Chapter 16

Complex numbers

Objectives

- To understand the **imaginary number** *i* and the set of **complex numbers** $\mathbb{C}$.
- To find the **real part** and the **imaginary part** of a complex number.
- To perform **addition**, **subtraction**, **multiplication** and **division** of complex numbers.
- To find the **conjugate** of a complex number.
- To represent complex numbers graphically on an **Argand diagram**.
- To work with complex numbers in **polar form**, and to understand the geometric interpretation of multiplication and division of complex numbers in this form.
- To factorise quadratic polynomials over $\mathbb{C}$.
- To solve quadratic equations over $\mathbb{C}$.

In this chapter we introduce a new set of numbers, called *complex numbers*. These numbers first arose in the search for solutions to polynomial equations.

In the sixteenth century, mathematicians including Girolamo Cardano began to consider square roots of negative numbers. Although these numbers were regarded as 'impossible', they arose in calculations to find real solutions of cubic equations.

For example, the cubic equation $x^3 - 15x - 4 = 0$ has three real solutions. Cardano's formula gives the solution

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}$$

which you can show equals 4.

Today complex numbers are widely used in physics and engineering, such as in the study of aerodynamics.

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16A Starting to build the complex numbers

Mathematicians in the eighteenth century introduced the imaginary number i with the property that

$$i^2 = -1$$

The equation $x^2 = -1$ has two solutions, namely i and $-i$.

By declaring that $i = \sqrt{-1}$, we can find square roots of all negative numbers.

For example:

$$\begin{aligned}\sqrt{-4} &= \sqrt{4 \times (-1)} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i\end{aligned}$$

Note: The identity $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds for positive real numbers a and b, but does not hold when both a and b are negative. In particular, $\sqrt{-1} \times \sqrt{-1} \neq \sqrt{(-1) \times (-1)}$.

Now consider the equation $x^2 + 2x + 3 = 0$. Using the quadratic formula gives

$$\begin{aligned}x &= \frac{-2 \pm \sqrt{4 - 12}}{2} \\ &= \frac{-2 \pm \sqrt{-8}}{2} \\ &= -1 \pm \sqrt{-2}\end{aligned}$$

This equation has no real solutions. However, using complex numbers we obtain solutions

$$x = -1 \pm \sqrt{2}i$$

▶ The set of complex numbers

A **complex number** is an expression of the form $a + bi$, where a and b are real numbers.

The set of all complex numbers is denoted by $\mathbb{C}$. That is,

$$\mathbb{C} = \{\, a + bi : a, b \in \mathbb{R} \,\}$$

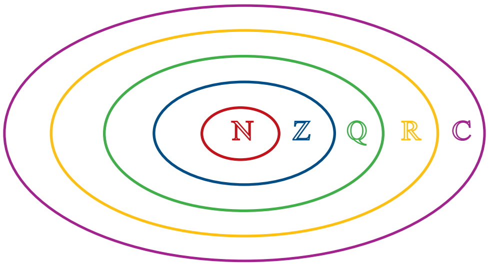
The letter often used to denote a complex number is z.

Therefore if $z \in \mathbb{C}$, then $z = a + bi$ for some $a, b \in \mathbb{R}$.

- If $a = 0$, then $z = bi$ is said to be an **imaginary number**.
- If $b = 0$, then $z = a$ is a **real number**.

The real numbers and the imaginary numbers are subsets of $\mathbb{C}$.

We can now extend the diagram from Chapter 2 to include the complex numbers.



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Real and imaginary parts

For a complex number $z = a + bi$, we define

$$\text{Re}(z) = a \quad \text{and} \quad \text{Im}(z) = b$$

where $\text{Re}(z)$ is called the **real part** of z and $\text{Im}(z)$ is called the **imaginary part** of z.

For example, for the complex number $z = 2 + 5i$, we have $\text{Re}(z) = 2$ and $\text{Im}(z) = 5$.

Note: Both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers. That is, $\text{Re}\colon \mathbb{C} \to \mathbb{R}$ and $\text{Im}\colon \mathbb{C} \to \mathbb{R}$.

Equality of complex numbers

Two complex numbers are defined to be **equal** if both their real parts and their imaginary parts are equal:

$$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

Example 1

If $4 - 3i = 2a + bi$, find the real values of a and b.

Solution

$$2a = 4 \quad \text{and} \quad b = -3$$

$$\therefore \quad a = 2 \quad \text{and} \quad b = -3$$

Example 2

Find the real values of a and b such that $(2a + 3b) + (a - 2b)i = -1 + 3i$.

Solution

$$2a + 3b = -1 \qquad (1)$$

$$a - 2b = 3 \qquad (2)$$

Multiply (2) by 2:

$$2a - 4b = 6 \qquad (3)$$

Subtract (3) from (1):

$$7b = -7$$

Therefore $b = -1$ and $a = 1$.

Operations on complex numbers

Addition and subtraction

Addition of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 + z_2 = (a + c) + (b + d)i$$

The **zero** of the complex numbers can be written as $0 = 0 + 0i$.

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If $z = a + bi$, then we define $-z = -a - bi$.

Subtraction of complex numbers

If $z_1 = a + bi$ and $z_2 = c + di$, then

$$z_1 - z_2 = z_1 + (-z_2) = (a - c) + (b - d)i$$

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 + z_2 = z_2 + z_1$
- $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$
- $z + 0 = z$
- $z + (-z) = 0$

Example 3

If $z_1 = 2 - 3i$ and $z_2 = -4 + 5i$, find:

a $z_1 + z_2$

b $z_1 - z_2$

Solution

a $z_1 + z_2 = (2 - 3i) + (-4 + 5i)$
$= (2 + (-4)) + (-3 + 5)i$
$= -2 + 2i$

b $z_1 - z_2 = (2 - 3i) - (-4 + 5i)$
$= (2 - (-4)) + (-3 - 5)i$
$= 6 - 8i$

Multiplication by a real constant

If $z = a + bi$ and $k \in \mathbb{R}$, then

$$kz = k(a + bi) = ka + kbi$$

For example, if $z = 3 - 6i$, then $3z = 9 - 18i$.

Powers of i

Successive multiplication by i gives the following:

- $i^0 = 1$
- $i^1 = i$
- $i^2 = -1$
- $i^3 = -i$
- $i^4 = (-1)^2 = 1$
- $i^5 = i$
- $i^6 = -1$
- $i^7 = -i$

In general, for $n = 0, 1, 2, 3, \dots$

- $i^{4n} = 1$
- $i^{4n+1} = i$
- $i^{4n+2} = -1$
- $i^{4n+3} = -i$

Example 4

Simplify:

a i^{13}

b $3i^4 \times (-2i)^3$

Solution

a $i^{13} = i^{4\times3+1}$
$= i$

b $3i^4 \times (-2i)^3 = 3 \times (-2)^3 \times i^4 \times i^3$
$= -24i^7$
$= 24i$

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Section summary

- The imaginary number i satisfies $i^2 = -1$.
- If a is a positive real number, then $\sqrt{-a} = i\sqrt{a}$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\text{Re}(z) = a$
  - the **imaginary part** of z is $\text{Im}(z) = b$.
- Equality of complex numbers:

  $$a + bi = c + di \quad \text{if and only if} \quad a = c \text{ and } b = d$$

- If $z_1 = a + bi$ and $z_2 = c + di$, then

  $$z_1 + z_2 = (a + c) + (b + d)i \quad \text{and} \quad z_1 - z_2 = (a - c) + (b - d)i$$

- When simplifying powers of i, remember that $i^4 = 1$.

