Chapter 20

20 Vectors

Objectives

- To understand the concept of a **vector** and to apply the basic operations on vectors.
- To recognise when two vectors are **parallel**.
- To use the unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ to represent vectors in two dimensions.
- To find the **scalar product** of two vectors.
- To use the scalar product to find the magnitude of the angle between two vectors.
- To use the scalar product to recognise when two vectors are **perpendicular**.
- To resolve a vector into **rectangular components**, where one component is parallel to a given vector and the other component is perpendicular.
- To use the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ to represent vectors in three dimensions.

In scientific experiments, some of the things that are measured are completely determined by their magnitude. Mass, length and time are determined by a number and an appropriate unit of measurement.

length	30 cm is the length of the page of a particular book
time	10 s is the time for one athlete to run 100 m

More is required to describe velocity, displacement or force. The direction must be recorded as well as the magnitude.

displacement	30 km in a direction north
velocity	60 km/h in a direction south-east

A quantity that has both a magnitude and a direction is called a **vector**. Our study of vectors will tie together different ideas from previous chapters, including geometry, trigonometry, complex numbers and transformations.

ISBN 978-1-107-56765-8

20A Introduction to vectors

Suppose that you are asked: ‘Where is your school in relation to your house?’

It is not enough to give an answer such as ‘four kilometres’. You need to specify a direction as well as a distance. You could give the answer ‘four kilometres north-east’.

Position is an example of a vector quantity.

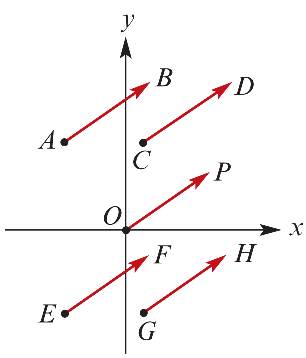
Directed line segments

A quantity that has a direction as well as a magnitude can be represented by an arrow:

- the arrow points in the direction of the action
- the length of the arrow gives the magnitude of the quantity in terms of a suitably chosen unit.

Arrows with the same length and direction are regarded as equivalent. These arrows are **directed line segments** and the sets of equivalent segments are called **vectors**.

The five directed line segments shown all have the same length and direction, and so they are equivalent.



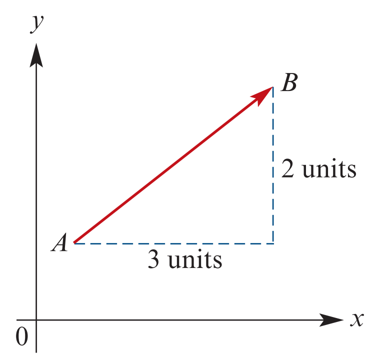
A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.

For simplicity of language, this is also called vector $\overrightarrow{AB}$. That is, the set of equivalent segments can be named through one member of the set.

Note: The five directed line segments in the diagram all name the same vector: $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF} = \overrightarrow{GH}$.

Column vectors

In Chapter 19, we introduced vectors in the context of translations of the plane. We represented each translation by a column of numbers, which was called a vector.



This is consistent with the approach here, as the column of numbers corresponds to a set of equivalent directed line segments.

For example, the column $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ corresponds to the directed line segments which go 3 across and 2 up.

Vector notation

A vector is often denoted by a single bold lowercase letter. The vector from A to B can be denoted by $\overrightarrow{AB}$ or by a single letter $\boldsymbol{v}$. That is, $\boldsymbol{v} = \overrightarrow{AB}$.

When a vector is handwritten, the notation is $\underset{\sim}{v}$.

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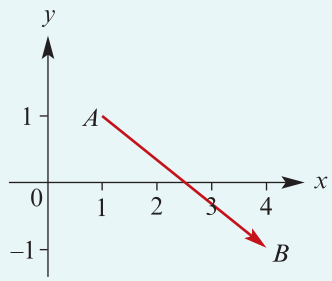
Example 1

Draw a directed line segment corresponding to $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Solution



Explanation

The vector $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$ is '3 across to the right and 2 down'.

Note: Here the segment starts at (1, 1) and goes to (4, −1). It can start at any point.

Example 2

The vector $\boldsymbol{u}$ is defined by the directed line segment from (2, 6) to (3, 1).

If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$, find a and b.

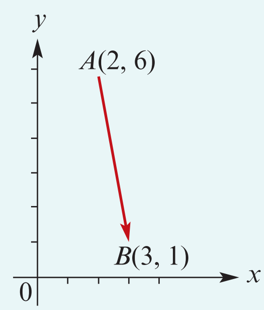
Solution

The vector is

$$\boldsymbol{u} = \begin{bmatrix} 3-2 \\ 1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Hence $a = 1$ and $b = -5$.

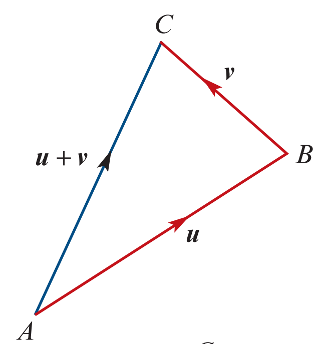
Explanation



Addition of vectors

Adding vectors geometrically

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ can be added geometrically by drawing a line segment representing $\boldsymbol{u}$ from A to B and then a line segment representing $\boldsymbol{v}$ from B to C.

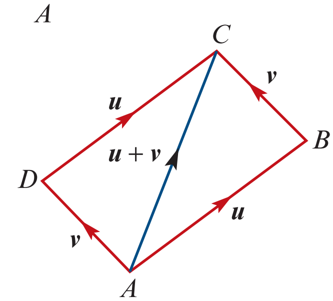


The sum $\boldsymbol{u} + \boldsymbol{v}$ is the vector from A to C. That is,

$$\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AC}$$

The same result is achieved if the order is reversed. This is represented in the diagram on the right:

$$\begin{aligned} \boldsymbol{u} + \boldsymbol{v} &= \overrightarrow{AC} \\ &= \boldsymbol{v} + \boldsymbol{u} \end{aligned}$$



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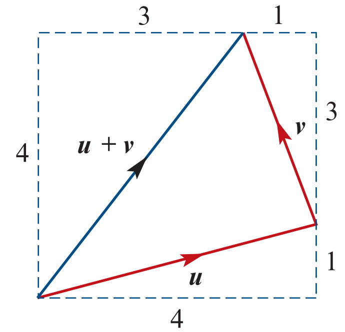
Adding column vectors

Two vectors can be added using column-vector notation.



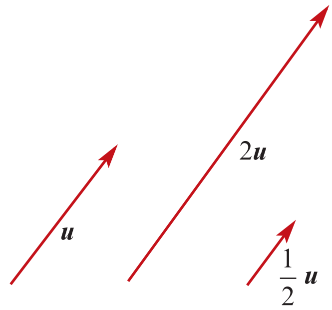
For example, if $\boldsymbol{u} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then

$$\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

▶ Scalar multiplication

Multiplication by a real number (scalar) changes the length of the vector. For example:

- $2\boldsymbol{u}$ is twice the length of $\boldsymbol{u}$
- $\frac{1}{2}\boldsymbol{u}$ is half the length of $\boldsymbol{u}$

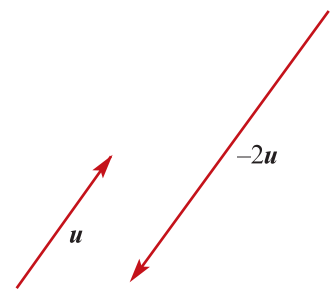


We have $2\boldsymbol{u} = \boldsymbol{u} + \boldsymbol{u}$ and $\frac{1}{2}\boldsymbol{u} + \frac{1}{2}\boldsymbol{u} = \boldsymbol{u}$.

In general, for $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.

When a vector is multiplied by -2, the vector's direction is reversed and the length is doubled.

When a vector is multiplied by -1, the vector's direction is reversed and the length remains the same.



If $\boldsymbol{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, then $-\boldsymbol{u} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$, $2\boldsymbol{u} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$ and $-2\boldsymbol{u} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$.

If $\boldsymbol{u} = \overrightarrow{AB}$, then

$$-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$$

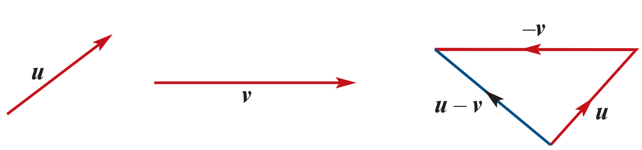
The directed line segment $-\overrightarrow{AB}$ goes from B to A.

▶ Zero vector

The **zero vector** is denoted by $\mathbf{0}$ and represents a line segment of zero length. The zero vector has no direction.

▶ Subtraction of vectors

To find $\boldsymbol{u} - \boldsymbol{v}$, we add $-\boldsymbol{v}$ to $\boldsymbol{u}$.



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Example 3

For the vectors $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, find $2\boldsymbol{u} + 3\boldsymbol{v}$.

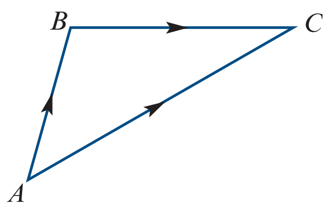
Solution

$$\begin{aligned} 2\boldsymbol{u} + 3\boldsymbol{v} &= 2\begin{bmatrix} 3 \\ -1 \end{bmatrix} + 3\begin{bmatrix} -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ -2 \end{bmatrix} + \begin{bmatrix} -6 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 4 \end{bmatrix} \end{aligned}$$

▶ Polygons of vectors

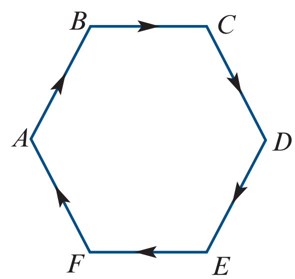
- For two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$



- For a polygon $ABCDEF$, we have

  $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DE} + \overrightarrow{EF} + \overrightarrow{FA} = \mathbf{0}$

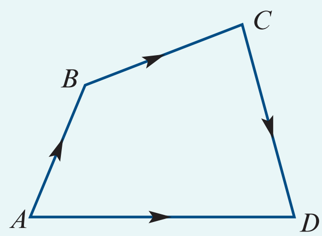


Example 4

Illustrate the vector sum $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, where A, B, C and D are points in the plane.

Solution

$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{AD}$



▶ Parallel vectors

Two parallel vectors have the same direction or opposite directions.

Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

For example, if $\boldsymbol{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$, then the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel as $\boldsymbol{v} = 3\boldsymbol{u}$.

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► Position vectors

We can use a point O, the origin, as a starting point for a vector to indicate the position of a point A in space relative to O.

For most of this chapter, we study vectors in two dimensions and the point O is the origin of the Cartesian plane. (Vectors in three dimensions are studied in Section 20F.)

For a point A, the **position vector** is $\overrightarrow{OA}$.

