Chapter 10

Circle geometry

Objectives

- To establish the following results and use them to solve problems:
  - The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
  - Angles in the same segment of a circle are equal.
  - A tangent to a circle is perpendicular to the radius drawn from the point of contact.
  - The two tangents drawn from an external point to a circle are the same length.
  - The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.
  - A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.
  - If AB and CD are two chords of a circle that cut at a point P, then $PA \cdot PB = PC \cdot PD$.

The two basic figures of geometry in the plane are the triangle and the circle. We have considered triangles and their properties in Chapter 9, and we will use the results of that chapter in establishing results involving circles.

A **circle** is the set of all points in the plane at a fixed distance r from a point O. Circles with the same radius are congruent to each other (and are said to be equal circles). We have seen in the previous chapter that all circles are similar to each other.

You may have come across the Cartesian equation of the circle in Mathematical Methods Units 1 & 2. For example, the circle with radius 1 and centre the origin has equation $x^2 + y^2 = 1$. In this chapter we take a different approach to the study of circles.

The theorems and related results in this chapter can be investigated through a geometry package such as GeoGebra or Cabri Geometry.

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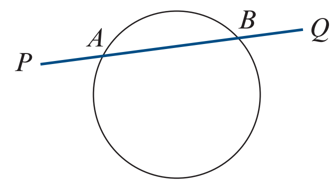
10A Angle properties of the circle



A line segment joining two points on a circle is called a **chord**. A line that cuts a circle at two distinct points is called a **secant**.

For example, in the diagram, the line PQ is a secant and the line segment AB is a chord.

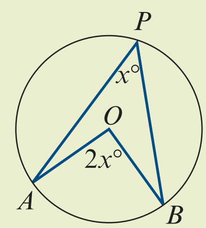
Suppose that we have a line segment or an arc AB and a point P not on AB. Then $\angle APB$ is the angle **subtended** by AB at the point P.

You should prove the following two results. The first proof uses the SSS congruence test and the second uses the SAS congruence test.

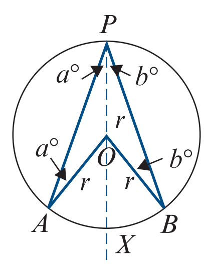
- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.

Theorem 1

The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



Proof Join points P and O and extend the line through O, as shown in the diagram on the right.



Note that $AO = BO = PO = r$, the radius of the circle. Therefore triangles PAO and PBO are isosceles.

Let $\angle APO = \angle PAO = a°$ and $\angle BPO = \angle PBO = b°$.

Then angle AOX is $2a°$ (exterior angle of a triangle) and angle BOX is $2b°$ (exterior angle of a triangle).

Hence $\angle AOB = 2a° + 2b° = 2(a + b)° = 2\angle APB$.

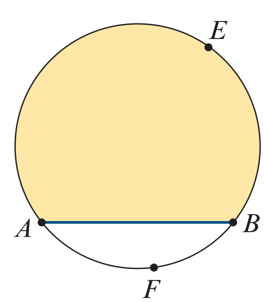
Note: In this proof, the centre O and point P are on the same side of chord AB. Slight variations of this proof can be used for other cases. The result is always true.

Converse of Theorem 1 Let A and B be points on a circle, centre O, and let P be a point on the same side of AB as O. If the angle APB is half the angle AOB, then P lies on the circle.

Segments

A **segment** of a circle is the part of the plane bounded by an arc and its chord. For example, in the diagram:

- Arc AEB and chord AB define a **major segment** (shaded).
- Arc AFB and chord AB define a **minor segment** (unshaded).



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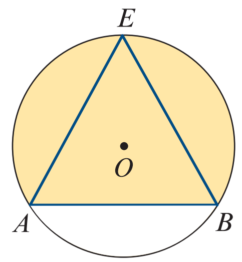
Angles in a segment

$\angle AEB$ is said to be an angle in segment AEB.



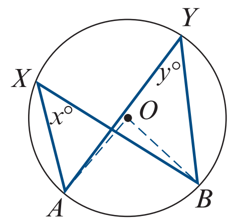
Theorem 2: Angles in the same segment

Angles in the same segment of a circle are equal.

Proof Let $\angle AXB = x^\circ$ and $\angle AYB = y^\circ$.

Then, by Theorem 1, $\angle AOB = 2x^\circ = 2y^\circ$.

Therefore $x = y$.

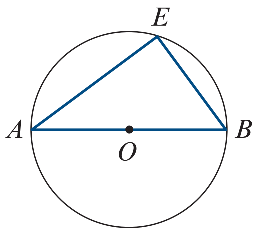


Note: A converse of this result is proved later in this section.

Theorem 3: Angle subtended by a diameter

The angle subtended by a diameter at the circumference is equal to a right angle (90°).

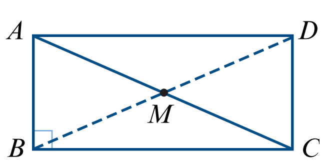
Proof The angle subtended at the centre is 180°, and so the result follows from Theorem 1.



We can give a straightforward proof of a converse of this result.

Converse of Theorem 3 The circle whose diameter is the hypotenuse of a right-angled triangle passes through all three vertices of the triangle.

Proof In the diagram, triangle ABC has a right angle at B. Let M be the midpoint of the hypotenuse AC.



We need to prove that $MA = MB = MC$.

Complete the rectangle $ABCD$.

