quad \sin A = \frac{3300 \sin 100^\circ}{AB}$$

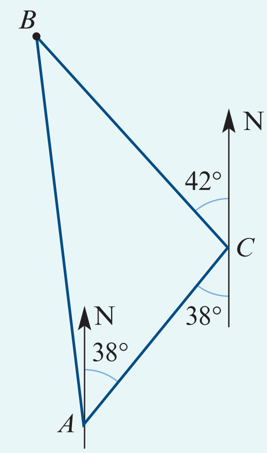
$$= 0.6728\ldots$$

$$\therefore \quad A = (42.288\ldots)^\circ$$

The bearing of B from A $= 360^\circ - (42.29^\circ - 38^\circ)$

$= 355.71^\circ$

The bearing of B from A is 356° to the nearest degree.



Exercise 13F

Example 11 **1** From the top of a vertical cliff 130 m high, the angle of depression of a buoy at sea is 18°. What is the distance of the buoy from the foot of the cliff?

Example 12 **2** The angle of elevation of the top of an old chimney stack at a point 40 m from its base is 41°. Find the height of the chimney.

3 A hiker standing on top of a mountain observes that the angle of depression to the base of a building is 41°. If the height of the hiker above the base of the building is 500 m, find the horizontal distance from the hiker to the building.

4 A person lying down on top of a cliff 40 m high observes the angle of depression to a buoy in the sea below to be 20°. If the person is in line with the buoy, find the distance between the buoy and the base of the cliff, which may be assumed to be vertical.

Example 13 **5** A person standing on top of a cliff 50 m high is in line with two buoys whose angles of depression are 18° and 20°. Calculate the distance between the buoys.

Example 14 **6** A ship sails 10 km north and then sails 15 km east. What is its bearing from the starting point?

7 A ship leaves port A and travels 15 km due east. It then turns and travels 22 km due north.

a What is the bearing of the ship from port A?

b What is the bearing of port A from the ship?

Example 15 **8** A yacht sails from point A on a bearing of 035° for 2000 m. It then alters course to a direction with a bearing of 320° and after sailing for 2500 m it reaches point B.

a Find the distance AB.

b Find the bearing of B from A.

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9 The bearing of a point A from a point B is 207°. What is the bearing of B from A?

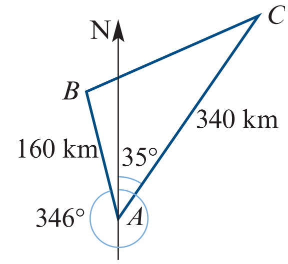
10 The bearing of a ship S from a lighthouse A is 055°. A second lighthouse B is due east of A. The bearing of S from B is 302°. Find the magnitude of angle ASB.

11 A yacht starts from L and sails 12 km due east to M. It then sails 9 km on a bearing of 142° to K. Find the magnitude of angle MLK.

12 The bearing of C from A is 035°. The bearing of B from A is 346°. The distance of C from A is 340 km. The distance of B from A is 160 km.

a Find the magnitude of angle BAC.

b Use the cosine rule to find the distance from B to C.



13 From a ship S, two other ships P and Q are on bearings 320° and 075° respectively. The distance PS is 7.5 km and the distance QS is 5 km. Find the distance PQ.

13G Problems in three dimensions

Some problems in three dimensions can be solved by picking out triangles from a main figure and finding lengths and angles through these triangles.

Example 16

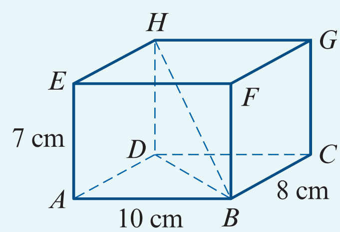
$ABCDEFGH$ is a cuboid. Find:

a the distance DB

b the distance HB

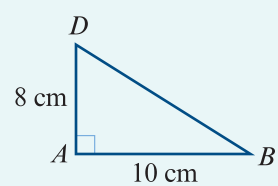
c the magnitude of angle HBD

d the magnitude of angle HBA.

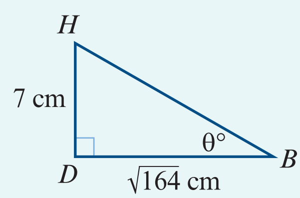


Solution

a
$$DB^2 = 8^2 + 10^2$$
$$= 164$$
$$\therefore\ DB = \sqrt{164}$$
$$= 12.81 \text{ cm} \qquad \text{(correct to two decimal places)}$$



b
$$HB^2 = HD^2 + DB^2$$
$$= 7^2 + 164$$
$$= 213$$
$$\therefore\ HB = \sqrt{213}$$
$$= 14.59 \text{ cm} \qquad \text{(correct to two decimal places)}$$



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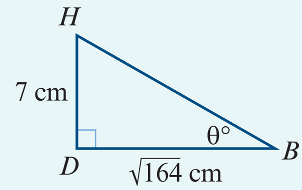
c $\tan\theta = \dfrac{HD}{BD}$