► Linear combinations of non-parallel vectors

If two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are not parallel, then

$$m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b} \quad \text{implies} \quad m = p \text{ and } n = q$$

Proof Assume that $m\boldsymbol{a} + n\boldsymbol{b} = p\boldsymbol{a} + q\boldsymbol{b}$. Then

$$m\boldsymbol{a} - p\boldsymbol{a} = q\boldsymbol{b} - n\boldsymbol{b}$$

$$\therefore \quad (m - p)\boldsymbol{a} = (q - n)\boldsymbol{b}$$

If $m \neq p$ or $n \neq q$, we could therefore write

$$\boldsymbol{a} = \frac{q-n}{m-p}\boldsymbol{b} \quad \text{or} \quad \boldsymbol{b} = \frac{m-p}{q-n}\boldsymbol{a}$$

But this is not possible, as $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors that are not parallel. Therefore $m = p$ and $n = q$.

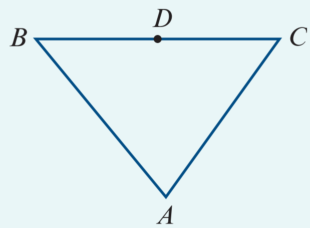
Example 5

Let A, B and C be the vertices of a triangle, and let D be the midpoint of BC.

Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$.

Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BD}$ **b** $\overrightarrow{DC}$ **c** $\overrightarrow{AC}$
d $\overrightarrow{AD}$ **e** $\overrightarrow{CA}$



Solution

a $\overrightarrow{BD} = \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\boldsymbol{b}$

b $\overrightarrow{DC} = \overrightarrow{BD} = \frac{1}{2}\boldsymbol{b}$

c $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \boldsymbol{a} + \boldsymbol{b}$

d $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \boldsymbol{a} + \frac{1}{2}\boldsymbol{b}$

e $\overrightarrow{CA} = -\overrightarrow{AC} = -(\boldsymbol{a} + \boldsymbol{b})$

Explanation

a: same direction and half the length

b: equivalent vectors

e: since $\overrightarrow{CA} + \overrightarrow{AC} = \boldsymbol{0}$

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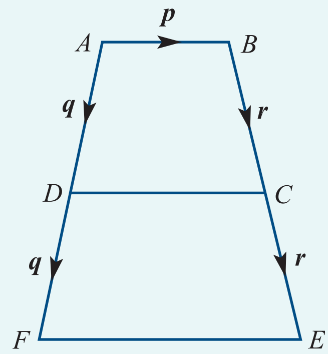
Example 6

In the figure, $\overrightarrow{DC} = k\boldsymbol{p}$ where $k \in \mathbb{R} \setminus \{0\}$.

a Express $\boldsymbol{p}$ in terms of k, $\boldsymbol{q}$ and $\boldsymbol{r}$.

b Express $\overrightarrow{FE}$ in terms of k and $\boldsymbol{p}$ to show that FE is parallel to DC.

c If $\overrightarrow{FE} = 4\overrightarrow{AB}$, find the value of k.



Solution

a $\boldsymbol{p} = \overrightarrow{AB}$

$= \overrightarrow{AD} + \overrightarrow{DC} + \overrightarrow{CB}$

$= \boldsymbol{q} + k\boldsymbol{p} - \boldsymbol{r}$

Therefore

$(1 - k)\boldsymbol{p} = \boldsymbol{q} - \boldsymbol{r}$

b $\overrightarrow{FE} = -2\boldsymbol{q} + \boldsymbol{p} + 2\boldsymbol{r}$

$= 2(\boldsymbol{r} - \boldsymbol{q}) + \boldsymbol{p}$

From part **a**, we have

$\boldsymbol{r} - \boldsymbol{q} = k\boldsymbol{p} - \boldsymbol{p}$

$= (k - 1)\boldsymbol{p}$

Therefore

$\overrightarrow{FE} = 2(k - 1)\boldsymbol{p} + \boldsymbol{p}$

$= 2k\boldsymbol{p} - 2\boldsymbol{p} + \boldsymbol{p}$

$= (2k - 1)\boldsymbol{p}$

c $\overrightarrow{FE} = 4\overrightarrow{AB}$

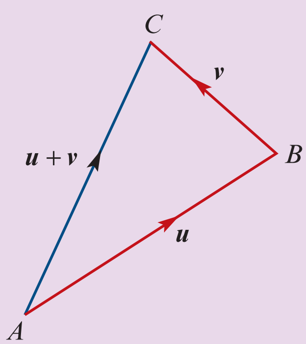
$(2k - 1)\boldsymbol{p} = 4\boldsymbol{p}$

$2k - 1 = 4$

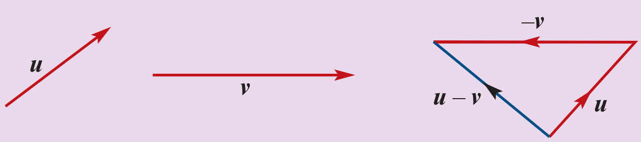
$\therefore \quad k = \dfrac{5}{2}$

Section summary

- A **vector** is a set of equivalent **directed line segments**.
- Addition of vectors
  If $\boldsymbol{u} = \overrightarrow{AB}$ and $\boldsymbol{v} = \overrightarrow{BC}$, then $\boldsymbol{u} + \boldsymbol{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.



- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - If $\boldsymbol{u} = \overrightarrow{AB}$, then $-\boldsymbol{u} = -\overrightarrow{AB} = \overrightarrow{BA}$.
- Zero vector
  The **zero vector**, denoted by $\boldsymbol{0}$, has zero length and has no direction.
- Subtraction of vectors
  $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$



- Parallel vectors
  Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there is some $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.

ISBN 978-1-107-56765-8

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Updated May 2021

Exercise 20A

Skillsheet

Example 1

1 On the same graph, draw arrows which represent the following vectors:

a $\begin{bmatrix}1\\5\end{bmatrix}$ **b** $\begin{bmatrix}0\\-2\end{bmatrix}$ **c** $\begin{bmatrix}-1\\-2\end{bmatrix}$ **d** $\begin{bmatrix}-4\\3\end{bmatrix}$

Example 2

2 The vector $\boldsymbol{u}$ is defined by the directed line segment from $(1, 5)$ to $(6, 6)$.
If $\boldsymbol{u} = \begin{bmatrix}a\\b\end{bmatrix}$, find a and b.

3 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(-1, 5)$ to $(2, -10)$.
If $\boldsymbol{v} = \begin{bmatrix}a\\b\end{bmatrix}$, find a and b.

4 Let $A = (1, -2)$, $B = (3, 0)$ and $C = (2, -3)$ and let O be the origin.
Express each of the following vectors in the form $\begin{bmatrix}a\\b\end{bmatrix}$:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

Example 3

5 Let $\boldsymbol{a} = \begin{bmatrix}1\\2\end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix}1\\-3\end{bmatrix}$ and $\boldsymbol{c} = \begin{bmatrix}-2\\1\end{bmatrix}$.

a Find:

i $\boldsymbol{a} + \boldsymbol{b}$ **ii** $2\boldsymbol{c} - \boldsymbol{a}$ **iii** $\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}$

b Show that $\boldsymbol{a} + \boldsymbol{b}$ is parallel to $\boldsymbol{c}$.

Example 4

6 If $A = (2, -3)$, $B = (4, 0)$, $C = (1, -4)$ and O is the origin, sketch the following vectors:

a $\overrightarrow{OA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{CO}$ **e** $\overrightarrow{CB}$

7 On graph paper, sketch the vectors joining the following pairs of points in the direction indicated:

a $(0, 0) \to (2, 1)$ **b** $(3, 4) \to (0, 0)$ **c** $(1, 3) \to (3, 4)$
d $(2, 4) \to (4, 3)$ **e** $(-2, 2) \to (5, -1)$ **f** $(-1, -3) \to (3, 0)$

8 Identify vectors from Question 7 which are parallel to each other.

9 **a** Plot the points $A(-1, 0)$, $B(1, 4)$, $C(4, 3)$ and $D(2, -1)$ on a set of coordinate axes.

b Sketch the vectors $\overrightarrow{AB}$, $\overrightarrow{BC}$, $\overrightarrow{AD}$ and $\overrightarrow{DC}$.

c Show that:

i $\overrightarrow{AB} = \overrightarrow{DC}$ **ii** $\overrightarrow{BC} = \overrightarrow{AD}$

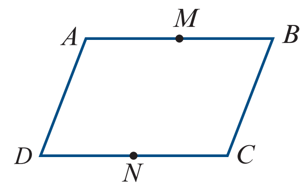
d Describe the shape of the quadrilateral $ABCD$.

10 Find the values of m and n such that $m\begin{bmatrix}3\\-3\end{bmatrix} + n\begin{bmatrix}2\\4\end{bmatrix} = \begin{bmatrix}-19\\61\end{bmatrix}$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

11 Points A, B, C, D are the vertices of a parallelogram, and M and N are the midpoints of AB and DC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AD}$.

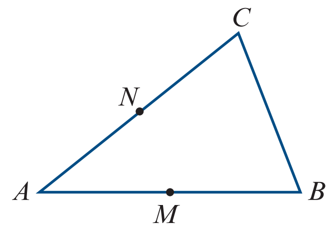


a Express the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{MD}$ **ii** $\overrightarrow{MN}$

b Find the relationship between $\overrightarrow{MN}$ and $\overrightarrow{AD}$.

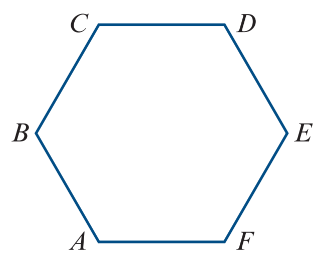
Example 5 **12** The figure represents the triangle ABC, where M and N are the midpoints of AB and AC respectively. Let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{AC}$.



a Express $\overrightarrow{CB}$ and $\overrightarrow{MN}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Hence describe the relation between the two vectors (or directed line segments).

Example 6 **13** The figure shows a regular hexagon $ABCDEF$. Let $\boldsymbol{a} = \overrightarrow{AF}$ and $\boldsymbol{b} = \overrightarrow{AB}$. Express the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:



a $\overrightarrow{CD}$ **b** $\overrightarrow{ED}$ **c** $\overrightarrow{BE}$ **d** $\overrightarrow{FC}$

e $\overrightarrow{FA}$ **f** $\overrightarrow{FB}$ **g** $\overrightarrow{FE}$

14 In parallelogram $ABCD$, let $\boldsymbol{a} = \overrightarrow{AB}$ and $\boldsymbol{b} = \overrightarrow{BC}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{DC}$ **b** $\overrightarrow{DA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{CA}$ **e** $\overrightarrow{BD}$

15 In triangle OAB, let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{b} = \overrightarrow{OB}$. The point P on AB is such that $\overrightarrow{AP} = 2\overrightarrow{PB}$ and the point Q is such that $\overrightarrow{OP} = 3\overrightarrow{PQ}$. Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

a $\overrightarrow{BA}$ **b** $\overrightarrow{PB}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{BQ}$

16 $PQRS$ is a quadrilateral in which $\overrightarrow{PQ} = \boldsymbol{u}$, $\overrightarrow{QR} = \boldsymbol{v}$ and $\overrightarrow{RS} = \boldsymbol{w}$. Express each of the following vectors in terms of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$:

a $\overrightarrow{PR}$ **b** $\overrightarrow{QS}$ **c** $\overrightarrow{PS}$

17 $OABC$ is a parallelogram. Let $\boldsymbol{u} = \overrightarrow{OA}$ and $\boldsymbol{v} = \overrightarrow{OC}$. Let M be the midpoint of AB.

a Express $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

b Express $\overrightarrow{CM}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

c If P is a point on CM and $\overrightarrow{CP} = \frac{2}{3}\overrightarrow{CM}$, express $\overrightarrow{CP}$ in terms of $\boldsymbol{u}$ and $\boldsymbol{v}$.

d Find $\overrightarrow{OP}$ and hence show that P lies on the line segment OB.

e Find the ratio $OP : PB$.