In Question 12 of Exercise 9C, you proved that the diagonals of a rectangle are equal and bisect each other.

Hence $AC = BD$ and M is also the midpoint of BD. It follows that $MA = MB = MC$.

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Cyclic polygons

- A set of points is said to be **concyclic** if they all lie on a common circle.
- A polygon is said to be **inscribed in a circle** if all its vertices lie on the circle. This implies that no part of the polygon lies outside the circle.
- A quadrilateral that can be inscribed in a circle is called a **cyclic quadrilateral**.

Theorem 4

The opposite angles of a quadrilateral inscribed in a circle sum to 180°.

That is, the opposite angles of a cyclic quadrilateral are supplementary.

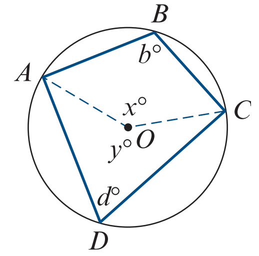
Proof In the diagram, the quadrilateral $ABCD$ is inscribed in a circle with centre O.

By Theorem 1, we have $x = 2d$ and $y = 2b$.

Now $\quad x + y = 360$

and so $\quad 2b + 2d = 360$

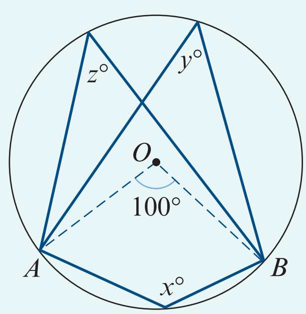
Hence $\quad b + d = 180$



Converse of Theorem 4 If opposite angles of a quadrilateral are supplementary, then the quadrilateral is cyclic.

Example 1

Find the value of each of the pronumerals in the diagram, where O is the centre of the circle and $\angle AOB = 100°$.



Solution

Theorem 1 gives $y = z = 50$.

The value of x can be found by observing either of the following:

1. Reflex angle AOB is 260°.
   Therefore $x = 130$ (by Theorem 1).
2. We have $x + y = 180$ (by Theorem 4).
   Therefore $x = 180 - 50 = 130$.

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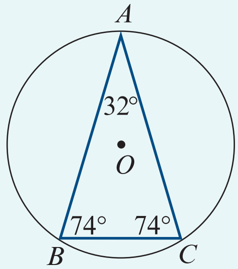
Example 2

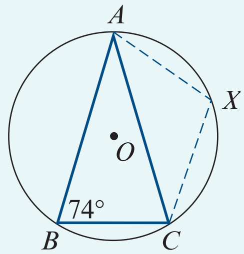
An isosceles triangle is inscribed in a circle as shown. Find the angles in the three minor segments of the circle cut off by the sides of this triangle.



Solution

To find $\angle AXC$, form the cyclic quadrilateral $AXCB$. Then $\angle AXC$ and $\angle ABC$ are supplementary. Therefore $\angle AXC = 106°$, and so all angles in the minor segment formed by AC have magnitude $106°$.

Similarly, it can be shown that all angles in the minor segment formed by AB have magnitude $106°$, and that all angles in the minor segment formed by BC have magnitude $148°$.



Example 3

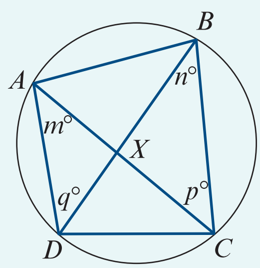
A, B, C and D are points on a circle. The diagonals of quadrilateral $ABCD$ meet at X. Prove that triangles ADX and BCX are similar.

Solution

$\angle DAC$ and $\angle DBC$ are in the same segment. Therefore $m = n$.

$\angle ADB$ and $\angle ACB$ are in the same segment. Therefore $q = p$.

Hence triangles ADX and BCX are similar (AAA).



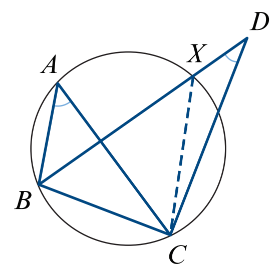
The converse theorems

We only prove a converse of Theorem 2 here, but the proofs of the converses of Theorems 1 and 4 use similar techniques. Try them for yourself.

Converse of Theorem 2 If a line segment subtends equal angles at two points on the same side of the line segment, then the two points and the endpoints of the line segment are concyclic.

Proof A circle is drawn through points A, B and C. (This can be done with any three non-collinear points.)

Assume that $\angle BAC = \angle BDC$ and that D lies outside the circle. (There is another case to consider when D is inside, but the proof is similar. If D lies on the circle, then we are finished.)



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Let X be the point of intersection of line BD with the circle. Then, by Theorem 2, $\angle BAC = \angle BXC$ and so $\angle BDC = \angle BXC$. But this is impossible. (You can use the equality of the angle sums of $\triangle BXC$ and $\triangle BDC$ to show this.)

Hence D lies on the same circle as A, B and C.

Section summary

- Equal chords of a circle subtend equal angles at the centre.
- If two chords subtend equal angles at the centre, then the chords are equal.
- **Theorem 1** The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.
- **Theorem 2** Angles in the same segment of a circle are equal.
- **Theorem 3** The angle subtended by a diameter at the circumference is 90°.
- **Theorem 4** Opposite angles of a cyclic quadrilateral sum to 180°.