$= \dfrac{7}{\sqrt{164}}$

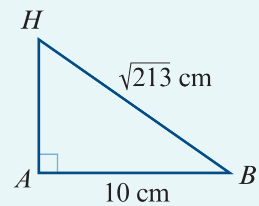
$\therefore\ \theta = 28.66^\circ$ (correct to two decimal places)



d From triangle HBA:

$\cos B = \dfrac{10}{\sqrt{213}}$

$\therefore\ B = 46.75^\circ$ (correct to two decimal places)



Example 17

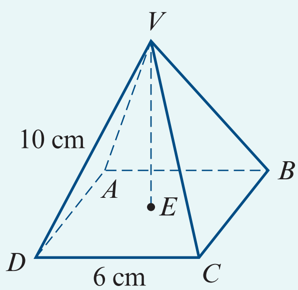
The figure shows a pyramid with a square base. The base has sides 6 cm long and the edges VA, VB, VC and VD are each 10 cm long.

a Find the length of DB.

b Find the length of BE.

c Find the length of VE.

d Find the magnitude of angle VBE.



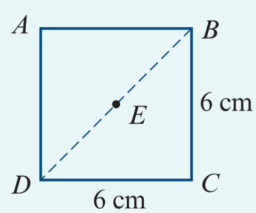
Solution

a $DB^2 = 6^2 + 6^2$

$= 72$

$\therefore\ DB = 6\sqrt{2}$

$= 8.4852\ldots$



The length of DB is 8.49 cm, correct to two decimal places.

b $BE = \dfrac{1}{2}DB$

$= 3\sqrt{2}$

$= 4.2426\ldots$

The length of BE is 4.24 cm, correct to two decimal places.

c $VE^2 = VB^2 - BE^2$

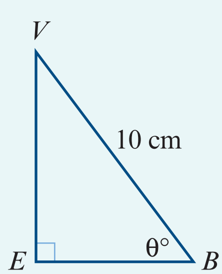
$= 10^2 - (3\sqrt{2})^2$

$= 100 - 18$

$= 82$

$\therefore\ VE = \sqrt{82}$

$= 9.0553\ldots$



The length of VE is 9.06 cm, correct to two decimal places.

d $\sin\theta = \dfrac{VE}{VB}$

$= \dfrac{\sqrt{82}}{10}$

$= 0.9055\ldots$

$\therefore\ \theta = (64.8959\ldots)^\circ$

The magnitude of $\angle VBE$ is 64.90°, correct to two decimal places.

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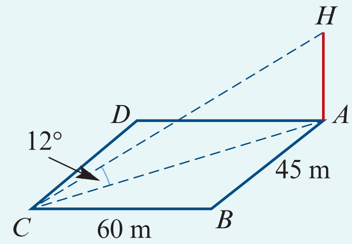
Example 18

A communications mast is erected at corner A of a rectangular courtyard $ABCD$ with side lengths 60 m and 45 m as shown. If the angle of elevation of the top of the mast from C is 12°, find:

a the height of the mast

b the angle of elevation of the top of the mast from B.

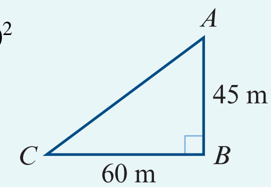


Solution

a $AC^2 = 45^2 + 60^2$

$= 5625$

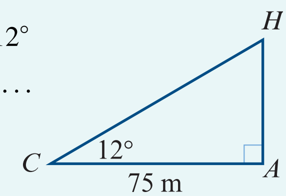
$\therefore \quad AC = 75$



$\dfrac{HA}{75} = \tan 12°$

$\therefore \quad HA = 75 \tan 12°$

$= 15.941\ldots$

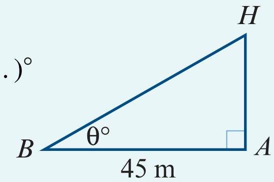


The height of the mast is 15.94 m, correct to two decimal places.

b $\tan\theta = \dfrac{HA}{45}$

$= 0.3542\ldots$

$\therefore \quad \theta = (19.507\ldots)°$



The angle of elevation of the top of the mast, H, from B is 19.51°, correct to two decimal places.

Exercise 13G

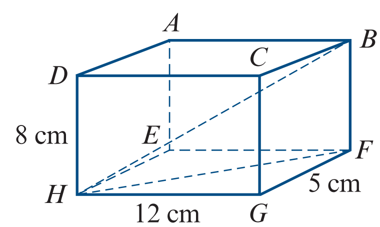
Example 16

1 $ABCDEFGH$ is a cuboid with dimensions as shown. Find:

a the length of FH **b** the length of BH

c the magnitude of angle BHF

d the magnitude of angle BHG.



Example 17

2 $VABCD$ is a right pyramid with a square base. The sides of the base are 8 cm in length. The height, VF, of the pyramid is 12 cm. If E is the midpoint of AD, find:

a the length of EF

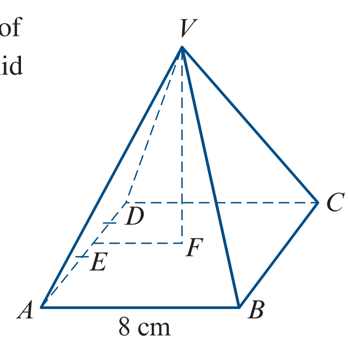
b the magnitude of angle VEF

c the length of VE

d the length of a sloping edge

e the magnitude of angle VAD

f the surface area of the pyramid.