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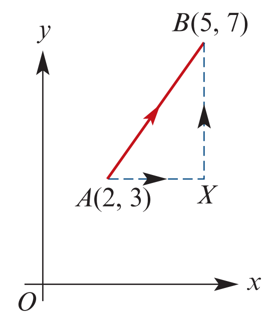
20B Components of vectors

The vector $\overrightarrow{AB}$ in the diagram is described by the column vector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.



From the diagram, we see that the vector $\overrightarrow{AB}$ can also be expressed as the sum

$$\overrightarrow{AB} = \overrightarrow{AX} + \overrightarrow{XB}$$

Using column-vector notation:

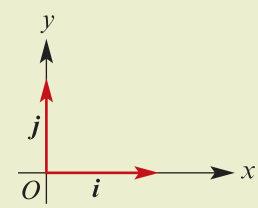
$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

This suggests the introduction of two important vectors.

Standard unit vectors in two dimensions

- Let $\boldsymbol{i}$ be the vector of unit length in the positive direction of the x-axis.
- Let $\boldsymbol{j}$ be the vector of unit length in the positive direction of the y-axis.

Using column-vector notation, we have $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



Note: These two vectors also played an important role in our study of linear transformations using matrices in Chapter 19.

For the example above, we have $\overrightarrow{AX} = 3\boldsymbol{i}$ and $\overrightarrow{XB} = 4\boldsymbol{j}$. Therefore

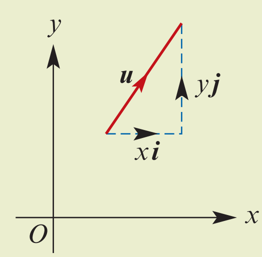
$$\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$$

It is possible to describe any two-dimensional vector in this way.

Component form

- We can write the vector $\boldsymbol{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$.

  We say that $\boldsymbol{u}$ is the sum of the two **components** $x\boldsymbol{i}$ and $y\boldsymbol{j}$.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is denoted by $|\boldsymbol{u}|$ and is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.



Operations with vectors now look more like basic algebra:

- $(x\boldsymbol{i} + y\boldsymbol{j}) + (m\boldsymbol{i} + n\boldsymbol{j}) = (x + m)\boldsymbol{i} + (y + n)\boldsymbol{j}$
- $k(x\boldsymbol{i} + y\boldsymbol{j}) = kx\boldsymbol{i} + ky\boldsymbol{j}$

Two vectors are equal if and only if their components are equal:

$$x\boldsymbol{i} + y\boldsymbol{j} = m\boldsymbol{i} + n\boldsymbol{j} \quad \text{if and only if} \quad x = m \text{ and } y = n$$

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Example 7

a Find $\overrightarrow{AB}$ if $\overrightarrow{OA} = 3\boldsymbol{i}$ and $\overrightarrow{OB} = 2\boldsymbol{i} - \boldsymbol{j}$.

b Find $|2\boldsymbol{i} - 3\boldsymbol{j}|$.

Solution

a
$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\\ &= -3\boldsymbol{i} + (2\boldsymbol{i} - \boldsymbol{j})\\ &= -\boldsymbol{i} - \boldsymbol{j}\end{aligned}$$

b
$$\begin{aligned}|2\boldsymbol{i} - 3\boldsymbol{j}| &= \sqrt{2^2 + (-3)^2}\\ &= \sqrt{4+9}\\ &= \sqrt{13}\end{aligned}$$

Example 8

Let A and B be points on the Cartesian plane such that $\overrightarrow{OA} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 3\boldsymbol{j}$.

Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

Solution

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB}\\ &= -\overrightarrow{OA} + \overrightarrow{OB}\end{aligned}$$

$$\begin{aligned}\therefore \quad \overrightarrow{AB} &= -(2\boldsymbol{i} + \boldsymbol{j}) + \boldsymbol{i} - 3\boldsymbol{j}\\ &= -\boldsymbol{i} - 4\boldsymbol{j}\end{aligned}$$

$$\therefore \quad |\overrightarrow{AB}| = \sqrt{1+16} = \sqrt{17}$$

Unit vectors

A **unit vector** is a vector of length one unit. For example, both $\boldsymbol{i}$ and $\boldsymbol{j}$ are unit vectors.

The unit vector in the direction of $\boldsymbol{a}$ is denoted by $\hat{\boldsymbol{a}}$. (We say ‘a hat’.)

Since $|\hat{\boldsymbol{a}}| = 1$, we have

$$|\boldsymbol{a}|\,\hat{\boldsymbol{a}} = \boldsymbol{a}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a}$$

Example 9

Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$.

Find $|\boldsymbol{a}|$, the magnitude of $\boldsymbol{a}$, and hence find the unit vector in the direction of $\boldsymbol{a}$.

Solution

$$|\boldsymbol{a}| = \sqrt{9+16} = 5$$

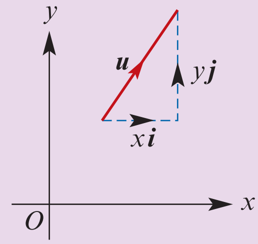
$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{|\boldsymbol{a}|}\boldsymbol{a} = \frac{1}{5}(3\boldsymbol{i} + 4\boldsymbol{j})$$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Section summary

- A **unit vector** is a vector of length one unit.
- Each vector $\boldsymbol{u}$ in the plane can be written in **component form** as $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.



- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Exercise 20B

Skillsheet
Example 7a

1 If A and B are points in the plane such that $\overrightarrow{OA} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\overrightarrow{OB} = 3\boldsymbol{i} - 5\boldsymbol{j}$, find $\overrightarrow{AB}$.

2 $OAPB$ is a rectangle with $\overrightarrow{OA} = 5\boldsymbol{i}$ and $\overrightarrow{OB} = 6\boldsymbol{j}$. Express each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

a $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BA}$

Example 7b

3 Determine the magnitude of each of the following vectors:

a $5\boldsymbol{i}$ **b** $-2\boldsymbol{j}$ **c** $3\boldsymbol{i} + 4\boldsymbol{j}$ **d** $-5\boldsymbol{i} + 12\boldsymbol{j}$

4 The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are given by $\boldsymbol{u} = 7\boldsymbol{i} + 8\boldsymbol{j}$ and $\boldsymbol{v} = 2\boldsymbol{i} - 4\boldsymbol{j}$.

a Find $|\boldsymbol{u} - \boldsymbol{v}|$. **b** Find constants x and y such that $x\boldsymbol{u} + y\boldsymbol{v} = 44\boldsymbol{j}$.

5 Points A and B have position vectors $\overrightarrow{OA} = 10\boldsymbol{i}$ and $\overrightarrow{OB} = 4\boldsymbol{i} + 5\boldsymbol{j}$. If M is the midpoint of AB, find $\overrightarrow{OM}$ in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$.

6 $OPAQ$ is a rectangle with $\overrightarrow{OP} = 2\boldsymbol{i}$ and $\overrightarrow{OQ} = \boldsymbol{j}$. Let M be the point on OP such that $OM = \frac{1}{5}OP$ and let N be the point on MQ such that $MN = \frac{1}{6}MQ$.

a Find each of the following vectors in terms of $\boldsymbol{i}$ and $\boldsymbol{j}$:

i $\overrightarrow{OM}$ **ii** $\overrightarrow{MQ}$ **iii** $\overrightarrow{MN}$ **iv** $\overrightarrow{ON}$ **v** $\overrightarrow{OA}$

b **i** Hence show that N is on the diagonal OA.

ii State the ratio of the lengths $ON : NA$.

7 The position vectors of A and B are given by $\overrightarrow{OA} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\overrightarrow{OB} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$. Find the distance between A and B.

8 Find the pronumerals in the following equations:

a $2 + 3\boldsymbol{j} = 2(\ell\boldsymbol{i} + k\boldsymbol{j})$ **b** $(x - 1)\boldsymbol{i} + y\boldsymbol{j} = 5\boldsymbol{i} + (x - 4)\boldsymbol{j}$

c $(x + y)\boldsymbol{i} + (x - y)\boldsymbol{j} = 6\boldsymbol{i}$ **d** $k(\boldsymbol{i} + \boldsymbol{j}) = 3\boldsymbol{i} - 2\boldsymbol{j} + \ell(2\boldsymbol{i} - \boldsymbol{j})$

Example 8

9 Let $A = (2, 3)$ and $B = (5, 1)$. Find $\overrightarrow{AB}$ and $|\overrightarrow{AB}|$.

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10 Let $\overrightarrow{OA} = 3\boldsymbol{i}$, $\overrightarrow{OB} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{OC} = -3\boldsymbol{i} + \boldsymbol{j}$. Find:

a $\overrightarrow{AB}$ **b** $\overrightarrow{AC}$ **c** $|\overrightarrow{BC}|$

11 Let $A = (5, 1)$, $B = (0, 4)$ and $C = (-1, 0)$. Find:

a D such that $\overrightarrow{AB} = \overrightarrow{CD}$ **b** F such that $\overrightarrow{AF} = \overrightarrow{BC}$ **c** G such that $\overrightarrow{AB} = 2\overrightarrow{GC}$

12 Let $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = -2\boldsymbol{i} + 2\boldsymbol{j}$. Points A, B and C are such that $\overrightarrow{AO} = \boldsymbol{a}$, $\overrightarrow{OB} = \boldsymbol{b}$ and $\overrightarrow{BC} = 2\boldsymbol{a}$, where O is the origin. Find the coordinates of A, B and C.

13 A, B, C and D are the vertices of a parallelogram and O is the origin. $A = (2, -1)$, $B = (-5, 4)$ and $C = (1, 7)$.

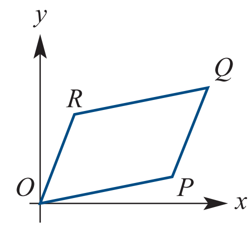
a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{OC}$ **iv** $\overrightarrow{BC}$ **v** $\overrightarrow{AD}$

b Hence find the coordinates of D.

14 The diagram shows a parallelogram $OPQR$. The points P and Q have coordinates $(12, 5)$ and $(18, 13)$ respectively. Find:

a $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ **b** $|\overrightarrow{RQ}|$ and $|\overrightarrow{OR}|$



15 $A(1, 6)$, $B(3, 1)$ and $C(13, 5)$ are the vertices of a triangle ABC.

a Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

b Hence show that ABC is a right-angled triangle.

