Chapter 14

14 Further trigonometry

Objectives

- To further explore the **symmetry** properties of circular functions.
- To further understand and sketch the **graphs** of circular functions.
- To solve equations involving circular functions.
- To evaluate simple trigonometric expressions using **trigonometric identities**.
- To prove simple trigonometric identities.
- To apply the **addition formulas** for circular functions.
- To apply the **double angle formulas** for circular functions.
- To simplify expressions of the form $a\cos x + b\sin x$.
- To sketch graphs of functions of the form $f(x) = a\cos x + b\sin x$.
- To solve equations of the form $a\cos x + b\sin x = c$.

There are many interesting and useful relationships between the trigonometric functions. The most fundamental is the Pythagorean identity:

$$\sin^2 A + \cos^2 A = 1$$

Some of these identities were discovered a very long time ago. For example, the following two results were discovered by the Indian mathematician Bhāskara II in the twelfth century:

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

They are of great importance in many areas of mathematics, including calculus.

Note: An introduction to sine, cosine and tangent as functions is given in Mathematical Methods Units 1 & 2 and also in an online chapter for this book, available in the Interactive Textbook.

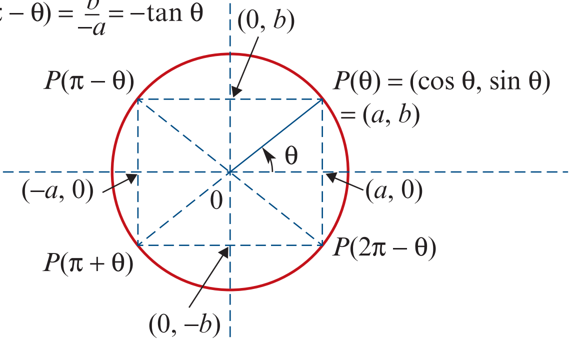
ISBN 978-1-107-56765-8

14A Symmetry properties

In this section we revise symmetry properties and exact values of sine and cosine.

Quadrant 2
By symmetry
$\sin(\pi - \theta) = b = \sin\theta$
$\cos(\pi - \theta) = -a = -\cos\theta$
$\tan(\pi - \theta) = \frac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3
$\sin(\pi + \theta) = -b = -\sin\theta$
$\cos(\pi + \theta) = -a = -\cos\theta$
$\tan(\pi + \theta) = \frac{-b}{-a} = \tan\theta$

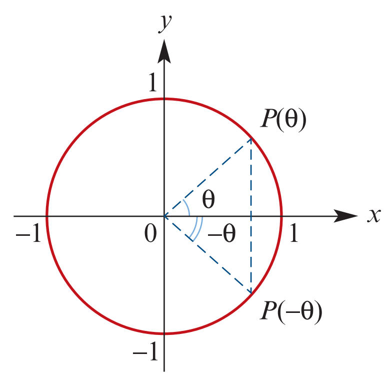
Quadrant 4
$\sin(2\pi - \theta) = -b = -\sin\theta$
$\cos(2\pi - \theta) = a = \cos\theta$
$\tan(2\pi - \theta) = \frac{-b}{a} = -\tan\theta$

Note: $\sin(2\pi + \theta) = \sin\theta$
$\cos(2\pi + \theta) = \cos\theta$
$\tan(2\pi + \theta) = \tan\theta$

Negative angles

By symmetry:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Exact values of sine and cosine

The values in this table can easily be determined from the graphs of sine and cosine or from the unit circle.

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	−1	0	1	0	−1	0
$\cos\theta$	−1	0	1	0	−1	0	1

The values in this table can easily be determined by drawing:

- an equilateral triangle of side length 2 and one median
- a square of side length 1 and one diagonal.

θ	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$

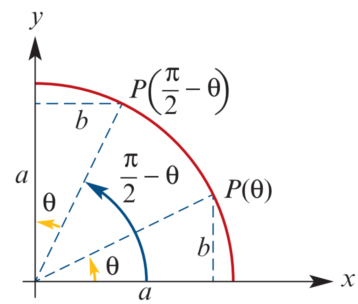
ISBN 978-1-107-56765-8

Complementary relationships

From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$

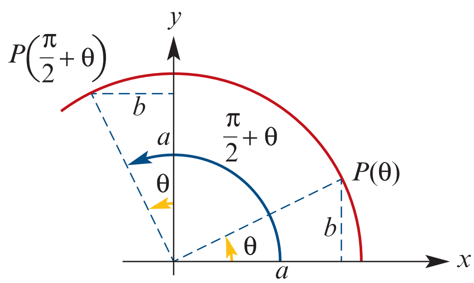
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 1

If $\sin\theta = 0.4$ and $\cos\alpha = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.4$

c $\sin(-\theta) = -\sin\theta$
$= -0.4$

Exercise 14A

1 Evaluate each of the following:

a $\cos\left(\frac{3\pi}{4}\right)$ **b** $\sin\left(\frac{5\pi}{4}\right)$ **c** $\sin\left(\frac{25\pi}{2}\right)$ **d** $\sin\left(\frac{15\pi}{6}\right)$ **e** $\cos\left(\frac{17\pi}{4}\right)$

f $\sin\left(-\frac{15\pi}{4}\right)$ **g** $\sin(27\pi)$ **h** $\sin\left(-\frac{17\pi}{3}\right)$ **i** $\cos\left(\frac{75\pi}{6}\right)$ **j** $\cos\left(-\frac{15\pi}{6}\right)$

k $\sin\left(-\frac{35\pi}{2}\right)$ **l** $\cos\left(-\frac{45\pi}{6}\right)$ **m** $\cos\left(\frac{16\pi}{3}\right)$ **n** $\sin\left(-\frac{105\pi}{2}\right)$ **o** $\cos(1035\pi)$

Example 1

2 If $\sin x = 0.3$ and $\cos\alpha = 0.6$, find the value of:

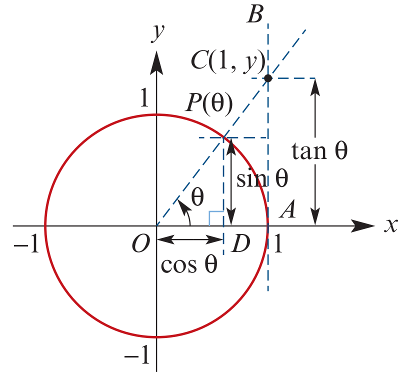
a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\cos\left(\frac{\pi}{2} - x\right)$ **d** $\sin(-x)$

e $\cos\left(\frac{\pi}{2} + x\right)$ **f** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **g** $\sin\left(\frac{3\pi}{2} + \alpha\right)$ **h** $\cos\left(\frac{3\pi}{2} - x\right)$

ISBN 978-1-107-56765-8

14B The tangent function

Consider the unit circle. If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

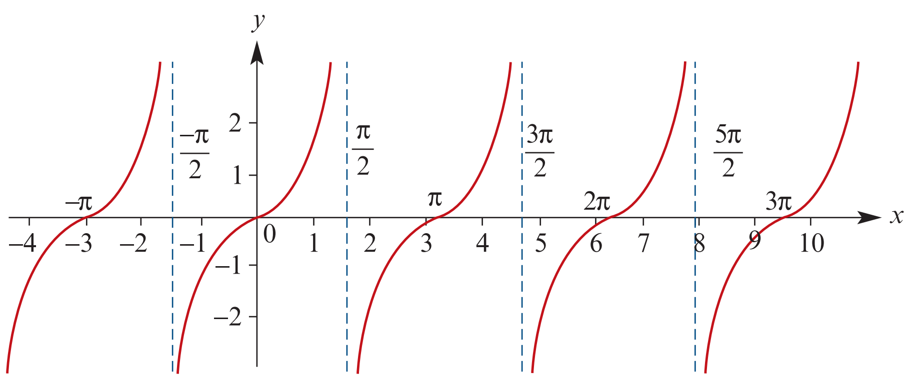
$$\therefore \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$.