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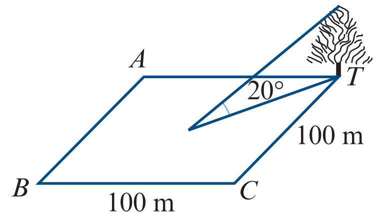
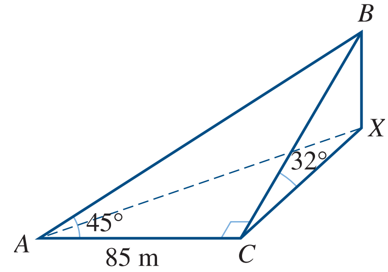
Example 18

3 A tree stands at a corner of a square playing field. Each side of the square is 100 m long. At the centre of the field, the tree subtends an angle of 20°. What angle does it subtend at each of the other three corners of the field?



4 Suppose that A, C and X are three points in a horizontal plane and that B is a point vertically above X. The length of AC is 85 m and the magnitudes of angles BAC, ACB and BCX are 45°, 90° and 32° respectively. Find:

a the distance CB **b** the height XB.

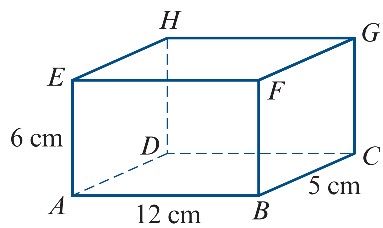


5 Standing due south of a tower 50 m high, the angle of elevation of the top is 26°. What is the angle of elevation after walking a distance 120 m due east?

6 From the top of a cliff 160 m high, two buoys are observed. Their bearings are 337° and 308°. Their respective angles of depression are 3° and 5°. Calculate the distance between the buoys.

7 Find the magnitude of each of the following angles for the cuboid shown:

a ACE **b** HDF **c** ECH



8 From a point A due north of a tower, the angle of elevation to the top of the tower is 45°. From point B, which is 100 m from A on a bearing of 120°, the angle of elevation is 26°. Find the height of the tower.

9 A and B are two positions on level ground. From an advertising balloon at a vertical height of 750 m, point A is observed in an easterly direction and point B at a bearing of 160°. The angles of depression of A and B, as viewed from the balloon, are 40° and 20° respectively. Find the distance between A and B.

10 A right pyramid, height 6 cm, stands on a square base of side length 5 cm. Find:

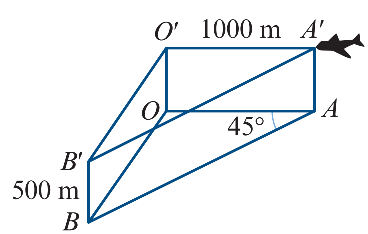
a the length of a sloping edge **b** the area of a triangular face.

11 A light aircraft flying at a height of 500 m above the ground is sighted at a point A' due east of an observer at a point O on the ground, measured horizontally to be 1 km from the plane. The aircraft is flying south-west (along $A'B'$) at 300 km/h.

a How far will it travel in one minute?

b Find its bearing from O (O') at this time.

c What will be its angle of elevation from O at this time?



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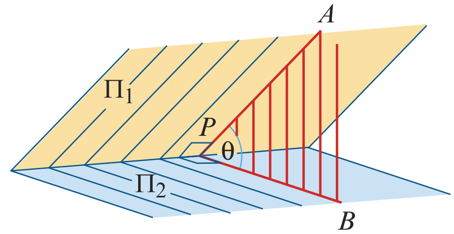
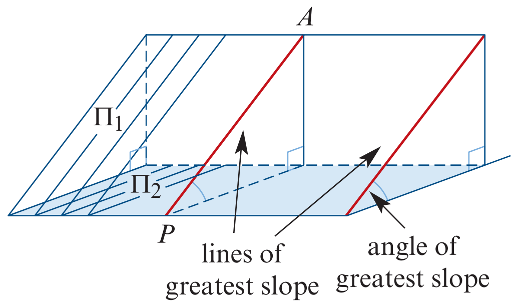
13H Angles between planes and more difficult 3D problems

▶ Angles between planes

Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between planes Π_1 and Π_2.



Note: If the plane Π_2 is horizontal, then PA is called a **line of greatest slope** in the plane Π_1.

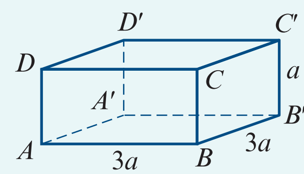


Example 19

For the cuboid shown in the diagram, find:

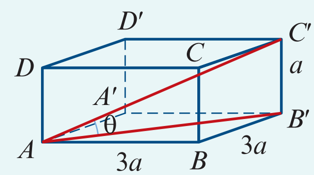
a the angle between AC' and the plane $ABB'A'$

b the angle between the planes ACD' and DCD'.



