9 Geometry in the plane and proof

Objectives

- To consider necessary and sufficient conditions for two lines to be **parallel**.
- To determine the **angle sum** of a polygon.
- To define **congruence** of two figures.
- To determine when two triangles are congruent.
- To write geometric proofs.
- To use **Pythagoras' theorem** and its converse.
- To apply **transformations** that are expansions from the origin.
- To define **similarity** of two figures.
- To determine when two triangles are similar.
- To determine and apply **similarity factors** for areas and volumes.
- To investigate properties of the **golden ratio**.

There are three main reasons for the study of geometry at school.

The first reason is that the properties of figures in two and three dimensions are helpful in other areas of mathematics. The second reason is that the subject provides a good setting to show how a large body of results may be deduced from a small number of assumptions. The third reason is that it gives you, the student, the opportunity to practise writing coherent, logical mathematical arguments.

In this chapter and the next, we use some of the proof techniques introduced in the previous chapter. Review of geometry from Years 9 and 10 is included, but in such a way that you can see the building of the results.

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9A Points, lines and angles

In this section we do not pretend to be fully rigorous, but aim to make you aware that assumptions are being made and that we base the proofs of the results on these assumptions. The assumptions do seem obvious to us, but there are ways of making the study of geometry even more rigorous. However, whatever we do, we will need to accept a set of results as our starting point.

▶ Points, lines and planes

We begin with a few basic concepts. No formal definitions are given.

Point In geometry, a point is used to indicate position.

Line In the physical world, we may illustrate the idea of a line as a tightly stretched wire or a fold in a piece of paper. A line has no width and is infinite in length.

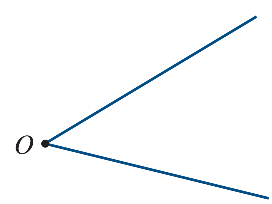
Plane A plane has no thickness and it extends infinitely in all directions.

We make the following assumptions about points and lines:

- Given a point and a line, the point may or may not lie on the line.
- Two distinct points are contained in exactly one line.
- Two distinct lines do not have more than one point in common.

▶ Angles

A **ray** is a portion of a line consisting of a point O and all the points on one side of O.

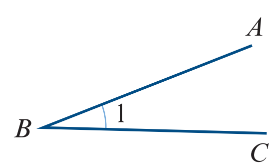


An **angle** is the figure formed by two distinct rays which have a common endpoint O. The common endpoint is called the **vertex** of the angle.

- If the two rays are part of one straight line, the angle is called a **straight angle** and measures 180°.
- A **right angle** is an angle of 90°.
- An **acute angle** is an angle which is less than 90°.
- An **obtuse angle** is an angle which is greater than 90° and less than 180°.
- **Supplementary angles** are two angles whose sum is 180°.
- **Complementary angles** are two angles whose sum is 90°.

Naming angles

The convention for naming an angle is to fully describe the rays of the angle and the endpoint where the rays meet.



The marked angle is denoted by $\angle ABC$.

When there is no chance of ambiguity, it can be written as $\angle B$.

Sometimes an angle can simply be numbered as shown, and in a proof we refer to the angle as $\angle 1$.

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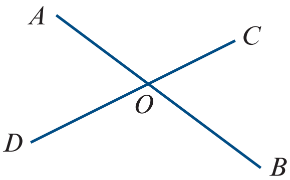
The important thing is that the writing of your argument must be clear and unambiguous. With complicated diagrams, the $\angle ABC$ notation is safest.

Theorem

If two straight lines intersect, then the opposite angles are equal in pairs.

Such angles are said to be **vertically opposite**.

Proof using angle names



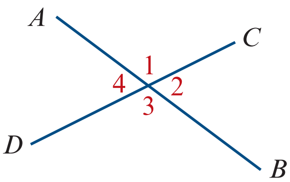
$\angle AOC$ and $\angle COB$ are supplementary.
That is, $\angle AOC + \angle COB = 180^\circ$.

Also, $\angle COB$ and $\angle BOD$ are supplementary.
That is, $\angle COB + \angle BOD = 180^\circ$.

Hence $\angle AOC = \angle BOD$.

The proof can also be presented with the labelling technique.

Proof using number labels



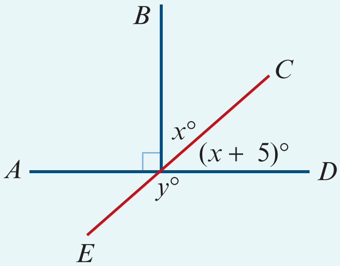
$\angle 1 + \angle 2 = 180^\circ$ (supplementary angles)

$\angle 2 + \angle 3 = 180^\circ$ (supplementary angles)

$\therefore \quad \angle 1 = \angle 3$

Example 1

Find the values of x and y in the diagram.



Solution

$x + (x + 5) = 90$ (complementary angles)

$2x = 85$

$\therefore \quad x = 42.5$

$y + (x + 5) = 180$ (supplementary angles)

$y + 47.5 = 180$

$\therefore \quad y = 132.5$

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Parallel lines

Given two distinct lines ℓ_1 and ℓ_2 in the plane, either the lines intersect in a single point or the lines have no point in common. In the latter case, the lines are said to be **parallel**. We can write this as $\ell_1 \parallel \ell_2$.

Here is another important assumption.

Playfair's axiom

Given any point P not on a line ℓ, there is only one line through P parallel to ℓ.

From this we have the following results for three distinct lines ℓ_1, ℓ_2 and ℓ_3 in the plane:

- If $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$, then $\ell_1 \parallel \ell_3$.
- If $\ell_1 \parallel \ell_2$ and ℓ_3 intersects ℓ_1, then ℓ_3 also intersects ℓ_2.

We prove the first of these and leave the other as an exercise. The proof is by contradiction.

Proof Let ℓ_1, ℓ_2 and ℓ_3 be three distinct lines in the plane such that $\ell_1 \parallel \ell_2$ and $\ell_2 \parallel \ell_3$. Now suppose that ℓ_1 is not parallel to ℓ_3. Then ℓ_1 and ℓ_3 meet at a point P. But by Playfair's axiom, there is only one line parallel to ℓ_2 passing through P. Therefore $\ell_1 = \ell_3$. But this gives a contradiction, as ℓ_1 and ℓ_3 are distinct by assumption.

Corresponding, alternate and co-interior angles

The following types of pairs of angles play an important role in considering parallel lines.

In the diagram, the lines ℓ_1 and ℓ_2 are crossed by a **transversal** ℓ_3.

Corresponding angles:

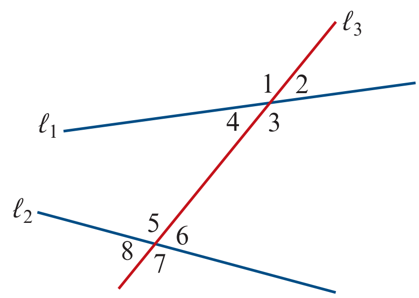
- Angles 1 and 5
- Angles 2 and 6
- Angles 3 and 7
- Angles 4 and 8

Alternate angles:

- Angles 3 and 5
- Angles 4 and 6

Co-interior angles:

- Angles 3 and 6
- Angles 4 and 5



The following result is easy to prove, and you should complete it as an exercise.

Theorem

When two lines are crossed by a transversal, any one of the following three conditions implies the other two:

- a pair of alternate angles are equal
- a pair of corresponding angles are equal
- a pair of co-interior angles are supplementary.

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The next result is important as it gives us the ability to establish properties of the angles associated with parallel lines crossed by a transversal, and it also gives us an easily applied method for proving that two lines are parallel.

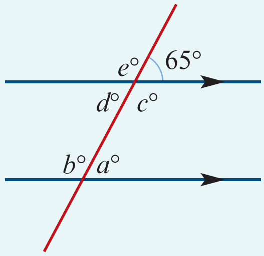
Theorem

- If two parallel lines are crossed by a transversal, then alternate angles are equal.
- Conversely, if two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Example 2

Find the values of the pronumerals.

Note: The arrows indicate that the two lines are parallel.



Solution

$a = 65$ (corresponding)

$d = 65$ (alternate with a)

$b = 115$ (co-interior with d)

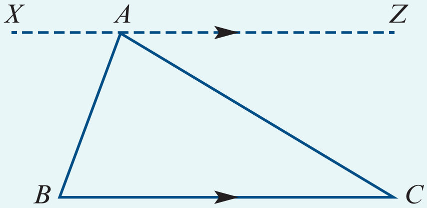
$e = 115$ (corresponding with b)

$c = 115$ (vertically opposite e)

Example 3

For $\triangle ABC$ shown in the diagram, the line XAZ is drawn through vertex A parallel to BC.

Use this construction to prove that the sum of the interior angles of a triangle is a straight angle (180°).



Solution

$\angle ABC = \angle XAB$ (alternate angles)

$\angle ACB = \angle ZAC$ (alternate angles)

$\angle XAB + \angle ZAC + \angle BAC$ is a straight angle.

Therefore $\angle ABC + \angle ACB + \angle BAC = 180°$.

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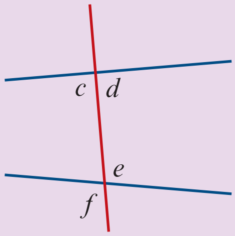
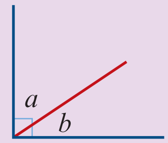
Section summary

- Pairs of angles
  - complementary (a and b)
  - supplementary (c and d)
  - vertically opposite (e and f)
  - alternate (c and e)
  - corresponding (c and f)
  - co-interior (d and e)





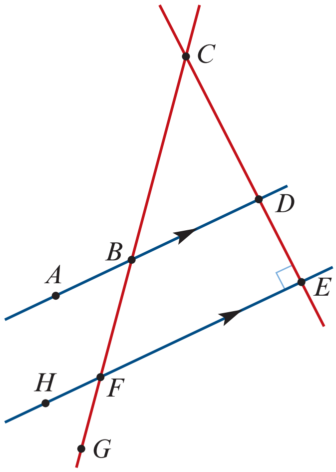
- Parallel lines

  If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.

  If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Exercise 9A

1 Consider the diagram shown.



a State whether each of the following angles is acute, obtuse, right or straight:

i $\angle ABC$ **ii** $\angle HFE$ **iii** $\angle CBD$ **iv** $\angle FED$

b State which angle is:

i corresponding to $\angle ABC$

ii alternate to $\angle ABF$

iii vertically opposite $\angle BFE$

iv co-interior to $\angle DBF$

c State which angles are:

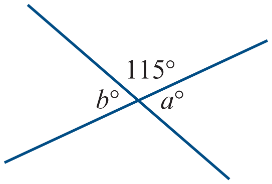
i complementary to $\angle BCD$

ii supplementary to $\angle CBD$

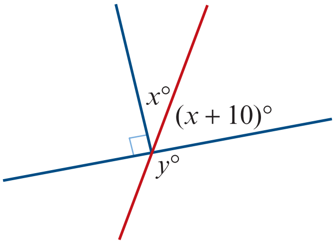
Example 1, 2

2 Calculate the values of the unknowns for each of the following. Give reasons.

a



b



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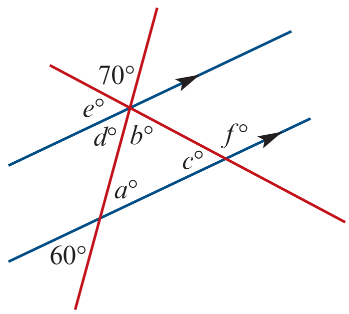
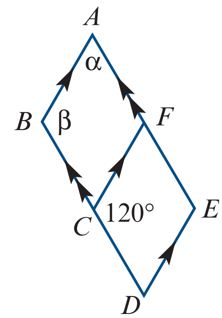
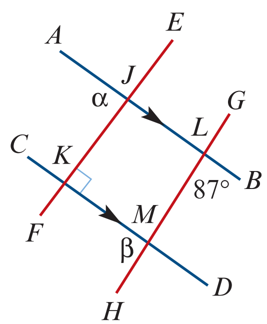
c



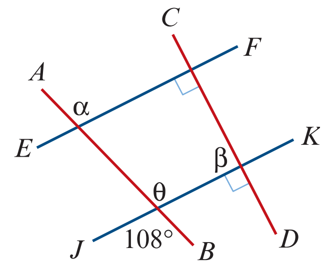
d



e

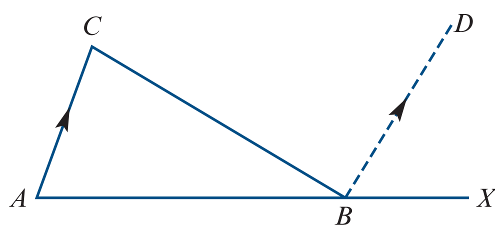


f

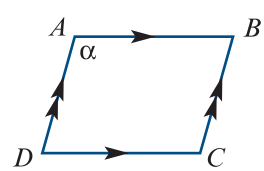


Example 3

3 Side AB of $\triangle ABC$ is extended to point X and line BD is drawn parallel to side AC.
Prove that the sum of two interior angles of a triangle is equal to the opposite exterior angle.
Hint: Using the diagram, this means showing that $\angle CAB + \angle ACB = \angle CBX$.



4 Recall that a parallelogram is a quadrilateral whose opposite sides are parallel. A parallelogram $ABCD$ is shown on the right. Let $\angle A = \alpha$.

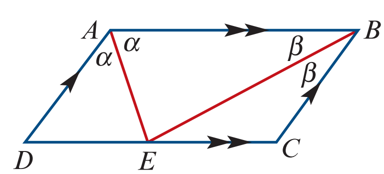


a Find the sizes of $\angle B$ and $\angle D$ in terms of α.

b Hence find the size of $\angle C$ in terms of α.

5 Prove the converse of the result in Question 4. That is, prove that if the opposite angles of a quadrilateral are equal, then the quadrilateral is a parallelogram.

6 Prove that AE is perpendicular to EB.



7 The lines PQ and RS are parallel. A transversal meets PQ at X and RS at Y. Lines XA and YB are bisectors of the angles PXY and XYS. Prove that XA is parallel to YB.

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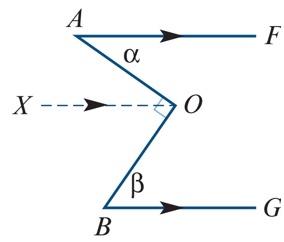
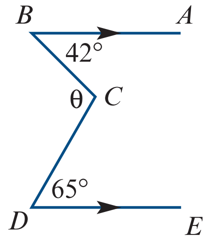
8 For the diagram on the right, show that $\alpha + \beta = 90°$.

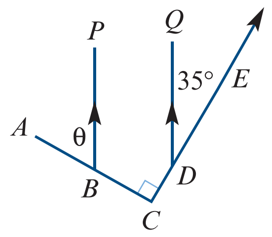


9 For each of the following, use a construction line to find the angle marked θ:

a



b



9B Triangles and polygons

We first define polygons.

A **line segment** AB is a portion of a line consisting of two distinct points A and B and all the points between them.

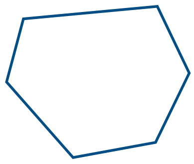
If distinct points $A_1, A_2, \ldots, A_n$ in the plane are connected in order by the line segments $A_1A_2, A_2A_3, \ldots, A_nA_1$, then the figure formed is a **polygon**. The points $A_1, A_2, \ldots, A_n$ are the vertices of the polygon, and the line segments $A_1A_2, A_2A_3, \ldots, A_nA_1$ are its sides.