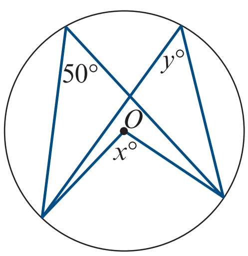
Exercise 10A

Skillsheet

1 Find the values of the pronumerals for each of the following, where O denotes the centre of the given circle:

Example 1

a



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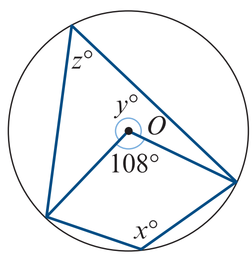
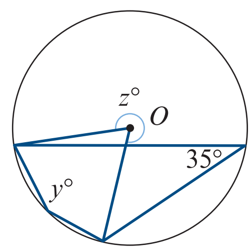
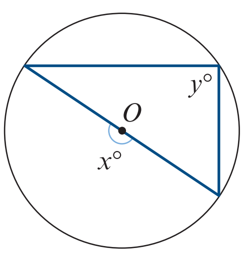


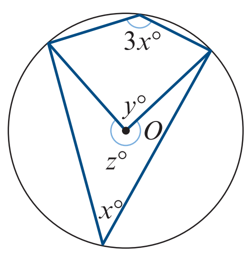
c



d

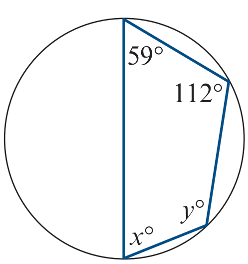


e

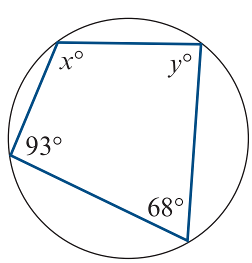


2 Find the values of the pronumerals for each of the following:

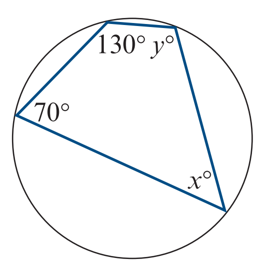
a



b



c



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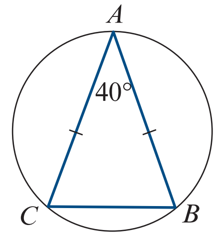
Example 2 **3** An isosceles triangle ABC is inscribed in a circle. (Inscribed means that all the vertices of the triangle lie on the circle.) What are the angles in the three minor segments cut off by the sides of this triangle?



4 $ABCDE$ is a pentagon inscribed in a circle. If $AE = DE$, $\angle BDC = 20°$, $\angle CAD = 28°$ and $\angle ABD = 70°$, find all the interior angles of the pentagon.

Example 3 **5** Prove that if two opposite sides of a cyclic quadrilateral are equal, then the other two sides are parallel.

6 $ABCD$ is a parallelogram. The circle through the points A, B and C cuts CD (extended if necessary) at E. Prove that $AE = AD$.

7 $ABCD$ is a cyclic quadrilateral and O is the centre of the circle through A, B, C and D. If $\angle AOC = 120°$, find the magnitude of $\angle ADC$.

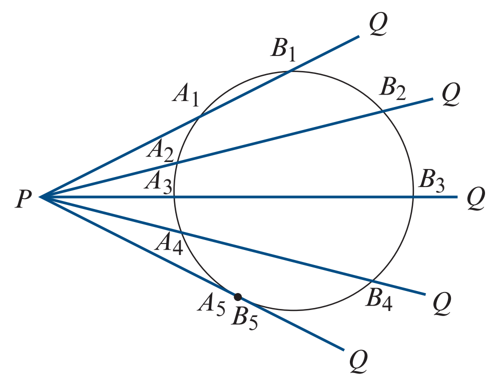
8 $PQRS$ is a cyclic quadrilateral with $\angle SQR = 36°$, $\angle PSQ = 64°$ and $\angle RSQ = 42°$. Find the interior angles of the quadrilateral.

9 Prove that if a parallelogram can be inscribed in a circle, then it must be a rectangle.

10 Prove that the bisectors of the four interior angles of a quadrilateral form a cyclic quadrilateral.

10B Tangents

Consider a point P outside a circle, as shown in the diagram. By rotating the secant PQ, with P as the pivot point, we obtain a sequence of pairs of points on the circle. As PQ moves towards the edge of the circle, the pairs of points become closer together, until they eventually coincide.



When PQ is in this final position (i.e. when the intersection points A and B coincide), it is called a **tangent** to the circle.

A tangent touches the circle at only one point, and this point is called the **point of contact**.

The **length of a tangent** from a point P outside the circle is the distance between P and the point of contact.

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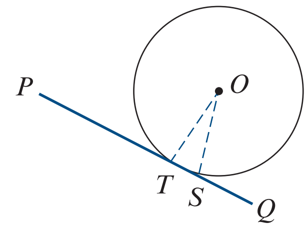
Theorem 5: Tangent is perpendicular to radius

A tangent to a circle is perpendicular to the radius drawn from the point of contact.

Proof This will be a proof by contradiction.



Let T be the point of contact of tangent PQ and suppose that $\angle OTP$ is not a right angle.

Let S be the point on PQ, not T, such that OSP is a right angle. Then triangle OST has a right angle at S.

Therefore $OT > OS$, as OT is the hypotenuse of triangle OST.