Solution

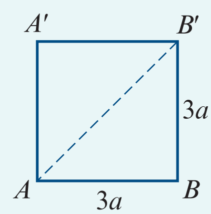
a To find the angle θ between AC' and the plane $ABB'A'$, we need the projection of AC' in the plane.



We drop a perpendicular from C' to the plane (line $C'B'$), and join the foot of the perpendicular to A (line $B'A$).

The required angle, θ, lies between $C'A$ and $B'A$.

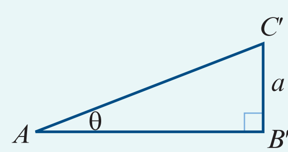
Draw separate diagrams showing the base and the section through A, C' and B'. Then we see that



$$AB' = \sqrt{(3a)^2 + (3a)^2} = 3a\sqrt{2}$$

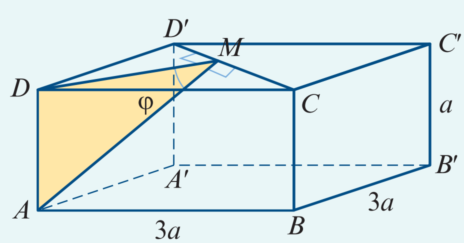
and $\quad \tan\theta = \dfrac{a}{3a\sqrt{2}} = \dfrac{1}{3\sqrt{2}}$

Hence the required angle, θ, is 13.26°.



b The line common to the planes ACD' and DCD' is CD'. Let M be the midpoint of the line segment CD'.

Then MD is perpendicular to CD' in the plane DCD', and MA is perpendicular to CD' in the plane ACD'.



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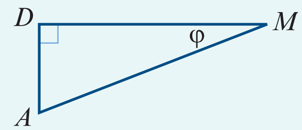
Thus φ is the angle between the planes DCD' and ACD'.



We have

$$DM = \frac{1}{2}DC' = \frac{1}{2}(3a\sqrt{2})$$

$$\therefore \quad \tan\varphi = a \div \left(\frac{3a\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{3}$$

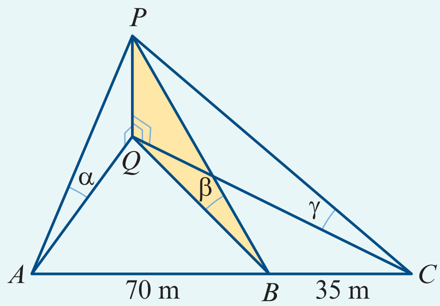
Hence the required angle is $\varphi = 25.24°$.

Example 20

Three points A, B and C are on a horizontal line such that $AB = 70$ m and $BC = 35$ m. The angles of elevation of the top of a tower are α, β and γ, where

$$\tan\alpha = \frac{1}{13}, \quad \tan\beta = \frac{1}{15}, \quad \tan\gamma = \frac{1}{20}$$

as shown in the diagram.



The base of the tower is at the same level as A, B and C. Find the height of the tower.

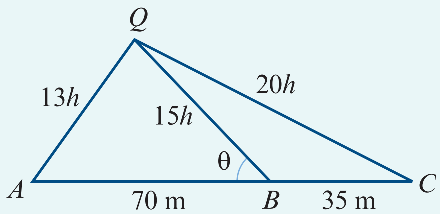
Solution

Let the height of the tower, PQ, be h m. Then

$$h = QA\tan\alpha = QB\tan\beta = QC\tan\gamma$$

which implies that

$$QA = 13h, \quad QB = 15h, \quad QC = 20h$$



Now consider the base triangle ACQ.

Using the cosine rule in $\triangle AQB$:

$$\cos\theta = \frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)}$$

Using the cosine rule in $\triangle CQB$:

$$-\cos\theta = \cos(180 - \theta) = \frac{(35)^2 + (15h)^2 - (20h)^2}{2(35)(15h)}$$

Hence

$$\frac{(70)^2 + (15h)^2 - (13h)^2}{2(70)(15h)} = \frac{(20h)^2 - (15h)^2 - (35)^2}{2(35)(15h)}$$

$$4900 + 56h^2 = 2(175h^2 - 1225)$$

$$7350 = 294h^2$$

$$\therefore \quad h = 5$$

The height of the tower is 5 m.

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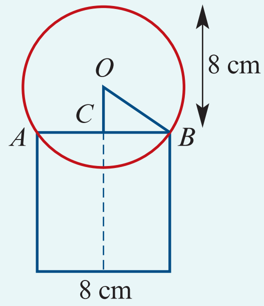
Example 21

A sphere rests on the top of a vertical cylinder which is open at the top. The inside diameter of the cylinder is 8 cm. The sphere projects 8 cm above the top of the cylinder. Find the radius length of the sphere.

Solution

This 3D problem can be represented by a 2D diagram without loss of information.