Types of polygons

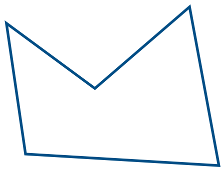
A **simple polygon** is a polygon such that no two sides have a point in common except a vertex.

A **convex polygon** is a polygon that contains each line segment connecting any pair of points on its boundary.

For example, the left-hand figure is convex, while the right-hand figure is not.



A convex polygon



A non-convex polygon

Note: In this chapter we will always assume that the polygons being considered are convex.

A **regular polygon** is a polygon in which all the angles are equal and all the sides are equal.

Names of polygons

- triangle (3 sides)
- quadrilateral (4 sides)
- pentagon (5 sides)
- hexagon (6 sides)
- heptagon (7 sides)
- octagon (8 sides)
- nonagon (9 sides)
- decagon (10 sides)
- dodecagon (12 sides)

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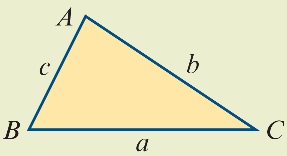
Triangles

A **triangle** is a figure formed by three line segments determined by a set of three points not on one line. If the three points are A, B and C, then the figure is called triangle ABC and commonly written $\triangle ABC$. The points A, B and C are called the vertices of the triangle.

Triangle inequality

An important property of a triangle is that any side is shorter than the sum of the other two.

In $\triangle ABC$: $\quad a < b + c,\ \ b < c + a$ and $c < a + b$.



Note: For $\triangle ABC$ labelled as shown, we have $c < b < a$ if and only if $\angle C < \angle B < \angle A$.

The following two results have been proved in Example 3 and in Question 3 of Exercise 9A.

Angles of a triangle

- The sum of the three interior angles of a triangle is 180°.
- The sum of two interior angles of a triangle is equal to the opposite exterior angle.

Classification of triangles

Equilateral triangle	a triangle in which all three sides are equal
Isosceles triangle	a triangle in which two sides are equal
Scalene triangle	a triangle in which all three sides are unequal

Important lines in a triangle

Median A **median** of a triangle is a line segment from a vertex to the midpoint of the opposite side.

Altitude An **altitude** of a triangle is a line segment from a vertex to the opposite side (possibly extended) which forms a right angle where it meets the opposite side.

Example 4

The sides of a triangle are $6 - x$, $4x + 1$ and $2x + 3$. Find the value of x for which the triangle is isosceles, and show that if it is isosceles, then it is equilateral.

Solution

$$6 - x = 4x + 1$$
$$\Rightarrow \quad 5x = 5$$
$$\Rightarrow \quad x = 1$$

When $x = 1$, we have $6 - x = 5$, $4x + 1 = 5$ and $2x + 3 = 5$. Hence the triangle is equilateral with each side of length 5 units.

Explanation

We want to show that if any two side lengths are equal, then the third length is the same.

It is enough to show that the three lines $y = 6 - x$, $y = 4x + 1$ and $y = 2x + 3$ intersect in a common point.

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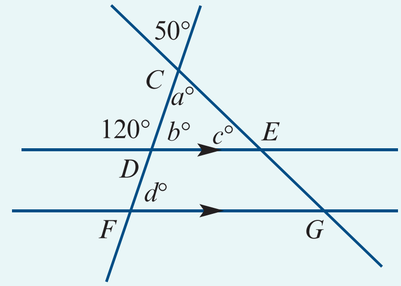
Example 5

Find the values of a, b, c and d, giving reasons.



Solution

$a = 50$ (vertically oposite angles)

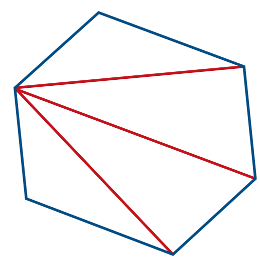
$b = 60$ (supplementary angles)

$c = 180 - (50 + 60) = 70$ (angle sum of a triangle)

$d = 60$ (corresponding angles $DE \parallel FG$)

Angle sum of a polygon

If a polygon has n sides, then we can draw $n - 3$ diagonals from a vertex. In this way, we can divide the polygon into $n - 2$ triangles, each with an angle sum of 180°.



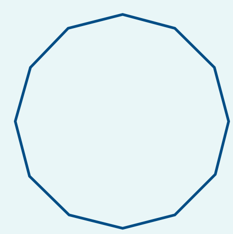
We have drawn a hexagon to illustrate this, but we could have used any polygon.

Angle sum of a polygon

- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- Each interior angle of a regular n-sided polygon has size $\dfrac{(n-2)}{n}180°$.

Example 6

A regular dodecagon is shown to the right.



a Find the sum of the interior angles of a dodecagon.

b Find the size of each interior angle of a regular dodecagon.

Solution

a The angle sum of a polygon with n sides is $(n - 2)180°$.
Therefore the angle sum of a dodecagon is 1800°.

b Each of the interior angles is $\dfrac{1800}{12} = 150°$.

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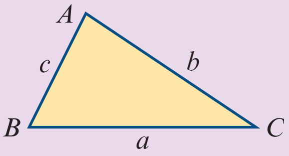
Section summary

- Polygons
  - The sum of the interior angles of an n-sided polygon is $(n-2)180°$.
  - In a **regular polygon**, all the angles are equal and all the sides are equal.
    Each interior angle of a regular n-sided polygon has size $\dfrac{(n-2)}{n}180°$.
- Triangles
  - An **equilateral triangle** is a triangle in which all three sides are equal.
  - An **isosceles triangle** is a triangle in which two sides are equal.
  - A **scalene triangle** is a triangle in which all three sides are unequal.
  - The sum of the three interior angles of a triangle is 180°.
  - The sum of two interior angles of a triangle is equal to the opposite exterior angle.

  In $\triangle ABC$:
  - $a < b + c$, $b < c + a$ and $c < a + b$
  - $c < b < a$ if and only if $\angle C < \angle B < \angle A$



Exercise 9B

1 Is it possible for a triangle to have sides of lengths:

a 12 cm, 9 cm, 20 cm
b 24 cm, 24 cm, 40 cm
c 5 cm, 5 cm, 5 cm
d 12 cm, 9 cm, 2 cm?

2 Describe each of the triangles in Question 1.

3 If a triangle has sides 10 cm and 20 cm, what can be said about the third side?

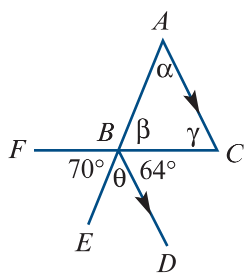
4 The sides of a triangle are $2n - 1$, $n + 5$ and $3n - 8$.

a Find the value(s) of n for which the triangle is isosceles.
b Is there a value of n which makes the triangle equilateral?

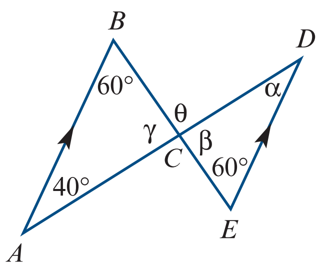
Example 4
5 The sides of a triangle are $2n - 1$, $n + 7$ and $3n - 9$. Prove that if the triangle is isosceles, then it is equilateral.

Example 5
6 Calculate the value of the unknowns for each of the following. Give reasons.

a



b



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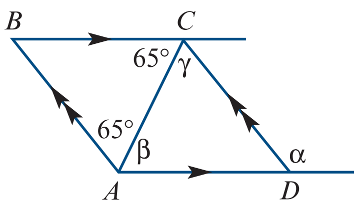

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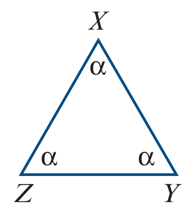

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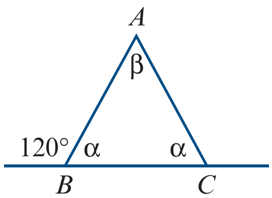

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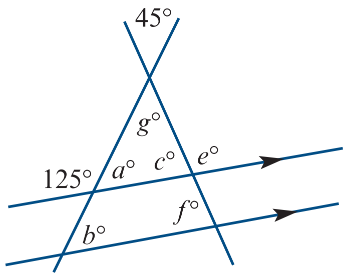

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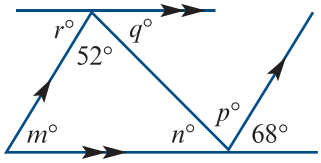
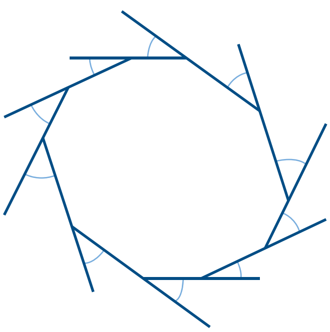


Example 6

7 Find the interior-angle sum and the size of each angle of a regular polygon with:

a 6 sides **b** 12 sides **c** 20 sides

8 In the decagon shown on the right, each side has been extended to form an exterior angle.



a Explain why the sum of the interior angles plus the sum of the exterior angles is 1800°.

b Hence find the sum of the decagon's 10 exterior angles.

9 Prove that the sum of the exterior angles of any polygon is 360°.

10 If the sum of the interior angles of a polygon is four times the sum of the exterior angles, how many sides does the polygon have?

11 Assume that the sum of the interior angles of a polygon is k times the sum of the exterior angles (where $k \in \mathbb{N}$). Prove that the polygon has $2(k + 1)$ sides.

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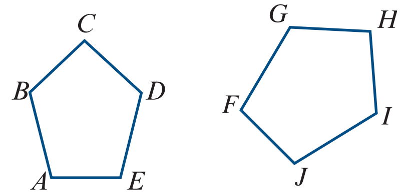
9C Congruence and proofs

Two plane figures are called **congruent** if one figure can be moved on top of the other figure, by a sequence of translations, rotations and reflections, so that they coincide exactly.

Congruent figures have exactly the same shape and size. For example, the two figures shown are congruent. We can write:

pentagon $ABCDE \equiv$ pentagon $FGHIJ$



When two figures are congruent, we can find a transformation that pairs up every part of one figure with the corresponding part of the other, so that:

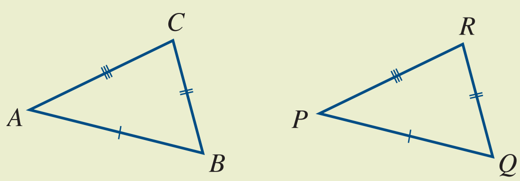
- paired angles have the same size
- paired intervals have the same length
- paired regions have the same area.

Congruent triangles

There are four standard tests for two triangles to be congruent.

- The SSS congruence test

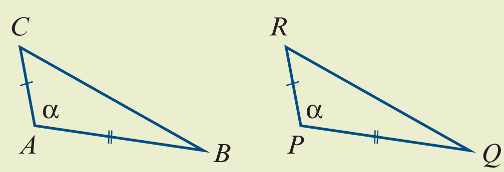
  If the three sides of one triangle are respectively equal to the three sides of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- The SAS congruence test

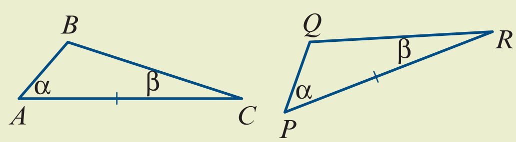
  If two sides and the included angle of one triangle are respectively equal to two sides and the included angle of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- The AAS congruence test

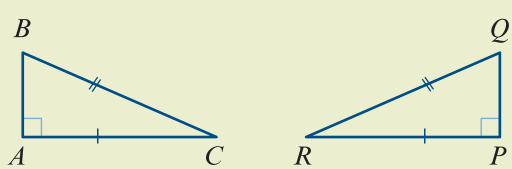
  If two angles and one side of one triangle are respectively equal to two angles and the matching side of another triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

- The RHS congruence test

  If the hypotenuse and one side of one right-angled triangle are respectively equal to the hypotenuse and one side of another right-angled triangle, then the two triangles are congruent.



$\triangle ABC \equiv \triangle PQR$

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Classification of quadrilaterals

Trapezium	a quadrilateral with at least one pair of opposite sides parallel
Parallelogram	a quadrilateral with both pairs of opposite sides parallel
Rhombus	a parallelogram with a pair of adjacent sides equal
Rectangle	a quadrilateral in which all angles are right angles
Square	a quadrilateral that is both a rectangle and a rhombus

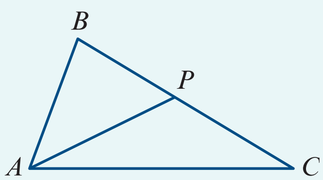
Proofs using congruence

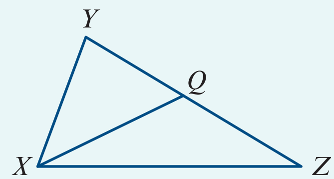
Example 7

Let $\triangle ABC$ and $\triangle XYZ$ be such that $\angle BAC = \angle YXZ$, $AB = XY$ and $AC = XZ$.

If P and Q are the midpoints of BC and YZ respectively, prove that $AP = XQ$.

Solution





From the given conditions, we have $\triangle ABC \equiv \triangle XYZ$ (SAS).

Therefore $\angle ABP = \angle XYQ$ and $BC = YZ$.

Thus $BP = YQ$, as P and Q are the midpoints of BC and YZ respectively.

Hence $\triangle ABP \equiv \triangle XYQ$ (SAS) and so $AP = XQ$.

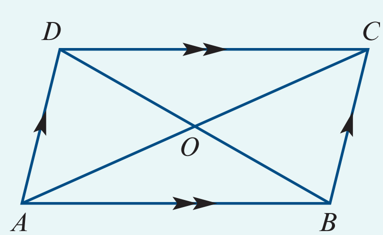
Example 8

a Prove that, in a parallelogram, the diagonals bisect each other.

b Prove that if the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

Solution

a Note that opposite sides of a parallelogram are equal. (See Question 8 of Exercise 9C.)



In triangles DOC and BOA:

$\angle ODC = \angle OBA$ (alternate angles $CD \parallel AB$)

$\angle OCD = \angle OAB$ (alternate angles $CD \parallel AB$)

$\angle AOB = \angle DOC$ (vertically opposite)

$AB = CD$ (opposite sides of parallelogram are equal)

$\triangle DOC \equiv \triangle BOA$ (AAS)

Hence $AO = OC$ and $DO = OB$.

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b

$OD = OB$ (diagonals bisect each other)

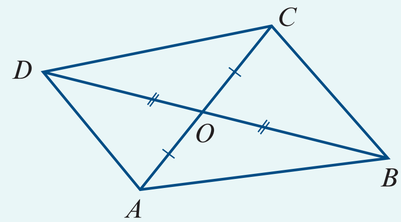
$OA = OC$ (diagonals bisect each other)

$\angle AOB = \angle DOC$ (vertically opposite)

$\angle DOA = \angle COB$ (vertically opposite)

$\triangle DOC \equiv \triangle BOA$ (SAS)

$\triangle DOA \equiv \triangle BOC$ (SAS)



Therefore $\angle ODC = \angle OBA$ and so $CD \parallel AB$, since alternate angles are equal. Similarly, we have $AD \parallel BC$. Hence $ABCD$ is a parallelogram.