Exercise 16A

1 State the values of $\text{Re}(z)$ and $\text{Im}(z)$ for each of the following:

a $z = 2 + 3i$ **b** $z = 4 + 5i$ **c** $z = \frac{1}{2} - \frac{3}{2}i$

d $z = -4$ **e** $z = 3i$ **f** $z = \sqrt{2} - 2\sqrt{2}i$

Example 1, 2

2 Find the real values of a and b in each of the following:

a $2a - 3bi = 4 + 6i$ **b** $a + b - 2abi = 5 - 12i$

c $2a + bi = 10$ **d** $3a + (a - b)i = 2 + i$

Example 3

3 Simplify:

a $(2 - 3i) + (4 - 5i)$ **b** $(4 + i) + (2 - 2i)$

c $(-3 - i) - (3 + i)$ **d** $(2 - \sqrt{2}i) + (5 - \sqrt{8}i)$

e $(1 - i) - (2i + 3)$ **f** $(2 + i) - (-2 - i)$

g $4(2 - 3i) - (2 - 8i)$ **h** $-(5 - 4i) + (1 + 2i)$

i $5(i + 4) + 3(2i - 7)$ **j** $\frac{1}{2}(4 - 3i) - \frac{3}{2}(2 - i)$

Example 4

4 Simplify:

a $\sqrt{-16}$ **b** $2\sqrt{-9}$ **c** $\sqrt{-2}$

d i^3 **e** i^{14} **f** i^{20}

g $-2i \times i^3$ **h** $4i^4 \times 3i^2$ **i** $\sqrt{8}i^5 \times \sqrt{-2}$

5 Simplify:

a $i(2 - i)$ **b** $i^2(3 - 4i)$

c $\sqrt{2}i(i - \sqrt{2})$ **d** $-\sqrt{3}(\sqrt{-3} + \sqrt{2})$

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16B Multiplication and division of complex numbers

In the previous section, we defined addition and subtraction of complex numbers. We begin this section by defining multiplication.

▶ Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\begin{aligned} z_1 \times z_2 &= (a + bi)(c + di) \\ &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (ad + bc)i \qquad \text{(since } i^2 = -1\text{)} \end{aligned}$$

We carried out this calculation with an assumption that we are in a system where all the usual rules of algebra apply. However, it should be understood that the following is a *definition* of multiplication for $\mathbb{C}$.

Multiplication of complex numbers

Let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$z_1 \times z_2 = (ac - bd) + (ad + bc)i$$

The multiplicative identity for $\mathbb{C}$ is $1 = 1 + 0i$.

It is easy to check that the following familiar properties of the real numbers extend to the complex numbers:

- $z_1 z_2 = z_2 z_1$
- $z \times 1 = z$
- $(z_1 z_2)z_3 = z_1(z_2 z_3)$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Example 5

If $z_1 = 3 - 2i$ and $z_2 = 1 + i$, find $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= (3 - 2i)(1 + i) \\ &= 3 - 2i + 3i - 2i^2 \\ &= 5 + i \end{aligned}$$

Explanation

Expand the brackets in the usual way.

Remember that $i^2 = -1$.

▶ The conjugate of a complex number

Let $z = a + bi$. The **conjugate** of z is denoted by $\bar{z}$ and is given by

$$\bar{z} = a - bi$$

For example, the conjugate of $-4 + 3i$ is $-4 - 3i$, and vice versa.

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For a complex number $z = a + bi$, we have

$$z\bar{z} = (a + bi)(a - bi)$$
$$= a^2 + abi - abi - b^2 i^2$$
$$= a^2 + b^2 \qquad \text{where } a^2 + b^2 \text{ is a real number}$$

The **modulus** of the complex number $z = a + bi$ is denoted by $|z|$ and is given by

$$|z| = \sqrt{a^2 + b^2}$$

The calculation above shows that

$$z\bar{z} = |z|^2$$

Note: In the case that z is a real number, this definition of $|z|$ agrees with the definition of the modulus of a real number given in Chapter 2.

Example 6

If $z_1 = 2 - 3i$ and $z_2 = -1 + 2i$, find:

a $\overline{z_1 + z_2}$ and $\overline{z_1} + \overline{z_2}$

b $\overline{z_1 \cdot z_2}$ and $\overline{z_1} \cdot \overline{z_2}$

Solution

We have $\overline{z_1} = 2 + 3i$ and $\overline{z_2} = -1 - 2i$.

a
$$z_1 + z_2 = (2 - 3i) + (-1 + 2i) = 1 - i$$
$$\overline{z_1 + z_2} = 1 + i$$
$$\overline{z_1} + \overline{z_2} = (2 + 3i) + (-1 - 2i) = 1 + i$$

b
$$z_1 \cdot z_2 = (2 - 3i)(-1 + 2i) = 4 + 7i$$
$$\overline{z_1 \cdot z_2} = 4 - 7i$$
$$\overline{z_1} \cdot \overline{z_2} = (2 + 3i)(-1 - 2i) = 4 - 7i$$

- The conjugate of a sum is equal to the sum of the conjugates:
$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$$
- The conjugate of a product is equal to the product of the conjugates:
$$\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

Division of complex numbers

Multiplicative inverse

We begin with some familiar algebra that will motivate the definition:

$$\frac{1}{a + bi} = \frac{1}{a + bi} \times \frac{a - bi}{a - bi} = \frac{a - bi}{(a + bi)(a - bi)} = \frac{a - bi}{a^2 + b^2}$$

We can see that

$$(a + bi) \times \frac{a - bi}{a^2 + b^2} = 1$$

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Although we have carried out this arithmetic, we have not yet defined what $\dfrac{1}{a+bi}$ means.

Multiplicative inverse of a complex number

If $z = a + bi$ with $z \neq 0$, then

$$z^{-1} = \frac{a - bi}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

Note: We can check that $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$.

Division

The formal definition of division in the complex numbers is via the multiplicative inverse:

Division of complex numbers

$$\frac{z_1}{z_2} = z_1 z_2^{-1} = \frac{z_1 \overline{z_2}}{|z_2|^2} \qquad (\text{for } z_2 \neq 0)$$

Here is the procedure that is used in practice:

Assume that $z_1 = a + bi$ and $z_2 = c + di$ (where $a, b, c, d \in \mathbb{R}$). Then

$$\frac{z_1}{z_2} = \frac{a + bi}{c + di}$$

Multiply the numerator and denominator by the conjugate of z_2:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a + bi}{c + di} \times \frac{c - di}{c - di}\\ &= \frac{(a + bi)(c - di)}{c^2 + d^2}\end{aligned}$$

Complete the division by simplifying. This process is demonstrated in the next example.

Example 7

If $z_1 = 2 - i$ and $z_2 = 3 + 2i$, find $\dfrac{z_1}{z_2}$.

Solution

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{2 - i}{3 + 2i} \times \frac{3 - 2i}{3 - 2i}\\ &= \frac{6 - 3i - 4i + 2i^2}{3^2 + 2^2}\\ &= \frac{4 - 7i}{13}\\ &= \frac{1}{13}(4 - 7i)\end{aligned}$$

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Example 8

Solve for z in the equation $(2 + 3i)z = -1 - 2i$.

Solution

$$(2 + 3i)z = -1 - 2i$$

$$\therefore \quad z = \frac{-1 - 2i}{2 + 3i}$$

$$= \frac{-1 - 2i}{2 + 3i} \times \frac{2 - 3i}{2 - 3i}$$

$$= \frac{-8 - i}{13}$$

$$= -\frac{1}{13}(8 + i)$$

There is an obvious similarity between the process for expressing a complex number with a real denominator and the process for rationalising the denominator of a surd expression.

Example 9

If $z = 2 - 5i$, find z^{-1} and express with a real denominator.