Let the radius length of the sphere be r cm. Then, in $\triangle OBC$, we have



$$OC = (8 - r)\text{ cm}, \quad BC = 4\text{ cm}, \quad OB = r\text{ cm}$$

Using Pythagoras' theorem:

$$(8 - r)^2 + 4^2 = r^2$$
$$64 - 16r + r^2 + 16 = r^2$$
$$-16r + 80 = 0$$
$$\therefore \quad r = 5$$

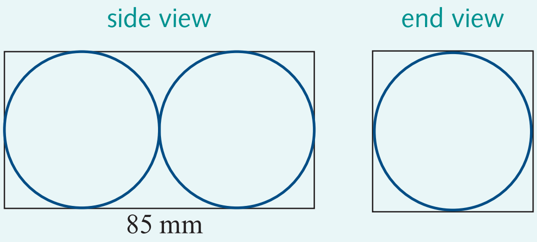
The radius length of the sphere is 5 cm.

Example 22

A box contains two standard golf balls that fit snugly inside. The box is 85 mm long. What percentage of the space inside the box is air?

Solution

Two 2D diagrams may be used to represent the 3D situation.



Let r mm be the radius length of a golf ball.

Length of box = 85 mm = $4r$ mm

Thus $r = \dfrac{85}{4}$, i.e. $r = 21.25$

So the box has dimensions 85 mm by 42.5 mm by 42.5 mm.

Now volume of box $= 42.5^2 \times 85$ using $V = Ah$

volume of two golf balls $= 2 \times \dfrac{4}{3} \times \pi \times 21.25^3$ using $V = \dfrac{4}{3}\pi r^3$

$= \dfrac{8}{3}\pi \times 21.25^3$

Hence percentage air $= \dfrac{100(42.5^2 \times 85 - \frac{8}{3}\pi \times 21.25^3)}{42.5^2 \times 85}$

$= 47.6\%$ to one decimal place

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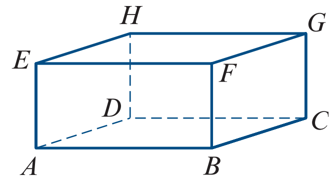
Exercise 13H

Example 19

1 The diagram shows a rectangular prism. Assume that $AB = 4a$ units, $BC = 3a$ units, $GC = a$ units.



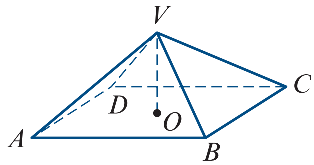
a Calculate the areas of the faces $ABFE$, $BCGF$ and $ABCD$.

b Calculate the magnitude of the angle which plane $GFAD$ makes with the base.

c Calculate the magnitude of the angle which plane $HGBA$ makes with the base.

d Calculate the magnitude of the angle which AG makes with the base.

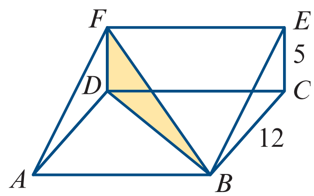
2 $VABCD$ is a right pyramid with square base $ABCD$, and with $AB = 2a$ and $OV = a$.



a Find the slope of the edge VA. That is, find the magnitude of $\angle VAO$.

b Find the slope of the face VBC.

3 A hill has gradient $\frac{5}{12}$. If BF makes an angle of 45° with the line of greatest slope, find:



a the gradient of BF

b the magnitude of $\angle FBD$.

4 The cross-section of a right prism is an isosceles triangle ABC with $AB = BC = 16$ cm and $\angle ABC = 58°$. The equal edges AD, BE and CF are parallel and of length 12 cm. Calculate:

a the length of AC

b the length of AE

c the magnitude of the angle between AE and EC.

Example 20

5 A vertical tower, AT, of height 50 m, stands at a point A on a horizontal plane. The points A, B and C lie on the same horizontal plane, where B is due west of A and C is due south of A. The angles of elevation of the top of the tower, T, from B and C are 25° and 30° respectively.

a Giving answers to the nearest metre, calculate the distances:

i AB **ii** AC **iii** BC

b Calculate the angle of elevation of T from the midpoint, M, of AB.

6 A right square pyramid, vertex O, stands on a square base $ABCD$. The height is 15 cm and the base side length is 10 cm. Find:

a the length of the slant edge

b the inclination of a slant edge to the base

c the inclination of a sloping face to the base

d the magnitude of the angle between two adjacent sloping faces.

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7 A post stands at one corner of a rectangular courtyard. The elevations of the top of the post from the nearest corners are $30°$ and $45°$. Find the elevation from the diagonally opposite corner.

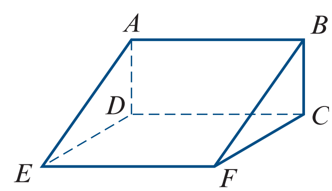
8 $VABC$ is a regular tetrahedron with base $\triangle ABC$. (All faces are equilateral triangles.) Find the magnitude of the angle between:

a a sloping edge and the base

b adjacent sloping faces.

9 An observer at a point A at sea level notes an aircraft due east at an elevation of $35°$. At the same time an observer at B, which is 2 km due south of A, reports the aircraft on a bearing of $50°$. Calculate the altitude of the aircraft.