16 $A(4, 4)$, $B(3, 1)$ and $C(7, 3)$ are the vertices of a triangle ABC.

a Find the vectors:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CA}$

b Find:

i $|\overrightarrow{AB}|$ **ii** $|\overrightarrow{BC}|$ **iii** $|\overrightarrow{CA}|$

c Hence show that triangle ABC is a right-angled isosceles triangle.

17 $A(-3, 2)$ and $B(0, 7)$ are points on the Cartesian plane, O is the origin and M is the midpoint of AB.

a Find:

i $\overrightarrow{OA}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{BA}$ **iv** $\overrightarrow{BM}$

b Hence find the coordinates of M. (Hint: $\overrightarrow{OM} = \overrightarrow{OB} + \overrightarrow{BM}$.)

Example 9 **18** Find the unit vector in the direction of each of the following vectors:

a $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ **b** $\boldsymbol{b} = 3\boldsymbol{i} - \boldsymbol{j}$ **c** $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{j}$

d $\boldsymbol{d} = \boldsymbol{i} - \boldsymbol{j}$ **e** $\boldsymbol{e} = \frac{1}{2}\boldsymbol{i} + \frac{1}{3}\boldsymbol{j}$ **f** $\boldsymbol{f} = 6\boldsymbol{i} - 4\boldsymbol{j}$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

20C Scalar product of vectors

The scalar product is an operation that takes two vectors and gives a real number.

Definition of the scalar product

We define the **scalar product** of two vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$$

For example:

$$(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} - 4\boldsymbol{j}) = 2 \times 1 + 3 \times (-4) = -10$$

The scalar product is often called the **dot product**.

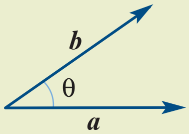
Note: If $\boldsymbol{a} = \boldsymbol{0}$ or $\boldsymbol{b} = \boldsymbol{0}$, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Geometric description of the scalar product

For vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$$

where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



Proof Let $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$. Then using the cosine rule in $\triangle OAB$ gives

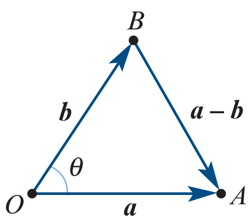
$$|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = |\boldsymbol{a} - \boldsymbol{b}|^2$$

$$(a_1^2 + a_2^2) + (b_1^2 + b_2^2) - 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta = (a_1 - b_1)^2 + (a_2 - b_2)^2$$

$$2(a_1b_1 + a_2b_2) = 2|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$a_1b_1 + a_2b_2 = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$

$$\therefore \quad \boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$$



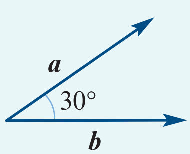
Example 10

a If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $30°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

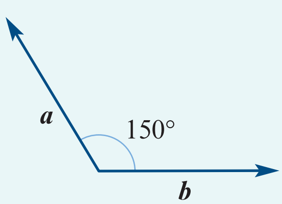
b If $|\boldsymbol{a}| = 4$, $|\boldsymbol{b}| = 5$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is $150°$, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

Solution

a
$$\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 30° = 20 \times \frac{\sqrt{3}}{2} = 10\sqrt{3}$$



b
$$\boldsymbol{a} \cdot \boldsymbol{b} = 4 \times 5 \times \cos 150° = 20 \times \frac{-\sqrt{3}}{2} = -10\sqrt{3}$$



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Cambridge University Press
Updated May 2021

Properties of the scalar product

- $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$
- $k(\boldsymbol{a} \cdot \boldsymbol{b}) = (k\boldsymbol{a}) \cdot \boldsymbol{b} = \boldsymbol{a} \cdot (k\boldsymbol{b})$
- $\boldsymbol{a} \cdot \boldsymbol{0} = 0$
- $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} \cdot \boldsymbol{b} + \boldsymbol{a} \cdot \boldsymbol{c}$
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$
- If the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular, then $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- If $\boldsymbol{a} \cdot \boldsymbol{b} = 0$ for non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, then the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular.
- For parallel vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, we have

$$\boldsymbol{a} \cdot \boldsymbol{b} = \begin{cases} |\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in the same direction} \\ -|\boldsymbol{a}|\,|\boldsymbol{b}| & \text{if } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel and in opposite directions} \end{cases}$$

Finding the magnitude of the angle between two vectors

The angle between two vectors can be found by using the two forms of the scalar product:

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta \quad \text{and} \quad \boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$$

Therefore

$$\cos\theta = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|\,|\boldsymbol{b}|} = \frac{a_1 b_1 + a_2 b_2}{|\boldsymbol{a}|\,|\boldsymbol{b}|}$$

Example 11

A, B and C are points defined by the position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively, where

$$\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}, \quad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} \quad \text{and} \quad \boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$$

Find the magnitude of $\angle ABC$.

Solution

$\angle ABC$ is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b} = -\boldsymbol{i} + 2\boldsymbol{j}$$

$$\overrightarrow{BC} = \boldsymbol{c} - \boldsymbol{b} = -\boldsymbol{i} - 3\boldsymbol{j}$$

We will apply the scalar product:

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = |\overrightarrow{BA}|\,|\overrightarrow{BC}| \cos(\angle ABC)$$

We have

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = (-\boldsymbol{i} + 2\boldsymbol{j}) \cdot (-\boldsymbol{i} - 3\boldsymbol{j}) = 1 - 6 = -5$$

$$|\overrightarrow{BA}| = \sqrt{1 + 4} = \sqrt{5}$$

$$|\overrightarrow{BC}| = \sqrt{1 + 9} = \sqrt{10}$$

Therefore

$$\cos(\angle ABC) = \frac{\overrightarrow{BA} \cdot \overrightarrow{BC}}{|\overrightarrow{BA}|\,|\overrightarrow{BC}|} = \frac{-5}{\sqrt{5}\sqrt{10}} = \frac{-1}{\sqrt{2}}$$

Hence $\angle ABC = \dfrac{3\pi}{4}$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

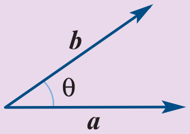
Section summary

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

  $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 b_1 + a_2 b_2$
- The scalar product can be described geometrically by

  $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}| \cos\theta$

  where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.



- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Exercise 20C

Skillsheet

1 Let $\boldsymbol{a} = \boldsymbol{i} - 4\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

2 Let $\boldsymbol{a} = 2\boldsymbol{i} - \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 3\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{a} \cdot \boldsymbol{b}$

d $\boldsymbol{a} \cdot \boldsymbol{c}$ **e** $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b})$

Example 10

3 **a** If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 45°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

b If $|\boldsymbol{a}| = 5$, $|\boldsymbol{b}| = 6$ and the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is 135°, find $\boldsymbol{a} \cdot \boldsymbol{b}$.

4 Expand and simplify:

a $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (\boldsymbol{a} + 2\boldsymbol{b})$ **b** $|\boldsymbol{a} + \boldsymbol{b}|^2 - |\boldsymbol{a} - \boldsymbol{b}|^2$

c $\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{b} \cdot (\boldsymbol{a} + \boldsymbol{b})$ **d** $\dfrac{\boldsymbol{a} \cdot (\boldsymbol{a} + \boldsymbol{b}) - \boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{a}|}$

5 If A and B are points defined by the position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ respectively, find:

a $\overrightarrow{AB}$

b $|\overrightarrow{AB}|$

c the magnitude of the angle between vectors $\overrightarrow{AB}$ and $\boldsymbol{a}$.

6 Let C and D be points with position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$ respectively. If $|\boldsymbol{c}| = 5$, $|\boldsymbol{d}| = 7$ and $\boldsymbol{c} \cdot \boldsymbol{d} = 4$, find $|\overrightarrow{CD}|$.

7 Solve each of the following equations:

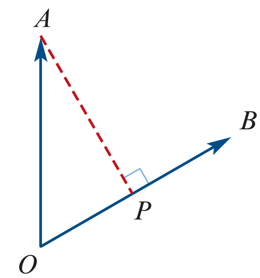
a $(\boldsymbol{i} + 2\boldsymbol{j}) \cdot (5\boldsymbol{i} + x\boldsymbol{j}) = -6$ **b** $(x\boldsymbol{i} + 7\boldsymbol{j}) \cdot (-4\boldsymbol{i} + x\boldsymbol{j}) = 10$

c $(x\boldsymbol{i} + \boldsymbol{j}) \cdot (-2\boldsymbol{i} - 3\boldsymbol{j}) = x$ **d** $x(2\boldsymbol{i} + 3\boldsymbol{j}) \cdot (\boldsymbol{i} + x\boldsymbol{j}) = 6$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

8 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 5\boldsymbol{j}$. Let P be the point on OB such that AP is perpendicular to OB. Then $\overrightarrow{OP} = q\boldsymbol{b}$, for a constant q.



a Express $\overrightarrow{AP}$ in terms of q, $\boldsymbol{a}$ and $\boldsymbol{b}$.

b Use the fact that $\overrightarrow{AP} \cdot \overrightarrow{OB} = 0$ to find the value of q.

c Find the coordinates of the point P.

Example 11

9 Find the angle, in degrees, between each of the following pairs of vectors, correct to two decimal places:

a $\boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{i} - 4\boldsymbol{j}$ **b** $-2\boldsymbol{i} + \boldsymbol{j}$ and $-2\boldsymbol{i} - 2\boldsymbol{j}$

c $2\boldsymbol{i} - \boldsymbol{j}$ and $4\boldsymbol{i}$ **d** $7\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{i} + \boldsymbol{j}$

10 Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be non-zero vectors such that $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. Use the geometric description of the scalar product to show that $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular vectors.

For Questions 11–12, find the angles in degrees correct to two decimal places.

11 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j}$ respectively. Let M be the midpoint of AB. Find:

a $\overrightarrow{OM}$ **b** $\angle AOM$ **c** $\angle BMO$

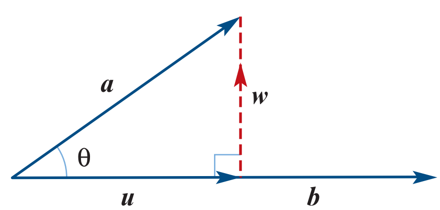
12 Let A, B and C be the points defined by the position vectors $3\boldsymbol{i}$, $4\boldsymbol{j}$ and $-2\boldsymbol{i} + 6\boldsymbol{j}$ respectively. Let M and N be the midpoints of $\overrightarrow{AB}$ and $\overrightarrow{AC}$ respectively. Find:

a **i** $\overrightarrow{OM}$ **ii** $\overrightarrow{ON}$ **b** $\angle MON$ **c** $\angle MOC$

20D Vector projections

It is often useful to decompose a vector $\boldsymbol{a}$ into a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

From the diagram, it can be seen that



$$\boldsymbol{a} = \boldsymbol{u} + \boldsymbol{w}$$

where $\boldsymbol{u} = k\boldsymbol{b}$ and so $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - k\boldsymbol{b}$.