Solution

$$z^{-1} = \frac{1}{z}$$

$$= \frac{1}{2 - 5i}$$

$$= \frac{1}{2 - 5i} \times \frac{2 + 5i}{2 + 5i}$$

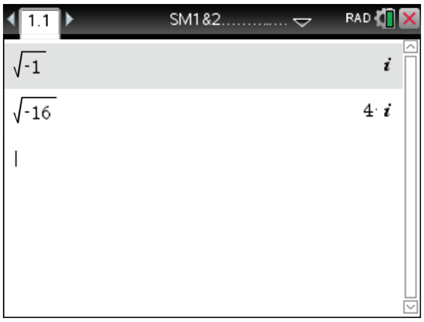
$$= \frac{2 + 5i}{29}$$

$$= \frac{1}{29}(2 + 5i)$$

Using the TI-Nspire

Set to complex mode using (⌂ on) > **Settings** > **Document Settings**. Select **Rectangular** from the **Real or Complex** field.

Note: The square root of a negative number can be found only in complex mode. But most computations with complex numbers can also be performed in real mode.



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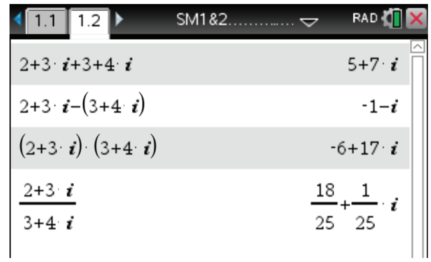
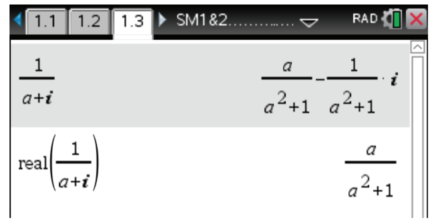
- The results of the arithmetic operations +, −, × and ÷ are illustrated using the two complex numbers $2 + 3i$ and $3 + 4i$.

Note: Do not use the text i for the imaginary constant. The symbol i is found using (π▸) or the Symbols palette ((ctrl) (📖)).



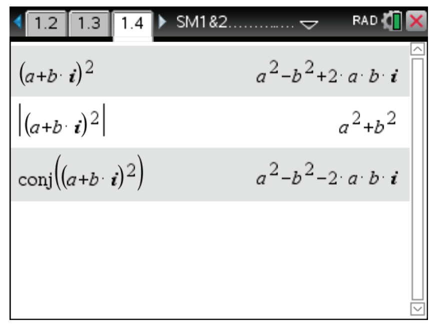
- To find the real part of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Real Part** as shown.

Hint: Type real(.



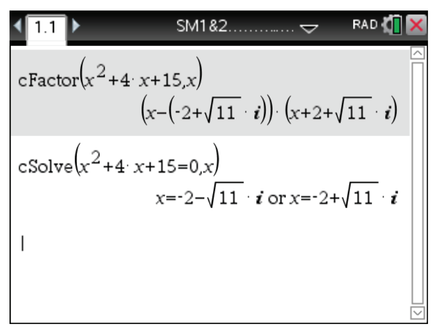
- To find the modulus of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Magnitude** as shown. Alternatively, use |□| from the 2D-template palette (template key) or type abs(.
- To find the conjugate of a complex number, use (menu) > **Number** > **Complex Number Tools** > **Complex Conjugate** as shown.

Hint: Type conj(.



There are also commands for factorising polynomials over the complex numbers and for solving polynomial equations over the complex numbers. These are available from (menu) > **Algebra** > **Complex**.

Note: You must use this menu even if the calculator is in complex mode. When using **cFactor**, you must include the variable as shown.

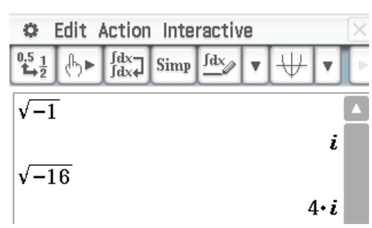


Using the Casio ClassPad

In Main √α, tap **Real** in the status bar at the bottom of the screen to change to **Cplx** mode.

- Enter $\sqrt{-1}$ and tap (EXE) to obtain the answer i.
- Enter $\sqrt{-16}$ to obtain the answer $4i$.

Note: The symbol i is found in both the (Math2) and the (Math3) keyboards.



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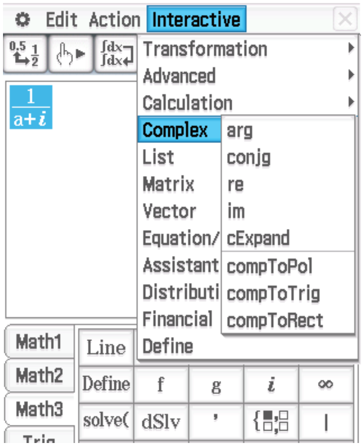
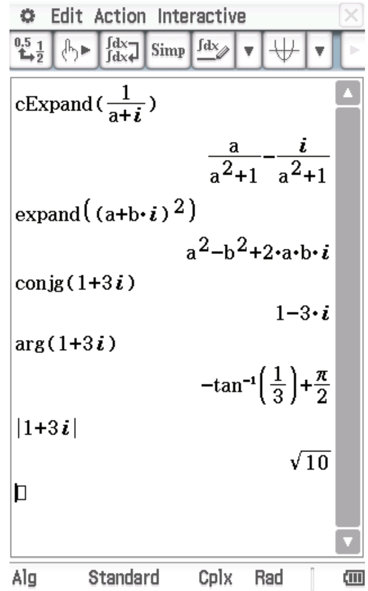
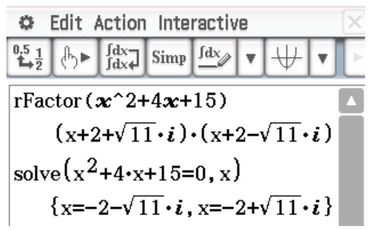
With the calculator set to complex mode, various operations on complex numbers can be carried out using options from **Interactive** > **Complex**.





Polynomials can be factorised over the complex numbers (**Interactive** > **Transformation** > **factor** > **rFactor**).

Equations can be solved over the complex numbers (**Interactive** > **Equation/Inequality** > **solve**).



Section summary

- **Multiplication** To find a product $(a + bi)(c + di)$, expand the brackets in the usual way, remembering that $i^2 = -1$.
- **Conjugate** If $z = a + bi$, then $\bar{z} = a - bi$.
- **Division** To perform a division, start with

$$\frac{a+bi}{c+di} = \frac{a+bi}{c+di} \times \frac{c-di}{c-di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

  and then simplify.
- **Multiplicative inverse** To find z^{-1}, calculate $\frac{1}{z}$.