A table of values for $y = \tan x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0

The graph of $y = \tan x$ is given below.



You must also know the following exact values:

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \qquad \tan\left(\frac{\pi}{4}\right) = 1, \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

Properties of the tangent function

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $x = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.

ISBN 978-1-107-56765-8

Cambridge University Press

Symmetry

Using symmetry properties of sine and cosine, we have

$$\tan(\pi - \theta) = \frac{\sin(\pi - \theta)}{\cos(\pi - \theta)} = \frac{\sin\theta}{-\cos\theta} = -\tan\theta$$

Similarly, we obtain:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(2\pi - \theta) = -\tan\theta$
- $\tan(-\theta) = -\tan\theta$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \frac{\cos\theta}{\sin\theta}$
- $\tan\left(\frac{\pi}{2} + \theta\right) = -\frac{\cos\theta}{\sin\theta}$

Note: We will see in Section 14E that $\frac{\cos\theta}{\sin\theta}$ can be written as $\cot\theta$.

Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function, which can be applied to finding the x-axis intercepts for graphs of the tangent function.

Example 2

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 3

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

$$\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$$

implies

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

Explanation

Note that

$$x \in [-2\pi, 2\pi] \iff x - \frac{\pi}{4} \in \left[-\frac{9\pi}{4}, \frac{7\pi}{4}\right]$$

$$\iff \frac{1}{2}\left(x - \frac{\pi}{4}\right) \in \left[-\frac{9\pi}{8}, \frac{7\pi}{8}\right]$$

ISBN 978-1-107-56765-8

Graphing the tangent function

When graphing a transformation of the tangent function:

- Find the period.
- Find the equations of the asymptotes.
- Find the intercepts with the axes.

Example 4

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$

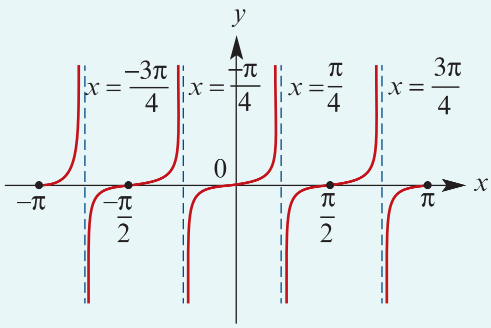
b $y = -2\tan(3x)$

Solution

a Period $= \dfrac{\pi}{n} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$

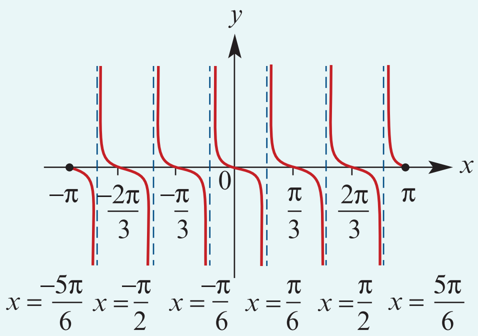
Axis intercepts: $x = \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$



b Period $= \dfrac{\pi}{n} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{3}$, $k \in \mathbb{Z}$



Section summary

- The tangent function is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
- The graph of $y = \tan x$:
  - The period is π.
  - The vertical asymptotes have equations $x = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $x = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

ISBN 978-1-107-56765-8

Exercise 14B

Example 2 **1** Solve each of the following equations for x in the stated interval:

a $\tan x = -1, \; x \in (0, 2\pi)$

b $\tan x = \sqrt{3}, \; x \in (0, 2\pi)$

c $\tan x = \frac{1}{\sqrt{3}}, \; x \in (0, 2\pi)$

d $\tan(2x) = 1, \; x \in (-\pi, \pi)$

e $\tan(2x) = \sqrt{3}, \; x \in (-\pi, \pi)$

f $\tan(2x) = -\frac{1}{\sqrt{3}}, \; x \in (-\pi, \pi)$

Example 3 **2** Solve each of the following equations for x in the stated interval:

a $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = 1, \; x \in (0, 2\pi)$

b $\tan\left(2\left(x - \frac{\pi}{4}\right)\right) = -1, \; x \in (-\pi, \pi)$

c $\tan\left(3\left(x - \frac{\pi}{6}\right)\right) = \sqrt{3}, \; x \in (-\pi, \pi)$

d $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{6}\right)\right) = -\frac{1}{\sqrt{3}}, \; x \in (-\pi, \pi)$

Example 4 **3** Sketch the graph of each of the following:

a $y = \tan(2x)$

b $y = \tan(3x)$

c $y = -\tan(2x)$

d $y = 3\tan x$

e $y = \tan\left(\frac{x}{2}\right)$

f $y = 2\tan\left(x + \frac{\pi}{4}\right)$

g $y = 3\tan x + 1$

h $y = 2\tan\left(x + \frac{\pi}{2}\right) + 1$

i $y = 3\tan\left(2\left(x - \frac{\pi}{4}\right)\right) - 2$

14C Reciprocal functions and the Pythagorean identity

In this section we introduce the reciprocals of the basic trigonometric functions. The graphs of these functions appear in Chapter 15, where reciprocal functions are studied in general. Here we use these functions in various forms of the Pythagorean identity.

▶ Reciprocal functions

The circular functions sine, cosine and tangent can be used to form three other functions, called the reciprocal circular functions.

Secant, cosecant and cotangent

- $\sec\theta = \frac{1}{\cos\theta}$ (for $\cos\theta \neq 0$)
- $\operatorname{cosec}\theta = \frac{1}{\sin\theta}$ (for $\sin\theta \neq 0$)
- $\cot\theta = \frac{\cos\theta}{\sin\theta}$ (for $\sin\theta \neq 0$)

Note: For $\cos\theta \neq 0$ and $\sin\theta \neq 0$, we have $\cot\theta = \frac{1}{\tan\theta}$ and $\tan\theta = \frac{1}{\cot\theta}$.

ISBN 978-1-107-56765-8

Example 5

Find the exact value of each of the following:

a $\sec\left(\frac{2\pi}{3}\right)$ **b** $\cot\left(\frac{5\pi}{4}\right)$ **c** $\operatorname{cosec}\left(\frac{7\pi}{4}\right)$

Solution

a
$$\sec\left(\frac{2\pi}{3}\right) = \frac{1}{\cos\left(\frac{2\pi}{3}\right)}$$
$$= \frac{1}{\cos\left(\pi - \frac{\pi}{3}\right)}$$
$$= \frac{1}{-\cos\left(\frac{\pi}{3}\right)}$$
$$= 1 \div \left(-\frac{1}{2}\right)$$
$$= -2$$

b
$$\cot\left(\frac{5\pi}{4}\right) = \frac{\cos\left(\frac{5\pi}{4}\right)}{\sin\left(\frac{5\pi}{4}\right)}$$
$$= \frac{\cos\left(\pi + \frac{\pi}{4}\right)}{\sin\left(\pi + \frac{\pi}{4}\right)}$$
$$= \frac{-1}{\sqrt{2}} \div \left(\frac{-1}{\sqrt{2}}\right)$$
$$= 1$$

c
$$\operatorname{cosec}\left(\frac{7\pi}{4}\right)$$
$$= \frac{1}{\sin\left(2\pi - \frac{\pi}{4}\right)}$$
$$= \frac{1}{-\sin\left(\frac{\pi}{4}\right)}$$
$$= 1 \div \left(-\frac{1}{\sqrt{2}}\right)$$
$$= -\sqrt{2}$$

Example 6

Find the values of x between 0 and 2π for which:

a $\sec x = -2$ **b** $\cot x = -1$

Solution

a
$$\sec x = -2$$
$$\frac{1}{\cos x} = -2$$
$$\cos x = \frac{-1}{2}$$
$$\therefore \quad x = \pi - \frac{\pi}{3} \quad \text{or} \quad x = \pi + \frac{\pi}{3}$$
$$\therefore \quad x = \frac{2\pi}{3} \quad \text{or} \quad x = \frac{4\pi}{3}$$