Example 9

Prove that the triangle formed by joining the midpoints of the three sides of an isosceles triangle (with the midpoints as the vertices of the new triangle) is also isosceles.

Solution

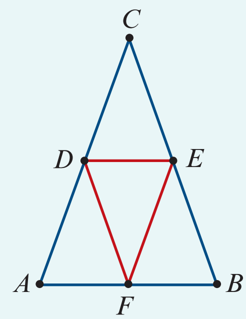
Assume $\triangle ABC$ is isosceles with $CA = CB$ and $\angle CAB = \angle CBA$. (See Question 3 of Exercise 9C.)

Then we have $DA = EB$, where D and E are the midpoints of CA and CB respectively.

We also have $AF = BF$, where F is the midpoint of AB.

Therefore $\triangle DAF \equiv \triangle EBF$ (SAS).

Hence $DF = EF$ and so $\triangle DEF$ is isosceles.



Section summary

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:
  - **SSS** the three sides of one triangle are equal to the three sides of the other triangle
  - **SAS** two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle
  - **AAS** two angles and one side of one triangle are equal to two angles and the matching side of the other triangle
  - **RHS** the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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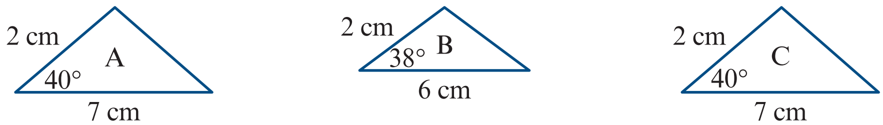
Exercise 9C

1 In each part, find pairs of congruent triangles. State the congruence tests used.

a



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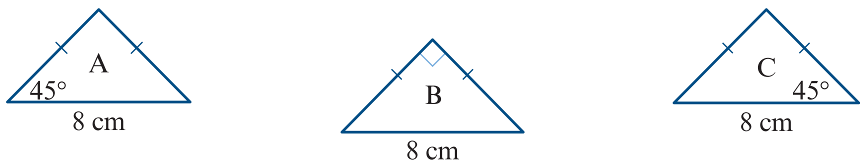

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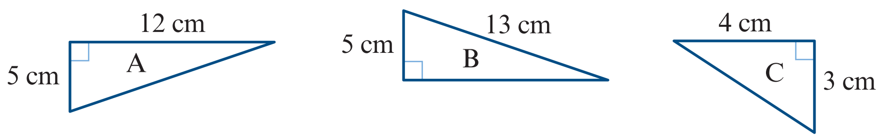
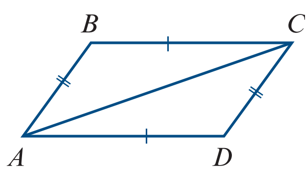


2 Name the congruent triangles and state the congruence test used:

a



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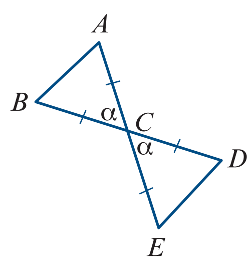

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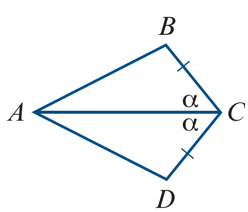

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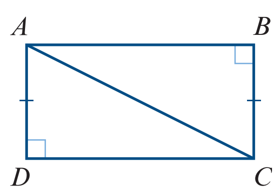

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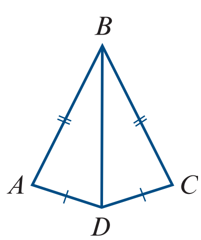

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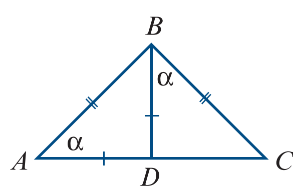

Example 7 **3** Prove that if $\triangle ABC$ is isosceles with $AB = AC$, then $\angle ABC = \angle ACB$.

4 Prove that if $\triangle ABC$ is such that $\angle ABC = \angle ACB$, then $\triangle ABC$ is isosceles. (This is the converse of Question 3.)

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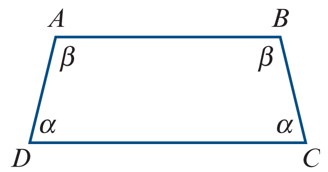
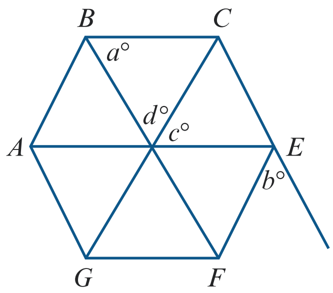
5 For the quadrilateral shown, prove that $AB \parallel CD$.



6 $ABCEFG$ is a regular hexagon.

a Find the values of a, b, c and d.

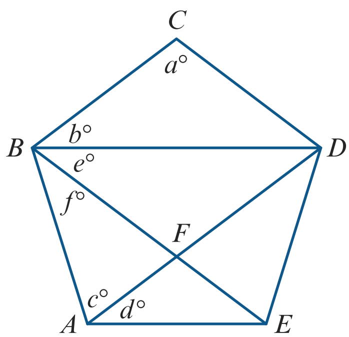
b Prove that $AE \parallel BC$ and $CG \parallel BA$.



7 $ABCDE$ is a regular pentagon.

a Find the values of a, b, c, d, e and f.

b Prove that $AE \parallel BD$ and $BE \parallel CD$.



Example 8

8 Proofs involving parallelograms Prove each of the following:

a In a parallelogram, opposite sides are equal and opposite angles are equal.

b If each side of a quadrilateral is equal to the opposite side, then the quadrilateral is a parallelogram.

c If each angle of a quadrilateral is equal to the opposite angle, then the quadrilateral is a parallelogram.

d If one side of a quadrilateral is equal and parallel to the opposite side, then the quadrilateral is a parallelogram.

9 Let $ABCD$ be a parallelogram and let P and Q be the midpoints of AB and DC respectively. Prove that $APCQ$ is a parallelogram.

10 Let $PQRS$ be a parallelogram whose diagonals meet at O. Let X, Y, Z and W be the midpoints of PO, QO, RO and SO respectively. Prove that $XYZW$ is a parallelogram.

11 Proofs involving rhombuses Prove each of the following:

a The diagonals of a rhombus bisect each other at right angles.

b The diagonals of a rhombus bisect the vertex angles through which they pass.

c If the diagonals of a quadrilateral bisect each other at right angles, then the quadrilateral is a rhombus.

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12 Proofs involving rectangles Prove each of the following:

a The diagonals of a rectangle are equal and bisect each other.

b A parallelogram with one right angle is a rectangle.

c If the diagonals of a quadrilateral are equal and bisect each other, then the quadrilateral is a rectangle.

Example 9 **13** $ABCDE$ is a pentagon in which all the sides are equal and diagonal AC is equal to diagonal AD. Prove that $\angle ABC = \angle AED$.

14 $\triangle ABC$ is equilateral and its sides are extended to points X, Y and Z so that AY, BZ and CX are all equal in length to the sides of $\triangle ABC$. Prove that $\triangle XYZ$ is also equilateral.

15 $ABCD$ is a quadrilateral in which $AB = BC$ and $AD = DC$. The diagonal BD is extended to a point K. Prove that $AK = CK$.

16 Prove that if the angle C of a triangle ABC is equal to the sum of the other two angles, then the length of side AB is equal to twice the length of the line segment joining C with the midpoint of AB.

17 Prove that if NO is the base of isosceles triangle MNO and if the perpendicular from N to MO meets MO at A, then angle ANO is equal to half of angle NMO.

18 If a median of a triangle is drawn, prove that the perpendiculars from the other vertices upon this median are equal. (The median may be extended.)

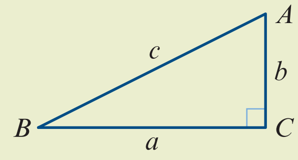
9D Pythagoras' theorem

Pythagoras' theorem

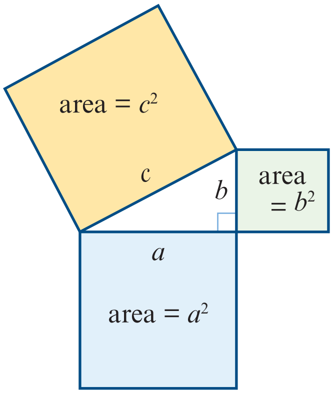
Let ABC be a triangle with side lengths a, b and c.

If $\angle C$ is a right angle, then

$$a^2 + b^2 = c^2$$



Pythagoras' theorem can be illustrated by the diagram shown here. The sum of the areas of the two smaller squares is equal to the area of the square on the longest side (hypotenuse).



There are many different proofs of Pythagoras' theorem. One was given at the start of Chapter 8. Here we give another proof, due to James A. Garfield, the 20th President of the United States.

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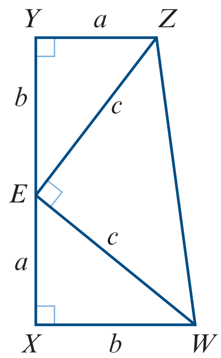
Proof The proof is based on the diagram shown on the right.



Area of trapezium $XYZW = \frac{1}{2}(a+b)(a+b)$

Area of $\triangle EWX$ + area of $\triangle EYZ$ + area of $\triangle EWZ$

$$= \tfrac{1}{2}ab + \tfrac{1}{2}ab + \tfrac{1}{2}c^2$$
$$= ab + \tfrac{1}{2}c^2$$

Thus $\frac{1}{2}(a+b)(a+b) = ab + \frac{1}{2}c^2$

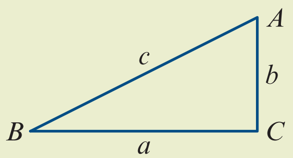
$$a^2 + 2ab + b^2 = 2ab + c^2$$

Hence $a^2 + b^2 = c^2$

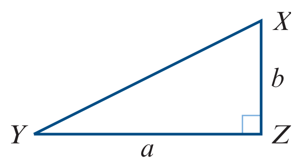
Converse of Pythagoras' theorem

Let ABC be a triangle with side lengths a, b and c.

If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



Proof Assume $\triangle ABC$ has side lengths $a = BC$, $b = CA$ and $c = AB$ such that $a^2 + b^2 = c^2$.



Construct a second triangle $\triangle XYZ$ with $YZ = a$ and $ZX = b$ such that $\angle XZY$ is a right angle.

By Pythagoras' theorem, the length of the hypotenuse of $\triangle XYZ$ is

$$\sqrt{a^2+b^2} = \sqrt{c^2}$$
$$= c$$

Therefore $\triangle ABC \equiv \triangle XYZ$ (SSS).

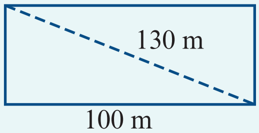
Hence $\angle C$ is a right angle.

Example 10

The diagonal of a soccer field is 130 m and the length of the long side of the field is 100 m. Find the length of the short side, correct to the nearest centimetre.

Solution

Let x be the length of the short side. Then



$$x^2 + 100^2 = 130^2$$
$$x^2 = 130^2 - 100^2$$
$$\therefore \quad x = \sqrt{6900}$$

Correct to the nearest centimetre, the length of the short side is 83.07 m.

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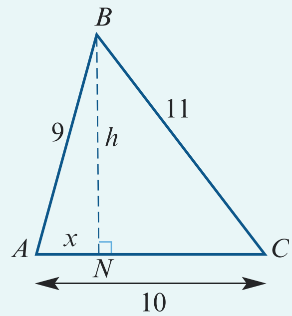
Example 11

Consider $\triangle ABC$ with $AB = 9$ cm, $BC = 11$ cm and $AC = 10$ cm. Find the length of the altitude of $\triangle ABC$ on AC.

Solution



Let BN be the altitude on AC as shown, with $BN = h$ cm.

Let $AN = x$ cm. Then $CN = (10 - x)$ cm.

In $\triangle ABN$: $\quad x^2 + h^2 = 81 \quad (1)$

In $\triangle CBN$: $\quad (10 - x)^2 + h^2 = 121 \quad (2)$

Expanding in equation (2) gives

$$100 - 20x + x^2 + h^2 = 121$$

Substituting for $x^2 + h^2$ from (1) gives

$$100 - 20x + 81 = 121$$

$$\therefore \quad x = 3$$

Substituting in (1), we have

$$9 + h^2 = 81$$

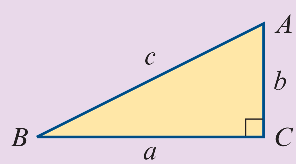
$$h^2 = 72$$

$$\therefore \quad h = 6\sqrt{2}$$

The length of altitude BN is $6\sqrt{2}$ cm.

Section summary

Pythagoras' theorem and its converse



Let ABC be a triangle with side lengths a, b and c.

- If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
- If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.

Exercise 9D

1. An 18 m ladder is 7 m away from the bottom of a vertical wall. How far up the wall does it reach?

Example 10

2. Find the length of the diagonal of a rectangle with dimensions 40 m by 9 m.

3. In a circle of centre O, a chord AB is of length 4 cm. The radius of the circle is 14 cm. Find the distance of the chord from O.

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4 A square has an area of 169 cm^2. What is the length of the diagonal?

5 Find the area of a square with a diagonal of length:

a 10 cm

b 8 cm

6 $ABCD$ is a square of side length 2 cm. If E is a point on AB extended and $CA = CE$, find the length of DE.

7 In a square of side length 2 cm, the midpoints of each side are joined to form a new square. Find the area of the new square.

Example 11

8 Consider $\triangle ABC$ with $AB = 7$ cm, $BC = 6$ cm and $AC = 5$ cm. Find the length of AN, the altitude on BC.

9 Which of the following are the three side lengths of a right-angled triangle?

a 5 cm, 6 cm, 7 cm

b 3.9 cm, 3.6 cm, 1.5 cm

c 2.4 cm, 2.4 cm, 4 cm

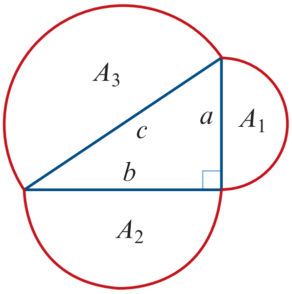
d 82 cm, 18 cm, 80 cm

10 Prove that a triangle with sides lengths $x^2 - 1$, $2x$ and $x^2 + 1$ is a right-angled triangle.

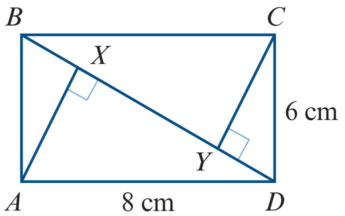
11 Consider $\triangle ABC$ such that $AB = 20$ cm, $AC = 15$ cm and the altitude AN has length 12 cm. Prove that $\triangle ABC$ is a right-angled triangle.

12 Find the length of an altitude in an equilateral triangle with side length 16 cm.

13 Three semicircles are drawn on the sides of this right-angled triangle. Let the areas of these semicircles be A_1, A_2 and A_3. Prove that $A_3 = A_1 + A_2$.