Exercise 16B

Example 5

1 Expand and simplify:

a $(4 + i)^2$ **b** $(2 - 2i)^2$ **c** $(3 + 2i)(2 + 4i)$

d $(-1 - i)^2$ **e** $(\sqrt{2} - \sqrt{3}i)(\sqrt{2} + \sqrt{3}i)$ **f** $(5 - 2i)(-2 + 3i)$

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2 Write down the conjugate of each of the following complex numbers:

a $2 - 5i$ **b** $-1 + 3i$ **c** $\sqrt{5} - 2i$ **d** $-5i$

Example 6

3 If $z_1 = 2 - i$ and $z_2 = -3 + 2i$, find:

a $\overline{z_1}$ **b** $\overline{z_2}$ **c** $z_1 \cdot z_2$ **d** $\overline{z_1 \cdot z_2}$
e $\overline{z_1} \cdot \overline{z_2}$ **f** $z_1 + z_2$ **g** $\overline{z_1 + z_2}$ **h** $\overline{z_1} + \overline{z_2}$

4 If $z = 2 - 4i$, express each of the following in the form $x + yi$:

a $\overline{z}$ **b** $z\overline{z}$ **c** $z + \overline{z}$ **d** $z(z + \overline{z})$

Example 9

e $z - \overline{z}$ **f** $i(z - \overline{z})$ **g** z^{-1} **h** $\dfrac{z}{i}$

5 Find the real values of a and b such that $(a + bi)(2 + 5i) = 3 - i$.

Example 7

6 Express in the form $x + yi$:

a $\dfrac{2 - i}{4 + i}$ **b** $\dfrac{3 + 2i}{2 - 3i}$ **c** $\dfrac{4 + 3i}{1 + i}$ **d** $\dfrac{2 - 2i}{4i}$ **e** $\dfrac{1}{2 - 3i}$ **f** $\dfrac{i}{2 + 6i}$

7 Find the real values of a and b if $(3 - i)(a + bi) = 6 - 7i$.

Example 8

8 Solve each of the following for z:

a $(2 - i)z = 42i$ **b** $(1 + 3i)z = -2 - i$ **c** $(3i + 5)z = 1 + i$
d $2(4 - 7i)z = 5 + 2i$ **e** $z(1 + i) = 4$

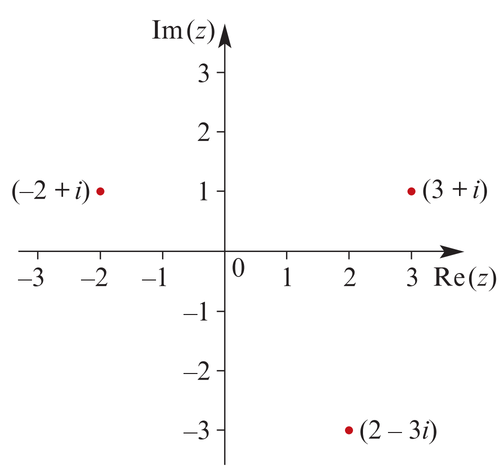
16C Argand diagrams

An **Argand diagram** is a geometric representation of the set of complex numbers. A complex number has two dimensions: the real part and the imaginary part. Therefore a plane is required to represent $\mathbb{C}$.

An Argand diagram is drawn with two perpendicular axes. The horizontal axis represents $\text{Re}(z)$, for $z \in \mathbb{C}$, and the vertical axis represents $\text{Im}(z)$, for $z \in \mathbb{C}$.

Each point on an Argand diagram represents a complex number. The complex number $a + bi$ is situated at the point (a, b) on the equivalent Cartesian axes, as shown by the examples in this figure.

A complex number written as $a + bi$ is said to be in **Cartesian form**.



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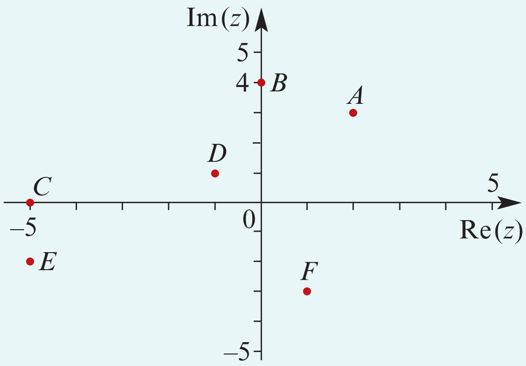
Example 10

Write down the complex number represented by each of the points shown on this Argand diagram.



Solution

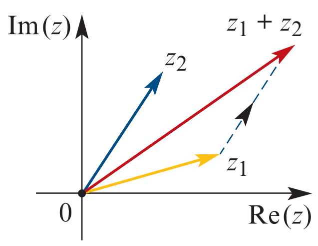
A $2 + 3i$ **B** $4i$ **C** -5

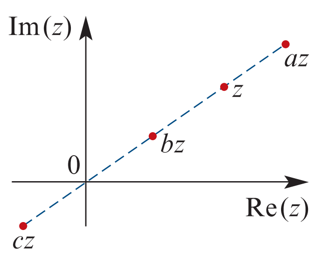
D $-1 + i$ **E** $-5 - 2i$ **F** $1 - 3i$

► Geometric representation of the basic operations on complex numbers

In an Argand diagram, the sum of two complex numbers z_1 and z_2 can be found geometrically by placing the 'tail' of z_2 on the 'tip' of z_1, as shown in the diagram on the left. We will see in Chapter 20 that this is analogous to vector addition.

When a complex number is multiplied by a real constant, it maintains the same 'direction', but its distance from the origin is scaled. This is shown in the diagram on the right.





$a > 1$

$0 < b < 1$

$c < 0$

The difference $z_1 - z_2$ is represented by the sum $z_1 + (-z_2)$.

Example 11

a Represent the following complex numbers as points on an Argand diagram:

i 2 **ii** $-3i$ **iii** $2 - i$ **iv** $-(2 + 3i)$ **v** $-1 + 2i$

b Let $z_1 = 2 + i$ and $z_2 = -1 + 3i$.
Represent the complex numbers z_1, z_2, $z_1 + z_2$ and $z_1 - z_2$ on an Argand diagram and show the geometric interpretation of the sum and difference.

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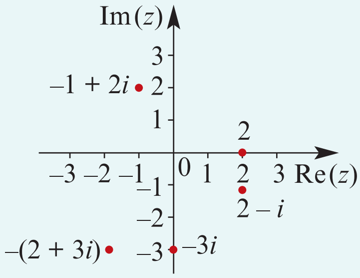
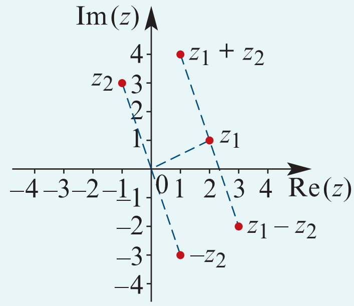
Solution

a



b



$$z_1 + z_2 = (2 + i) + (-1 + 3i) = 1 + 4i$$

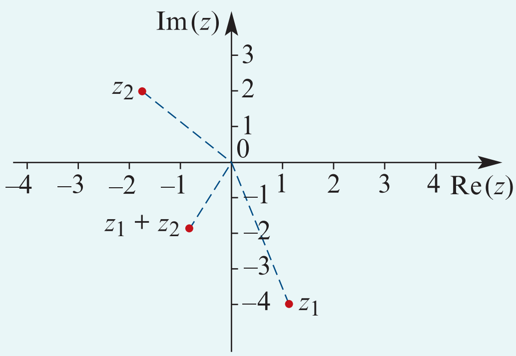
$$z_1 - z_2 = (2 + i) - (-1 + 3i) = 3 - 2i$$

Example 12

Let $z_1 = 1 - 4i$ and $z_2 = -2 + 2i$. Find $z_1 + z_2$ algebraically and illustrate $z_1 + z_2$ on an Argand diagram.

Solution

$$\begin{aligned} z_1 + z_2 &= (1 - 4i) + (-2 + 2i) \\ &= -1 - 2i \end{aligned}$$

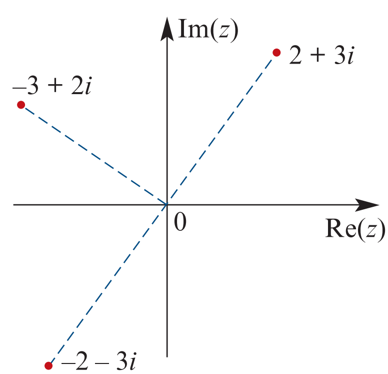


Rotation about the origin

When the complex number $2 + 3i$ is multiplied by -1, the result is $-2 - 3i$. This is achieved through a rotation of 180° about the origin.