For $\boldsymbol{w}$ to be perpendicular to $\boldsymbol{b}$, we must have

$$\boldsymbol{w} \cdot \boldsymbol{b} = 0$$

$$(\boldsymbol{a} - k\boldsymbol{b}) \cdot \boldsymbol{b} = 0$$

$$\boldsymbol{a} \cdot \boldsymbol{b} - k(\boldsymbol{b} \cdot \boldsymbol{b}) = 0$$

Hence $k = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}$ and therefore $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

This vector $\boldsymbol{u}$ is called the **vector projection** (or **vector resolute**) of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Vector resolute

The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ can be expressed in any one of the following equivalent forms:

$$\boldsymbol{u} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|^2}\boldsymbol{b} = \left(\boldsymbol{a} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right)\left(\frac{\boldsymbol{b}}{|\boldsymbol{b}|}\right) = (\boldsymbol{a} \cdot \hat{\boldsymbol{b}})\,\hat{\boldsymbol{b}}$$

Note: The quantity $\boldsymbol{a} \cdot \hat{\boldsymbol{b}} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is called the **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

Note that, from our previous calculation, we have $\boldsymbol{w} = \boldsymbol{a} - \boldsymbol{u} = \boldsymbol{a} - \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

Expressing $\boldsymbol{a}$ as the sum of the two components, the first parallel to $\boldsymbol{b}$ and the second perpendicular to $\boldsymbol{b}$, gives

$$\boldsymbol{a} = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} + \left(\boldsymbol{a} - \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}\right)$$

This is sometimes described as resolving the vector $\boldsymbol{a}$ into **rectangular components**, one parallel to $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.

Example 12

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$. Find the vector resolute of:

a $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$.

Solution

a $\boldsymbol{a} \cdot \boldsymbol{b} = 1 - 3 = -2$

$\boldsymbol{b} \cdot \boldsymbol{b} = 1 + 1 = 2$

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b} = \frac{-2}{2}(\boldsymbol{i} - \boldsymbol{j})$$
$$= -1(\boldsymbol{i} - \boldsymbol{j})$$
$$= -\boldsymbol{i} + \boldsymbol{j}$$

b $\boldsymbol{b} \cdot \boldsymbol{a} = \boldsymbol{a} \cdot \boldsymbol{b} = -2$

$\boldsymbol{a} \cdot \boldsymbol{a} = 1 + 9 = 10$

The vector resolute of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ is

$$\frac{\boldsymbol{b} \cdot \boldsymbol{a}}{\boldsymbol{a} \cdot \boldsymbol{a}}\boldsymbol{a} = \frac{-2}{10}(\boldsymbol{i} + 3\boldsymbol{j})$$
$$= -\frac{1}{5}(\boldsymbol{i} + 3\boldsymbol{j})$$

Example 13

Find the scalar resolute of $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$ in the direction of $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$.

Solution

$\boldsymbol{a} \cdot \boldsymbol{b} = -2 + 6 = 4$

$|\boldsymbol{b}| = \sqrt{1 + 9} = \sqrt{10}$

The scalar resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is

$$\frac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|} = \frac{4}{\sqrt{10}} = \frac{2\sqrt{10}}{5}$$

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Example 14

Resolve $\boldsymbol{i} + 3\boldsymbol{j}$ into rectangular components, one of which is parallel to $2\boldsymbol{i} - 2\boldsymbol{j}$.

Solution

Let $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} - 2\boldsymbol{j}$.

The vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.

We have

$$\boldsymbol{a} \cdot \boldsymbol{b} = 2 - 6 = -4$$
$$\boldsymbol{b} \cdot \boldsymbol{b} = 4 + 4 = 8$$

Therefore the vector resolute is

$$\frac{-4}{8}(2\boldsymbol{i} - 2\boldsymbol{j}) = -\frac{1}{2}(2\boldsymbol{i} - 2\boldsymbol{j})$$
$$= -\boldsymbol{i} + \boldsymbol{j}$$

The perpendicular component is

$$\boldsymbol{a} - (-\boldsymbol{i} + \boldsymbol{j}) = (\boldsymbol{i} + 3\boldsymbol{j}) - (-\boldsymbol{i} + \boldsymbol{j})$$
$$= 2\boldsymbol{i} + 2\boldsymbol{j}$$

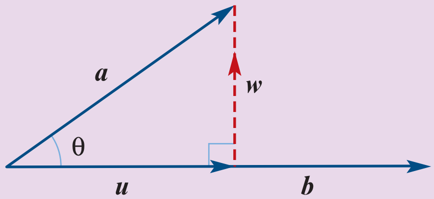
Hence we can write

$$\boldsymbol{i} + 3\boldsymbol{j} = (-\boldsymbol{i} + \boldsymbol{j}) + (2\boldsymbol{i} + 2\boldsymbol{j})$$

Check: We can check our calculation by verifying that the second component is indeed perpendicular to $\boldsymbol{b}$. We have $(2\boldsymbol{i} + 2\boldsymbol{j}) \cdot (2\boldsymbol{i} - 2\boldsymbol{j}) = 4 - 4 = 0$, as expected.

Section summary

- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is given by $\boldsymbol{u} = \dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is the 'signed length' of the vector resolute $\boldsymbol{u}$ and is given by $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.



Exercise 20D

1 Points A and B are defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$.

a Find $\hat{\boldsymbol{a}}$. **b** Find $\hat{\boldsymbol{b}}$. **c** Find $\hat{\boldsymbol{c}}$, where $\boldsymbol{c} = \overrightarrow{AB}$.

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2 Let $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $|\boldsymbol{b}|$

b Find the vector with the same magnitude as $\boldsymbol{b}$ and with the same direction as $\boldsymbol{a}$.

3 Points A and B are defined by the position vectors $\boldsymbol{a} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} + 12\boldsymbol{j}$.

a Find:

i $\hat{\boldsymbol{a}}$ **ii** $\hat{\boldsymbol{b}}$

b Find the unit vector which bisects $\angle AOB$.

Example 12

4 For each pair of vectors, find the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$:

a $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

b $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - 4\boldsymbol{j}$

c $\boldsymbol{a} = 4\boldsymbol{i} - \boldsymbol{j}$ and $\boldsymbol{b} = 4\boldsymbol{i}$

Example 13

5 For each of the following pairs of vectors, find the scalar resolute of the first vector in the direction of the second vector:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = \boldsymbol{i} - 2\boldsymbol{j}$

c $\boldsymbol{b} = 2\boldsymbol{j}$ and $\boldsymbol{a} = 2\boldsymbol{i} + \sqrt{3}\boldsymbol{j}$

d $\boldsymbol{b} = \boldsymbol{i} - \sqrt{5}\boldsymbol{j}$ and $\boldsymbol{c} = -\boldsymbol{i} + 4\boldsymbol{j}$

Example 14

6 For each of the following pairs of vectors, find the resolution of the vector $\boldsymbol{a}$ into rectangular components, one of which is parallel to $\boldsymbol{b}$:

a $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 5\boldsymbol{i}$

b $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$

c $\boldsymbol{a} = -\boldsymbol{i} + \boldsymbol{j},\ \boldsymbol{b} = 2\boldsymbol{i} + 2\boldsymbol{j}$

7 Let A and B be the points defined by the position vectors $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b a unit vector through A perpendicular to OB

8 Let A and B be the points defined by the position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$ respectively. Find:

a the vector resolute of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$

b the vector component of $\boldsymbol{a}$ perpendicular to $\boldsymbol{b}$

c the shortest distance from A to line OB

9 Points A, B and C have position vectors $\boldsymbol{a} = \boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to line AC

d the area of triangle ABC

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

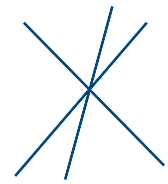
20E Geometric proofs

In this section we see how vectors can be used as a tool for proving geometric results.

We require the following two definitions.

Collinear points Three or more points are collinear if they all lie on a single line.

Concurrent lines Three or more lines are concurrent if they all pass through a single point.



Here are some properties of vectors that will be useful:

- For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{a}$ is in the same direction as $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$, and the vector $-k\boldsymbol{a}$ is in the opposite direction to $\boldsymbol{a}$ and has magnitude $k|\boldsymbol{a}|$.
- If vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel, then $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$. Conversely, if $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-zero vectors such that $\boldsymbol{b} = k\boldsymbol{a}$ for some $k \in \mathbb{R} \setminus \{0\}$, then $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel.
- If $\boldsymbol{a}$ and $\boldsymbol{b}$ are parallel with at least one point in common, then $\boldsymbol{a}$ and $\boldsymbol{b}$ lie on the same straight line. For example, if $\overrightarrow{AB} = k\overrightarrow{BC}$ for some $k \in \mathbb{R} \setminus \{0\}$, then A, B and C are collinear.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$

Example 15

Three points P, Q and R have position vectors $\boldsymbol{p}$, $\boldsymbol{q}$ and $k(2\boldsymbol{p} + \boldsymbol{q})$ respectively, relative to a fixed origin O. The points O, P and Q are not collinear.

Find the value of k if:

a $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$ **b** $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$ **c** P, Q and R are collinear.

Solution

a
$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR}\\ &= -\boldsymbol{q} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= 2k\boldsymbol{p} + (k-1)\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{QR}$ is parallel to $\boldsymbol{p}$, then there is some $\lambda \in \mathbb{R} \setminus \{0\}$ such that

$$2k\boldsymbol{p} + (k-1)\boldsymbol{q} = \lambda\boldsymbol{p}$$

This implies that

$$2k = \lambda \quad \text{and} \quad k - 1 = 0$$

Hence $k = 1$.

b
$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PO} + \overrightarrow{OR}\\ &= -\boldsymbol{p} + k(2\boldsymbol{p} + \boldsymbol{q})\\ &= (2k-1)\boldsymbol{p} + k\boldsymbol{q}\end{aligned}$$

If $\overrightarrow{PR}$ is parallel to $\boldsymbol{q}$, then there is some $m \in \mathbb{R} \setminus \{0\}$ such that

$$(2k-1)\boldsymbol{p} + k\boldsymbol{q} = m\boldsymbol{q}$$

This implies that

$$2k - 1 = 0 \quad \text{and} \quad k = m$$

Hence $k = \frac{1}{2}$.

Note: Since points O, P and Q are not collinear, the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are not parallel.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

c If points P, Q and R are collinear, then there exists $n \in \mathbb{R} \setminus \{0\}$ such that

$$n\overrightarrow{PQ} = \overrightarrow{QR}$$

$$\therefore \quad n(-\boldsymbol{p} + \boldsymbol{q}) = 2k\boldsymbol{p} + (k-1)\boldsymbol{q}$$

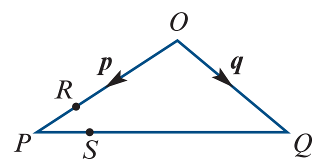
This implies that

$$-n = 2k \quad \text{and} \quad n = k - 1$$

Therefore $3k - 1 = 0$ and so $k = \frac{1}{3}$.

Exercise 20E

1 In the diagram, $OR = \dfrac{4}{5}OP$, $\boldsymbol{p} = \overrightarrow{OP}$, $\boldsymbol{q} = \overrightarrow{OQ}$ and $PS : SQ = 1 : 4$.



a Express each of the following in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$:

i $\overrightarrow{OR}$ **ii** $\overrightarrow{RP}$ **iii** $\overrightarrow{PO}$ **iv** $\overrightarrow{PS}$ **v** $\overrightarrow{RS}$

b What can be said about line segments RS and OQ?

c What type of quadrilateral is $ORSQ$?

d The area of triangle PRS is 5 cm^2. What is the area of $ORSQ$?