14 Rectangle $ABCD$ has $CD = 6$ cm and $AD = 8$ cm. Line segments CY and AX are drawn such that points X and Y lie on BD and $\angle AXD = \angle CYD = 90°$. Find the length of XY.



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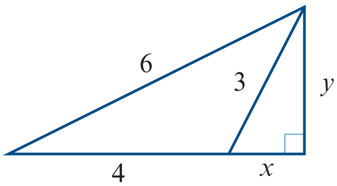
15 Find the values of x and y.



16 If P is a point in rectangle $ABCD$ such that $PA = 3$ cm, $PB = 4$ cm and $PC = 5$ cm, find the length of PD.

17 Let AQ be an altitude of $\triangle ABC$, where Q lies between B and C. Let P be the midpoint of BC. Prove that $AB^2 + AC^2 = 2PB^2 + 2AP^2$.

18 For a parallelogram $ABCD$, prove that $2AB^2 + 2BC^2 = AC^2 + BD^2$.

9E Ratios

This section is revision of work of previous years.

Example 12

Divide 300 in the ratio 3 : 2.

Solution

$$\begin{aligned} \text{one part} &= 300 \div 5 = 60 \\ \therefore \quad \text{two parts} &= 60 \times 2 = 120 \\ \therefore \quad \text{three parts} &= 60 \times 3 = 180 \end{aligned}$$

Example 13

Divide 3000 in the ratio 3 : 2 : 1.

Solution

$$\begin{aligned} \text{one part} &= 3000 \div 6 = 500 \\ \therefore \quad \text{two parts} &= 500 \times 2 = 1000 \\ \therefore \quad \text{three parts} &= 500 \times 3 = 1500 \end{aligned}$$

Exercise 9E

Skillsheet

Example 12

1 Divide 9000 in the ratio 2 : 7.

Example 13

2 Divide 15 000 in the ratio 2 : 2 : 1.

3 $x : 6 = 9 : 15$. Find x.

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4 The ratio of the numbers of orange flowers to pink flowers in a garden is 6 : 11. There are 144 orange flowers. How many pink flowers are there?

5 $15 : 2 = x : 3$. Find x.

6 The angles of a triangle are in the ratio 6 : 5 : 7. Find the sizes of the three angles.

7 Three men X, Y and Z share an amount of money in the ratio 2 : 3 : 7. If Y receives \$2 more than X, how much does Z receive?

8 An alloy consists of copper, zinc and tin in the ratio 1 : 3 : 4 (by weight). If there is 10 g of copper in the alloy, find the weights of zinc and tin.

9 The ratio of red beads to white beads to green beads in a bag is 7 : 2 : 1. If there are 56 red beads, how many white beads and how many green beads are there?

10 On a map, the length of a road is represented by 45 mm. If the scale is 1 : 125 000, find the actual length of the road.

11 Five thousand two hundred dollars was divided between a mother and daughter in the ratio 8 : 5. Find the difference between the sums they received.

12 Points A, B, C and D are placed in that order on a line so that $AB = 2BC = CD$. Express BD as a fraction of AD.

13 If the radius of a circle is increased by two units, find the ratio of the new circumference to the new diameter.

14 In a class of 30 students, the ratio of boys to girls is 2 : 3. If six boys join the class, find the new ratio of boys to girls in the class.

15 If $a : b = 3 : 4$ and $a : (b + c) = 2 : 5$, find the ratio $a : c$.

16 The scale of a map is 1 : 250 000. Find the distance, in kilometres, between two towns which are 3.5 cm apart on the map.

17 Prove that if $\dfrac{a-c}{b-d} = \dfrac{c}{d}$, then $\dfrac{a}{b} = \dfrac{c}{d}$.

18 Prove that if $\dfrac{a}{x} = \dfrac{b}{y} = \dfrac{c}{z} = \dfrac{2}{3}$, then $\dfrac{a+b+c}{x+y+z} = \dfrac{2}{3}$.

19 Prove that if $\dfrac{x}{y} = \dfrac{m}{n}$, then $\dfrac{x+y}{x-y} = \dfrac{m+n}{m-n}$.

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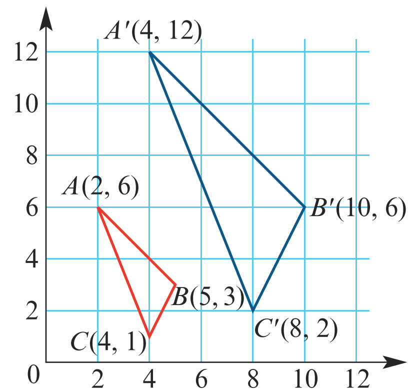
9F An introduction to similarity

The two triangles ABC and $A'B'C'$ shown in the diagram are similar.

Note: $OA' = 2OA$, $OB' = 2OB$, $OC' = 2OC$

Triangle $A'B'C'$ can be considered as the image of triangle ABC under a mapping of the plane in which the coordinates are multiplied by 2.



This mapping is called an **expansion** from the origin of factor 2. From now on we will call this factor the **similarity factor**.

The rule for this mapping can be written in transformation notation as $(x, y) \to (2x, 2y)$.

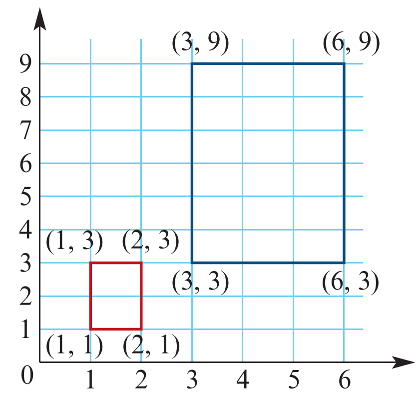
There is also a mapping from $\triangle A'B'C'$ to $\triangle ABC$, which is an expansion from the origin of factor $\frac{1}{2}$. The rule for this is $(x, y) \to (\frac{1}{2}x, \frac{1}{2}y)$.

Two figures are called **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.

- Matching lengths of similar figures are in the same ratio.
- Matching angles of similar figures are equal.

For example, the rectangle with side lengths 1 and 2 is similar to the rectangle with side lengths 3 and 6.

Here the similarity factor is 3 and the rule for the mapping is $(x, y) \to (3x, 3y)$.



Notes:

- Any two circles are similar.
- Any two squares are similar.
- Any two equilateral triangles are similar.

► Similar triangles

If triangle ABC is similar to triangle $A'B'C'$, we can write this as

$$\triangle ABC \sim \triangle A'B'C'$$

The triangles are named so that angles of equal magnitude hold the same position. That is, A matches to A', B matches to B' and C matches to C'. So we have

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = k$$

where k is the **similarity factor**.

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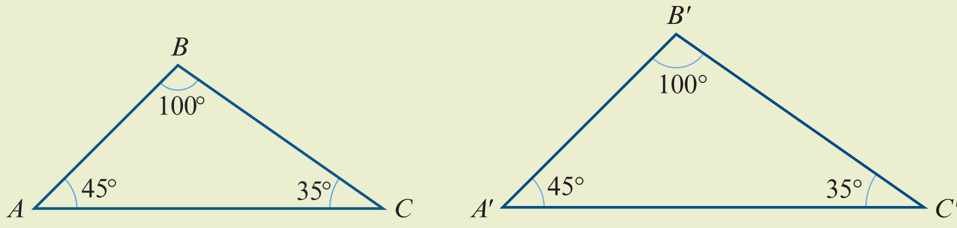
There are four standard tests for two triangles to be similar.

- **The AAA similarity test**

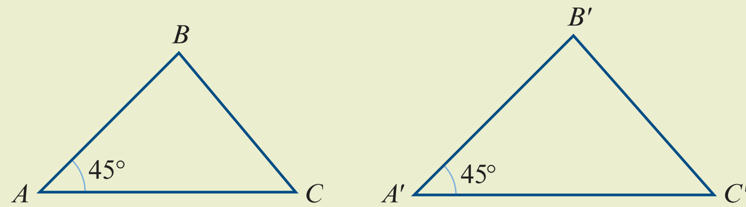
  If two angles of one triangle are respectively equal to two angles of another triangle, then the two triangles are similar.



- **The SAS similarity test**

  If the ratios of two pairs of matching sides are equal and the included angles are equal, then the two triangles are similar.

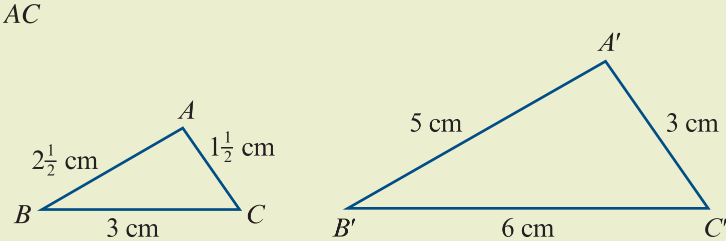
$$\frac{A'B'}{AB} = \frac{A'C'}{AC}$$



- **The SSS similarity test**

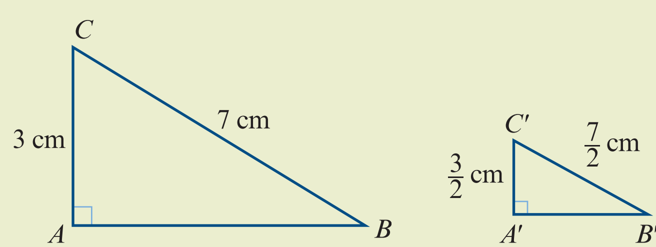
  If the sides of one triangle can be matched up with the sides of another triangle so that the ratio of matching lengths is constant, then the two triangles are similar.

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{A'C'}{AC}$$



- **The RHS similarity test**

  If the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides, then the two triangles are similar.



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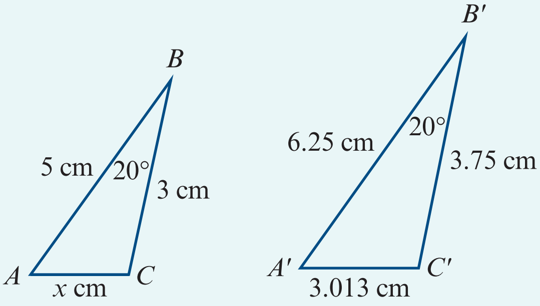
Example 14

a Give the reason for triangle ABC being similar to triangle $A'B'C'$.

b Find the value of x.



Solution

a Triangle ABC is similar to triangle $A'B'C'$ by SAS, since

$$\frac{5}{6.25} = 0.8 = \frac{3}{3.75}$$

and $\angle ABC = 20^\circ = \angle A'B'C$

b $$\frac{x}{3.013} = \frac{5}{6.25}$$

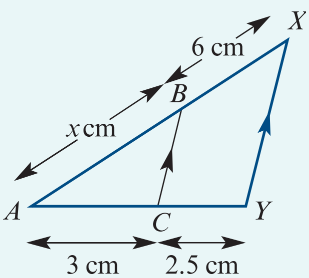
$$\therefore \quad x = \frac{5}{6.25} \times 3.013$$

$$= 2.4104$$

Example 15

a Give the reason for triangle ABC being similar to triangle AXY.

b Find the value of x.



Solution

a Corresponding angles are of equal magnitude (AAA).

b $$\frac{AB}{AX} = \frac{AC}{AY}$$

$$\frac{x}{x+6} = \frac{3}{5.5}$$

$$5.5x = 3(x+6)$$

$$2.5x = 18$$

$$\therefore \quad x = 7.2$$

Section summary

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.

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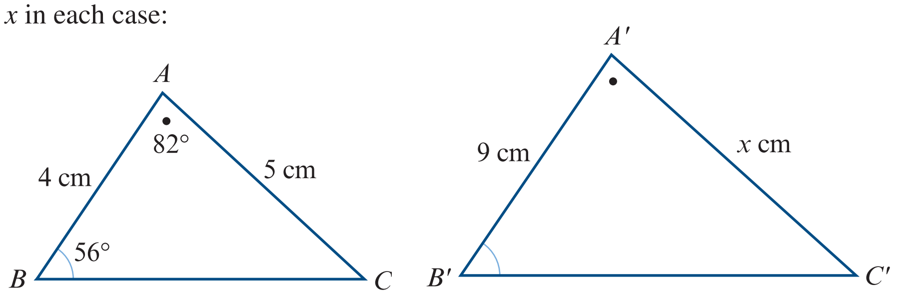
- Two triangles are similar provided any one of the following four conditions holds:
  - **AAA** two angles of one triangle are equal to two angles of the other triangle
  - **SAS** the ratios of two pairs of matching sides are equal and the included angles are equal
  - **SSS** the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant
  - **RHS** the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.

Exercise 9F

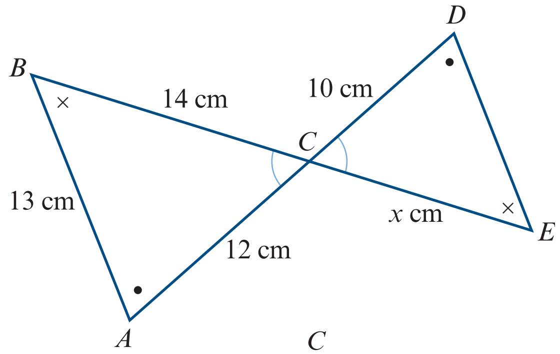
Example 14

1 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:

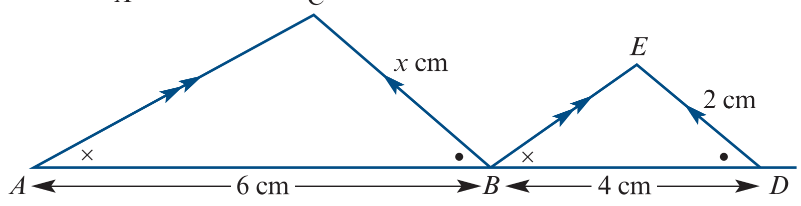
a



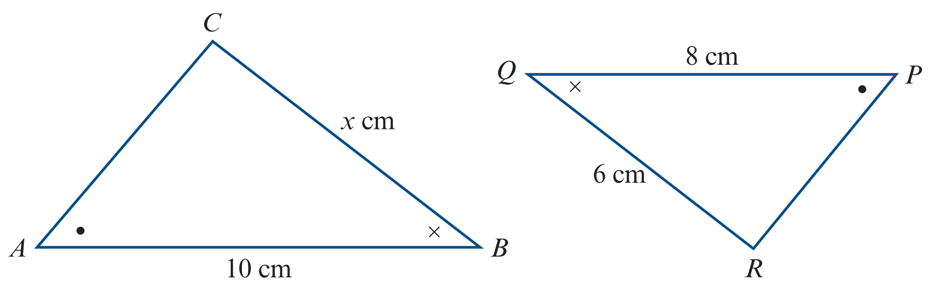
b



c



d



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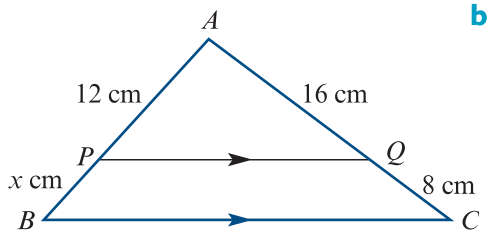
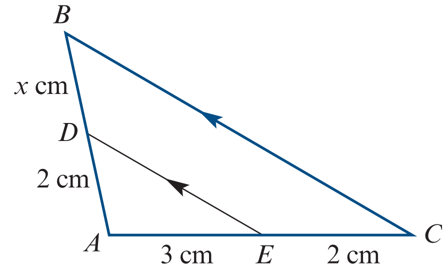
Example 15