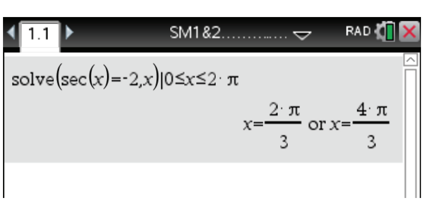
b
$$\cot x = -1$$
$$\tan x = -1$$
$$\therefore \quad x = \pi - \frac{\pi}{4} \quad \text{or} \quad x = 2\pi - \frac{\pi}{4}$$
$$\therefore \quad x = \frac{3\pi}{4} \quad \text{or} \quad x = \frac{7\pi}{4}$$

Using the TI-Nspire

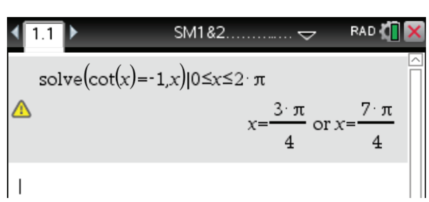
Check that your calculator is in radian mode. Use (menu) > **Algebra** > **Solve** as shown.

Note: Access sec and cot using (trig). Access ≤ using (ctrl) (=).

a



b



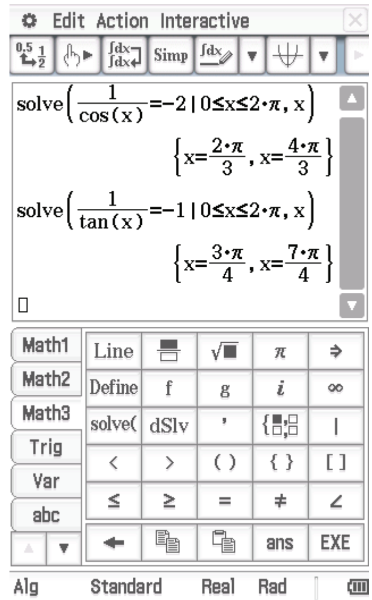
ISBN 978-1-107-56765-8

Using the Casio ClassPad

The ClassPad does not recognise sec x, cosec x and cot x. These functions must be entered as reciprocals of cos x, sin x and tan x respectively.

a
- Enter and highlight: $\frac{1}{\cos(x)} = -2 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to x and tap OK.

b
- Enter and highlight: $\frac{1}{\tan(x)} = -1 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve**, ensure the variable is set to x and tap OK.



Note: The 'for' operator | is found in the Math3 keyboard and is used to specify a condition. In this case, the condition is the domain restriction.

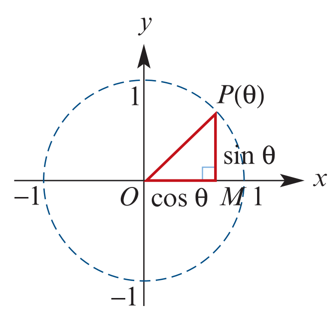
The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

By Pythagoras' theorem:

$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$



Since this is true for all values of θ, it is called an identity.

We can write $(\cos\theta)^2$ and $(\sin\theta)^2$ as $\cos^2\theta$ and $\sin^2\theta$, and therefore we obtain:

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

We can derive other forms of this identity:

- Dividing both sides by $\cos^2\theta$ gives

$$\frac{\cos^2\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

$$\therefore \quad 1 + \tan^2\theta = \sec^2\theta$$

- Dividing both sides by $\sin^2\theta$ gives

$$\frac{\cos^2\theta}{\sin^2\theta} + \frac{\sin^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$\therefore \quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

ISBN 978-1-107-56765-8

Example 7

a If $\operatorname{cosec} x = \frac{7}{4}$, find $\cos x$.

b If $\sec x = -\frac{3}{2}$ and $\frac{\pi}{2} \le x \le \pi$, find $\sin x$.

Solution

a Since $\operatorname{cosec} x = \frac{7}{4}$, we have $\sin x = \frac{4}{7}$.

Now
$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \cos^2 x + \frac{16}{49} &= 1\\ \cos^2 x &= \frac{33}{49}\\ \therefore \quad \cos x &= \pm\frac{\sqrt{33}}{7}\end{aligned}$$

b Since $\sec x = -\frac{3}{2}$, we have $\cos x = -\frac{2}{3}$.

Now
$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \frac{4}{9} + \sin^2 x &= 1\\ \therefore \quad \sin x &= \pm\frac{\sqrt{5}}{3}\end{aligned}$$

But $\sin x$ is positive for $P(x)$ in the 2nd quadrant, and so $\sin x = \frac{\sqrt{5}}{3}$.

Example 8

If $\sin\theta = \frac{3}{5}$ and $\frac{\pi}{2} < \theta < \pi$, find the values of $\cos\theta$ and $\tan\theta$.

Solution

$$\begin{aligned}\cos^2\theta + \sin^2\theta &= 1\\ \cos^2\theta + \frac{9}{25} &= 1\\ \therefore \quad \cos^2\theta &= \frac{16}{25}\end{aligned}$$

Thus $\cos\theta = -\frac{4}{5}$, since $\frac{\pi}{2} < \theta < \pi$, and therefore $\tan\theta = \frac{\sin\theta}{\cos\theta} = -\frac{3}{4}$.

Example 9

Prove the identity $\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta$.

Solution

$$\begin{aligned}\text{LHS} &= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}\\ &= \frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta}\\ &= \frac{2}{1-\cos^2\theta}\\ &= \frac{2}{\sin^2\theta}\\ &= 2\operatorname{cosec}^2\theta\\ &= \text{RHS}\end{aligned}$$

ISBN 978-1-107-56765-8

Section summary

- Reciprocal functions

$$\sec\theta = \frac{1}{\cos\theta} \quad (\text{for } \cos\theta \neq 0)$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} \quad (\text{for } \sin\theta \neq 0)$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

Exercise 14C

Example 5

1 Find the exact value of each of the following:

a $\cot\left(\frac{3\pi}{4}\right)$ **b** $\operatorname{cosec}\left(\frac{5\pi}{4}\right)$ **c** $\sec\left(\frac{5\pi}{6}\right)$ **d** $\operatorname{cosec}\left(\frac{\pi}{2}\right)$

e $\sec\left(\frac{4\pi}{3}\right)$ **f** $\operatorname{cosec}\left(\frac{13\pi}{6}\right)$ **g** $\cot\left(\frac{7\pi}{3}\right)$ **h** $\sec\left(\frac{5\pi}{3}\right)$

2 Without using a calculator, write down the exact value of each of the following:

a $\cot 135°$ **b** $\sec 150°$ **c** $\operatorname{cosec} 90°$ **d** $\cot 240°$ **e** $\operatorname{cosec} 225°$

f $\sec 330°$ **g** $\cot 315°$ **h** $\operatorname{cosec} 300°$ **i** $\cot 420°$

Example 6

3 Find the values of x between 0 and 2π for which:

a $\operatorname{cosec} x = 2$ **b** $\cot x = \sqrt{3}$ **c** $\sec x + \sqrt{2} = 0$ **d** $\operatorname{cosec} x = \sec x$

Example 7, 8

4 If $\sec\theta = \frac{-17}{8}$ and $\frac{\pi}{2} < \theta < \pi$, find:

a $\cos\theta$ **b** $\sin\theta$ **c** $\tan\theta$

5 If $\tan\theta = \frac{-7}{24}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find $\cos\theta$ and $\sin\theta$.

6 Find the value of $\sec\theta$ if $\tan\theta = 0.4$ and θ is not in the 1st quadrant.

7 If $\tan\theta = \frac{4}{3}$ and $\pi < \theta < \frac{3\pi}{2}$, evaluate $\frac{\sin\theta - 2\cos\theta}{\cot\theta - \sin\theta}$.