When the complex number $2 + 3i$ is multiplied by i, we obtain

$$\begin{aligned} i(2 + 3i) &= 2i + 3i^2 \\ &= 2i - 3 \\ &= -3 + 2i \end{aligned}$$



The result is achieved through a rotation of 90° anticlockwise about the origin.

If $-3 + 2i$ is multiplied by i, the result is $-2 - 3i$. This is again achieved through a rotation of 90° anticlockwise about the origin.

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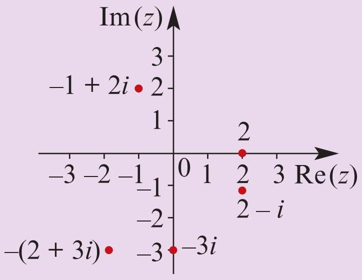
Section summary

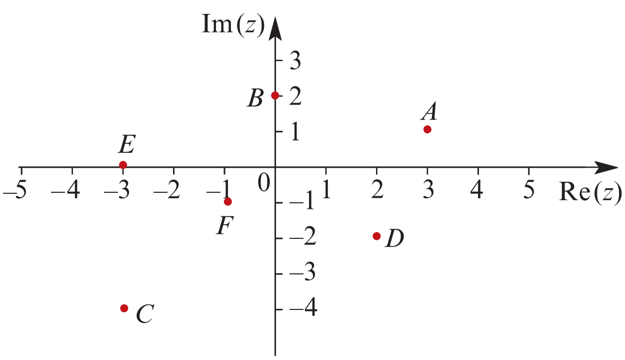
- An **Argand diagram** is a geometric representation of the set of complex numbers.
- The horizontal axis represents Re(z) and the vertical axis represents Im(z), for $z \in \mathbb{C}$.
- The operations of addition, subtraction and multiplication by a real constant all have geometric interpretations on an Argand diagram.
- Multiplication of a complex number by i corresponds to a rotation of 90° anticlockwise about the origin.



Exercise 16C

Example 10

1 Write down the complex numbers represented on this Argand diagram.



Example 11

2 Represent each of the following complex numbers as points on an Argand diagram:

a $3 - 4i$ **b** $-4 + i$ **c** $4 + i$

d -3 **e** $-2i$ **f** $-5 - 2i$

Example 12

3 If $z_1 = 6 - 5i$ and $z_2 = -3 + 4i$, represent each of the following on an Argand diagram:

a $z_1 + z_2$ **b** $z_1 - z_2$

4 If $z = 1 + 3i$, represent each of the following on an Argand diagram:

a z **b** $\bar{z}$ **c** z^2

d $-z$ **e** $\dfrac{1}{z}$

5 If $z = 2 - 5i$, represent each of the following on an Argand diagram:

a z **b** zi **c** zi^2

d zi^3 **e** zi^4

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16D Solving equations over the complex numbers

Quadratic equations with a negative discriminant have no real solutions. The introduction of complex numbers enables us to solve such quadratic equations.

Sum of two squares

Since $i^2 = -1$, we can rewrite a sum of two squares as a difference of two squares:

$$\begin{aligned} z^2 + a^2 &= z^2 - (ai)^2 \\ &= (z + ai)(z - ai) \end{aligned}$$

This allows us to solve equations of the form $z^2 + a^2 = 0$.

Example 13

Solve the equations:

a $z^2 + 16 = 0$

b $2z^2 + 6 = 0$

Solution

a
$$\begin{aligned} z^2 + 16 &= 0 \\ z^2 - 16i^2 &= 0 \\ (z + 4i)(z - 4i) &= 0 \\ \therefore \quad z &= \pm 4i \end{aligned}$$

b
$$\begin{aligned} 2z^2 + 6 &= 0 \\ z^2 + 3 &= 0 \\ z^2 - 3i^2 &= 0 \\ (z + \sqrt{3}i)(z - \sqrt{3}i) &= 0 \\ \therefore \quad z &= \pm\sqrt{3}i \end{aligned}$$

Solution of quadratic equations

To solve quadratic equations which have a negative discriminant, we can use the quadratic formula in the usual way.

Example 14

Solve the equation $3z^2 + 5z + 3 = 0$.

Solution

Using the quadratic formula:

$$\begin{aligned} z &= \frac{-5 \pm \sqrt{25 - 36}}{6} \\ &= \frac{-5 \pm \sqrt{-11}}{6} \\ &= \frac{1}{6}(-5 \pm \sqrt{11}i) \end{aligned}$$

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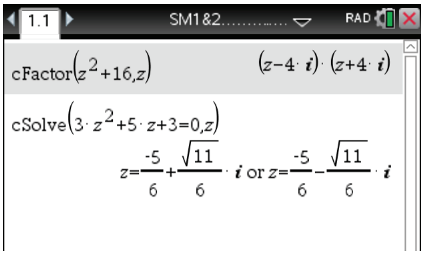
Using the TI-Nspire

- To factorise polynomials over the complex numbers, use (menu) > **Algebra** > **Complex** > **Factor** as shown.
- To solve polynomial equations over the complex numbers, use (menu) > **Algebra** > **Complex** > **Solve** as shown.



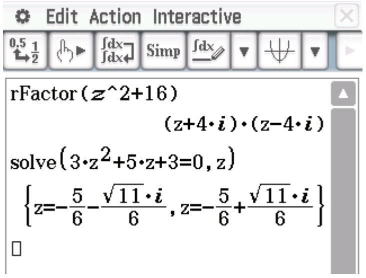
Using the Casio ClassPad

To factorise:

- Ensure the mode is set to **Cplx**.
- Enter and highlight the expression $z^2 + 16$.
- Select **Interactive** > **Transformation** > **factor** > **rFactor**.

To solve:

- Enter and highlight $3z^2 + 5z + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve**.
- Ensure the variable selected is z.



Section summary

- Quadratic equations can be solved over the complex numbers using the same techniques as for the real numbers.
- Two properties of complex numbers that are useful when solving equations:
  - $z^2 + a^2 = z^2 - (ai)^2 = (z + ai)(z - ai)$
  - $\sqrt{-a} = i\sqrt{a}$, where a is a positive real number.

Exercise 16D

Skillsheet

Example 13, 14

1 Solve each of the following equations over $\mathbb{C}$:

a $z^2 + 4 = 0$ **b** $2z^2 + 18 = 0$

c $3z^2 = -15$ **d** $(z - 2)^2 + 16 = 0$

e $(z + 1)^2 = -49$ **f** $z^2 - 2z + 3 = 0$

g $z^2 + 3z + 3 = 0$ **h** $2z^2 + 5z + 4 = 0$

i $3z^2 = z - 2$ **j** $2z = z^2 + 5$

k $2z^2 - 6z = -10$ **l** $z^2 - 6z = -14$

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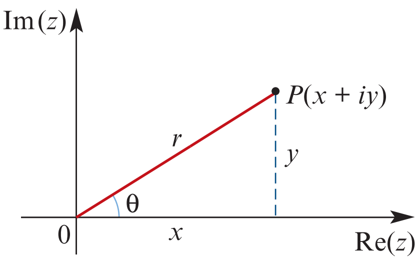
16E Polar form of a complex number

Polar coordinates for points in the plane were introduced in Chapter 15. Similarly, each complex number may be described by an angle and a distance from the origin. In this section, we will see that this is a very useful way to describe complex numbers.