2 Give reasons why each of the following pairs of triangles are similar and find the value of x in each case:

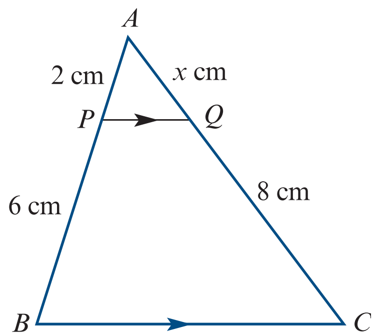
a



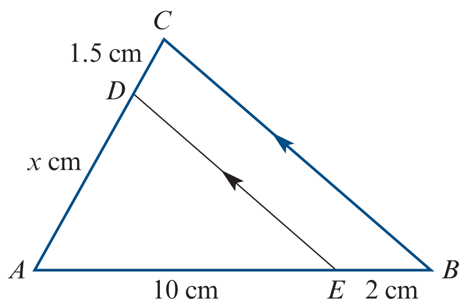
b



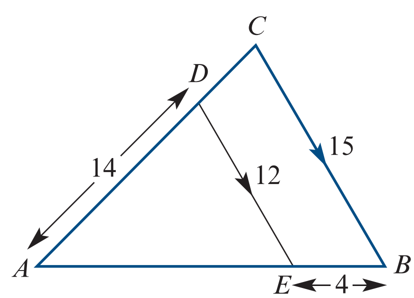
c



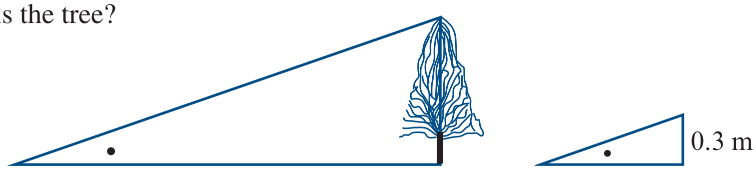
d



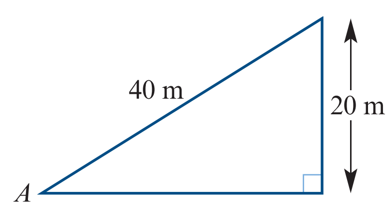
3 Given that $AD = 14$, $DE = 12$, $BC = 15$ and $EB = 4$, find AC, AE and AB.



4 A tree casts a shadow of 33 m and at the same time a stick 30 cm long casts a shadow of 224 cm. How tall is the tree?



5 A 20 m high neon sign is supported by a 40 m steel cable as shown. An ant crawls along the cable starting at A. How high is the ant when it is 15 m from A?



6 A hill has a gradient of 1 in 20, i.e. for every 20 m horizontally there is a 1 m increase in height. If you go 300 m horizontally, how high up will you be?

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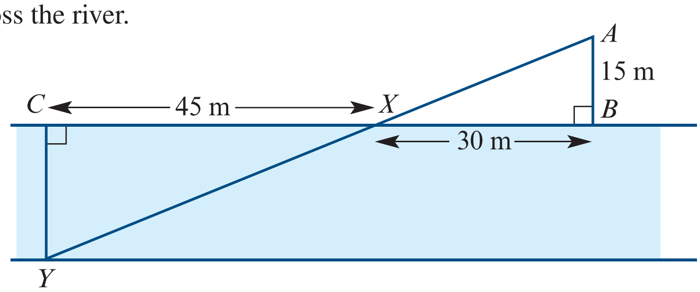
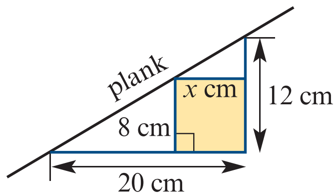
7 A man stands at A and looks at point Y across the river. He gets a friend to place a stone at X so that the three points A, X and Y are collinear (that is, they all lie on a single line). He then measures AB, BX and XC to be 15 m, 30 m and 45 m respectively. Find CY, the distance across the river.

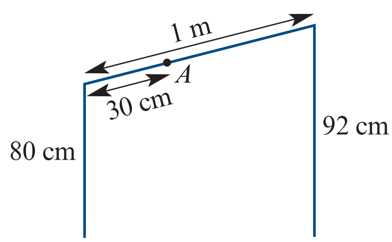


8 Find the height, h m, of a tree that casts a shadow 32 m long at the same time that a vertical straight stick 2 m long casts a shadow 6.2 m long.

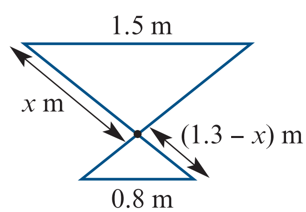
9 A plank is placed straight up stairs that are 20 cm wide and 12 cm deep. Find x, where x cm is the width of the widest rectangular box of height 8 cm that can be placed on a stair under the plank.



10 The sloping edge of a technical drawing table is 1 m from front to back. Calculate the height above the ground of a point A, which is 30 cm from the front edge.



11 Two similar rods 1.3 m long have to be hinged together to support a table 1.5 m wide. The rods have been fixed to the floor 0.8 m apart. Find the position of the hinge by finding the value of x.



12 A man whose eyes are 1.7 m from the ground, when standing 3.5 m in front of a wall 3 m high, can just see the top of a tower that is 100 m away from the wall. Find the height of the tower.

13 A man is 8 m up a 10 m ladder, the top of which leans against a vertical wall and touches it at a height of 9 m above the ground. Find the height of the man above the ground.

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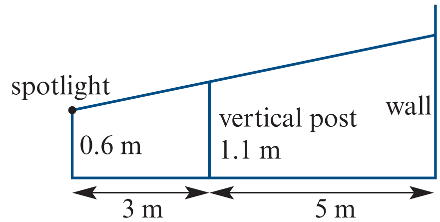
14 A spotlight is at a height of 0.6 m above ground level. A vertical post 1.1 m high stands 3 m away and 5 m further away there is a vertical wall. How high up the wall does the shadow reach?



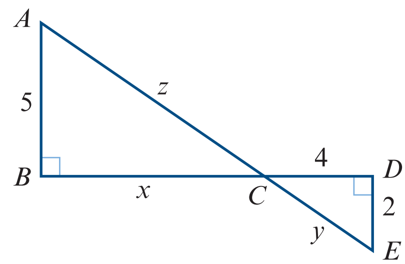
15 Consider the diagram on the right.

a Prove that $\triangle ABC \sim \triangle EDC$.

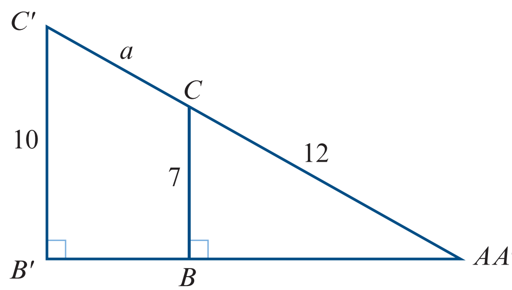
b Find x.

c Use Pythagoras' theorem to find y and z.

d Verify that $y : z = ED : AB$.

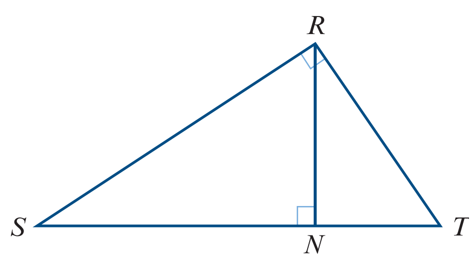


16 Find a.



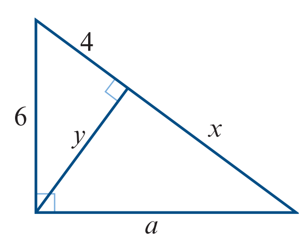
17 A man who is 1.8 m tall casts a shadow of 0.76 m in length. If at the same time a telephone pole casts a 3 m shadow, find the height of the pole.

18 In the diagram shown, $RT = 4$ cm and $ST = 10$ cm. Find the length NT.



19 ABC is a triangular frame with $AB = 14$ m, $BC = 10$ m and $CA = 7$ m. A point P on AB, 1.5 m from A, is linked by a rod to a point Q on AC, 3 m from A. Calculate the length PQ.

20 Using this diagram, find a, x and y.



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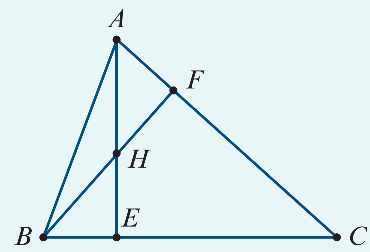
9G Proofs involving similarity

Example 16

The altitudes AE and BF of $\triangle ABC$ intersect at H.
Prove that

$$\frac{AE}{BF} = \frac{AC}{BC}$$



Solution

$$\angle CEA = \angle CFB \qquad (AE \text{ and } BF \text{ are altitudes})$$

$$\angle ACE = \angle BCF \qquad (\text{common})$$

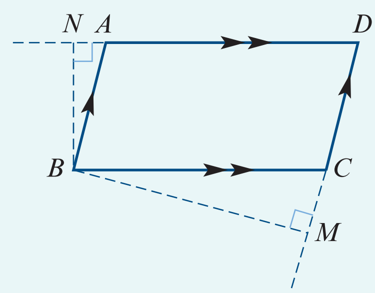
$$\therefore \quad \triangle CAE \sim \triangle CBF \qquad (\text{AAA})$$

$$\therefore \quad \frac{AE}{BF} = \frac{AC}{BC}$$

Example 17

$ABCD$ is a parallelogram with $\angle ABC$ acute.
BM is perpendicular to DC extended, and BN is perpendicular to DA extended. Prove that

$$DC \times CM = DA \times AN$$



Solution

$$\angle BCM = \angle ABC \qquad (\text{alternate angles } AB \parallel CD)$$

$$\angle BAN = \angle ABC \qquad (\text{alternate angles } BC \parallel AD)$$

$$\therefore \quad \angle BCM = \angle BAN$$

$$\angle BNA = \angle BMC = 90^\circ \qquad (\text{given})$$

$$\therefore \quad \triangle BCM \sim \triangle BAN$$

Hence $\dfrac{CM}{AN} = \dfrac{BC}{AB}$

But $AB = CD$ and $BC = AD$, giving

$$\frac{CM}{AN} = \frac{AD}{CD}$$

Hence $DC \times CM = DA \times AN$.

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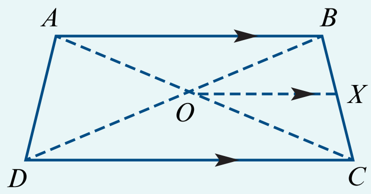
Example 18

$ABCD$ is a trapezium with diagonals intersecting at O. A line through O, parallel to the base CD, meets BC at X. Prove that $BX \times DC = XC \times AB$.

Solution



$\triangle ABC \sim \triangle OXC$ (AAA)

$\triangle DCB \sim \triangle OXB$ (AAA)

Thus $\dfrac{BX}{BC} = \dfrac{OX}{DC}$ (1)

and $\dfrac{XC}{BC} = \dfrac{OX}{AB}$ (2)

Divide (1) by (2):

$$\frac{BX}{XC} = \frac{AB}{DC}$$

$\therefore \quad BX \times DC = XC \times AB$

Exercise 9G

Skillsheet

Example 16

1 Let M be the midpoint of a line segment AB. Assume that AXB and MYB are equilateral triangles on opposite sides of AB and that XY cuts AB at Z. Prove that $\triangle AXZ \sim \triangle BYZ$ and hence prove that $AZ = 2ZB$.

Example 17

2 $ABCD$ is a rectangle. Assume that P, Q and R are points on AB, BC and CD respectively such that $\angle PQR$ is a right angle. Prove that $BQ \times QC = PB \times CR$.

3 **a** AC is a diagonal of a regular pentagon $ABCDE$. Find the sizes of $\angle BAC$ and $\angle CAE$.

b AC, AD and BD are diagonals of a regular pentagon $ABCDE$, with AC and BD meeting at X. Prove that $(AB)^2 = BX \times BD$.

4 $\triangle ABC$ has a right angle at A, and AD is the altitude to BC.

a Prove that $AD \times BC = AB \times AC$.

b Prove that $(DA)^2 = DB \times DC$.

c Prove that $(BA)^2 = BD \times BC$.

Example 18

5 $ABCD$ is a trapezium with AB one of the parallel sides. The diagonals meet at O. OX is the perpendicular from O to AB, and XO extended meets CD at Y.

Prove that $\dfrac{OX}{OY} = \dfrac{OA}{OC} = \dfrac{AB}{CD}$.

6 P is the point on side AB of $\triangle ABC$ such that $AP : AB = 1 : 3$, and Q is the point on BC such that $CQ : CB = 1 : 3$. The line segments AQ and CP intersect at X. Prove that $AX : AQ = 3 : 5$.

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7 P and Q are points on sides AB and AC respectively of $\triangle ABC$ such that $PQ \parallel BC$. The median AD meets PQ at M. Prove that $PM = MQ$.

8 $ABCD$ is a straight line and $AB = BC = CD$. An equilateral triangle $\triangle BCP$ is drawn with base BC. Prove that $(AP)^2 = AB \times AD$.

9 $ABCD$ is a quadrilateral such that $\angle BAD = \angle DBC$ and $\dfrac{DA}{AB} = \dfrac{DB}{BC}$. Prove that DB bisects $\angle ADC$.

10 $\triangle ABC$ has a right angle at C. The bisector of $\angle BCA$ meets AB at D, and DE is the perpendicular from D to AC. Prove that $\dfrac{1}{BC} + \dfrac{1}{AC} = \dfrac{1}{DE}$.

11 Proportions in a right-angled triangle

- **a** Prove that, for a right-angled triangle, the altitude on its hypotenuse forms two triangles which are similar to the original triangle, and hence to each other.
- **b** Prove Pythagoras' theorem by using part **a** (or by using similar triangles directly).

9H Areas, volumes and similarity

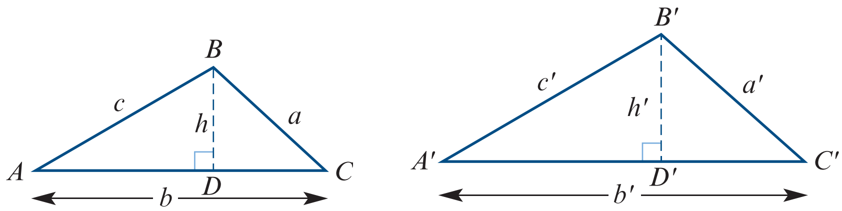
In this section we look at the areas of similar shapes and the volumes of similar solids.