8 If $\cos\theta = \frac{2}{3}$ and θ is in the 4th quadrant, express $\frac{\tan\theta - 3\sin\theta}{\cos\theta - 2\cot\theta}$ in simplest surd form.

Example 9

9 Prove each of the following identities for suitable values of θ and φ:

a $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$

b $\cos^2\theta\tan^2\theta + \sin^2\theta\cot^2\theta = 1$

c $\frac{\tan\theta}{\tan\varphi} = \frac{\tan\theta + \cot\varphi}{\cot\theta + \tan\varphi}$

d $(\sin\theta + \cos\theta)^2 + (\sin\theta - \cos\theta)^2 = 2$

e $\frac{1 + \cot^2\theta}{\cot\theta\operatorname{cosec}\theta} = \sec\theta$

f $\sec\theta + \tan\theta = \frac{\cos\theta}{1 - \sin\theta}$

ISBN 978-1-107-56765-8

14D Addition formulas and double angle formulas

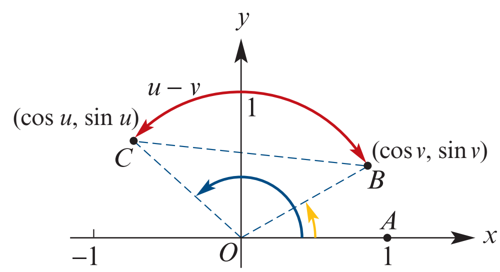
Addition formulas

Addition formulas for cosine

1 $\cos(u + v) = \cos u \cos v - \sin u \sin v$

2 $\cos(u - v) = \cos u \cos v + \sin u \sin v$

Proof Consider a unit circle as shown:



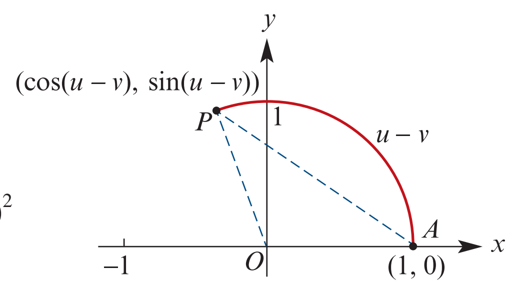
arc length $AB = v$ units

arc length $AC = u$ units

arc length $BC = u - v$ units

Rotate ΔOCB so that B is coincident with A. Then C is moved to

$P(\cos(u - v), \sin(u - v))$



Since the triangles CBO and PAO are congruent, we have $CB = PA$.

Using the coordinate distance formula:

$$\begin{aligned} CB^2 &= (\cos u - \cos v)^2 + (\sin u - \sin v)^2 \\ &= 2 - 2(\cos u \cos v + \sin u \sin v) \end{aligned}$$

$$\begin{aligned} PA^2 &= (\cos(u - v) - 1)^2 + (\sin(u - v) - 0)^2 \\ &= 2 - 2\cos(u - v) \end{aligned}$$

Since $CB = PA$, this gives

$$2 - 2\cos(u - v) = 2 - 2(\cos u \cos v + \sin u \sin v)$$

$$\therefore \quad \cos(u - v) = \cos u \cos v + \sin u \sin v$$

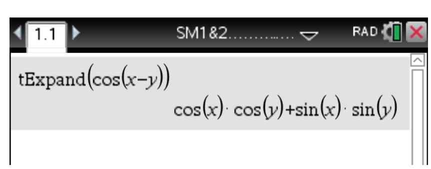
We can now obtain the first formula from the second by replacing v with $-v$:

$$\begin{aligned} \cos(u + v) &= \cos(u - (-v)) \\ &= \cos u \cos(-v) + \sin u \sin(-v) \\ &= \cos u \cos v - \sin u \sin v \end{aligned}$$

Note: Here we used $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.

Using the TI-Nspire

Access the **tExpand( )** command from menu > **Algebra** > **Trigonometry** > **Expand** and complete as shown.

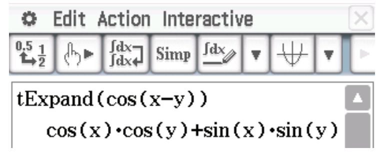


ISBN 978-1-107-56765-8

Cambridge University Press

Using the Casio ClassPad

- In $\sqrt[\text{Main}]{\alpha}$, enter and highlight $\cos(x - y)$.
- Go to **Interactive** > **Transformation** > **tExpand** and tap OK.



Example 10

Evaluate cos 75°.

Solution

$$\begin{aligned}
\cos 75^\circ &= \cos(45^\circ + 30^\circ)\\
&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ\\
&= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2}\\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}}\\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}\\
&= \frac{\sqrt{6} - \sqrt{2}}{4}
\end{aligned}$$

Addition formulas for sine

1 $\sin(u + v) = \sin u \cos v + \cos u \sin v$

2 $\sin(u - v) = \sin u \cos v - \cos u \sin v$

Proof We use the symmetry properties $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$ and $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$:

$$\begin{aligned}
\sin(u + v) &= \cos\left(\frac{\pi}{2} - (u + v)\right)\\
&= \cos\left(\left(\frac{\pi}{2} - u\right) - v\right)\\
&= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v\\
&= \sin u \cos v + \cos u \sin v
\end{aligned}$$

We can now obtain the second formula from the first by replacing v with $-v$:

$$\begin{aligned}
\sin(u - v) &= \sin u \cos(-v) + \cos u \sin(-v)\\
&= \sin u \cos v - \cos u \sin v
\end{aligned}$$

ISBN 978-1-107-56765-8

Example 11

Evaluate:

a $\sin 75^\circ$ **b** $\sin 15^\circ$

Solution

a
$$\begin{aligned}\sin 75^\circ &= \sin(30^\circ + 45^\circ)\\ &= \sin 30^\circ \cos 45^\circ + \cos 30^\circ \sin 45^\circ\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{2}+\sqrt{6}}{4}\end{aligned}$$

b
$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\cdot\frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}\\ &= \frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Addition formulas for tangent

1 $\tan(u+v) = \dfrac{\tan u + \tan v}{1 - \tan u \tan v}$ **2** $\tan(u-v) = \dfrac{\tan u - \tan v}{1 + \tan u \tan v}$

Proof To obtain the first formula, we write

$$\tan(u+v) = \frac{\sin(u+v)}{\cos(u+v)} = \frac{\sin u \cos v + \cos u \sin v}{\cos u \cos v - \sin u \sin v}$$

Now divide the numerator and denominator by $\cos u \cos v$. The second formula can be obtained from the first by using $\tan(-\theta) = -\tan\theta$.

Example 12

If u and v are acute angles such that $\tan u = 4$ and $\tan v = \dfrac{3}{5}$, show that $u - v = \dfrac{\pi}{4}$.

Solution

$$\begin{aligned}\tan(u-v) &= \frac{\tan u - \tan v}{1 + \tan u \tan v}\\ &= \frac{4 - \frac{3}{5}}{1 + 4\times\frac{3}{5}}\\ &= \frac{20-3}{5+4\times 3}\\ &= 1\end{aligned}$$

$$\therefore \quad u - v = \frac{\pi}{4}$$

Note: The function $\tan\theta$ is one-to-one for $0 < \theta < \dfrac{\pi}{2}$.

ISBN 978-1-107-56765-8

Double angle formulas

Using the addition formulas, we can easily derive useful expressions for $\sin(2u)$, $\cos(2u)$ and $\tan(2u)$.