10 $ABFE$ represents a section of a ski run which has a uniform inclination of $30°$ to the horizontal, with $AE = 100$ m and $AB = 100$ m. A skier traverses the slope from A to F. Calculate:



a the distance that the skier has traversed

b the inclination of the skier's path to the horizontal.

Example 21 **11** A sphere of radius length 8 cm rests on the top of a hollow inverted cone of height 15 cm whose vertical angle is $60°$. Find the height of the centre of the sphere above the vertex of the cone.

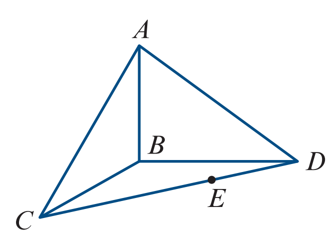
Example 22 **12** Four congruent spheres, radius length 10 cm, are placed on a horizontal table so that each touches two others and their centres form a square. A fifth congruent sphere rests on top of them. Find the height of the top of this fifth sphere above the table.

13 A cube has edge length a cm. What is the radius length, in terms of a, of:

a the sphere that just contains the cube

b the sphere that just fits inside the cube?

14 In the diagram, the edge AB is vertical, $\triangle BCD$ is horizontal, $\angle CBD$ is a right angle and $AB = 20$ m, $BD = 40$ m, $BC = 30$ m. Calculate the inclination to the horizontal of:



a AD

b AE, where AE is the line of greatest slope

c AE, where E is the midpoint of CD.

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Chapter summary

AS

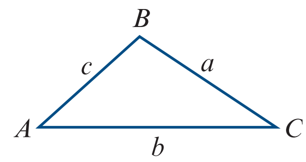
Nrich

Triangles

■ Labelling triangles

- Interior angles are denoted by uppercase letters.
- The length of the side opposite an angle is denoted by the corresponding lowercase letter.

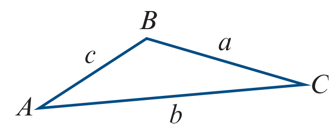
For example, the magnitude of angle BAC is denoted by A, and the length of side BC by a.



■ Sine rule

For triangle ABC:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



The sine rule is used to find unknown quantities in a triangle in the following cases:

- one side and two angles are given
- two sides and a non-included angle are given.

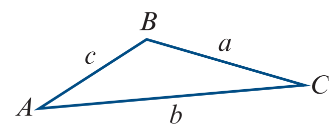
In the first case, the triangle is uniquely defined. But in the second case, there may be two triangles.

■ Cosine rule

For triangle ABC:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



The symmetrical results also hold:

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

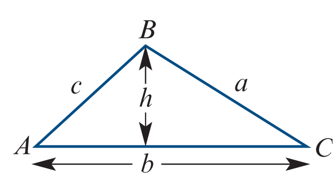
The cosine rule is used to find unknown quantities in a triangle in the following cases:

- two sides and the included angle are given
- three sides are given.

■ Area of a triangle

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}bc\sin A$$



That is, the area of a triangle is half the product of the lengths of two sides and the sine of the angle included between them.

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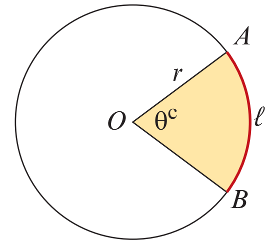
Circles

- Length of minor arc AB (red curve) is given by

$$\ell = r\theta$$

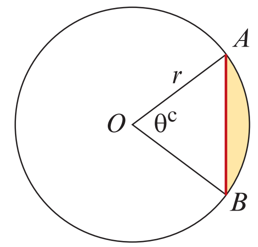


- Area of sector AOB (shaded) is given by

$$\text{Area} = \frac{1}{2}r^2\theta$$

- Length of chord AB (red line) is given by

$$\ell = 2r\sin\left(\frac{\theta}{2}\right)$$

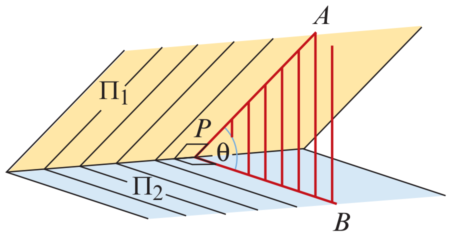


- Area of segment (shaded) is given by

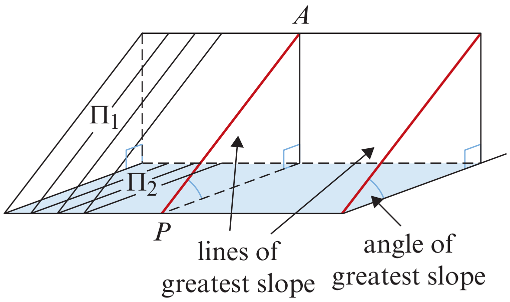
$$\text{Area} = \frac{1}{2}r^2(\theta - \sin\theta)$$

Angle between planes

- Consider any point P on the common line of two planes Π_1 and Π_2. If lines PA and PB are drawn at right angles to the common line so that PA is in Π_1 and PB is in Π_2, then $\angle APB$ is the angle between Π_1 and Π_2.