► Similarity and area

> If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then
>
> area of similar shape $= k^2 \times$ area of original shape

For example, if triangles ABC and $A'B'C'$ are similar with $A'B' = kAB$, then

area of $\triangle A'B'C' = k^2 \times$ area of $\triangle ABC$



This can be shown by observing that, since $\triangle ABC \sim \triangle A'B'C'$, we have

$$\begin{aligned}\text{area of } \triangle A'B'C' &= \frac{1}{2} b'h' \\ &= \frac{1}{2}(kb)(kh) \\ &= k^2\left(\frac{1}{2} bh\right) \\ &= k^2 \times \text{area of } \triangle ABC\end{aligned}$$

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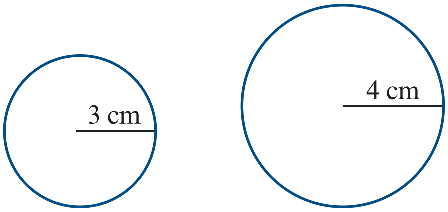
Here are some more examples of similar shapes and the ratio of their areas.

Similar circles

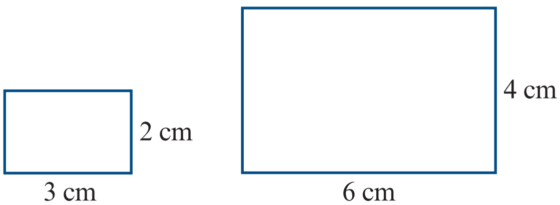


Area = 9π cm^2 Area = 16π cm^2

Similarity factor $= \dfrac{4}{3}$

Ratio of areas $= \dfrac{16\pi}{9\pi} = \left(\dfrac{4}{3}\right)^2$

Similar rectangles

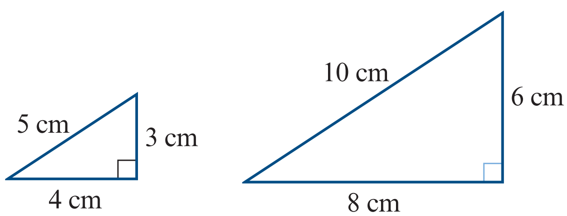


Area = 6 cm^2 Area = 24 cm^2

Similarity factor = 2

Ratio of areas $= \dfrac{24}{6} = 4 = 2^2$

Similar triangles



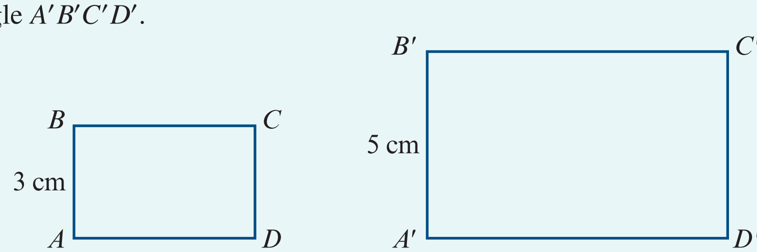
Area = 6 cm^2 Area = 24 cm^2

Similarity factor = 2

Ratio of areas $= \dfrac{24}{6} = 4 = 2^2$

Example 19

The two rectangles shown below are similar. The area of rectangle $ABCD$ is 20 cm^2. Find the area of rectangle $A'B'C'D'$.



Solution

The ratio of their side lengths is $\dfrac{A'B'}{AB} = \dfrac{5}{3}$.

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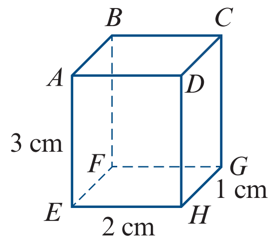
The ratio of their areas is $\dfrac{\text{Area of } A'B'C'D'}{\text{Area of } ABCD} = \left(\dfrac{5}{3}\right)^2 = \dfrac{25}{9}$.

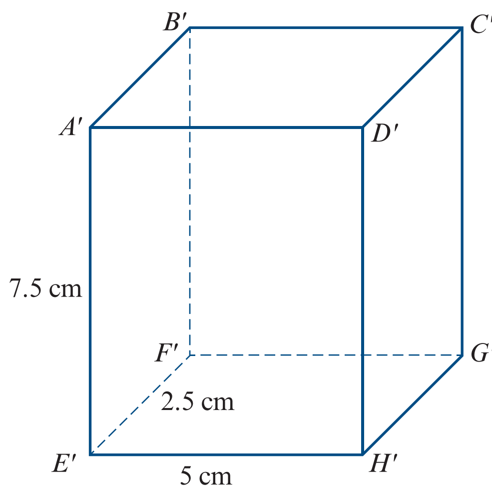
$$\therefore \quad \text{Area of } A'B'C'D' = \frac{25}{9} \times 20 = 55\tfrac{5}{9}\ \text{cm}^2$$

Similarity and volume

Two solids are considered to be similar if they have the same shape and the ratios of their corresponding linear dimensions are equal.

For example, the two cuboids $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ shown are similar, with similarity factor 2.5.





If two solids are similar and the similarity factor is k, then

volume of similar solid = k^3 × volume of original solid

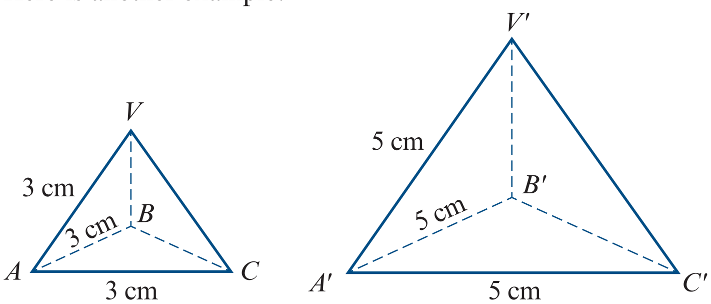
For example, for the two cuboids shown, we have

Volume of $ABCDEFGH = 2 \times 1 \times 3 = 6\ \text{cm}^3$

Volume of $A'B'C'D'E'F'G'H' = 5 \times 2.5 \times 7.5 = 93.75\ \text{cm}^3$

$$\therefore \quad \text{Ratio of volumes} = \frac{93.75}{6} = 15.625 = 2.5^3$$

Here is another example:



Similarity factor $= \dfrac{5}{3}$

Ratio of volumes $= \left(\dfrac{5}{3}\right)^3$

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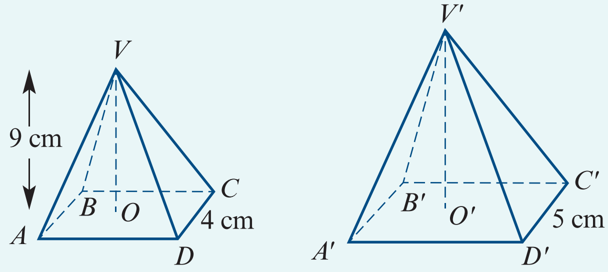
Example 20

The two square pyramids are similar and $VO = 9$ cm.

a Find the ratio of the lengths of their bases, and hence find the height $V'O'$ of pyramid $V'A'B'C'D'$.



b The volume of $VABCD$ is 48 cm^3. Find the ratio of their volumes, and hence find the volume of $V'A'B'C'D'$.

Solution

a The ratio of the length of their bases is

$$\frac{C'D'}{CD} = \frac{5}{4}$$

$$\therefore\ V'O' = \frac{5}{4} \times 9$$

$$= 11.25 \text{ cm}$$

b The ratio of their volumes is

$$\frac{\text{Volume of } V'A'B'C'D'}{\text{Volume of } VABCD} = \left(\frac{5}{4}\right)^3 = \frac{125}{64}$$

$$\therefore\ \text{Volume of } V'A'B'C'D' = \frac{125}{64} \times 48$$

$$= 93.75 \text{ cm}^3$$

Section summary

- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

  area of similar shape $= k^2 \times$ area of original shape

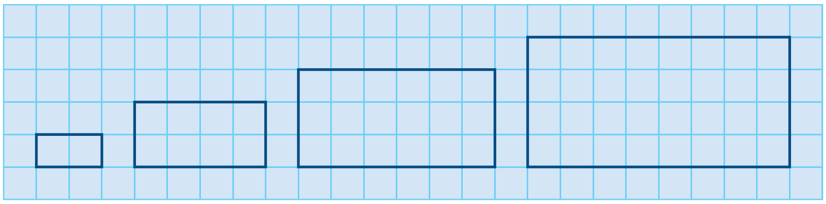
- If two solids are similar and the similarity factor is k, then

  volume of similar solid $= k^3 \times$ volume of original solid

Exercise 9H

Skillsheet

1 These four rectangles are similar:



a Write down the ratio of the lengths of their bases.

b By counting rectangles, write down the ratio of their areas.

c Is there a relationship between these two ratios?

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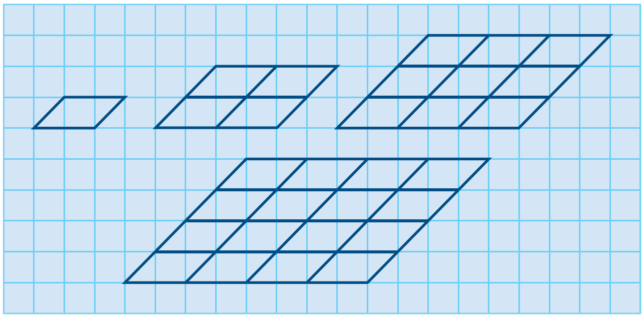
2 These four parallelograms are similar:



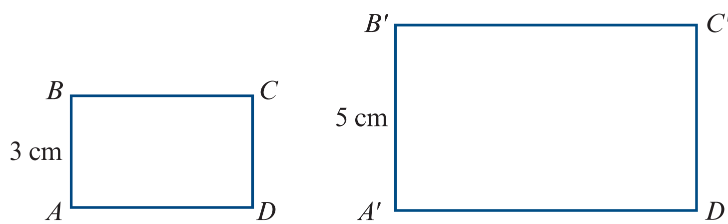
a Write down the ratio of the lengths of their bases.

b By counting parallelograms, write down the ratio of their areas.

c Is there a relationship between these two ratios?

Example 19

3 The two rectangles shown are similar. The area of rectangle $ABCD$ is 7 cm^2. Find the area of rectangle $A'B'C'D'$.



4 Triangle ABC is similar to triangle XYZ with

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{ZX}{CA} = 2.1$$

The area of triangle XYZ is 20 cm^2. Find the area of triangle ABC.

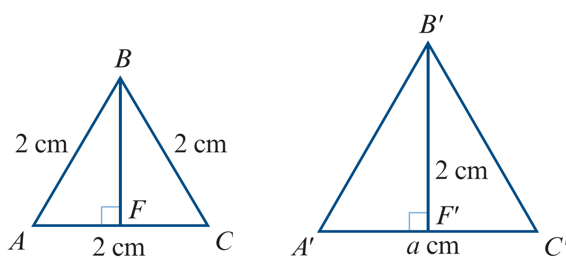
5 Triangles ABC and $A'B'C'$ are equilateral triangles.

a Find the length of BF.

b Find a.

c Find the ratio

$$\frac{\text{Area of } \triangle A'B'C'}{\text{Area of } \triangle ABC}$$



6 The areas of two similar triangles are 16 and 25. What is the ratio of a pair of corresponding sides?

7 The areas of two similar triangles are 144 and 81. If the base of the larger triangle is 30, what is the corresponding base of the smaller triangle?

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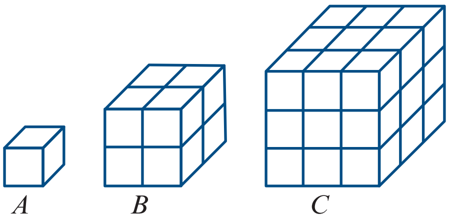
8 These three solids are similar.



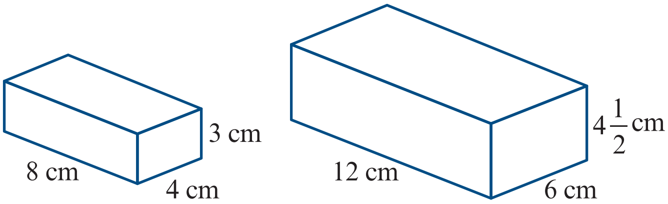
a Write down the ratio of the lengths of the bases.

b Write down the ratio of the lengths of the heights.

c By counting cuboids equal in shape and size to cuboid A, write down the ratio of the volumes.

d Is there a relationship between the answers to **a**, **b** and **c**?

9 These are two similar rectangular blocks.



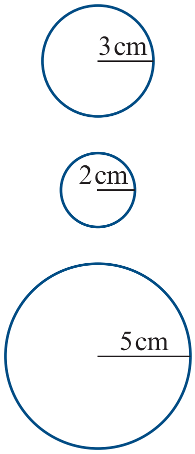
a Write down the ratio of their:

i longest edges **ii** depths **iii** heights.

b By counting cubes of side length 1 cm, write down the ratio of their volumes.

c Is there any relationship between the ratios in **a** and **b**?

Example 20 **10** These three solids are spheres.



a Write down the ratio of the radii of the three spheres.

b The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

Express the volume of each sphere as a multiple of π.

Hence write down the ratio of their volumes.

c Is there any relationship between the ratios found in **a** and **b**?

In each of Questions 11–20, the objects are mathematically similar.

11 The sides of two cubes are in the ratio 2 : 1. What is the ratio of their volumes?

12 The radii of two spheres are in the ratio 3 : 4. What is the ratio of their volumes?

13 Two regular tetrahedrons have volumes in the ratio 8 : 27. What is the ratio of their sides?

14 Two right cones have volumes in the ratio 64 : 27. What is the ratio of:

a their heights **b** their base radii?

15 Two similar bottles are such that one is twice as high as the other. What is the ratio of:

a their surface areas **b** their capacities?

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16 Each linear dimension of a model car is $\frac{1}{10}$ of the corresponding car dimension. Find the ratio of:

a the areas of their windscreens
b the capacities of their boots
c the widths of the cars
d the number of wheels they have.

17 Three similar jugs have heights 8 cm, 12 cm and 16 cm. If the smallest jug holds $\frac{1}{2}$ litre, find the capacities of the other two.

18 Three similar drinking glasses have heights 7.5 cm, 9 cm and 10.5 cm. If the tallest glass holds 343 millilitres, find the capacities of the other two.

19 A toy manufacturer produces model cars which are similar in every way to the actual cars. If the ratio of the door area of the model to the door area of the car is 1 : 2500, find:

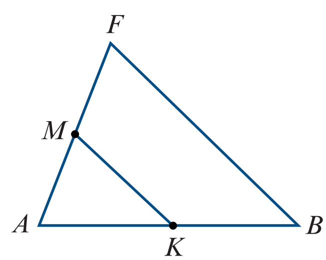
a the ratio of their lengths
b the ratio of the capacities of their petrol tanks
c the width of the model, if the actual car is 150 cm wide
d the area of the rear window of the actual car if the area of the rear window of the model is 3 cm^2.

20 The ratio of the areas of two similar labels on two similar jars of coffee is 144 : 169. Find the ratio of:

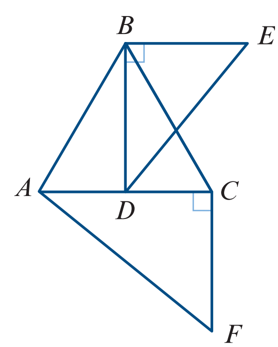
a the heights of the two jars
b their capacities.

21 **a** In the figure, if M is the midpoint of AF and K is the midpoint of AB, then how many times larger is the area of $\triangle ABF$ than the area of $\triangle AKM$?

b If the area of $\triangle ABF$ is 15, find the area of $\triangle AKM$.