Double angle formulas for cosine

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u \\ &= 2\cos^2 u - 1 && (\text{since } \sin^2 u = 1 - \cos^2 u) \\ &= 1 - 2\sin^2 u && (\text{since } \cos^2 u = 1 - \sin^2 u)\end{aligned}$$

Proof $\cos(u + u) = \cos u \cos u - \sin u \sin u$

$$= \cos^2 u - \sin^2 u$$

Double angle formula for sine

$$\sin(2u) = 2\sin u \cos u$$

Proof $\sin(u + u) = \sin u \cos u + \cos u \sin u$

$$= 2\sin u \cos u$$

Double angle formula for tangent

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

Proof $\tan(u + u) = \dfrac{\tan u + \tan u}{1 - \tan u \tan u}$

$$= \frac{2\tan u}{1 - \tan^2 u}$$

Example 13

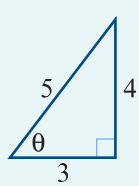
If $\tan\theta = \frac{4}{3}$ and $0 < \theta < \frac{\pi}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\tan(2\theta)$

Solution

a $\sin\theta = \frac{4}{5}$ and $\cos\theta = \frac{3}{5}$

$$\begin{aligned}\therefore \sin(2\theta) &= 2\sin\theta\cos\theta \\ &= 2 \times \frac{4}{5} \times \frac{3}{5} \\ &= \frac{24}{25}\end{aligned}$$



b
$$\begin{aligned}\tan(2\theta) &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2 \times \frac{4}{3}}{1 - \frac{16}{9}} \\ &= \frac{2 \times 4 \times 3}{9 - 16} \\ &= -\frac{24}{7}\end{aligned}$$

ISBN 978-1-107-56765-8

Example 14

Prove each of the following identities:

a $\dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan(2\theta)$

b $\dfrac{\sin\theta}{\sin\varphi} + \dfrac{\cos\theta}{\cos\varphi} = \dfrac{2\sin(\theta+\varphi)}{\sin(2\varphi)}$

c $\dfrac{1}{\cos\theta + \sin\theta} + \dfrac{1}{\cos\theta - \sin\theta} = \tan(2\theta)\operatorname{cosec}\theta$

Solution

a
$$\begin{aligned}\text{LHS} &= \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)}\\ &= \tan(2\theta)\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$.

b
$$\begin{aligned}\text{LHS} &= \frac{\sin\theta}{\sin\varphi} + \frac{\cos\theta}{\cos\varphi}\\ &= \frac{\sin\theta\cos\varphi + \cos\theta\sin\varphi}{\sin\varphi\cos\varphi}\\ &= \frac{\sin(\theta+\varphi)}{\frac{1}{2}\sin(2\varphi)}\\ &= \frac{2\sin(\theta+\varphi)}{\sin(2\varphi)}\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\sin(2\varphi) \neq 0$.

c
$$\begin{aligned}\text{LHS} &= \frac{1}{\cos\theta + \sin\theta} + \frac{1}{\cos\theta - \sin\theta}\\ &= \frac{\cos\theta - \sin\theta + \cos\theta + \sin\theta}{\cos^2\theta - \sin^2\theta}\\ &= \frac{2\cos\theta}{\cos(2\theta)}\\ &= \frac{2\cos\theta}{\cos(2\theta)} \cdot \frac{\sin\theta}{\sin\theta}\\ &= \frac{\sin(2\theta)}{\cos(2\theta)\sin\theta}\\ &= \frac{\tan(2\theta)}{\sin\theta}\\ &= \tan(2\theta)\operatorname{cosec}\theta\\ &= \text{RHS}\end{aligned}$$

Note: Identity holds when $\cos(2\theta) \neq 0$ and $\sin\theta \neq 0$.

Sometimes the easiest way to prove that two expressions are equal is to simplify each of them separately. This is demonstrated in the following example.

ISBN 978-1-107-56765-8

Example 15

Prove that $(\sec A - \cos A)(\operatorname{cosec} A - \sin A) = \dfrac{1}{\tan A + \cot A}$.

Solution

$$\begin{aligned}\text{LHS} &= (\sec A - \cos A)(\operatorname{cosec} A - \sin A)\\ &= \left(\frac{1}{\cos A} - \cos A\right)\left(\frac{1}{\sin A} - \sin A\right)\\ &= \frac{1-\cos^2 A}{\cos A} \times \frac{1-\sin^2 A}{\sin A}\\ &= \frac{\sin^2 A\cos^2 A}{\cos A\sin A}\\ &= \cos A\sin A\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{1}{\tan A + \cot A}\\ &= \frac{1}{\dfrac{\sin A}{\cos A} + \dfrac{\cos A}{\sin A}}\\ &= \frac{\cos A\sin A}{\sin^2 A + \cos^2 A}\\ &= \cos A\sin A\end{aligned}$$

We have shown that LHS = RHS.

Section summary

- **Addition formulas**

$$\cos(u+v) = \cos u\cos v - \sin u\sin v$$
$$\cos(u-v) = \cos u\cos v + \sin u\sin v$$
$$\sin(u+v) = \sin u\cos v + \cos u\sin v$$
$$\sin(u-v) = \sin u\cos v - \cos u\sin v$$
$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u\tan v}$$
$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u\tan v}$$

- **Double angle formulas**

$$\begin{aligned}\cos(2u) &= \cos^2 u - \sin^2 u\\ &= 2\cos^2 u - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$
$$\sin(2u) = 2\sin u\cos u$$
$$\tan(2u) = \frac{2\tan u}{1-\tan^2 u}$$

Exercise 14D

Skillsheet

Example 10

1 By using the appropriate addition formulas, find exact values for the following:

a $\cos 15°$ **b** $\cos 105°$

Example 11

2 By using the appropriate addition formulas, find exact values for the following:

a $\sin 165°$ **b** $\tan 75°$

3 Find the exact value of:

a $\cos\left(\frac{5\pi}{12}\right)$ **b** $\sin\left(\frac{11\pi}{12}\right)$ **c** $\tan\left(\frac{-\pi}{12}\right)$

ISBN 978-1-107-56765-8

Example 12 **4** If $\sin u = \frac{12}{13}$ and $\sin v = \frac{3}{5}$, evaluate $\sin(u + v)$. (Note: There is more than one answer.)

5 Simplify the following:

a $\sin\left(\theta + \frac{\pi}{6}\right)$ **b** $\cos\left(\varphi - \frac{\pi}{4}\right)$ **c** $\tan\left(\theta + \frac{\pi}{3}\right)$ **d** $\sin\left(\theta - \frac{\pi}{4}\right)$

6 Simplify:

a $\cos(u - v)\sin v + \sin(u - v)\cos v$ **b** $\sin(u + v)\sin v + \cos(u + v)\cos v$

Example 13 **7** If $\sin\theta = \frac{-3}{5}$, with θ in the 3rd quadrant, and $\cos\varphi = \frac{-5}{13}$, with φ in the 2nd quadrant, evaluate each of the following without using a calculator:

a $\cos(2\varphi)$ **b** $\sin(2\theta)$ **c** $\tan(2\theta)$ **d** $\sec(2\varphi)$
e $\sin(\theta + \varphi)$ **f** $\cos(\theta - \varphi)$ **g** $\operatorname{cosec}(\theta + \varphi)$ **h** $\cot(2\theta)$

8 For acute angles u and v such that $\tan u = \frac{4}{3}$ and $\tan v = \frac{5}{12}$, evaluate:

a $\tan(u + v)$ **b** $\tan(2u)$ **c** $\cos(u - v)$ **d** $\sin(2u)$

9 If $\sin\alpha = \frac{3}{5}$ and $\sin\beta = \frac{24}{25}$, with $\frac{\pi}{2} < \beta < \alpha < \pi$, evaluate:

a $\cos(2\alpha)$ **b** $\sin(\alpha - \beta)$ **c** $\tan(\alpha + \beta)$ **d** $\sin(2\beta)$