2 The position vectors of three points A, B and C relative to an origin O are $\boldsymbol{a}$, $\boldsymbol{b}$ and $k\boldsymbol{a}$ respectively. The point P lies on AB and is such that $AP = 2PB$. The point Q lies on BC and is such that $CQ = 6QB$.

a Find in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i the position vector of P

ii the position vector of Q

b Given that OPQ is a straight line, find:

i the value of k

ii the ratio $\dfrac{OP}{PQ}$

c The position vector of a point R is $\dfrac{7}{3}\boldsymbol{a}$. Show that PR is parallel to BC.

Example 15

3 The position vectors of two points A and B relative to an origin O are $3\boldsymbol{i} + 3.5\boldsymbol{j}$ and $6\boldsymbol{i} - 1.5\boldsymbol{j}$ respectively.

a **i** Given that $\overrightarrow{OD} = \dfrac{1}{3}\overrightarrow{OB}$ and $\overrightarrow{AE} = \dfrac{1}{4}\overrightarrow{AB}$, write down the position vectors of D and E.

ii Hence find $|\overrightarrow{ED}|$.

b Given that OE and AD intersect at X and that $\overrightarrow{OX} = p\overrightarrow{OE}$ and $\overrightarrow{XD} = q\overrightarrow{AD}$, find the position vector of X in terms of:

i p **ii** q

c Hence determine the values of p and q.

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4 Points P and Q have position vectors $\boldsymbol{p}$ and $\boldsymbol{q}$, with reference to an origin O, and M is the point on PQ such that

$$\beta\overrightarrow{PM} = \alpha\overrightarrow{MQ}$$

a Prove that the position vector of M is given by $\boldsymbol{m} = \dfrac{\beta\boldsymbol{p} + \alpha\boldsymbol{q}}{\alpha + \beta}$.

b Write the position vectors of P and Q as $\boldsymbol{p} = k\boldsymbol{a}$ and $\boldsymbol{q} = \ell\boldsymbol{b}$, where k and ℓ are positive real numbers and $\boldsymbol{a}$ and $\boldsymbol{b}$ are unit vectors.

i Prove that the position vector of any point on the internal bisector of $\angle POQ$ has the form $\lambda(\boldsymbol{a} + \boldsymbol{b})$.

ii If M is the point where the internal bisector of $\angle POQ$ meets PQ, show that

$$\frac{\alpha}{\beta} = \frac{k}{\ell}$$

5 Suppose that $OABC$ is a rhombus. Let $\boldsymbol{a} = \overrightarrow{OA}$ and $\boldsymbol{c} = \overrightarrow{OC}$.

a Express each of the following vectors in terms of $\boldsymbol{a}$ and $\boldsymbol{c}$:

i $\overrightarrow{AB}$ **ii** $\overrightarrow{OB}$ **iii** $\overrightarrow{AC}$

b Find $\overrightarrow{OB} \cdot \overrightarrow{AC}$.

c Prove that the diagonals of a rhombus intersect at right angles.

6 Suppose that $ORST$ is a parallelogram, where O is the origin. Let U be the midpoint of RS and let V be the midpoint of ST. Denote the position vectors of R, S, T, U and V by $\boldsymbol{r}$, $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively.

a Express $\boldsymbol{s}$ in terms of $\boldsymbol{r}$ and $\boldsymbol{t}$.

b Express $\boldsymbol{v}$ in terms of $\boldsymbol{s}$ and $\boldsymbol{t}$.

c Hence, or otherwise, show that $4(\boldsymbol{u} + \boldsymbol{v}) = 3(\boldsymbol{r} + \boldsymbol{s} + \boldsymbol{t})$.

7 Prove that the midpoints of any quadrilateral are the vertices of a parallelogram.

8 Prove that the diagonals of a square are of equal length and bisect each other.

9 Prove that the diagonals of a parallelogram bisect each other.

10 Prove that the altitudes of a triangle are concurrent. That is, they meet at a point.

11 Apollonius' theorem

For $\triangle OAB$, the point C is the midpoint of side AB. Prove that:

a $4\overrightarrow{OC} \cdot \overrightarrow{OC} = OA^2 + OB^2 + 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

b $4\overrightarrow{AC} \cdot \overrightarrow{AC} = OA^2 + OB^2 - 2\overrightarrow{OA} \cdot \overrightarrow{OB}$

c $2OC^2 + 2AC^2 = OA^2 + OB^2$

12 If P is any point in the plane of rectangle $ABCD$, prove that

$$PA^2 + PC^2 = PB^2 + PD^2$$

ISBN 978-1-107-56765-8

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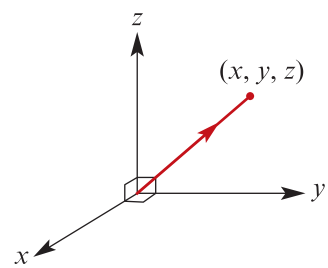
13 Prove that the medians bisecting the equal sides of an isosceles triangle are equal.

14 **a** Prove that if $(\boldsymbol{c} - \boldsymbol{b}) \cdot \boldsymbol{a} = 0$ and $(\boldsymbol{c} - \boldsymbol{a}) \cdot \boldsymbol{b} = 0$, then $(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{c} = 0$.

b Use part **a** to prove that the altitudes of a triangle meet at a point.

20F Vectors in three dimensions

Points in three dimensions are represented using three perpendicular axes as shown.

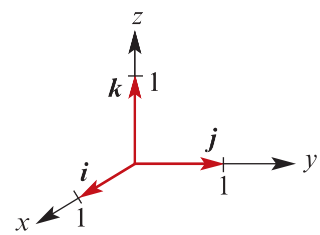


Vectors in three dimensions are of the form

$$\boldsymbol{a} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

where $\boldsymbol{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\boldsymbol{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\boldsymbol{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are the standard unit vectors for three dimensions.



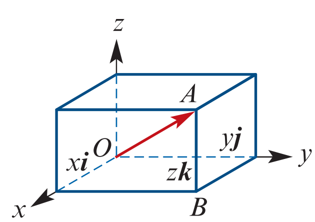
The position vector for point $A(x, y, z)$ is

$$\overrightarrow{OA} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Using Pythagoras' theorem twice:

$$\begin{aligned} OA^2 &= OB^2 + BA^2 \\ &= OB^2 + z^2 \\ &= x^2 + y^2 + z^2 \end{aligned}$$

$$\therefore \quad |\overrightarrow{OA}| = \sqrt{x^2 + y^2 + z^2}$$



Example 16

Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{b} = \boldsymbol{i} + 7\boldsymbol{k}$. Find:

a $\boldsymbol{a} + \boldsymbol{b}$ **b** $\boldsymbol{b} - 3\boldsymbol{a}$ **c** $|\boldsymbol{a}|$

Solution

a $$\begin{aligned} &\boldsymbol{a} + \boldsymbol{b} \\ &= \boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k} + \boldsymbol{i} + 7\boldsymbol{k} \\ &= 2\boldsymbol{i} + \boldsymbol{j} + 6\boldsymbol{k} \end{aligned}$$

b $$\begin{aligned} &\boldsymbol{b} - 3\boldsymbol{a} \\ &= \boldsymbol{i} + 7\boldsymbol{k} - 3(\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}) \\ &= -2\boldsymbol{i} - 3\boldsymbol{j} + 10\boldsymbol{k} \end{aligned}$$

c $$\begin{aligned} |\boldsymbol{a}| &= \sqrt{1^2 + 1^2 + (-1)^2} \\ &= \sqrt{3} \end{aligned}$$

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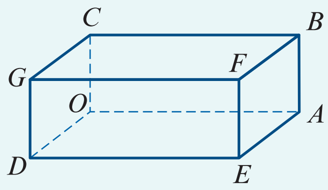
Example 17

$OABCDEFG$ is a cuboid such that $\overrightarrow{OA} = 3\boldsymbol{j}$, $\overrightarrow{OC} = \boldsymbol{k}$ and $\overrightarrow{OD} = \boldsymbol{i}$.



a Express each of the following in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

i $\overrightarrow{OE}$ **ii** $\overrightarrow{OF}$ **iii** $\overrightarrow{GF}$ **iv** $\overrightarrow{GB}$

b Let M and N be the midpoints of OD and GF respectively. Find MN.

Solution

a **i** $\overrightarrow{OE} = \overrightarrow{OA} + \overrightarrow{AE} = 3\boldsymbol{j} + \boldsymbol{i}$ (as $\overrightarrow{AE} = \overrightarrow{OD}$)

ii $\overrightarrow{OF} = \overrightarrow{OE} + \overrightarrow{EF} = 3\boldsymbol{j} + \boldsymbol{i} + \boldsymbol{k}$ (as $\overrightarrow{EF} = \overrightarrow{OC}$)

iii $\overrightarrow{GF} = \overrightarrow{OA} = 3\boldsymbol{j}$

iv $\overrightarrow{GB} = \overrightarrow{DA} = \overrightarrow{DO} + \overrightarrow{OA} = -\boldsymbol{i} + 3\boldsymbol{j}$

b
$$\begin{aligned}\overrightarrow{MN} &= \overrightarrow{MD} + \overrightarrow{DG} + \overrightarrow{GN}\\ &= \frac{1}{2}\overrightarrow{OD} + \overrightarrow{OC} + \frac{1}{2}\overrightarrow{OA}\\ &= \frac{1}{2}\boldsymbol{i} + \boldsymbol{k} + \frac{3}{2}\boldsymbol{j}\end{aligned}$$

$$|\overrightarrow{MN}| = \sqrt{\frac{1}{4} + 1 + \frac{9}{4}} = \frac{\sqrt{14}}{2}$$

Example 18

If $\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, find $\hat{\boldsymbol{a}}$.

Solution

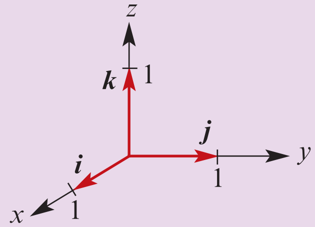
$$|\boldsymbol{a}| = \sqrt{9 + 4 + 4} = \sqrt{17}$$

$$\therefore \quad \hat{\boldsymbol{a}} = \frac{1}{\sqrt{17}}(3\boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k})$$

Section summary

In three dimensions:

- The standard unit vectors are $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$.
- Each vector can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.
- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.