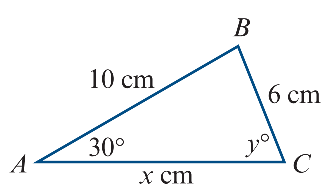
- If plane Π_2 is horizontal, then PA is called a **line of greatest slope** in plane Π_1.



Technology-free questions

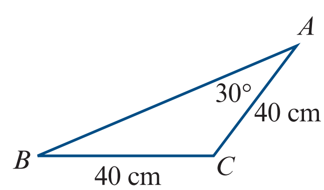
1 **a** Find x.

b Find y.



2 **a** Find AH, where AH is the altitude.

b Find CM, where CM is the median.



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3 From a port P, a ship Q is 20 km away on a bearing of 112°, and a ship R is 12 km away on a bearing of 052°. Find the distance between the two ships.

4 In a quadrilateral $ABCD$, $AB = 5$ cm, $BC = 5$ cm, $CD = 7$ cm, $B = 120°$ and $C = 90°$. Find:

a the length of the diagonal AC

b the area of triangle ABC

c the area of triangle ADC

d the area of the quadrilateral.

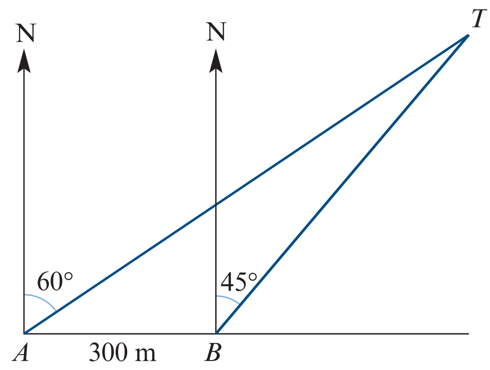
5 If $\sin x = \sin 37°$ and x is obtuse, find x.

6 A point T is 10 km due north of a point S. A point R, which is east of the straight line joining T and S, is 8 km from T and 7 km from S. Calculate the cosine of the bearing of R from S.

7 In $\triangle ABC$, $AB = 5$ cm, $\angle BAC = 60°$ and $AC = 6$ cm. Calculate the sine of $\angle ABC$.

8 The area of a sector of a circle with radius 6 cm is 33 cm^2. Calculate the angle of the sector.

9 The diagram shows two survey points, A and B, which are on an east–west line on level ground. From point A, the bearing of a tower T is 060°, while from point B, the bearing of the tower is 045°.



a **i** Find the magnitude of $\angle TAB$.

ii Find the magnitude of $\angle ATB$.

b Given that $\sin 15° = \dfrac{\sqrt{6} - \sqrt{2}}{4}$, find the distances AT and BT.

10 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour?

11 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 150° to a checkpoint B. It then flies due east to its base C.

a If the bearing of C from A is 120°, find the distances AC and BC.

b The helicopter flies at a constant speed throughout and takes five minutes to fly from A to C. Find its speed.

12 A sector of a circle has an arc length of 30 cm. If the radius of the circle is 12 cm, find the area of the sector.

13 A chord PQ of a circle, radius 5 cm, subtends an angle of 2 radians at the centre of the circle. Taking π to be 3.14, calculate the length of the major arc PQ, correct to one decimal place.

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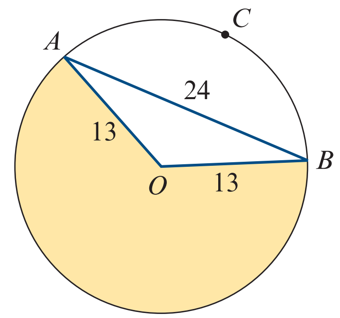
14 The diagram shows a circle of radius length 13 cm and a chord AB of length 24 cm. Calculate:

a the length of arc ACB

b the area of the shaded region.



15 From a cliff top 11 m above sea level, two boats are observed. One has an angle of depression of 45° and is due east, the other an angle of depression of 30° on a bearing of 120°. Calculate the distance between the boats.

Multiple-choice questions

1 In a triangle XYZ, $x = 21$ cm, $y = 18$ cm and $\angle YXZ = 62°$. The magnitude of $\angle XYZ$, correct to one decimal place, is

A 0.4° **B** 0.8° **C** 1.0° **D** 49.2° **E** 53.1°

2 In a triangle ABC, $a = 30$, $b = 21$ and $\cos C = \frac{51}{53}$. The value of c, to the nearest whole number, is

A 9 **B** 10 **C** 11 **D** 81 **E** 129

3 In a triangle ABC, $a = 5.2$ cm, $b = 6.8$ cm and $c = 7.3$ cm. The magnitude of $\angle ACB$, correct to the nearest degree, is

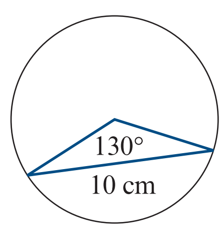
A 43° **B** 63° **C** 74° **D** 82° **E** 98°