The diagram shows the point P corresponding to the complex number $z = x + yi$. We see that $x = r\cos\theta$ and $y = r\sin\theta$, and so we can write

$$\begin{aligned} z &= x + yi \\ &= r\cos\theta + (r\sin\theta)\,i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$



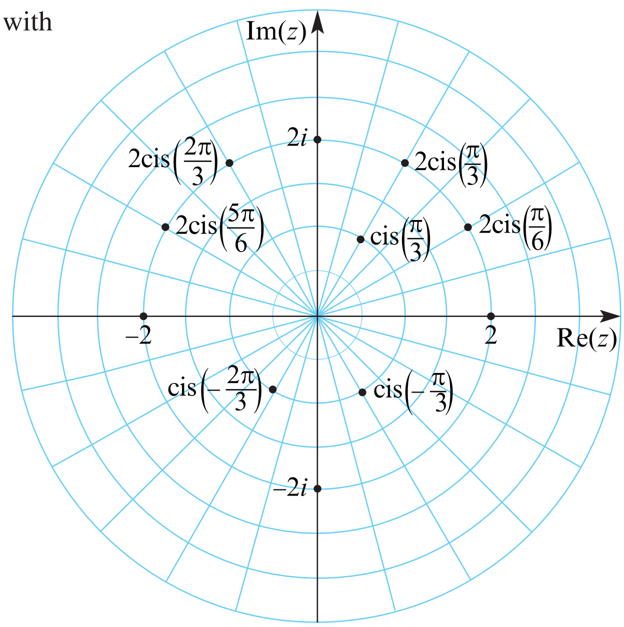
This is called the **polar form** of the complex number. The polar form is abbreviated to

$$z = r\operatorname{cis}\theta$$

- The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
- The angle θ, measured anticlockwise from the horizontal axis, is called the **argument** of z.

Polar form for complex numbers is also called **modulus–argument form**.

This Argand diagram uses a polar grid with rays at intervals of $\frac{\pi}{12} = 15°$.



Non-uniqueness of polar form

Each complex number has more than one representation in polar form.

Since $\cos\theta = \cos(\theta + 2n\pi)$ and $\sin\theta = \sin(\theta + 2n\pi)$, for all $n \in \mathbb{Z}$, we can write

$$z = r\operatorname{cis}\theta = r\operatorname{cis}(\theta + 2n\pi) \quad \text{for all } n \in \mathbb{Z}$$

The convention is to use the angle θ such that $-\pi < \theta \le \pi$. This value of θ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$. That is,

$$-\pi < \operatorname{Arg} z \le \pi$$

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Example 15

Express each of the following complex numbers in polar form:

a $z = 1 + \sqrt{3}i$

b $z = 2 - 2i$

Solution

a We have $x = 1$ and $y = \sqrt{3}$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{1 + 3} \\ &= 2 \end{aligned}$$

The point $z = 1 + \sqrt{3}i$ is in the 1st quadrant, and so $0 < \theta < \frac{\pi}{2}$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{2}$$

and $$\sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

Hence $\theta = \frac{\pi}{3}$ and therefore

$$\begin{aligned} z &= 1 + \sqrt{3}i \\ &= 2\operatorname{cis}\left(\frac{\pi}{3}\right) \end{aligned}$$

b We have $x = 2$ and $y = -2$, giving

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{4 + 4} \\ &= 2\sqrt{2} \end{aligned}$$

The point $z = 2 - 2i$ is in the 4th quadrant, and so $-\frac{\pi}{2} < \theta < 0$.

We know that

$$\cos\theta = \frac{x}{r} = \frac{1}{\sqrt{2}}$$

and $$\sin\theta = \frac{y}{r} = \frac{-1}{\sqrt{2}}$$

Hence $\theta = -\frac{\pi}{4}$ and therefore

$$\begin{aligned} z &= 2 - 2i \\ &= 2\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right) \end{aligned}$$

Example 16

Express $z = 2\operatorname{cis}\left(\frac{-2\pi}{3}\right)$ in Cartesian form.

Solution

$$\begin{aligned} x &= r\cos\theta \\ &= 2\cos\left(\frac{-2\pi}{3}\right) \\ &= 2 \times \left(-\frac{1}{2}\right) \\ &= -1 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 2\sin\left(\frac{-2\pi}{3}\right) \\ &= 2 \times \left(\frac{-\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Hence $z = 2\operatorname{cis}\left(\frac{-2\pi}{3}\right) = -1 - \sqrt{3}i$.

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Multiplication and division in polar form

We can give a simple geometric interpretation of multiplication and division of complex numbers in polar form.

If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \qquad \text{(multiply the moduli and add the angles)}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2) \qquad \text{(divide the moduli and subtract the angles)}$$

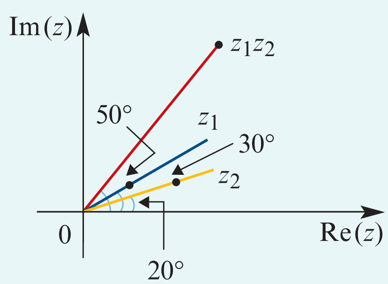
These two results can be proved using the addition formulas for sine and cosine established in Chapter 14.

Example 17

Let $z_1 = 2 \operatorname{cis} 30^\circ$ and $z_2 = 4 \operatorname{cis} 20^\circ$. Find the product $z_1 z_2$ and represent it on an Argand diagram.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 2 \times 4 \operatorname{cis}(30^\circ + 20^\circ) \\ &= 8 \operatorname{cis} 50^\circ \end{aligned}$$



Example 18

Let $z_1 = 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ and $z_2 = 2 \operatorname{cis}\left(\frac{5\pi}{6}\right)$. Find the product $z_1 z_2$.

Solution

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \\ &= 6 \operatorname{cis}\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \\ &= 6 \operatorname{cis}\left(\frac{4\pi}{3}\right) \end{aligned}$$

$$\therefore \quad z_1 z_2 = 6 \operatorname{cis}\left(\frac{-2\pi}{3}\right) \qquad \text{since } -\pi < \operatorname{Arg} z \leq \pi$$

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Example 19

If $z_1 = -\sqrt{3} + i$ and $z_2 = 2\sqrt{3} + 2i$, find the quotient $\dfrac{z_1}{z_2}$ and express in Cartesian form.

Solution

First express z_1 and z_1 in polar form:

- $|z_1| = \sqrt{3+1} = 2$

  Let $\theta_1 = \operatorname{Arg} z_1$. Then $\cos\theta_1 = \dfrac{-\sqrt{3}}{2}$ and $\sin\theta_1 = \dfrac{1}{2}$, giving $\theta_1 = \dfrac{5\pi}{6}$.

- $|z_2| = \sqrt{12+4} = 4$

  Let $\theta_2 = \operatorname{Arg} z_2$. Then $\cos\theta_2 = \dfrac{\sqrt{3}}{2}$ and $\sin\theta_2 = \dfrac{1}{2}$, giving $\theta_2 = \dfrac{\pi}{6}$.

Hence $z_1 = 2\operatorname{cis}\left(\dfrac{5\pi}{6}\right)$ and $z_2 = 4\operatorname{cis}\left(\dfrac{\pi}{6}\right)$. Therefore

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1}{r_2}\operatorname{cis}(\theta_1 - \theta_2)\\ &= \frac{2}{4}\operatorname{cis}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)\\ &= \frac{1}{2}\operatorname{cis}\left(\frac{2\pi}{3}\right)\end{aligned}$$

In Cartesian form:

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{1}{2}\cos\left(\frac{2\pi}{3}\right) + \frac{1}{2}\sin\left(\frac{2\pi}{3}\right)i\\ &= \frac{1}{2}\left(\frac{-1}{2}\right) + \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)i\\ &= -\frac{1}{4}(1 - \sqrt{3}i)\end{aligned}$$

Section summary

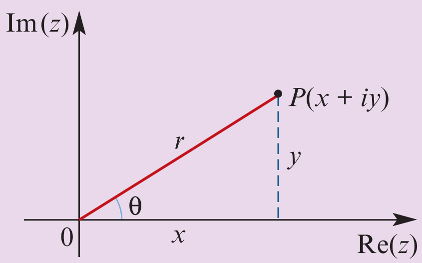
- **Polar form**

  A complex number in Cartesian form

  $$z = x + yi$$

  can be written in polar form as

  $$\begin{aligned} z &= r(\cos\theta + i\sin\theta)\\ &= r\operatorname{cis}\theta\end{aligned}$$



  - The distance $r = \sqrt{x^2 + y^2}$ is called the **modulus** of z and is denoted by $|z|$.
  - The angle θ, measured anticlockwise from the horizontal axis, is called the **argument** of z.