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Exercise 20F

Skillsheet

Example 16

1 Let $\boldsymbol{a} = \boldsymbol{i} + \boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} + \boldsymbol{k}$. Find:

a $\boldsymbol{a} - \boldsymbol{b}$ **b** $3\boldsymbol{b} - 2\boldsymbol{a} + \boldsymbol{c}$ **c** $|\boldsymbol{b}|$

d $|\boldsymbol{b} + \boldsymbol{c}|$ **e** $3(\boldsymbol{a} - \boldsymbol{b}) + 2\boldsymbol{c}$

Example 17

2 $OABCDEFG$ is a cuboid such that

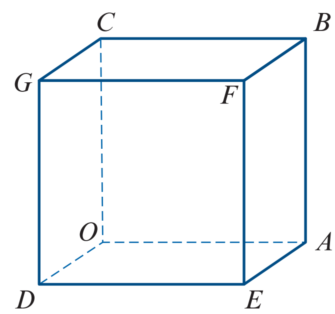
$$\overrightarrow{OA} = 2\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$

Express the following vectors in terms of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

a $\overrightarrow{OB}$ **b** $\overrightarrow{OE}$ **c** $\overrightarrow{OG}$

d $\overrightarrow{OF}$ **e** $\overrightarrow{ED}$ **f** $\overrightarrow{EG}$

g $\overrightarrow{CE}$ **h** $\overrightarrow{BD}$



Example 18

3 Let $\boldsymbol{a} = 3\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$.

a **i** Find $\hat{\boldsymbol{a}}$.

ii Find $-2\hat{\boldsymbol{a}}$.

b Find the vector $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{b}| = 5$.

4 If $\boldsymbol{a} = \boldsymbol{i} - \boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - 3\boldsymbol{k}$, find the vector $\boldsymbol{c}$ in the direction of $\boldsymbol{a}$ such that $|\boldsymbol{c}| = |\boldsymbol{b}|$.

5 Let P and Q be the points defined by the position vectors $\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$ and $2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ respectively. Let M be the midpoint of PQ. Find:

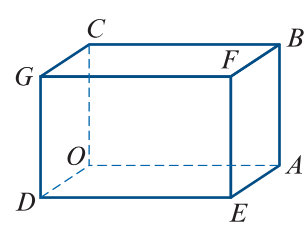
a $\overrightarrow{PQ}$ **b** $|\overrightarrow{PQ}|$ **c** $\overrightarrow{OM}$

6 $OABCDEFG$ is a cuboid such that

$$\overrightarrow{OA} = 3\boldsymbol{j}, \quad \overrightarrow{OC} = 2\boldsymbol{k} \quad \text{and} \quad \overrightarrow{OD} = \boldsymbol{i}$$

The point M is such that $\overrightarrow{OM} = \frac{1}{3}\overrightarrow{OE}$, and N is the midpoint of BF. Find:

a $\overrightarrow{MN}$ **b** $|\overrightarrow{MN}|$



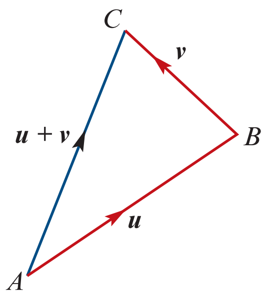
ISBN 978-1-107-56765-8

Chapter summary

AS
Nrich

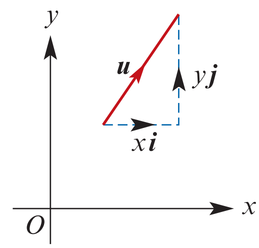
- A **vector** is a set of equivalent **directed line segments**.
- A directed line segment from a point A to a point B is denoted by $\overrightarrow{AB}$.
- The **position vector** of a point A is the vector $\overrightarrow{OA}$, where O is the origin.
- A vector can be written as a column of numbers. The vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ is '2 across and 3 up'.

Basic operations on vectors



- Addition
  - If $\boldsymbol{u} = \begin{bmatrix} a \\ b \end{bmatrix}$ and $\boldsymbol{v} = \begin{bmatrix} c \\ d \end{bmatrix}$, then $\boldsymbol{u} + \boldsymbol{v} = \begin{bmatrix} a+c \\ b+d \end{bmatrix}$.
  - The sum $\boldsymbol{u} + \boldsymbol{v}$ can also be obtained geometrically as shown.
- Scalar multiplication
  - For $k \in \mathbb{R}^+$, the vector $k\boldsymbol{u}$ has the same direction as $\boldsymbol{u}$, but its length is multiplied by a factor of k.
  - The vector $-\boldsymbol{v}$ has the same length as $\boldsymbol{v}$, but the opposite direction.
  - Two non-zero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are **parallel** if there exists $k \in \mathbb{R} \setminus \{0\}$ such that $\boldsymbol{u} = k\boldsymbol{v}$.
- Subtraction $\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{u} + (-\boldsymbol{v})$

Component form

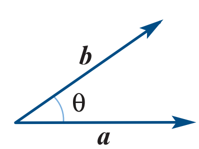


- In two dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$, where
  - $\boldsymbol{i}$ is the unit vector in the positive direction of the x-axis
  - $\boldsymbol{j}$ is the unit vector in the positive direction of the y-axis.
- The **magnitude** of vector $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j}$ is given by $|\boldsymbol{u}| = \sqrt{x^2 + y^2}$.
- The unit vector in the direction of vector $\boldsymbol{a}$ is given by $\hat{\boldsymbol{a}} = \dfrac{1}{|\boldsymbol{a}|}\boldsymbol{a}$.

Scalar product and vector projections

- The **scalar product** of vectors $\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j}$ and $\boldsymbol{b} = b_1\boldsymbol{i} + b_2\boldsymbol{j}$ is given by

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2$$



- The scalar product is described geometrically by $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$, where θ is the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.
- Therefore $\boldsymbol{a} \cdot \boldsymbol{a} = |\boldsymbol{a}|^2$.
- Two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are **perpendicular** if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.
- Resolving a vector $\boldsymbol{a}$ into rectangular components is expressing the vector $\boldsymbol{a}$ as a sum of two vectors, one parallel to a given vector $\boldsymbol{b}$ and the other perpendicular to $\boldsymbol{b}$.
- The **vector resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{\boldsymbol{b} \cdot \boldsymbol{b}}\boldsymbol{b}$.
- The **scalar resolute** of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$ is $\dfrac{\boldsymbol{a} \cdot \boldsymbol{b}}{|\boldsymbol{b}|}$.

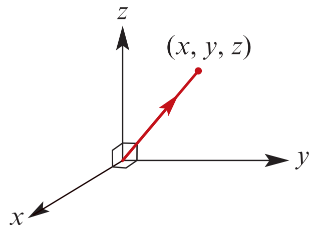
ISBN 978-1-107-56765-8

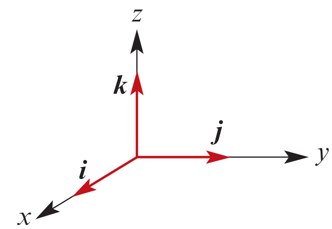
Cambridge University Press Updated May 2021

Review

Vectors in three dimensions

- In three dimensions, each vector $\boldsymbol{u}$ can be written in the form $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are unit vectors as shown.





- If $\boldsymbol{u} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then $|\boldsymbol{u}| = \sqrt{x^2 + y^2 + z^2}$.

Technology-free questions

1 Given that $\boldsymbol{a} = 7\boldsymbol{i} + 6\boldsymbol{j}$ and $\boldsymbol{b} = 2\boldsymbol{i} + x\boldsymbol{j}$, find the values of x for which:

a $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ **b** $\boldsymbol{a}$ and $\boldsymbol{b}$ have the same magnitude.

2 $ABCD$ is a parallelogram where $\overrightarrow{OA} = 2\boldsymbol{i} - \boldsymbol{j}$, $\overrightarrow{AB} = 3\boldsymbol{i} + 4\boldsymbol{j}$ and $\overrightarrow{AD} = -2\boldsymbol{i} + 5\boldsymbol{j}$. Find the coordinates of the four vertices of the parallelogram.

3 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - 4\boldsymbol{j} + 5\boldsymbol{k}$ and $\boldsymbol{c} = -\boldsymbol{i} - 4\boldsymbol{j} + 2\boldsymbol{k}$. Find the values of p and q such that $\boldsymbol{a} + p\boldsymbol{b} + q\boldsymbol{c}$ is parallel to the x-axis.

4 The position vectors of P and Q are $2\boldsymbol{i} - 2\boldsymbol{j} + 4\boldsymbol{k}$ and $3\boldsymbol{i} - 7\boldsymbol{j} + 12\boldsymbol{k}$ respectively.

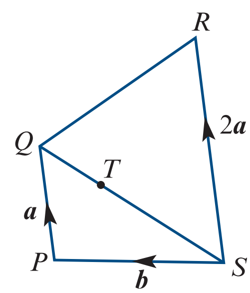
a Find $|\overrightarrow{PQ}|$. **b** Find the unit vector in the direction of $\overrightarrow{PQ}$.

5 The position vectors of A, B and C are $2\boldsymbol{j} + 2\boldsymbol{k}$, $4\boldsymbol{i} + 10\boldsymbol{j} + 18\boldsymbol{k}$ and $x\boldsymbol{i} + 14\boldsymbol{j} + 26\boldsymbol{k}$ respectively. Find x if A, B and C are collinear.

6 $\overrightarrow{OA} = 4\boldsymbol{i} + 3\boldsymbol{j}$ and C is a point on OA such that $|\overrightarrow{OC}| = \dfrac{16}{5}$.

a Find the unit vector in the direction of $\overrightarrow{OA}$.

b Hence find $\overrightarrow{OC}$.



7 In the diagram, $ST = 2TQ$, $\overrightarrow{PQ} = \boldsymbol{a}$, $\overrightarrow{SR} = 2\boldsymbol{a}$ and $\overrightarrow{SP} = \boldsymbol{b}$.

a Find each of the following in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$:

i $\overrightarrow{SQ}$ **ii** $\overrightarrow{TQ}$ **iii** $\overrightarrow{RQ}$ **iv** $\overrightarrow{PT}$ **v** $\overrightarrow{TR}$

b Show that P, T and R are collinear.

8 If $\boldsymbol{a} = 5\boldsymbol{i} - s\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{b} = t\boldsymbol{i} + 2\boldsymbol{j} + u\boldsymbol{k}$ are equal vectors.

a Find s, t and u. **b** Find $|\boldsymbol{a}|$.

9 The vector $\boldsymbol{p}$ has magnitude 7 units and bearing 050° and the vector $\boldsymbol{q}$ has magnitude 12 units and bearing 170°. (These are compass bearings on the horizontal plane.) Draw a diagram (not to scale) showing $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{p} + \boldsymbol{q}$. Calculate the magnitude of $\boldsymbol{p} + \boldsymbol{q}$.

10 If $\boldsymbol{a} = 5\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{b} = 3\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}$, find:

a $\boldsymbol{a} + 2\boldsymbol{b}$ **b** $|\boldsymbol{a}|$ **c** $\hat{\boldsymbol{a}}$ **d** $\boldsymbol{a} - \boldsymbol{b}$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

11 Let O, A and B be the points $(0, 0)$, $(3, 4)$ and $(4, -6)$ respectively.

a If C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$, find the coordinates of C.

b If D is the point $(1, 24)$ and $\overrightarrow{OD} = h\overrightarrow{OA} + k\overrightarrow{OB}$, find the values of h and k.

12 Let $\boldsymbol{p} = 3\boldsymbol{i} + 7\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} - 5\boldsymbol{j}$. Find the values of m and n such that $m\boldsymbol{p} + n\boldsymbol{q} = 8\boldsymbol{i} + 9\boldsymbol{j}$.