4 The area of the triangle ABC, where $b = 5$ cm, $c = 3$ cm, $\angle A = 30°$ and $\angle B = 70°$, is

A 2.75 cm^2 **B** 3.75 cm^2 **C** 6.5 cm^2 **D** 7.5 cm^2 **E** 8 cm^2

5 The length of the radius of the circle shown, correct to two decimal places, is

A 5.52 cm **B** 8.36 cm **C** 9.01 cm
D 12.18 cm **E** 18.13 cm



6 A chord of length 5 cm is drawn in a circle of radius 6 cm. The area of the smaller region inside the circle cut off by the chord, correct to one decimal place, is

A 1.8 cm^2 **B** 2.3 cm^2 **C** 3.9 cm^2 **D** 13.6 cm^2 **E** 15.5 cm^2

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7 From a point on a cliff 500 m above sea level, the angle of depression to a boat is 20°. The distance from the foot of the cliff to the boat, to the nearest metre, is

A 182 m **B** 193 m **C** 210 m **D** 1374 m **E** 1834 m

8 A tower 80 m high is 1.3 km away from a point on the ground. The angle of elevation to the top of the tower from this point, correct to the nearest degree, is

A 1° **B** 4° **C** 53° **D** 86° **E** 89°

9 A man walks 5 km due east followed by 7 km due south. The bearing he must take to return to the start is

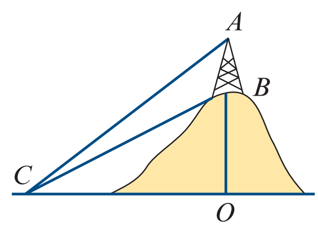
A 036° **B** 306° **C** 324° **D** 332° **E** 348°

10 A boat sails at a bearing of 215° from A to B. The bearing it must take from B to return to A is

A 035° **B** 055° **C** 090° **D** 215° **E** 250°

Extended-response questions

1 AB is a tower 60 m high on top of a hill. The magnitude of $\angle ACO$ is 49° and the magnitude of $\angle BCO$ is 37°.

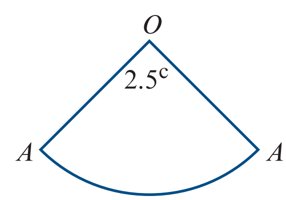


a Find the magnitudes of $\angle ACB$, $\angle CBO$ and $\angle CBA$.

b Find the length of BC.

c Find the height of the hill, i.e. the length of OB.

2 The angle of a sector of a circle, centre O and radius length 12 cm, has magnitude 2.5 radians. The sector is folded so that OA and OA' are joined to form a cone. Calculate:

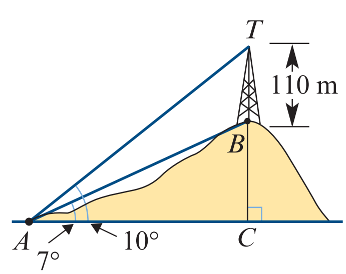


a the base radius length of the cone

b the curved surface area of the cone

c the shortest distance between two points diametrically opposed on the edge of the base.

3 A tower 110 m high stands on the top of a hill. From a point A at the foot of the hill, the angle of elevation of the bottom of the tower is 7° and that of the top is 10°.



a Find the magnitudes of angles TAB, ABT and ATB.

b Use the sine rule to find the length of AB.

c Find CB, the height of the hill.

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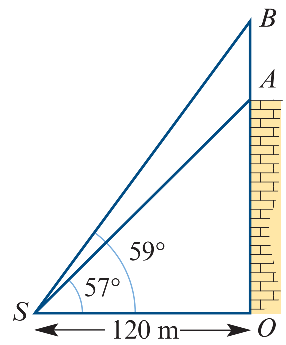
Review

4 Point S is a distance of 120 m from the base of a building. On the building is an aerial, AB. The angle of elevation from S to A is 57°. The angle of elevation from S to B is 59°. Find:

a the distance OA

b the distance OB

c the distance AB.

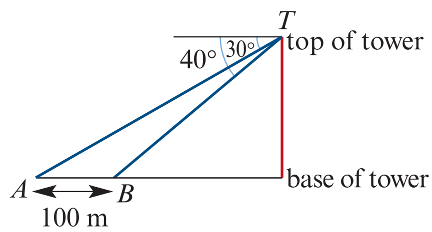


5 From the top of a communications tower, T, the angles of depression of two points A and B on a horizontal line through the base of the tower are 30° and 40°. The distance between the points is 100 m. Find:

a the distance AT

b the distance BT

c the height of the tower.



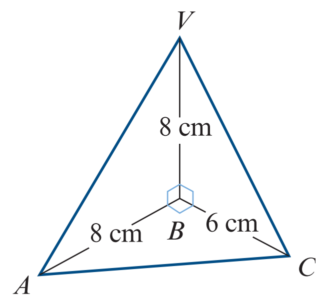
6 Angles VBA, VBC and ABC are right angles. Find:

a the distance VA

b the distance VC

c the distance AC

d the magnitude of angle VCA.



7 The perimeter of a triangle ABC is L metres. Find the area of the triangle in terms of L and the triangle's angles α, β and γ.

Hint: Let $AB = x$. Using the sine rule, first find the other side lengths in terms of x.

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