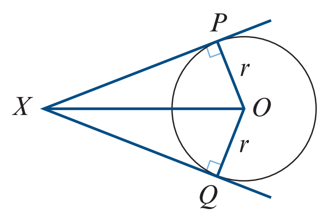
This implies that S is inside the circle, as OT is a radius.

Thus the line through T and S must cut the circle again. But PQ is a tangent, and so this is a contradiction. Hence we have shown that $\angle OTP$ is a right angle.

Theorem 6: Two tangents from the same point

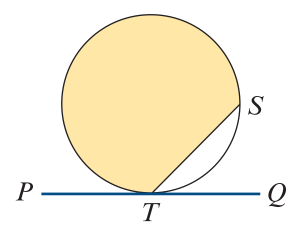
The two tangents drawn from an external point to a circle are the same length.

Proof We can see that $\triangle XPO$ is congruent to $\triangle XQO$ using the RHS test, as $\angle XPO = \angle XQO = 90^\circ$, the side XO is common and $OP = OQ$ (radii).



Therefore $XP = XQ$.

The alternate segment theorem



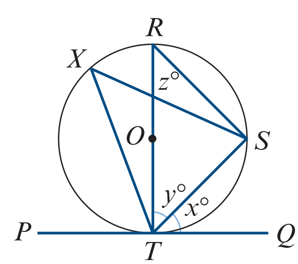
In the diagram:

- The shaded segment is called the **alternate segment** in relation to $\angle STQ$.
- The unshaded segment is alternate to $\angle STP$.

Theorem 7: Alternate segment theorem

The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.

Proof Let $\angle STQ = x^\circ$, $\angle RTS = y^\circ$ and $\angle TRS = z^\circ$, where RT is a diameter.



Then $\angle RST = 90^\circ$ (Theorem 3, angle subtended by a diameter), and therefore $y + z = 90$.

Also $\angle RTQ = 90^\circ$ (Theorem 5, tangent is perpendicular to radius), and therefore $x + y = 90$.

Thus $x = z$. But $\angle TXS$ is in the same segment as $\angle TRS$ and so $\angle TXS = x^\circ$.

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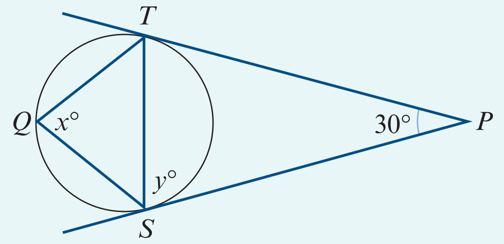
Example 4

Find the magnitudes of the angles x and y in the diagram.



Solution

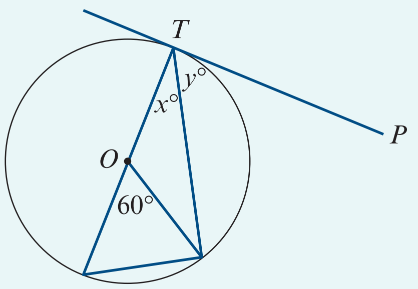
Triangle PST is isosceles (Theorem 6, two tangents from the same point).

Therefore $\angle PST = \angle PTS$ and so $y = 75$.

The alternate segment theorem gives $x = y = 75$.

Example 5

Find the values of x and y, where PT is tangent to the circle centre O.

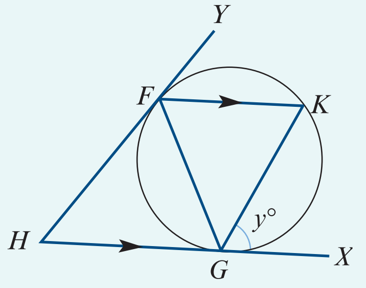


Solution

$x = 30$ as the angle at the circumference is half the angle subtended at the centre, and so $y = 60$ as $\angle OTP$ is a right angle.

Example 6

The tangents to a circle at F and G meet at H. A chord FK is drawn parallel to HG. Prove that triangle FGK is isosceles.



Solution

Let $\angle XGK = y°$.

Then $\angle GFK = y°$ (alternate segment theorem) and $\angle GKF = y°$ (alternate angles).

Therefore triangle FGK is isosceles with $FG = KG$.

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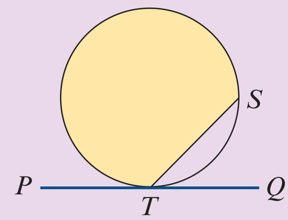
Section summary

- A tangent to a circle is perpendicular to the radius drawn from the point of contact.
- The two tangents drawn from an external point to a circle are the same length.
- In the diagram, the **alternate segment** to $\angle STQ$ is shaded, and the alternate segment to $\angle STP$ is unshaded.
- Alternate segment theorem The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



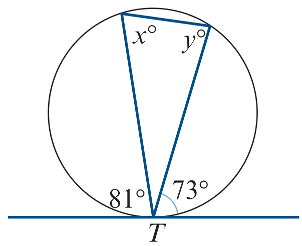
Exercise 10B

Skillsheet

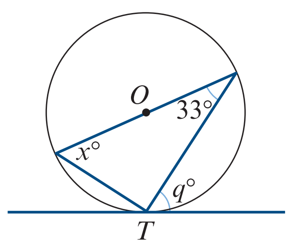
1 Find the values of the pronumerals for each of the following, where T is the point of contact of the tangent and O is the centre of the circle:

Example 4

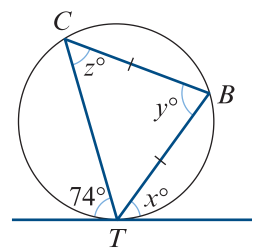
a



b

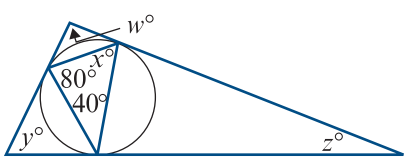


c

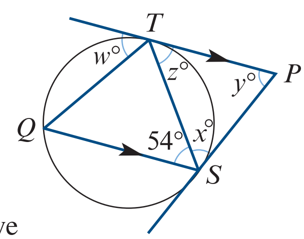


$BC = BT$

d



e



Note: In the diagram for part **e**, the two tangents from P have points of contact at S and T, and TP is parallel to QS.