13 The points A, B and C have position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ relative to an origin O. Write down an equation connecting $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ for each of the following cases:

a $OABC$ is a parallelogram

b B divides AC in the ratio $3 : 2$. That is, $AB : BC = 3 : 2$.

14 Let $\boldsymbol{a} = 2\boldsymbol{i} - 3\boldsymbol{j}$, $\boldsymbol{b} = -\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{c} = -2\boldsymbol{i} - 2\boldsymbol{j}$. Find:

a $\boldsymbol{a} \cdot \boldsymbol{a}$ **b** $\boldsymbol{b} \cdot \boldsymbol{b}$ **c** $\boldsymbol{c} \cdot \boldsymbol{c}$ **d** $\boldsymbol{a} \cdot \boldsymbol{b}$

e $\boldsymbol{a} \cdot (\boldsymbol{b} + \boldsymbol{c})$ **f** $(\boldsymbol{a} + \boldsymbol{b}) \cdot (\boldsymbol{a} + \boldsymbol{c})$ **g** $(\boldsymbol{a} + 2\boldsymbol{b}) \cdot (3\boldsymbol{c} - \boldsymbol{b})$

15 Points A, B and C have position vectors $\boldsymbol{a} = 4\boldsymbol{i} + \boldsymbol{j}$, $\boldsymbol{b} = 3\boldsymbol{i} + 5\boldsymbol{j}$ and $\boldsymbol{c} = -5\boldsymbol{i} + 3\boldsymbol{j}$ respectively. Evaluate $\overrightarrow{AB} \cdot \overrightarrow{BC}$ and hence show that $\triangle ABC$ is right-angled at B.

16 Given the vectors $\boldsymbol{p} = 5\boldsymbol{i} + 3\boldsymbol{j}$ and $\boldsymbol{q} = 2\boldsymbol{i} + t\boldsymbol{j}$, find the values of t for which:

a $\boldsymbol{p} + \boldsymbol{q}$ is parallel to $\boldsymbol{p} - \boldsymbol{q}$ **b** $\boldsymbol{p} - 2\boldsymbol{q}$ is perpendicular to $\boldsymbol{p} + 2\boldsymbol{q}$ **c** $|\boldsymbol{p} - \boldsymbol{q}| = |\boldsymbol{q}|$

17 Points A, B and C have position vectors $\boldsymbol{a} = 2\boldsymbol{i} + 2\boldsymbol{j}$, $\boldsymbol{b} = \boldsymbol{i} + 2\boldsymbol{j}$ and $\boldsymbol{c} = 2\boldsymbol{i} - 3\boldsymbol{j}$. Find:

a **i** $\overrightarrow{AB}$ **ii** $\overrightarrow{AC}$

b the vector resolute of $\overrightarrow{AB}$ in the direction of $\overrightarrow{AC}$

c the shortest distance from B to the line AC.

Multiple-choice questions

1 The vector $\boldsymbol{v}$ is defined by the directed line segment from $(1, 1)$ to $(3, 5)$.
If $\boldsymbol{v} = a\boldsymbol{i} + b\boldsymbol{j}$, then

A $a = 3$ and $b = 5$ **B** $a = -2$ and $b = -4$ **C** $a = 2$ and $b = 4$

D $a = 2$ and $b = 3$ **E** $a = 4$ and $b = 2$

2 If vector $\overrightarrow{AB} = \boldsymbol{u}$ and vector $\overrightarrow{AC} = \boldsymbol{v}$, then vector $\overrightarrow{CB}$ is equal to

A $\boldsymbol{u} + \boldsymbol{v}$ **B** $\boldsymbol{v} - \boldsymbol{u}$ **C** $\boldsymbol{u} - \boldsymbol{v}$ **D** $\boldsymbol{u} \times \boldsymbol{v}$ **E** $\boldsymbol{v} + \boldsymbol{u}$

3 If vector $\boldsymbol{a} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then $\boldsymbol{a} + \boldsymbol{b} =$

A $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$ **B** $\begin{bmatrix} 1 \\ 5 \end{bmatrix}$ **C** $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ **D** $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

4 If vector $\boldsymbol{a} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and vector $\boldsymbol{b} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, then $2\boldsymbol{a} - 3\boldsymbol{b} =$

A $\begin{bmatrix} 9 \\ -13 \end{bmatrix}$ **B** $\begin{bmatrix} 9 \\ 7 \end{bmatrix}$ **C** $\begin{bmatrix} 9 \\ -7 \end{bmatrix}$ **D** $\begin{bmatrix} 3 \\ -13 \end{bmatrix}$ **E** $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$

ISBN 978-1-107-56765-8

Cambridge University Press Updated May 2021

5 $PQRS$ is a parallelogram. If $\overrightarrow{PQ} = \boldsymbol{p}$ and $\overrightarrow{QR} = \boldsymbol{q}$, then $\overrightarrow{SQ}$ is equal to

A $\boldsymbol{p} + \boldsymbol{q}$ **B** $\boldsymbol{p} - \boldsymbol{q}$ **C** $\boldsymbol{q} - \boldsymbol{p}$ **D** $2\boldsymbol{q}$ **E** $2\boldsymbol{p}$

6 $|3\boldsymbol{i} - 5\boldsymbol{j}| =$

A 2 **B** $\sqrt{34}$ **C** 34 **D** 8 **E** -16

7 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $\overrightarrow{AB}$ equals

A $-\boldsymbol{i} - 5\boldsymbol{j}$ **B** $-\boldsymbol{i} + 5\boldsymbol{j}$ **C** $-\boldsymbol{i} - \boldsymbol{j}$ **D** $-\boldsymbol{i} + \boldsymbol{j}$ **E** $\boldsymbol{i} + \boldsymbol{j}$

8 If $\overrightarrow{OA} = 2\boldsymbol{i} + 3\boldsymbol{j}$ and $\overrightarrow{OB} = \boldsymbol{i} - 2\boldsymbol{j}$, then $|\overrightarrow{AB}|$ equals

A 6 **B** 26 **C** $\sqrt{26}$ **D** $\sqrt{24}$ **E** 36

9 If $\boldsymbol{a} = 2\boldsymbol{i} + 3\boldsymbol{j}$, then the unit vector in the direction of $\boldsymbol{a}$ is

A $2\boldsymbol{i} + 3\boldsymbol{j}$ **B** $\frac{1}{13}(2\boldsymbol{i} + 3\boldsymbol{j})$ **C** $\frac{1}{\sqrt{5}}(2\boldsymbol{i} + 3\boldsymbol{j})$

D $\frac{1}{\sqrt{13}}(2\boldsymbol{i} + 3\boldsymbol{j})$ **E** $\sqrt{13}(2\boldsymbol{i} + 3\boldsymbol{j})$

10 If $\boldsymbol{a} = -3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$, then $\hat{\boldsymbol{a}}$ is

A $\frac{1}{7}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **B** $\frac{1}{\sqrt{7}}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **C** $\frac{1}{\sqrt{19}}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

D $\frac{1}{19}(-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$ **E** $-3\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$

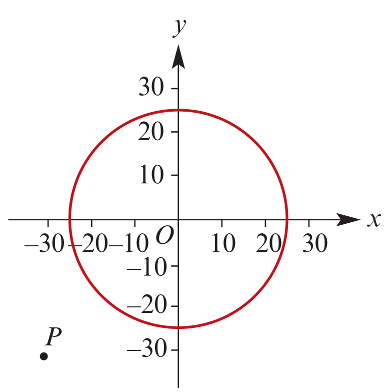
Extended-response questions

1 Let $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ represent a displacement 1 km due east.

Let $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ represent a displacement 1 km due north.

The diagram shows a circle of radius 25 km with centre at $O(0, 0)$. A lighthouse entirely surrounded by sea is located at O. The lighthouse is not visible from points outside the circle.

A ship is initially at point P, which 31 km west and 32 km south of the lighthouse.



a Write down the vector $\overrightarrow{OP}$.

The ship is travelling in the direction of vector $\boldsymbol{u} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ with speed 20 km/h.

An hour after leaving P, the ship is at point R.

b Show that $\overrightarrow{PR} = \begin{bmatrix} 16 \\ 12 \end{bmatrix}$ and hence find the vector $\overrightarrow{OR}$.

c Show that the lighthouse first becomes visible when the ship reaches R.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

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2 Given that $\boldsymbol{p} = 3\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{q} = -2\boldsymbol{i} + 4\boldsymbol{j}$, find:

a $|\boldsymbol{p} - \boldsymbol{q}|$

b $|\boldsymbol{p}| - |\boldsymbol{q}|$

c $\boldsymbol{r}$ such that $\boldsymbol{p} + 2\boldsymbol{q} + \boldsymbol{r} = \boldsymbol{0}$

3 Let $\boldsymbol{a} = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$, $\boldsymbol{b} = \begin{bmatrix} 11 \\ 7 \\ 3 \end{bmatrix}$, $\boldsymbol{c} = \begin{bmatrix} 7 \\ 9 \\ 7 \end{bmatrix}$ and $\boldsymbol{d} = \begin{bmatrix} 26 \\ 12 \\ 2 \end{bmatrix}$.

a Find the value of the scalar k such that $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = k\boldsymbol{d}$.

b Find the scalars x and y such that $x\boldsymbol{a} + y\boldsymbol{b} = \boldsymbol{d}$.

c Use your answers to **a** and **b** to find scalars p, q and r (not all zero) such that $p\boldsymbol{a} + q\boldsymbol{b} - r\boldsymbol{c} = \boldsymbol{0}$.

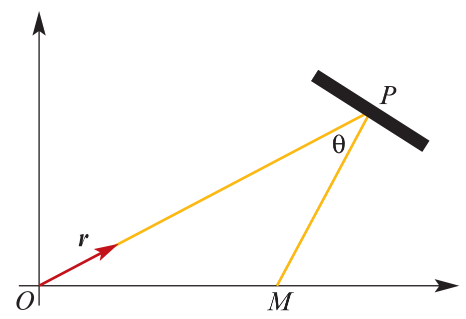
4 The quadrilateral $PQRS$ is a parallelogram. The point P has coordinates $(5, 8)$, the point R has coordinates $(32, 17)$ and the vector $\overrightarrow{PQ}$ is given by $\overrightarrow{PQ} = \begin{bmatrix} 20 \\ -15 \end{bmatrix}$.

a Find the coordinates of Q and write down the vector $\overrightarrow{QR}$.

b Write down the vector $\overrightarrow{RS}$ and show that the coordinates of S are $(12, 32)$.

5 The diagram shows the path of a light beam from its source at O in the direction of the vector $\boldsymbol{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

At point P, the beam is reflected by an adjustable mirror and meets the x-axis at M. The position of M varies, depending on the adjustment of the mirror at P.



a Given that $\overrightarrow{OP} = 4\boldsymbol{r}$, find the coordinates of P.

b The point M has coordinates $(k, 0)$. Find an expression, in terms of k, for vector $\overrightarrow{PM}$.

c Find the magnitudes of vectors $\overrightarrow{OP}$, $\overrightarrow{OM}$ and $\overrightarrow{PM}$, and hence find the value of k for which θ is equal to $90°$.

d Find the value θ for which M has coordinates $(9, 0)$.

ISBN 978-1-107-56765-8