22 In the diagram, $\triangle ABC$ is equilateral, $\angle BDE = \angle CAF$ and D is the midpoint of AC. Find the ratio of the area of $\triangle BDE$ to the area of $\triangle ACF$.



23 The areas of two similar triangles are 144 cm^2 and 81 cm^2. If the length of one side of the first triangle is 6 cm, what is the length of the corresponding side of the second?

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9I The golden ratio

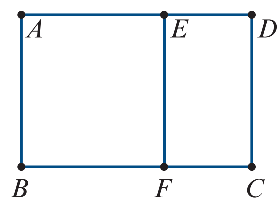
The golden ratio is the irrational number

$$\varphi = \frac{1+\sqrt{5}}{2} = 1.618\,033\,988\ldots$$

This number is mentioned in the books of Euclid, where it is used in the construction of a regular pentagon. (See Extended-response question 2.) The golden ratio also arises naturally in connection with the sequence of Fibonacci numbers.

Golden rectangles

A **golden rectangle** is a rectangle that can be cut up into a square and a rectangle that is similar to the original one.



Let $ABCD$ be a rectangle with $AB < BC$.

Then there is a point E on AD and a point F on BC such that $ABFE$ is a square. We say that $ABCD$ is a golden rectangle if $FCDE$ is similar to $ABCD$.

Theorem

All golden rectangles are similar, with ratio of length to width given by

$$\frac{1+\sqrt{5}}{2} : 1$$

Proof Assume that $ABCD$, as shown in the diagram above, is a golden rectangle. Let $AD = \ell$ and $CD = w$. Then $ED = \ell - w$.

As the two rectangles are similar, we have

$$\frac{AD}{CD} = \frac{CD}{ED} = k$$

where k is the ratio that we want to find. Thus

$$\frac{\ell}{w} = \frac{w}{\ell - w} = k$$

and therefore

$$\ell^2 - \ell w = w^2$$

Substitute $\ell = kw$:

$$(kw)^2 - kw^2 = w^2$$

$$k^2 - k - 1 = 0$$

Using the quadratic formula gives $k = \dfrac{1+\sqrt{5}}{2}$ since $k > 0$.

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The **golden ratio** is denoted by φ and is given by

$$\varphi = \frac{1+\sqrt{5}}{2}$$

Alternatively, the golden ratio can be defined as the unique positive number φ such that

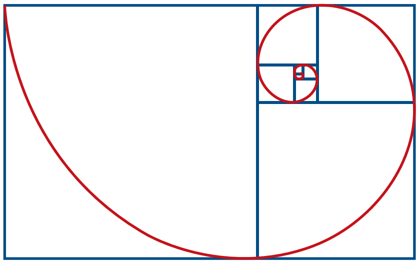
$$\varphi^2 - \varphi - 1 = 0$$

Note: A rectangle of length φ and width 1 is indeed a golden rectangle, since $\varphi^2 - \varphi - 1 = 0$ implies that $\frac{\varphi}{1} = \frac{1}{\varphi - 1}$.

A sequence of golden rectangles

Starting from any golden rectangle, we can remove a square to form another golden rectangle. Therefore we can remove another square to form yet another golden rectangle.

Continuing in this way, we can create the spiral shown.



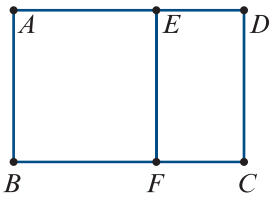
▶ The golden ratio and the geometric mean

We met the geometric mean of two numbers in Chapter 4. You may have noticed that this idea arose in our consideration of the golden rectangle.

Recall that, if a, b and c are positive numbers such that

$$\frac{c}{a} = \frac{b}{c}$$

then we say that c is the **geometric mean** of a and b.



Again, let $ABCD$ be a rectangle with $AB < BC$.

Choose points E and F as shown such that $ABFE$ is a square. Then $CD = AE$.

Therefore $ABCD$ is a golden rectangle if and only if

$$\frac{AD}{AE} = \frac{AE}{ED}$$

This is precisely the requirement that AE is the geometric mean of AD and ED.

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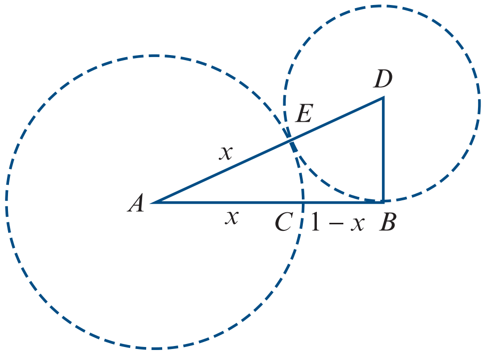
▶ Construction of the golden ratio

We can construct the golden ratio as follows.



1 Start with a line segment AB of unit length.
2 Draw BD perpendicular to AB with length $\frac{AB}{2} = \frac{1}{2}$.
3 Draw line segment AD.
4 Draw a circle with centre D and radius DB, cutting AD at E.
5 Draw a circle with centre A and radius AE, cutting AB at C.
6 Use Pythagoras' theorem to show that $x = AE = \frac{\sqrt{5}-1}{2}$. Then $\frac{AB}{AC} = \frac{1}{x} = \varphi$.

▶ Irrationality of the golden ratio

One way to prove that the golden ratio is irrational is first to prove that $\sqrt{5}$ is irrational. However, we can give a more direct proof as follows.

Theorem

The golden ratio φ is irrational.

Proof Suppose that the golden ratio is rational. Then we can write

$$\varphi = \frac{m}{n} \qquad \text{for some } m, n \in \mathbb{N}$$

We can assume that m and n have no common factors, and hence the numerator m is as small as possible. Note that $\varphi > 1$ and so $m > n$.

Since $\varphi^2 - \varphi - 1 = 0$, we have

$$\begin{aligned}\varphi &= \frac{1}{\varphi - 1}\\ &= \frac{1}{\frac{m}{n} - 1}\\ &= \frac{n}{m-n}\end{aligned}$$

As $m > n$, we have now expressed φ as a fraction with a numerator smaller than m. But this contradicts our initial assumption. Hence φ is irrational.

Section summary

- The **golden ratio** is $\varphi = \frac{1+\sqrt{5}}{2}$.
- All golden rectangles are similar, with the ratio of length to width $\varphi : 1$.
- The golden ratio is the unique positive number φ such that $\varphi^2 - \varphi - 1 = 0$.

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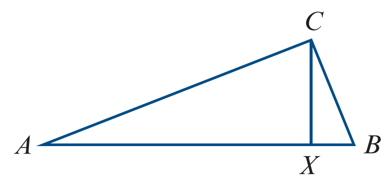
Exercise 9I

1 For the golden ratio φ show that:

a $\varphi - 1 = \dfrac{1}{\varphi}$ **b** $\varphi^3 = 2\varphi + 1$ **c** $2 - \varphi = (\varphi - 1)^2 = \dfrac{1}{\varphi^2}$

2 ABC is a right-angled triangle with the right angle at C, and CX is the altitude of the triangle from C.



a Prove that $\dfrac{AX}{CX} = \dfrac{CX}{XB}$.

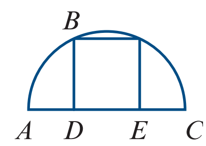
Note: This shows that the length CX is the geometric mean of lengths AX and XB.

b Find CX if:

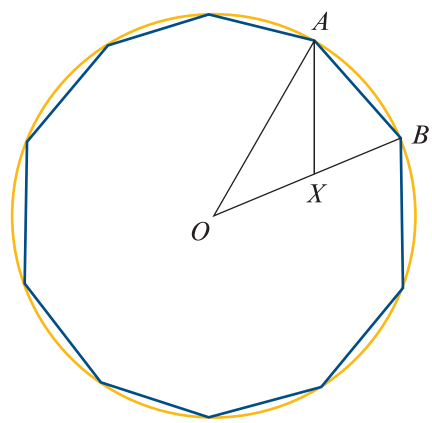
i $AX = 2$ and $XB = 8$ **ii** $AX = 1$ and $XB = 10$.

3 A square is inscribed in a semicircle as shown. Prove that

$$\frac{AD}{BD} = \frac{BD}{CD} = \varphi - 1$$



4 A regular decagon is inscribed in a circle with unit radius as shown.



a Find the magnitude of angle:

i AOB **ii** OAB

b The line AX bisects angle OAB. Prove that:

i triangle AXB is isosceles

ii triangle AXO is isosceles

iii triangle AOB is similar to triangle BXA.

c Find the length of AB, correct to two decimal places.

5 Calculate $\varphi^0, \varphi^1, \varphi^2, \varphi^3, \varphi^4$ and $\varphi^{-1}, \varphi^{-2}, \varphi^{-3}, \varphi^{-4}$. Show that each power of φ is equal to the sum of the two powers before it. That is, show that $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$.

6 The Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ is defined by $t_1 = t_2 = 1$ and $t_{n+1} = t_{n-1} + t_n$. Consider the sequence

$$\frac{t_2}{t_1}, \frac{t_3}{t_2}, \frac{t_4}{t_3}, \frac{t_5}{t_4}, \ldots$$

Show numerically that, as n gets very large, the ratio $\dfrac{t_{n+1}}{t_n}$ approaches φ.

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Chapter summary

AS

Nrich

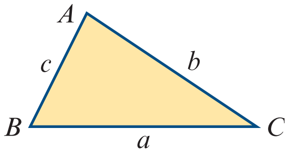
Parallel lines

- If two parallel lines are crossed by a transversal, then:
  - alternate angles are equal
  - corresponding angles are equal
  - co-interior angles are supplementary.
- If two lines crossed by a transversal form an equal pair of alternate angles, then the two lines are parallel.

Polygons

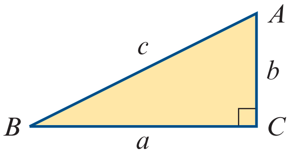
- The sum of the interior angles of an n-sided polygon is $(n - 2)180°$.
- A **regular polygon** is a polygon in which all angles are equal and all sides are equal.
- Triangle inequality

  In $\triangle ABC$: $a < b + c$, $b < c + a$ and $c < a + b$.



- Pythagoras' theorem and its converse

  Let ABC be a triangle with side lengths a, b and c.
  - If $\angle C$ is a right angle, then $a^2 + b^2 = c^2$.
  - If $a^2 + b^2 = c^2$, then $\angle C$ is a right angle.



- Classification of quadrilaterals:
  - A **trapezium** is a quadrilateral with at least one pair of opposite sides parallel.
  - A **parallelogram** is a quadrilateral with both pairs of opposite sides parallel.
  - A **rhombus** is a parallelogram with a pair of adjacent sides equal.
  - A **rectangle** is a parallelogram in which one angle is a right angle.
  - A **square** is a rectangle with a pair of adjacent sides equal.

Congruence

- **Congruent figures** have exactly the same shape and size.
- If triangle ABC is congruent to triangle XYZ, this can be written as $\triangle ABC \equiv \triangle XYZ$.
- Two triangles are congruent provided any one of the following four conditions holds:

  SSS the three sides of one triangle are equal to the three sides of the other triangle

  SAS two sides and the included angle of one triangle are equal to two sides and the included angle of the other triangle

  AAS two angles and one side of one triangle are equal to two angles and the matching side of the other triangle

  RHS the hypotenuse and one side of a right-angled triangle are equal to the hypotenuse and one side of another right-angled triangle.

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Similarity

- Two figures are **similar** if we can enlarge one figure so that its enlargement is congruent to the other figure.
  - Matching lengths of similar figures are in the same ratio.
  - Matching angles of similar figures are equal.
- If triangle ABC is similar to triangle XYZ, this can be written as $\triangle ABC \sim \triangle XYZ$.
- Two triangles are similar provided any one of the following four conditions holds:

  AAA two angles of one triangle are equal to two angles of the other triangle

  SAS the ratios of two pairs of matching sides are equal and the included angles are equal

  SSS the sides of one triangle can be matched up with the sides of the other triangle so that the ratio of matching lengths is constant

  RHS the ratio of the hypotenuses of two right-angled triangles equals the ratio of another pair of sides.
- If two shapes are similar and the similarity factor is k (that is, if for any length AB of one shape, the corresponding length $A'B'$ of the similar shape is kAB), then

  $$\text{area of similar shape} = k^2 \times \text{area of original shape}$$
- If two solids are similar and the similarity factor is k, then

  $$\text{volume of similar solid} = k^3 \times \text{volume of original solid}$$

Technology-free questions

1 $ABCD$ is a rhombus with $AB = 16$ cm. The midpoints of its sides are joined to form a quadrilateral.

  a Describe the quadrilateral formed.

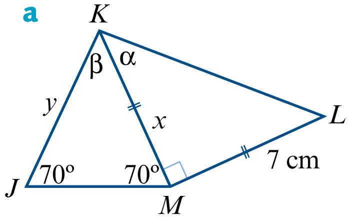
  b What is the length of the diagonal of this quadrilateral?

2 Prove that a triangle with sides $x^2 - y^2$, $x^2 + y^2$ and $2xy$ is a right-angled triangle.

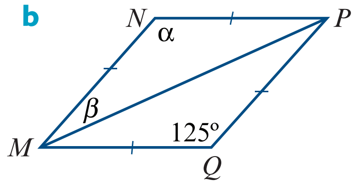
3 Find the side length of a rhombus whose diagonals are 6 cm and 10 cm.

4 Find the values of the unknowns (x, y, α, β, γ and θ) for each of the following:

a



b



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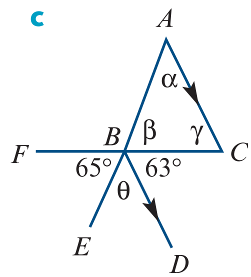

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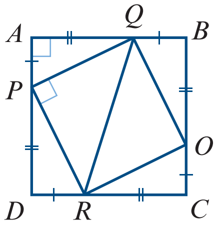
Review

5 A 25 m pole is leaning against a vertical wall with the foot of the pole 20 m from the wall. If its foot slips a further 4 m from the wall, find the distance that the top of the pole has slipped down the wall.

6 **a** Prove that $\triangle PAQ \equiv \triangle QBO$.
b Prove that $\triangle PQR \equiv \triangle ORQ$.



7 Let XYZ be a triangle with a point P on XY and a point Q on XZ such that PQ is parallel to YZ.

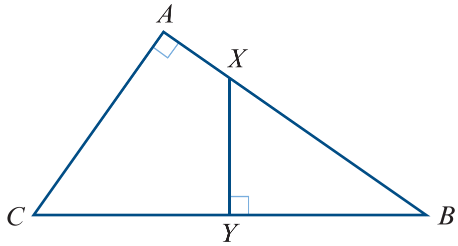
a Show that the two triangles XYZ and XPQ are similar.
b If $XY = 36$ cm, $XZ = 30$ cm and $XP = 24$ cm, find:
i XQ **ii** QZ
c Write down the values of $XP : PY$ and $PQ : YZ$.

8 Triangles ABC and DEF are similar. If the area of triangle ABC is 12.5 cm^2, the area of triangle DEF is 4.5 cm^2 and $AB = 5$ cm, find:

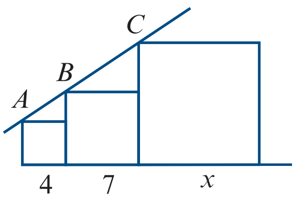
a the length of DE **b** the value of $AC : DF$ **c** the value of $EF : BC$.