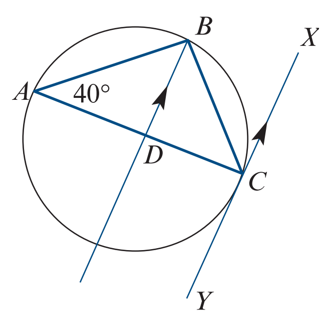
Example 5

2 A triangle ABC is inscribed in a circle, and the tangent to the circle at C is parallel to the bisector of angle ABC.

a Find the magnitude of $\angle BCX$.

b Find the magnitude of $\angle CBD$, where D is the point of intersection of the bisector of angle ABC with AC.

c Find the magnitude of $\angle ABC$.



3 Assume that AB and AC are two tangents to a circle, touching the circle at B and C, and that $\angle BAC = 116°$. Find the magnitudes of the angles in the two segments into which BC divides the circle.

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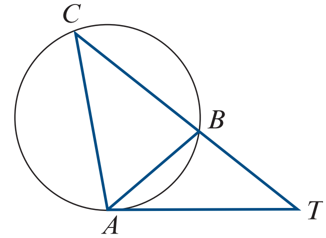
4 AT is a tangent at A and TBC is a secant to the circle. Given that $\angle CTA = 30°$ and $\angle CAT = 110°$, find the magnitude of angles ACB, ABC and BAT.



Example 6 **5** From a point A outside a circle, a secant ABC is drawn cutting the circle at B and C, and a tangent AD touching it at D. A chord DE is drawn equal in length to chord DB. Prove that triangles ABD and CDE are similar.

6 Assume that AB is a chord of a circle and that CT, the tangent at C, is parallel to AB. Prove that $CA = CB$.

7 Through a point T, a tangent TA and a secant TPQ are drawn to a circle APQ. The chord AB is drawn parallel to PQ. Prove that the triangles PAT and BAQ are similar.

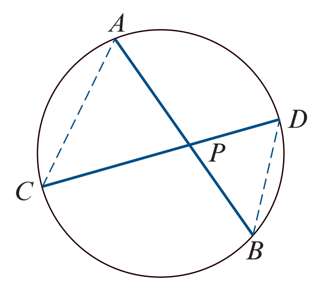
8 PQ is a diameter of a circle and AB is a perpendicular chord cutting it at N. Prove that PN is equal in length to the perpendicular from P onto the tangent at A.

10C Chords in circles

Theorem 8

If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.

Proof Case 1: The intersection point P is inside the circle.



Consider triangles APC and DPB:

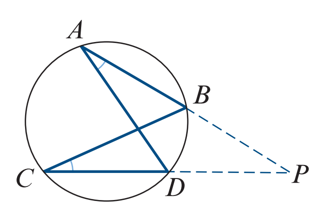
$$\angle APC = \angle DPB \quad \text{(vertically opposite)}$$
$$\angle CAB = \angle BDC \quad \text{(angles in the same segment)}$$

Thus triangle APC is similar to triangle DPB. This gives

$$\frac{PA}{PD} = \frac{PC}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$

Case 2: The intersection point P is outside the circle.



Show that triangle APD is similar to triangle CPB. This gives

$$\frac{PA}{PC} = \frac{PD}{PB}$$

$$\therefore \quad PA \cdot PB = PC \cdot PD$$

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A converse of Theorem 8 is:

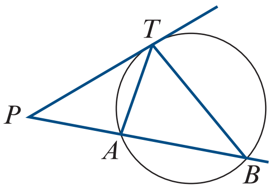
> If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.

This is proved in Extended-response question 2.

Theorem 9

If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Proof Consider triangles PAT and PTB:



$\angle ATP = \angle TBA$ (alternate segment theorem)

$\angle PAT = \angle PTB$ (angle sum of a triangle)

Therefore triangle PAT is similar to triangle PTB.

This gives

$$\frac{PA}{PT} = \frac{PT}{PB}$$

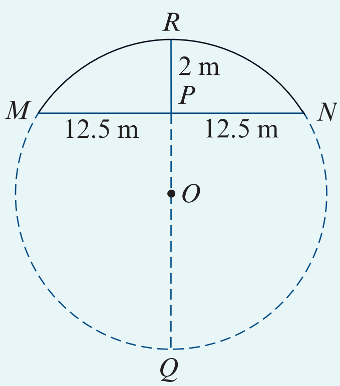
$$\therefore \quad PT^2 = PA \cdot PB$$

Example 7

The arch of a bridge is to be in the form of an arc of a circle. The span of the bridge is to be 25 m and the height in the middle 2 m. Find the radius of the circle.