10 If $\sin\theta = -\frac{\sqrt{3}}{2}$ and $\cos\theta = \frac{1}{2}$, evaluate:

a $\sin(2\theta)$ **b** $\cos(2\theta)$

11 Simplify each of the following expressions:

a $(\sin\theta - \cos\theta)^2$ **b** $\cos^4\theta - \sin^4\theta$

Example 14, 15 **12** Prove the following identities:

a $\sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right) = \sin\theta - \cos\theta$ **b** $\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) = \cos\theta$

c $\tan\left(\theta + \frac{\pi}{4}\right)\tan\left(\theta - \frac{\pi}{4}\right) = -1$ **d** $\cos\left(\theta + \frac{\pi}{6}\right) + \sin\left(\theta + \frac{\pi}{3}\right) = \sqrt{3}\cos\theta$

e $\tan\left(\theta + \frac{\pi}{4}\right) = \frac{1 + \tan\theta}{1 - \tan\theta}$ **f** $\frac{\sin(u + v)}{\cos u \cos v} = \tan v + \tan u$

g $\frac{\tan u + \tan v}{\tan u - \tan v} = \frac{\sin(u + v)}{\sin(u - v)}$ **h** $\cos(2\theta) + 2\sin^2\theta = 1$

i $\sin(4\theta) = 4\sin\theta\cos^3\theta - 4\cos\theta\sin^3\theta$ **j** $\frac{1 - \sin(2\theta)}{\sin\theta - \cos\theta} = \sin\theta - \cos\theta$

ISBN 978-1-107-56765-8

14E Simplifying $a\cos x + b\sin x$

In this section, we see how to rewrite the rule of a function $f(x) = a\cos x + b\sin x$ in terms of a single circular function.

$$a\cos x + b\sin x = r\cos(x - \alpha) \quad \text{where } r = \sqrt{a^2 + b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

Proof Let $r = \sqrt{a^2 + b^2}$. Consider the point $P\left(\frac{a}{r}, \frac{b}{r}\right)$ and its distance from the origin O:

$$OP^2 = \left(\frac{a}{r}\right)^2 + \left(\frac{b}{r}\right)^2 = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} = 1$$

The point P is on the unit circle, and so $\frac{a}{r} = \cos\alpha$ and $\frac{b}{r} = \sin\alpha$, for some angle α.

We can now write

$$\begin{aligned} a\cos x + b\sin x &= r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right) \\ &= r(\cos\alpha\ \cos x + \sin\alpha\ \sin x) \\ &= r\cos(x - \alpha) \end{aligned}$$

Similarly, it may be shown that

$$a\cos x + b\sin x = r\sin(x + \beta) \quad \text{where } r = \sqrt{a^2 + b^2},\ \sin\beta = \frac{a}{r},\ \cos\beta = \frac{b}{r}$$

Example 16

Express $\cos x - \sqrt{3}\sin x$ in the form $r\cos(x - \alpha)$. Hence find the range of the function f with rule $f(x) = \cos x - \sqrt{3}\sin x$ and find the maximum and minimum values of f.

Solution

Here $a = 1$ and $b = -\sqrt{3}$. Therefore

$$r = \sqrt{1 + 3} = 2, \quad \cos\alpha = \frac{a}{r} = \frac{1}{2} \quad \text{and} \quad \sin\alpha = \frac{b}{r} = \frac{-\sqrt{3}}{2}$$

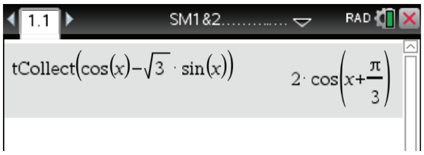
We see that $\alpha = -\frac{\pi}{3}$ and so

$$f(x) = \cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

Thus the range of f is $[-2, 2]$, the maximum value is 2 and the minimum value is -2.

Using the TI-Nspire

Access the **tCollect( )** command from (menu) > **Algebra** > **Trigonometry** > **Collect** and complete as shown.



ISBN 978-1-107-56765-8

Example 17

Solve $\cos x - \sqrt{3}\sin x = 1$ for $x \in [0, 2\pi]$.

Solution

From Example 16, we have

$$\cos x - \sqrt{3}\sin x = 2\cos\left(x + \frac{\pi}{3}\right)$$

$$\therefore \quad 2\cos\left(x + \frac{\pi}{3}\right) = 1$$

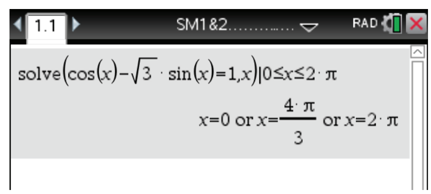
$$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3} \text{ or } \frac{7\pi}{3}$$

$$x = 0, \frac{4\pi}{3} \text{ or } 2\pi$$

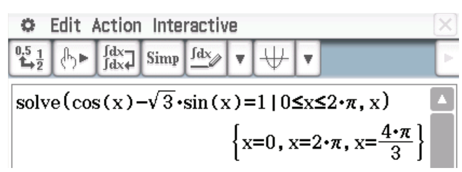
Using the TI-Nspire

Use **solve( )** from the **Algebra** menu as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $\cos(x) - \sqrt{3}\sin(x) = 1 \mid 0 \le x \le 2\pi$
- Select **Interactive** > **Equation/Inequality** > **solve** and tap OK.



Example 18

Express $\sqrt{3}\sin(2x) - \cos(2x)$ in the form $r\sin(2x + \alpha)$.

Solution

A slightly different technique is used. Assume that

$$\sqrt{3}\sin(2x) - \cos(2x) = r\sin(2x + \alpha)$$

$$= r(\sin(2x)\cos\alpha + \cos(2x)\sin\alpha)$$

This is to hold for all x.

For $x = \frac{\pi}{4}$: $\quad \sqrt{3} = r\cos\alpha \qquad (1)$

For $x = 0$: $\quad -1 = r\sin\alpha \qquad (2)$

ISBN 978-1-107-56765-8

Squaring and adding (1) and (2) gives

$$r^2\cos^2\alpha + r^2\sin^2\alpha = 4$$
$$r^2 = 4$$
$$\therefore \quad r = \pm 2$$

We take the positive solution. Substituting in (1) and (2) gives

$$\frac{\sqrt{3}}{2} = \cos\alpha \quad \text{and} \quad -\frac{1}{2} = \sin\alpha$$

Thus $\alpha = -\frac{\pi}{6}$ and hence

$$\sqrt{3}\sin(2x) - \cos(2x) = 2\sin\left(2x - \frac{\pi}{6}\right)$$

Check: Expand the right-hand side of the equation using an addition formula.

Section summary

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$
- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2 + b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

Exercise 14E

Skillsheet
Example 16

1 Find the maximum and minimum values of the following:

a $4\cos x + 3\sin x$
b $\sqrt{3}\cos x + \sin x$
c $\cos x - \sin x$
d $\cos x + \sin x$
e $3\cos x + \sqrt{3}\sin x$
f $\sin x - \sqrt{3}\cos x$
g $\cos x - \sqrt{3}\sin x + 2$
h $5 + 3\sin x - 2\cos x$

Example 17

2 Solve each of the following for $x \in [0, 2\pi]$ or for $\theta \in [0, 360]$:

a $\sin x - \cos x = 1$
b $\sqrt{3}\sin x + \cos x = 1$
c $\sin x - \sqrt{3}\cos x = -1$
d $3\cos x - \sqrt{3}\sin x = 3$
e $4\sin\theta^\circ + 3\cos\theta^\circ = 5$
f $2\sqrt{2}\sin\theta^\circ - 2\cos\theta^\circ = 3$

3 Write $\sqrt{3}\cos(2x) - \sin(2x)$ in the form $r\cos(2x + \alpha)$.

Example 18

4 Write $\cos(3x) - \sin(3x)$ in the form $r\sin(3x - \alpha)$.

5 Sketch the graph of each of the following, showing one cycle:

a $f(x) = \sin x - \cos x$
b $f(x) = \sqrt{3}\sin x + \cos x$
c $f(x) = \sin x + \cos x$
d $f(x) = \sin x - \sqrt{3}\cos x$