9 If a 1 m stake casts a shadow 2.3 m long, find the height (in metres) of a tree which casts a shadow 21 m long.

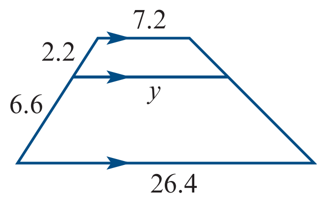
10 ABC is a right-angled triangle with $AB = 4$ and $AC = 3$. If the triangle is folded along the line XY, then vertex C coincides with vertex B. Find the length of XY.



11 Points A, B and C lie on a straight line. The squares are adjacent and have side lengths 4, 7 and x. Find the value of x.



12 Find the value of y in the diagram on the right.



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13 An alloy is produced by mixing metal X with metal Y in the ratio of $5 : 3$ by volume. The mass of 1 cm^3 of metal X is $\frac{8}{5}$ g and the mass of 1 cm^3 of metal Y is $\frac{4}{3}$ g. Calculate:

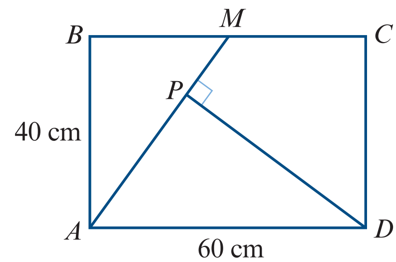
a the mass of a solid cube of alloy with edge length 4 cm

b the ratio by mass, in the form $n : 1$, of metal X to metal Y in the alloy

c the volume, to the nearest cm^3, of a cubic block of alloy with a mass of 1.5 kg

d the length, in mm, of the edge of this cubic block.

14 $ABCD$ is a rectangle in which $AB = 40$ cm and $AD = 60$ cm. The midpoint of BC is M, and DP is perpendicular to AM.



a Prove that the triangles BMA and PAD are similar.

b Calculate the ratio of the areas of the triangles BMA and PAD.

c Calculate the length of PD.

15 A sculptor is commissioned to create a bronze statue 2 m high. He begins by making a clay model 30 cm high.

a Express, in simplest form, the ratio of the height of the completed bronze statue to the height of the clay model.

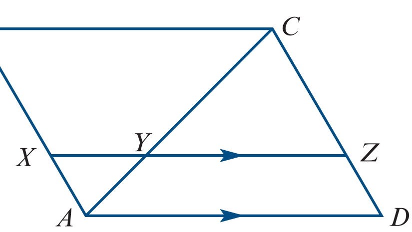
b If the surface area of the model is 360 cm^2, find the surface area of the statue.

c If the volume of the model is 1000 cm^3, find the volume of the statue.

16 The radius of a spherical soap bubble increases by 1%. Find, correct to the nearest whole number, the percentage increase in:

a its surface area **b** its volume.

17 AC is the diagonal of a rhombus $ABCD$. The line XYZ is parallel to AD, and $AX = 3$ cm and $AB = 9$ cm. Find:



a $\dfrac{XY}{BC}$ **b** $\dfrac{AY}{AC}$ **c** $\dfrac{CY}{AC}$

d $\dfrac{YZ}{AD}$ **e** $\dfrac{\text{area } \triangle AXY}{\text{area } \triangle ABC}$ **f** $\dfrac{\text{area } \triangle CYZ}{\text{area } \triangle ACD}$

18 AB and DC are parallel sides of a trapezium and $DC = 3AB$. The diagonals AC and DB intersect at O. Prove that $AO = \frac{1}{4}AC$.

19 Triangles ABC and PQR are similar. The medians AX and PY are drawn, where X is the midpoint of BC and Y is the midpoint of QR. Prove that:

a triangles ABX and PQY are similar **b** $\dfrac{AX}{PY} = \dfrac{BC}{QR}$

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Multiple-choice questions

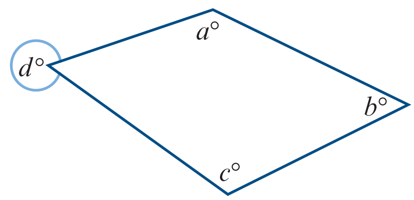
1 One angle of a triangle is twice the size of the second angle, and the third angle is 66°. The smallest angle is

A 66° **B** 53° **C** 38° **D** 24° **E** 22°

2 Three of the angles of a pentagon are right angles. The other two angles are of equal size $x°$. The angle $x°$ is equal to

A 45° **B** 135° **C** 120° **D** 150° **E** 108°

3 d is equal to

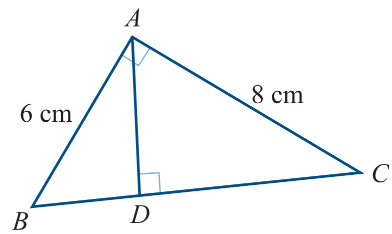


A $360 - (a + b + c)$ **B** $a + b + c$

C $b - a + c$ **D** $a - b + c$

E $a + b - c$

4 In the figure, $\angle BAC$ is a right angle and AD is perpendicular to BC. If $AB = 6$ cm and $AC = 8$ cm, then the length of AD is



A 4 cm **B** $\frac{24}{5}$ cm **C** $\frac{17}{3}$ cm

D $\frac{13}{2}$ cm **E** 7 cm

5 Two sides of a triangle have lengths 14 and 18. Which of the following *cannot* be the length of the third side?

A 2 **B** 6 **C** 7 **D** 28 **E** 30

6 If $5 : 3 = 7 : x$, then x is equal to

A 12 **B** $\frac{35}{3}$ **C** 5 **D** $\frac{21}{5}$ **E** $\frac{5}{21}$

7 Brass is composed of a mixture of copper and zinc. If the ratio of copper to zinc is 85 : 15, then the amount of copper in 400 kg of brass is

A 60 kg **B** 340 kg **C** 360 kg **D** 380 kg **E** 150 kg

8 If the total cost of P articles is Q dollars, then the cost of R articles of the same type is

A PQR **B** $\frac{P}{QR}$ **C** $\frac{PQ}{R}$ **D** $\frac{QR}{P}$ **E** $\frac{R}{PQ}$

9 A car is 3.2 m long. The length of a model of the car at the scale 1 : 100 is

A 0.032 cm **B** 0.32 cm **C** 3.2 cm **D** 320 cm **E** 32 cm

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10 An athlete runs 75 m in 9 s. If she were to maintain the same average speed for 100 m, then her time for 100 m would be

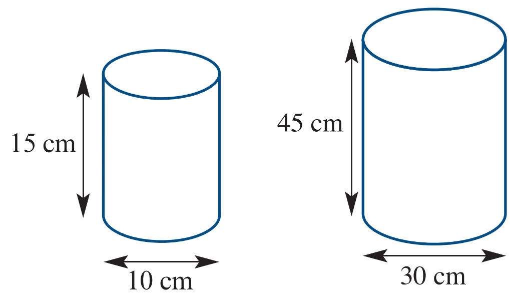
A 11.6 s **B** 12.0 s **C** 11.8 s **D** 12.2 s **E** 12.4 s

11 If 50 is divided into three parts in the ratio 1 : 3 : 6, then the largest part is

A 5 **B** 15 **C** $\frac{50}{3}$ **D** 30 **E** 3

12 Two similar cylinders are shown. The ratio of the volume of the smaller cylinder to the volume of the larger cylinder is

A 1 : 3 **B** 1 : 9
C 1 : 27 **D** 1 : 5
E 2 : 9

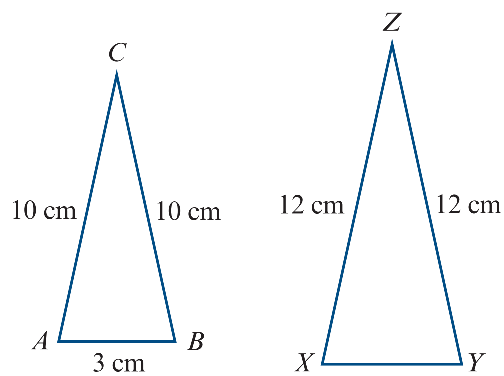


13 The radius of sphere A is $\frac{4}{5}$ times the radius of sphere B. Hence, the ratio of the volume of sphere A to the volume of sphere B is

A 16 : 25 **B** 4 : 5 **C** 5 : 4 **D** 25 : 16 **E** 64 : 125

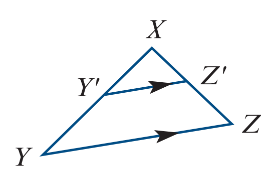
14 Triangles ABC and XYZ are similar isosceles triangles. The length of XY is

A 4 cm **B** 5 cm
C 4.2 cm **D** 2.5 cm
E 3.6 cm



15 YZ is parallel to $Y'Z'$ and $Y'Y = \frac{1}{3}YX$. The area of triangle XYZ is 60 cm^2. The area of triangle $XY'Z'$ is

A 20 cm^2 **B** 30 cm^2 **C** $\frac{20}{9}$ cm^2
D $\frac{20}{3}$ cm^2 **E** $\frac{80}{3}$ cm^2



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Review

Extended-response questions

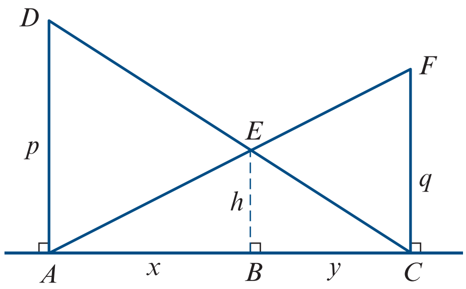
1 **a** In this diagram, which other triangle is similar to $\triangle DAC$?

b Explain why $\frac{h}{p} = \frac{y}{x+y}$.

c Use another pair of similar triangles to write down an expression for $\frac{h}{q}$ in terms of x and y.

d Explain why $h \cdot \left(\frac{1}{p} + \frac{1}{q}\right) = 1$.

e Calculate h when $p = 4$ and $q = 5$.



2 $ABCDE$ is a regular pentagon whose sides are each 1 unit long. Each diagonal is of length d units. In a regular pentagon, each diagonal is parallel to one of the sides of the pentagon.

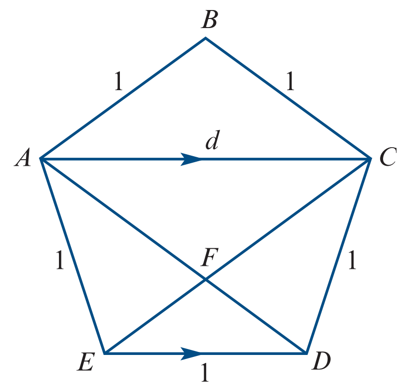
a What kind of shape is $ABCF$ and what is the length of CF?

b Explain why the length of EF is $d - 1$.

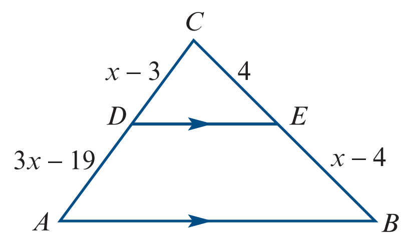
c Which triangle is similar to $\triangle EFD$?

d Use the pair of similar triangles to write an equation for d and show that the equation can be rewritten as $d^2 - d - 1 = 0$.

e Find d.



3 Place conditions upon x such that DE is parallel to AB given that $CD = x - 3$, $DA = 3x - 19$, $CE = 4$ and $EB = x - 4$.



4 **a** If BR, CS and DT are perpendicular to BD, name the pairs of similar triangles.

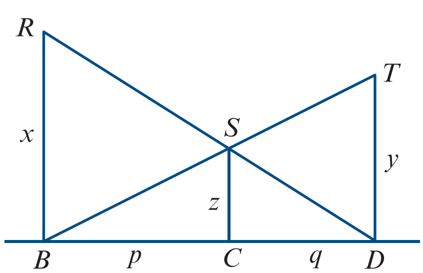
b Which of the following is correct?

$$\frac{z}{y} = \frac{p}{q} \quad \text{or} \quad \frac{z}{y} = \frac{p}{p+q}$$

c Which of the following is correct?

$$\frac{z}{x} = \frac{q}{p} \quad \text{or} \quad \frac{z}{x} = \frac{q}{p+q}$$

d Show that $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$.



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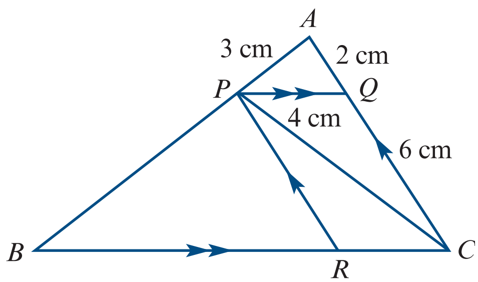
5 In the diagram, PQ is parallel to BC and PR is parallel to AC, with $AQ = 2$ cm, $QC = 6$ cm, $AP = 3$ cm and $PQ = 4$ cm.



a Calculate:

i PB **ii** BR

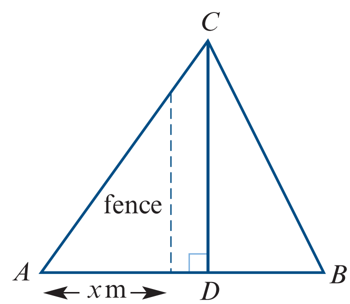
iii $\dfrac{\text{area } \triangle APQ}{\text{area } \triangle ABC}$ **iv** $\dfrac{\text{area } \triangle BPR}{\text{area } \triangle ABC}$

b If the area of triangle APQ is a cm^2, express in terms of a:

i area $\triangle ABC$ **ii** area $\triangle CPQ$

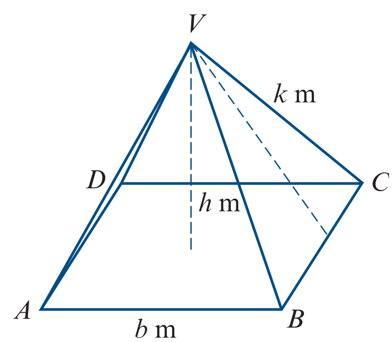
6 Construct a triangle ABC such that $BC = 10$ cm, $AC = 9$ cm and $AB = 6$ cm. Find a point D on AB and a point E on AC such that DE is parallel to BC and the area of $\triangle ADE$ is one-ninth the area of $\triangle ABC$.

7 A triangular lot has boundaries of lengths $AB = 130$ m, $BC = 40\sqrt{10}$ m and $CA = 150$ m. The length of CD is 120 m. A fence is to be erected which runs at right angles to AB. If the lot is to be divided into two equal areas, find x.



8 The Greek historian Herodotus wrote that the proportions of the great pyramid at Giza in Egypt were chosen so that the area of a square, for which the side lengths are equal to the height of the great pyramid, is equal to the area of one of the triangular faces.

Let h m be the height of the pyramid, let k m be the altitude of one of the face triangles, and let b m be the length of a side of the square base.



Show that Herodotus' definition gives $k : \dfrac{b}{2} = \varphi : 1$, where φ is the golden ratio.

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