Solution

Let r be the radius of the circle. Then $PQ = 2r - 2$.



Use Theorem 8 with the chords RQ and MN:

$$RP \cdot PQ = MP \cdot PN$$

Therefore

$$2PQ = 12.5^2$$

$$PQ = \frac{12.5^2}{2}$$

$$2r - 2 = \frac{12.5^2}{2} \qquad \text{as } PQ = 2r - 2$$

$$\therefore \quad r = \frac{1}{2}\left(\frac{12.5^2}{2} + 2\right)$$

$$= \frac{641}{16} \text{ m}$$

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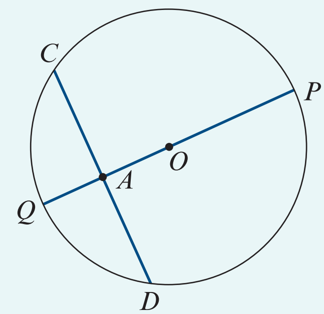
Example 8

Let A be any point inside a circle with radius r and centre O. Show that, if CD is a chord through A, then $CA \cdot AD = r^2 - OA^2$.

Solution

Let PQ be a diameter through A as shown.



By Theorem 8:

$$CA \cdot AD = QA \cdot AP$$

Since $QA = r - OA$ and $AP = r + OA$, this gives

$$\begin{aligned} CA \cdot AD &= (r - OA)(r + OA) \\ &= r^2 - OA^2 \end{aligned}$$

Section summary

- **Theorem 8** If AB and CD are two chords of a circle that cut at a point P (which may be inside or outside the circle), then $PA \cdot PB = PC \cdot PD$.
- **Theorem 9** If P is a point outside a circle and T, A, B are points on the circle such that PT is a tangent and PAB is a secant, then $PT^2 = PA \cdot PB$.

Exercise 10C

Skillsheet
Example 7

1 Two chords AB and CD intersect at a point P within a circle.

  a Given that $AP = 5$ cm, $PB = 4$ cm, $CP = 2$ cm, find PD.

  b Given that $AP = 4$ cm, $CP = 3$ cm, $PD = 8$ cm, find PB.

2 If AB is a chord and P is a point on AB such that $AP = 8$ cm, $PB = 5$ cm and P is 3 cm from the centre of the circle, find the radius.

3 If AB is a chord of a circle with centre O and P is a point on AB such that $BP = 4PA$, $OP = 5$ cm and the radius of the circle is 7 cm, find AB.

Example 8

4 Two circles intersect at points A and B. From any point P on the line AB, tangents PQ and PR are drawn to the circles. Prove that $PQ = PR$.

5 PQ is a variable chord of the smaller of two fixed concentric circles, and PQ extended meets the circumference of the larger circle at R. Prove that the product $RP \cdot RQ$ is constant for all positions and lengths of PQ.

6 ABC is an isosceles triangle with $AB = AC$. A line through A meets BC at D and the circumcircle of the triangle at E. Prove that $AB^2 = AD \cdot AE$.

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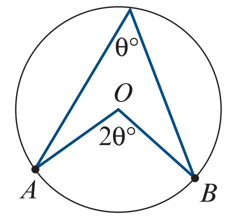
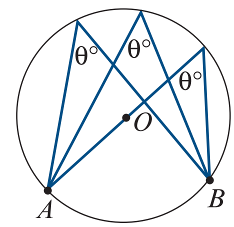
Chapter summary

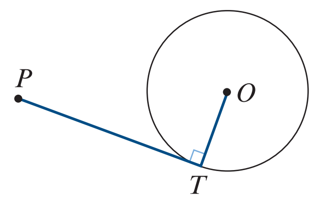
- The angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.



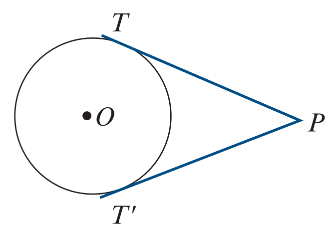
- Angles in the same segment of a circle are equal.



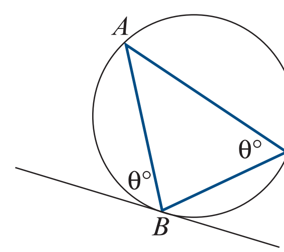
- A tangent to a circle is perpendicular to the radius drawn from the point of contact.



- The two tangents drawn from an external point are the same length, i.e. $PT = PT'$.



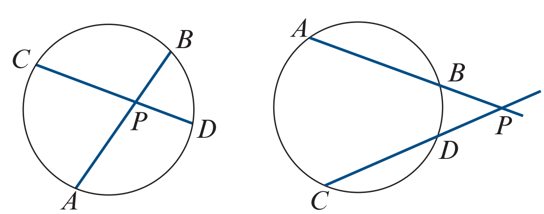
- The angle between a tangent and a chord drawn from the point of contact is equal to any angle in the alternate segment.



- A quadrilateral is cyclic if and only if the sum of each pair of opposite angles is 180°.

- If AB and CD are two chords of a circle that cut at a point P, then

$$PA \cdot PB = PC \cdot PD$$



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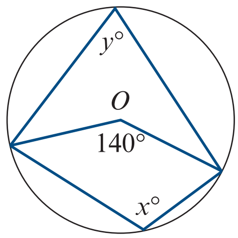
Technology-free questions

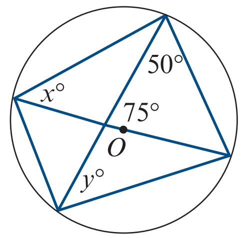
1 $\triangle ABC$ has $\angle A = 36^\circ$ and $\angle C = 90^\circ$. M is the midpoint of AB and CN is the altitude on AB. Find the size of $\angle MCN$.