ISBN 978-1-107-56765-8

Review

AS

Nrich

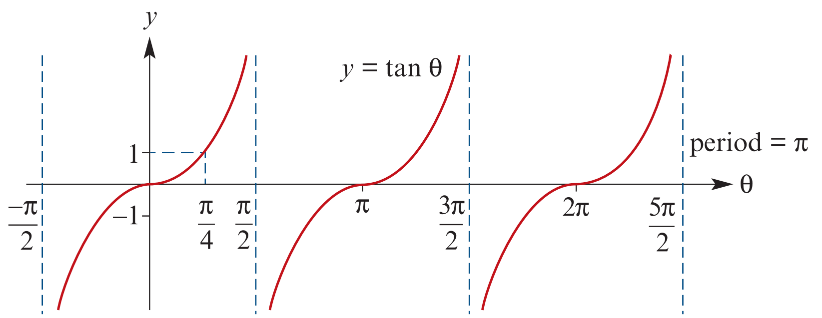
Chapter summary

- Complementary relationships

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- Graph of the tangent function



- Reciprocal circular functions

$$\sec\theta = \frac{1}{\cos\theta} \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$= \frac{1}{\tan\theta} \qquad (\text{if } \cos\theta \neq 0)$$

- Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$\cot^2\theta + 1 = \operatorname{cosec}^2\theta$$

- Addition formulas

$$\cos(u+v) = \cos u \cos v - \sin u \sin v$$

$$\cos(u-v) = \cos u \cos v + \sin u \sin v$$

$$\sin(u+v) = \sin u \cos v + \cos u \sin v$$

$$\sin(u-v) = \sin u \cos v - \cos u \sin v$$

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}$$

$$\tan(u-v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$$

- Double angle formulas

$$\cos(2u) = \cos^2 u - \sin^2 u$$

$$= 2\cos^2 u - 1$$

$$= 1 - 2\sin^2 u$$

$$\sin(2u) = 2\sin u \cos u$$

$$\tan(2u) = \frac{2\tan u}{1 - \tan^2 u}$$

- $a\cos x + b\sin x = r\cos(x - \alpha)$ where $r = \sqrt{a^2+b^2}$, $\cos\alpha = \frac{a}{r}$, $\sin\alpha = \frac{b}{r}$

- $a\cos x + b\sin x = r\sin(x + \beta)$ where $r = \sqrt{a^2+b^2}$, $\sin\beta = \frac{a}{r}$, $\cos\beta = \frac{b}{r}$

ISBN 978-1-107-56765-8

Technology-free questions

1 Prove each of the following identities:

a $\sec\theta + \operatorname{cosec}\theta\cot\theta = \sec\theta\operatorname{cosec}^2\theta$ **b** $\sec\theta - \sin\theta = \dfrac{\tan^2\theta + \cos^2\theta}{\sec\theta + \sin\theta}$

2 Find the maximum and minimum values of each of the following:

a $3 + 2\sin\theta$ **b** $1 - 3\cos\theta$ **c** $4\sin\left(\dfrac{3\theta}{2}\right)$ **d** $2\sin^2\left(\dfrac{\theta}{2}\right)$ **e** $\dfrac{1}{2 + \cos\theta}$

3 Find the values of $\theta \in [0, 2\pi]$ for which:

a $\sin^2\theta = \dfrac{1}{4}$ **b** $\sin(2\theta) = \dfrac{1}{2}$ **c** $\cos(3\theta) = \dfrac{\sqrt{3}}{2}$ **d** $\sin^2(2\theta) = 1$

e $\tan^2\theta = \dfrac{1}{3}$ **f** $\tan(2\theta) = -1$ **g** $\sin(3\theta) = -1$ **h** $\sec(2\theta) = \sqrt{2}$

4 Solve the equation $\tan(\theta^\circ) = 2\sin(\theta^\circ)$ for values of θ° from 0° to 360°.

5 If $\sin A = \dfrac{5}{13}$ and $\sin B = \dfrac{8}{17}$, where A and B are acute, find:

a $\cos(A + B)$ **b** $\sin(A - B)$ **c** $\tan(A + B)$

6 Find:

a $\cos 80^\circ \cos 20^\circ + \sin 80^\circ \sin 20^\circ$ **b** $\dfrac{\tan 15^\circ + \tan 30^\circ}{1 - \tan 15^\circ \tan 30^\circ}$

7 If $A + B = \dfrac{\pi}{2}$, find the value of:

a $\sin A\cos B + \cos A\sin B$ **b** $\cos A\cos B - \sin A\sin B$

8 Find the maximum and minimum values of the function with rule:

a $3 + 2\sin\theta$ **b** $4 - 5\cos\theta$

9 Prove each of the following:

a $\sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A - \sin^2 B$

b $\dfrac{\sin\theta}{1 + \cos\theta} + \dfrac{1 + \cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$ **c** $\dfrac{\sin\theta - 2\sin^3\theta}{2\cos^3\theta - \cos\theta} = \tan\theta$

10 Given that $\sin A = \dfrac{\sqrt{5}}{3}$ and that A is obtuse, find the value of:

a $\cos(2A)$ **b** $\sin(2A)$ **c** $\sin(4A)$

11 Prove:

a $\dfrac{1 - \tan^2 A}{1 + \tan^2 A} = \cos(2A)$ **b** $\sqrt{2r^2(1 - \cos\theta)} = 2r\sin\left(\dfrac{\theta}{2}\right)$ for $r > 0$ and θ acute

12 **a** Find $\tan 15^\circ$ in simplest surd form.

b Using the identities for $\sin(u \pm v)$, express $2\sin x\cos y$ as the sum of two sines.

ISBN 978-1-107-56765-8

Cambridge University Press

Review

13 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sin x + \cos x = 1$

b $\sin\left(\frac{1}{2}x\right)\cos\left(\frac{1}{2}x\right) = -\frac{1}{4}$

c $3\tan(2x) = 2\tan x$

d $\sin^2 x = \cos^2 x + 1$

e $\sin(3x)\cos x - \cos(3x)\sin x = \frac{\sqrt{3}}{2}$

f $2\cos\left(2x - \frac{\pi}{3}\right) = -\sqrt{3}$

14 Sketch the graph of:

a $y = 2\cos^2 x$

b $y = 1 - 2\sin\left(\frac{\pi}{2} - \frac{x}{2}\right)$

c $f(x) = \tan(2x)$

15 If $\tan A = 2$ and $\tan(\theta + A) = 4$, find the exact value of $\tan\theta$.

16 **a** Express $2\cos\theta + 9\sin\theta$ in the form $r\cos(\theta - \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$.

b **i** Give the maximum value of $2\cos\theta + 9\sin\theta$.

ii Give the cosine of θ for which this maximum occurs.

iii Find the smallest positive solution of the equation $2\cos\theta + 9\sin\theta = 1$.