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- The polar form of a complex number is not unique. The argument θ of z such that $-\pi < \theta \le \pi$ is called the **principal value** of the argument of z and is denoted by $\operatorname{Arg} z$.
- Multiplication and division in polar form

  If $z_1 = r_1 \operatorname{cis} \theta_1$ and $z_2 = r_2 \operatorname{cis} \theta_2$, then

$$z_1 z_2 = r_1 r_2 \operatorname{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} \operatorname{cis}(\theta_1 - \theta_2)$$

Exercise 16E

Skillsheet
Example 15

1 Express each of the following in polar form $r \operatorname{cis} \theta$ with $-\pi < \theta \le \pi$:

a $1 + \sqrt{3}i$ **b** $1 - i$ **c** $-2\sqrt{3} + 2i$

d $-4 - 4i$ **e** $12 - 12\sqrt{3}i$ **f** $-\frac{1}{2} + \frac{1}{2}i$

Example 16

2 Express each of the following in the form $x + yi$:

a $3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ **b** $\sqrt{2} \operatorname{cis}\left(\frac{\pi}{3}\right)$ **c** $2 \operatorname{cis}\left(\frac{\pi}{6}\right)$

d $5 \operatorname{cis}\left(\frac{3\pi}{4}\right)$ **e** $12 \operatorname{cis}\left(\frac{5\pi}{6}\right)$ **f** $3\sqrt{2} \operatorname{cis}\left(\frac{-\pi}{4}\right)$

g $5 \operatorname{cis}\left(\frac{4\pi}{3}\right)$ **h** $5 \operatorname{cis}\left(\frac{-2\pi}{3}\right)$

Example 17, 18

3 Simplify the following and express the answers in Cartesian form:

a $2 \operatorname{cis}\left(\frac{\pi}{6}\right) \cdot 3 \operatorname{cis}\left(\frac{\pi}{12}\right)$ **b** $4 \operatorname{cis}\left(\frac{\pi}{12}\right) \cdot 3 \operatorname{cis}\left(\frac{\pi}{4}\right)$

c $\operatorname{cis}\left(\frac{\pi}{4}\right) \cdot 5 \operatorname{cis}\left(\frac{5\pi}{12}\right)$ **d** $12 \operatorname{cis}\left(\frac{-\pi}{3}\right) \cdot 3 \operatorname{cis}\left(\frac{2\pi}{3}\right)$

e $12 \operatorname{cis}\left(\frac{5\pi}{6}\right) \cdot 3 \operatorname{cis}\left(\frac{\pi}{2}\right)$ **f** $(\sqrt{2} \operatorname{cis} \pi) \cdot \sqrt{3} \operatorname{cis}\left(\frac{-3\pi}{4}\right)$

Example 19

g $\dfrac{10 \operatorname{cis}\left(\frac{\pi}{4}\right)}{5 \operatorname{cis}\left(\frac{\pi}{12}\right)}$ **h** $\dfrac{12 \operatorname{cis}\left(\frac{-\pi}{3}\right)}{3 \operatorname{cis}\left(\frac{2\pi}{3}\right)}$ **i** $\dfrac{12\sqrt{8} \operatorname{cis}\left(\frac{3\pi}{4}\right)}{3\sqrt{2} \operatorname{cis}\left(\frac{\pi}{12}\right)}$ **j** $\dfrac{20 \operatorname{cis}\left(\frac{-\pi}{6}\right)}{8 \operatorname{cis}\left(\frac{5\pi}{6}\right)}$

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Review

Chapter summary

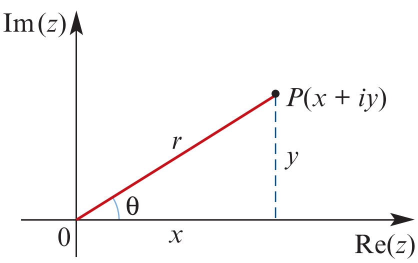
AS Nrich

- The imaginary number i has the property $i^2 = -1$.
- The set of **complex numbers** is $\mathbb{C} = \{ a + bi : a, b \in \mathbb{R} \}$.
- For a complex number $z = a + bi$:
  - the **real part** of z is $\mathrm{Re}(z) = a$
  - the **imaginary part** of z is $\mathrm{Im}(z) = b$.
- Complex numbers z_1 and z_2 are equal if and only if $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.
- An **Argand diagram** is a geometric representation of $\mathbb{C}$.
- The **modulus** of z, denoted by $|z|$, is the distance from the origin to the point representing z in an Argand diagram. Thus $|a + bi| = \sqrt{a^2 + b^2}$.
- The **argument** of z is an angle measured anticlockwise about the origin from the positive direction of the x-axis to the line joining the origin to z.
- The **principal value** of the argument, denoted by $\mathrm{Arg}\, z$, is the angle in the interval $(-\pi, \pi]$.
- The complex number $z = x + yi$ can be expressed in **polar form** as

$$z = r(\cos\theta + i\sin\theta)$$
$$= r\,\mathrm{cis}\,\theta$$

where $r = |z| = \sqrt{x^2 + y^2}$, $\cos\theta = \dfrac{x}{r}$, $\sin\theta = \dfrac{y}{r}$.

This is also called modulus–argument form.



- The **complex conjugate** of $z = a + bi$ is given by $\bar{z} = a - bi$. Note that $z\bar{z} = |z|^2$.
- Division of complex numbers:

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \times \frac{\bar{z}_2}{\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$$

- Multiplication and division in polar form:
  Let $z_1 = r_1 \,\mathrm{cis}\,\theta_1$ and $z_2 = r_2 \,\mathrm{cis}\,\theta_2$. Then

$$z_1 z_2 = r_1 r_2 \,\mathrm{cis}(\theta_1 + \theta_2) \quad \text{and} \quad \frac{z_1}{z_2} = \frac{r_1}{r_2}\,\mathrm{cis}(\theta_1 - \theta_2)$$

Technology-free questions

1 For $z_1 = m + ni$ and $z_2 = p + qi$, express each of the following in the form $a + bi$:

a $2z_1 + 3z_2$ **b** $\bar{z}_2$ **c** $z_1\bar{z}_2$

d $\dfrac{z_1}{z_2}$ **e** $z_1 + \bar{z}_1$ **f** $(z_1 + z_2)(z_1 - z_2)$

g $\dfrac{1}{z_1}$ **h** $\dfrac{z_2}{z_1}$ **i** $\dfrac{3z_1}{z_2}$

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2 Let $z = 1 - \sqrt{3}i$. For each of the following, express in the form $a + bi$ and mark on an Argand diagram:

a z **b** z^2 **c** z^3 **d** $\frac{1}{z}$ **e** $\bar{z}$ **f** $\frac{1}{\bar{z}}$

3 Write each of the following in polar form:

a $1 + i$ **b** $1 - \sqrt{3}i$ **c** $2\sqrt{3} + i$

d $3\sqrt{2} + 3\sqrt{2}i$ **e** $-3\sqrt{2} - 3\sqrt{2}i$ **f** $\sqrt{3} - i$

4 Write each of the following in Cartesian form:

a $-2\operatorname{cis}\left(\frac{\pi}{3}\right)$ **b** $3\operatorname{cis}\left(\frac{\pi}{4}\right)$ **c** $3\operatorname{cis}\left(\frac{3\pi}{4}\right)$

d $-3\operatorname{cis}\left(\frac{-3\pi}{4}\right)$ **e** $3\operatorname{cis}\left(\frac{-5\pi}{6}\right)$ **f** $\sqrt{2}\operatorname{cis}\left(\frac{-\pi}{4}\right)$