2 Find the values of the pronumerals in each of the following:

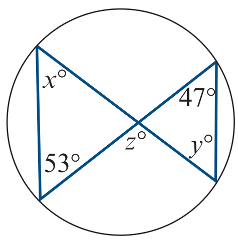
a



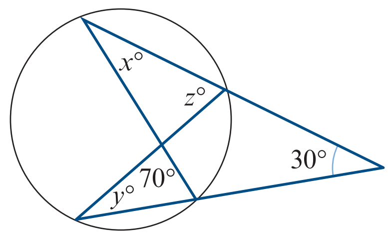
b



c



d



3 Let OP be a radius of a circle with centre O. A chord BA is drawn parallel to OP. The lines OA and BP intersect at C. Prove that:

a $\angle CAB = 2\angle CBA$

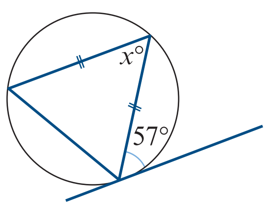
b $\angle PCA = 3\angle PBA$

4 A chord AB of a circle, centre O, is extended to C. The straight line bisecting $\angle OAB$ meets the circle at E. Prove that EB bisects $\angle OBC$.

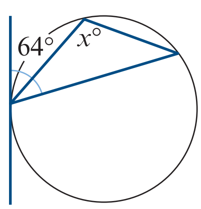
5 Two circles intersect at A and B. The tangent at B to one circle meets the second again at D, and a straight line through A meets the first circle at P and the second at Q. Prove that BP is parallel to DQ.

6 Find the values of the pronumerals for each of the following:

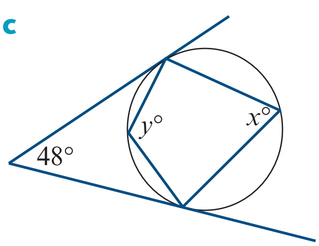
a



b



c



7 Two circles intersect at M and N. The tangent to the first circle at M meets the second circle at P, while the tangent to the second at N meets the first at Q. Prove that $MN^2 = NP \cdot QM$.

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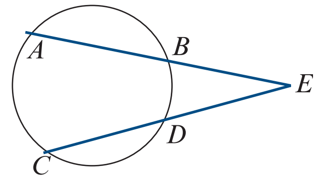
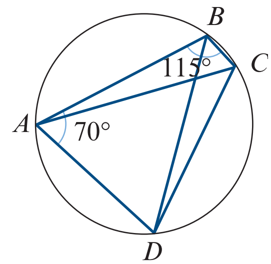
8 If $AB = 10$ cm, $BE = 5$ cm and $CE = 25$ cm, find DE.



Multiple-choice questions

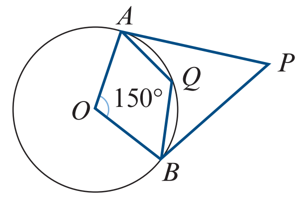
1 In the diagram, the points A, B, C and D lie on a circle, $\angle ABC = 115°$, $\angle BAD = 70°$ and $AB = AD$. The magnitude of $\angle ACD$ is

A 45° **B** 55° **C** 40° **D** 70° **E** 50°



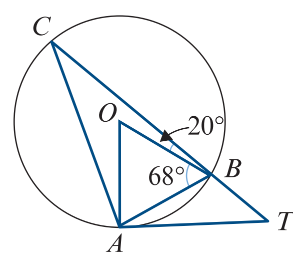
2 In the diagram, PA and PB are tangents to the circle centre O. Given that Q is a point on the minor arc AB and that $\angle AOB = 150°$, the magnitudes of $\angle APB$ and $\angle AQB$ are

A $\angle APB = 30°$ and $\angle AQB = 105°$
B $\angle APB = 40°$ and $\angle AQB = 110°$
C $\angle APB = 25°$ and $\angle AQB = 105°$
D $\angle APB = 30°$ and $\angle AQB = 110°$
E $\angle APB = 25°$ and $\angle AQB = 100°$



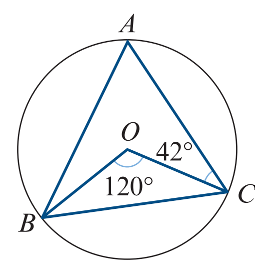
3 A circle with centre O passes through A, B and C. The line AT is the tangent to the circle at A, and CBT is a straight line. Given that $\angle ABO = 68°$ and $\angle OBC = 20°$, the magnitude of $\angle ATB$ is

A 60° **B** 64° **C** 65° **D** 70° **E** 66°



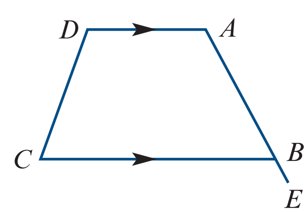
4 In the diagram, the points A, B and C lie on a circle centre O. If $\angle BOC = 120°$ and $\angle ACO = 42°$, then the magnitude of $\angle ABO$ is