Multiple-choice questions

1 $\operatorname{cosec} x - \sin x$ is equal to

A $\cos x \cot x$ **B** $\operatorname{cosec} x \tan x$ **C** $1 - \sin^2 x$

D $\sin x \operatorname{cosec} x$ **E** $\frac{1 - \sin x}{\sin x}$

2 If $\cos x = \frac{-1}{3}$, then the possible values of $\sin x$ are

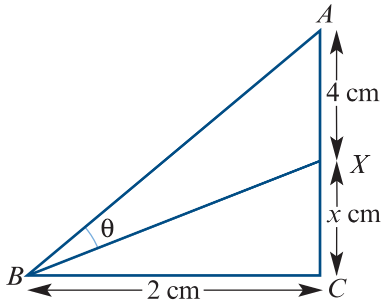
A $\frac{-2\sqrt{2}}{3}, \frac{2\sqrt{2}}{3}$ **B** $\frac{-2}{3}, \frac{2}{3}$ **C** $\frac{-8}{9}, \frac{8}{9}$

D $\frac{-\sqrt{2}}{3}, \frac{\sqrt{2}}{3}$ **E** $\frac{1}{2}, \frac{-1}{2}$

3 If $\cos\theta = \frac{a}{b}$ and $0 < \theta < \frac{\pi}{2}$, then $\tan\theta$ is equal to

A $\frac{\sqrt{a^2 + b^2}}{b}$ **B** $\frac{\sqrt{b^2 - a^2}}{a}$ **C** $\frac{a}{\sqrt{b^2 - a^2}}$ **D** $\frac{a}{\sqrt{b^2 + a^2}}$ **E** $\frac{a}{b\sqrt{b^2 + a^2}}$

4 In the diagram, the magnitude of $\angle ABX$ is θ, $AX = 4$ cm, $XC = x$ cm and $BC = 2$ cm. Therefore $\tan\theta$ is equal to



A $\frac{8}{(x + 2)^2}$ **B** $\frac{4}{x}$ **C** $8 - x$

D $8 + x$ **E** $\frac{8}{\sqrt{x^2 + 4}}$

ISBN 978-1-107-56765-8

Review

5 If $\frac{\pi}{2} < A < \pi$ and $\pi < B < \frac{3\pi}{2}$ with $\cos A = t$ and $\sin B = t$, then $\sin(B + A)$ equals

A 0 **B** 1 **C** $2t^2 - 1$ **D** $1 - 2t^2$ **E** -1

6 $\dfrac{\sin(2A)}{\cos(2A) - 1}$ is equal to

A $\cot(2A) - 1$ **B** $\sin(2A) + \sec(2A)$ **C** $\dfrac{\sin A}{\cos A - 1}$
D $\sin(2A) - \tan(2A)$ **E** $-\cot A$

7 $\sin\left(\frac{\pi}{2} - x\right)$ is not equal to

A $\cos(2\pi - x)$ **B** $-\sin\left(\frac{3\pi}{2} + x\right)$ **C** $\sin x$
D $\cos(-x)$ **E** $\sin\left(\frac{\pi}{2} + x\right)$

8 $(1 + \cot x)^2 + (1 - \cot x)^2$ is equal to

A $2 + \cot(x) + 2\cot(2x)$ **B** 2 **C** $-4\cot x$
D $2 + \cot(2x)$ **E** $2\operatorname{cosec}^2 x$

9 If $\sin(2A) = m$ and $\cos A = n$, then $\tan A$ is equal to

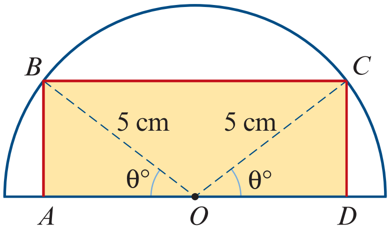
A $\dfrac{m}{2n^2}$ **B** $\dfrac{n}{m}$ **C** $\dfrac{2n}{m^2}$ **D** $\dfrac{2n}{m}$ **E** $\dfrac{2n^2}{m}$

10 Expressing $-\cos x + \sin x$ in the form $r\sin(x + \alpha)$, where $r > 0$, gives

A $\sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$ **B** $-\sin\left(x + \frac{\pi}{4}\right)$ **C** $\sqrt{2}\sin\left(x + \frac{5\pi}{4}\right)$
D $\sqrt{2}\sin\left(x + \frac{7\pi}{4}\right)$ **E** $\sqrt{2}\sin\left(x + \frac{3\pi}{4}\right)$

Extended-response questions

1 The diagram shows a rectangle $ABCD$ inside a semicircle, centre O and radius 5 cm, with $\angle BOA = \angle COD = \theta°$.



a Show that the perimeter, P cm, of the rectangle is given by $P = 20\cos\theta + 10\sin\theta$.

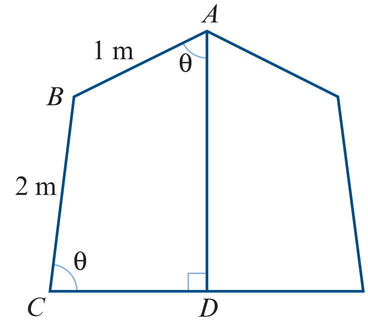
b Express P in the form $r\cos(\theta - \alpha)$ and hence find the value of θ for which $P = 16$.

c Find the value of k for which the area of the rectangle is $k\sin(2\theta)$ cm^2.

d Find the value of θ for which the area is a maximum.

ISBN 978-1-107-56765-8

2 The diagram shows a vertical section through a tent in which $AB = 1$ m, $BC = 2$ m and $\angle BAD = \angle BCD = \theta$. The line CD is horizontal, and the diagram is symmetrical about the vertical AD.



a Obtain an expression for AD in terms of θ.

b Express AD in the form $r\cos(\theta - \alpha)$, where r is positive.

c State the maximum length of AD and the corresponding value of θ.

d Given that $AD = 2.15$ m, find the value of θ for which $\theta > \alpha$.

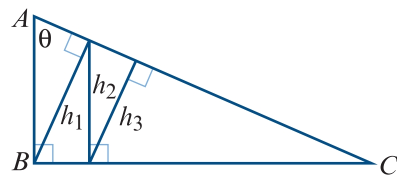
3 **a** Prove the identity $\cos(2\theta) = \dfrac{1 - \tan^2\theta}{1 + \tan^2\theta}$.

b **i** Use the result of **a** to show that $1 + x^2 = \sqrt{2}x^2 - \sqrt{2}$, where $x = \tan(67\frac{1}{2})°$.

ii Hence find the values of integers a and b such that $\tan(67\frac{1}{2})° = a + b\sqrt{2}$.

c Find the value of $\tan(7\frac{1}{2})°$.

4 In the diagram, $\triangle ABC$ has a right angle at B, the length of BC is 1 unit and $\angle BAC = \theta$.



a Find in terms of θ:

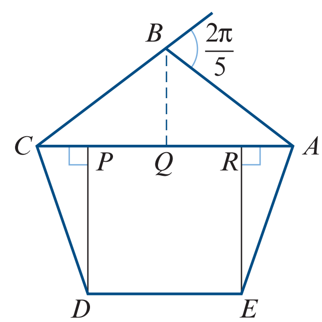
i h_1 **ii** h_2 **iii** h_3 **iv** h_n

b Show that the infinite sum is given by

$$h_1 + h_2 + h_3 + \cdots = \frac{\cos\theta}{1 - \sin\theta}$$

c If the value of the infinite sum is $\sqrt{2}$, find θ.

5 $ABCDE$ is a regular pentagon with side length one unit. The exterior angles of a regular pentagon each have magnitude $\dfrac{2\pi}{5}$.



a **i** Show that the magnitude of $\angle BCA$ is $\dfrac{\pi}{5}$.

ii Find the length of CA.

b **i** Show the magnitude of $\angle DCP$ is $\dfrac{2\pi}{5}$.

ii Use the fact that $AC = 2CQ = 2CP + PR$ to show that $2\cos\left(\dfrac{\pi}{5}\right) = 2\cos\left(\dfrac{2\pi}{5}\right) + 1$.

iii Use the identity $\cos(2\theta) = 2\cos^2\theta - 1$ to form a quadratic equation in terms of $\cos\left(\dfrac{\pi}{5}\right)$.

iv Find the exact value of $\cos\left(\dfrac{\pi}{5}\right)$.

6 **a** Prove each of the identities:

i $\cos\theta = \dfrac{1 - \tan^2(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$ **ii** $\sin\theta = \dfrac{2\tan(\frac{1}{2}\theta)}{1 + \tan^2(\frac{1}{2}\theta)}$

b Use the results of **a** to find the value of $\tan(\frac{1}{2}\theta)$, given that $8\cos\theta - \sin\theta = 4$.

ISBN 978-1-107-56765-8