5 Let $z = \operatorname{cis}\left(\frac{\pi}{3}\right)$. On an Argand diagram, carefully plot:

a z^2 **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $\operatorname{cis}\left(\frac{2\pi}{3}\right)$

6 Let $z = \operatorname{cis}\left(\frac{\pi}{4}\right)$. On an Argand diagram, carefully plot:

a iz **b** $\bar{z}$ **c** $\frac{1}{z}$ **d** $-iz$

Multiple-choice questions

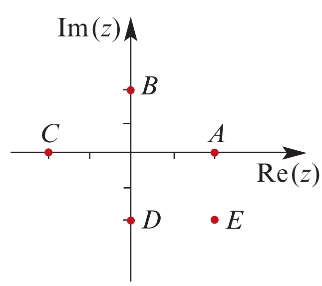
1 If $u = 1 + i$, then $\frac{1}{2 - u}$ is equal to

A $-\frac{1}{2} - \frac{1}{2}i$ **B** $\frac{1}{5} + \frac{2}{5}i$ **C** $\frac{1}{2} + \frac{1}{2}i$ **D** $-\frac{1}{2} + \frac{1}{5}i$ **E** $1 + 5i$

2 The point C on the Argand diagram represents the complex number z. Which point represents the complex number $i \times z$?

A A **B** B **C** C

D D **E** E



3 If $|z| = 5$, then $\left|\frac{1}{z}\right| =$

A $\frac{1}{\sqrt{5}}$ **B** $-\frac{1}{\sqrt{5}}$ **C** $\frac{1}{5}$ **D** $-\frac{1}{5}$ **E** $\sqrt{5}$

4 If $(x + yi)^2 = -32i$ for real values of x and y, then

A $x = 4, y = 4$ **B** $x = -4, y = 4$

C $x = 4, y = -4$ **D** $x = 4, y = -4$ or $x = -4, y = 4$

E $x = 4, y = 4$ or $x = -4, y = -4$

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5 The linear factors of $z^2 + 6z + 10$ over $\mathbb{C}$ are

A $(z + 3 + i)^2$ **B** $(z + 3 - i)^2$ **C** $(z + 3 + i)(z - 3 + i)$
D $(z + 3 - i)(z + 3 + i)$ **E** $(z + 3 + i)(z - 3 - i)$

6 Let $z = \frac{1}{1 - i}$. If $r = |z|$ and $\theta = \text{Arg}\, z$, then

A $r = 2$ and $\theta = \frac{\pi}{4}$ **B** $r = \frac{1}{2}$ and $\theta = \frac{\pi}{4}$ **C** $r = \sqrt{2}$ and $\theta = -\frac{\pi}{4}$
D $r = \frac{1}{\sqrt{2}}$ and $\theta = -\frac{\pi}{4}$ **E** $r = \frac{1}{\sqrt{2}}$ and $\theta = \frac{\pi}{4}$

7 The solution of the equation $\dfrac{z - 2i}{z - (3 - 2i)} = 2$, where $z \in \mathbb{C}$, is

A $z = 6 + 2i$ **B** $z = 6 - 2i$ **C** $z = -6 - 6i$ **D** $z = 6 - 6i$ **E** $z = -6 + 2i$

8 Let $z = a + bi$, where $a, b \in \mathbb{R}$. If $z^2(1 + i) = 2 - 2i$, then the Cartesian form of one value of z could be

A $\sqrt{2i}$ **B** $-\sqrt{2i}$ **C** $-1 - i$ **D** $-1 + i$ **E** $\sqrt{-2}$

9 The value of the discriminant for the quadratic expression $(2 + 2i)z^2 + 8iz - 4(1 - i)$ is

A -32 **B** 0 **C** 64 **D** 32 **E** -64

10 If $\text{Arg}(ai + 1) = \frac{\pi}{6}$, then the real number a is

A $\sqrt{3}$ **B** $-\sqrt{3}$ **C** 1 **D** $\frac{1}{\sqrt{3}}$ **E** $-\frac{1}{\sqrt{3}}$

Extended-response questions

1 **a** Find the exact solutions in $\mathbb{C}$ for the equation $z^2 - 2\sqrt{3}z + 4 = 0$.

b **i** Plot the two solutions from part **a** on an Argand diagram.

ii Find the equation of the circle, with centre the origin, which passes through these two points.

iii Find the value of $a \in \mathbb{Z}$ such that the circle passes through $(0, \pm a)$.

2 Let z be a complex number with $|z| = 6$. Let A be the point representing z and let B be the point representing $(1 + i)z$.

a Find:

i $|(1 + i)z|$

ii $|(1 + i)z - z|$

b Prove that OAB is a right-angled isosceles triangle.

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3 Let $z = \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{2}}i$.

a On an Argand diagram, the points O, A, Z, P and Q represent the complex numbers 0, 1, z, $1 + z$ and $1 - z$ respectively. Show these points on a diagram.

b Prove that the magnitude of $\angle POQ$ is $\dfrac{\pi}{2}$. Find the ratio $\dfrac{OP}{OQ}$.

4 Let z_1 and z_2 be two complex numbers. Prove the following:

a $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\overline{z_2} + \overline{z_1}z_2$

b $|z_1 - z_2|^2 = |z_1|^2 + |z_2|^2 - (z_1\overline{z_2} + \overline{z_1}z_2)$

c $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

State a geometric theorem from the result of **c**.

5 Let z_1 and z_2 be two complex numbers.

a Prove the following:

i $\overline{\overline{z_1}z_2} = z_1\overline{z_2}$

ii $z_1\overline{z_2} + \overline{z_1}z_2$ is a real number

iii $z_1\overline{z_2} - \overline{z_1}z_2$ is an imaginary number

iv $(z_1\overline{z_2} + \overline{z_1}z_2)^2 - (z_1\overline{z_2} - \overline{z_1}z_2)^2 = 4|z_1z_2|^2$

b Use the results from part **a** and Question 4 to prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.
Hint: Show that $(|z_1| + |z_2|)^2 - |z_1 + z_2|^2 \geq 0$.

c Hence prove that $|z_1 - z_2| \geq |z_1| - |z_2|$.

6 Assume that $|z| = 1$ and that the argument of z is θ, where $0 < \theta < \pi$. Find the modulus and argument of:

a $z + 1$ **b** $z - 1$ **c** $\dfrac{z - 1}{z + 1}$

7 The quadratic expression $ax^2 + bx + c$ has real coefficients.

a Find the discriminant of $ax^2 + bx + c$.

b Find the condition in terms of a, b and c for which the equation $ax^2 + bx + c = 0$ has no real solutions.

c If this condition is fulfilled, let z_1 and z_2 be the complex solutions of the equation and let P_1 and P_2 the corresponding points on an Argand diagram.

i Find $z_1 + z_2$ and $|z_1|$ in terms of a, b and c.

ii Find $\cos(\angle P_1OP_2)$ in terms of a, b and c.

8 Let z_1 and z_2 be the solutions of the quadratic equation $z^2 + z + 1 = 0$.

a Find z_1 and z_2.

b Prove that $z_1 = z_2^2$ and $z_2 = z_1^2$.

c Find the modulus and the principal value of the argument of z_1 and z_2.

d Let P_1 and P_2 be the points on an Argand diagram corresponding to z_1 and z_2. Find the area of triangle P_1OP_2.

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