A 18° **B** 20° **C** 22° **D** 24° **E** 26°



5 $ABCD$ is a cyclic quadrilateral with AD parallel to BC, and $\angle DCB = 65°$. The magnitude of $\angle CBE$ is

A 100° **B** 110° **C** 115° **D** 120° **E** 122°



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Review

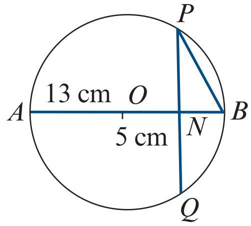
6 A chord AB of a circle subtends an angle of $50°$ at a point on the circumference of the circle. The acute angle between the tangents at A and B has magnitude

A $80°$ **B** $65°$ **C** $75°$ **D** $85°$ **E** $82°$

7 Chords AB and CD of a circle intersect at P. If $AP = 12$ cm, $PB = 6$ cm and $CP = 2$ cm, then the length of PD in centimetres is

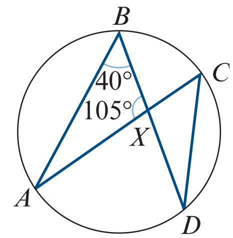
A 12 **B** 24 **C** 36 **D** 48 **E** 56

8 In the diagram, AB is the diameter of a circle with centre O and radius 13 cm. The chord PQ is perpendicular to AB, and N is the point of intersection of AB and PQ, with $ON = 5$ cm. The length of chord PB, in centimetres, is



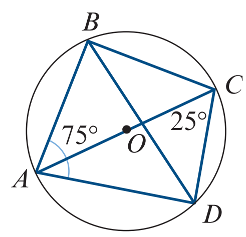
A 12 **B** $4\sqrt{13}$ **C** $2\sqrt{13}$
D 14 **E** 8

9 A, B, C and D are points on a circle, with $\angle ABD = 40°$ and $\angle AXB = 105°$. The magnitude of $\angle XDC$ is



A $35°$ **B** $40°$ **C** $45°$ **D** $50°$ **E** $55°$

10 A, B, C and D are points on a circle, centre O, such that AC is a diameter of the circle. If $\angle BAD = 75°$ and $\angle ACD = 25°$, then the magnitude of $\angle BDC$ is



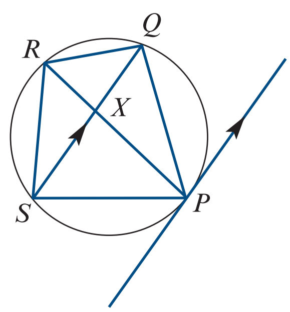
A $10°$ **B** $15°$ **C** $20°$ **D** $25°$ **E** $30°$

Extended-response questions

1 The diagonals PR and QS of a cyclic quadrilateral $PQRS$ intersect at X. The tangent at P is parallel to QS.
Prove that:

a $PQ = PS$

b PR bisects $\angle QRS$



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Review

2 *If line segments AB and CD intersect at a point M and $AM \cdot BM = CM \cdot DM$, then the points A, B, C and D are concyclic.*

To prove this claim, show that:

a $\dfrac{AM}{CM} = \dfrac{DM}{BM}$ **b** $\triangle AMC \sim \triangle DMB$

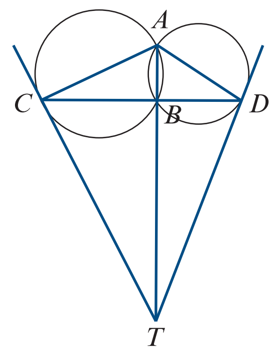
c $\angle CAM = \angle BDM$ **d** $ABCD$ is cyclic

3 Two circles intersect at A and B. The tangents at C and D intersect at T on the extension of AB. Prove that, if CBD is a straight line, then:

a $TCAD$ is a cyclic quadrilateral

b $\angle TAC = \angle TAD$

c $TC = TD$.

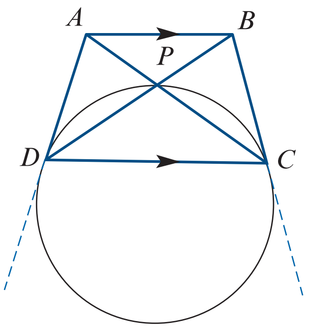


4 $ABCD$ is a trapezium in which AB is parallel to DC and the diagonals meet at P. The circle through D, P and C touches AD and BC at D and C respectively.
Prove that:

a $\angle BAC = \angle ADB$

b the circle through A, P and D touches BA at A

c $ABCD$ is a cyclic quadrilateral.



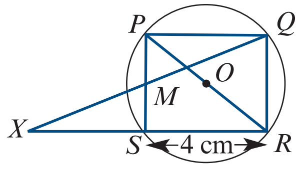
5 $PQRS$ is a square of side length 4 cm inscribed in a circle with centre O. The midpoint of the side PS is M. The line segments QM and RS are extended to meet at X.

a Prove that:

  i $\triangle XPR$ is isosceles

  ii PX is the tangent to the circle at P.

b Calculate the area of trapezium $PQRX$.



6 **a** An isosceles triangle ABC, with $AB = AC$, is inscribed in a circle. A chord AD intersects BC at E. Prove that

$$AB^2 - AE^2 = BE \cdot CE$$

b Diameter AB of a circle with centre O is extended to C, and from C a line is drawn tangent to the circle at P. The line PT is drawn perpendicular to AB at T. Prove that

$$CA \cdot CB - TA \cdot TB = CT^2$$

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