Chapter 15

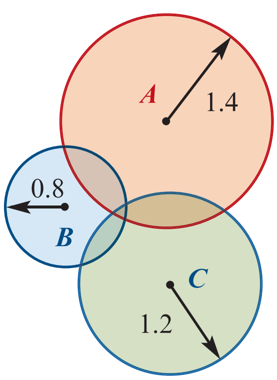
15 Graphing techniques

Objectives

- To sketch graphs of **reciprocal functions**, including those of polynomial functions and circular functions.
- To give **locus definitions** of lines, circles, parabolas, ellipses and hyperbolas, and to find the **Cartesian equations** of these curves.
- To use **parametric equations** to describe curves in the plane.
- To understand **polar coordinates** and their relationship to **Cartesian coordinates**.
- To sketch graphs in polar form.

The extensive use of mobile phones has led to an increased awareness of potential threats to the privacy of their users. For example, a little basic mathematics can be employed to track the movements of someone in possession of a mobile phone.

Suppose that there are three transmission towers within range of your mobile phone. By measuring the time taken for signals to travel between your phone and each transmission tower, it is possible to estimate the distance from your phone to each tower.



In the diagram, there are transmission towers at points A, B and C. If it is estimated that a person is no more than 1.4 km from A, no more than 0.8 km from B and no more than 1.2 km from C, then the person can be located in the intersection of the three circles.

In this chapter, we will look at different ways of describing circles and various other interesting figures.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

15A Reciprocal functions

► Reciprocals of polynomials

You have learned in previous years that the **reciprocal** of a non-zero number a is $\frac{1}{a}$. Likewise, we have the following definition.

If $y = f(x)$ is a polynomial function, then its **reciprocal function** is defined by the rule

$$y = \frac{1}{f(x)}$$

For example, the reciprocal of the function $y = x^3$ is $y = \frac{1}{x^3}$.

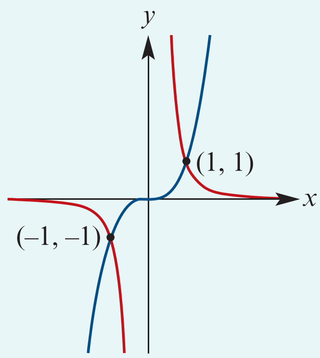
In this section, we will find relationships between the graph of a function and the graph of its reciprocal. Let's consider some specific examples, from which we will draw general conclusions.

Example 1

Sketch the graphs of $y = x^3$ and $y = \frac{1}{x^3}$ on the same set of axes.

Solution

We first sketch the graph of $y = x^3$. This is shown in blue.



Horizontal asymptotes

If $x \to \pm\infty$, then $\frac{1}{x^3} \to 0$. Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes

Notice that $x^3 = 0$ when $x = 0$.

If x is a small positive number, then $\frac{1}{x^3}$ is a large positive number.

If x is a small negative number, then $\frac{1}{x^3}$ is a large negative number.

Therefore the line $x = 0$ is a vertical asymptote of the reciprocal function.

Observations from the example

This example highlights behaviour typical of reciprocal functions:

- If $y = f(x)$ is a non-zero polynomial function, then the graph of $y = \frac{1}{f(x)}$ will have vertical asymptotes where $f(x) = 0$.
- The graphs of a function and its reciprocal are always on the same side of the x-axis.
- If the graphs of a function and its reciprocal intersect, then it must be where $f(x) = \pm 1$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

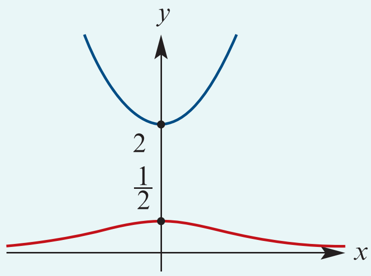
The following example is perhaps easier, because the reciprocal graph has no vertical asymptotes. This time we are interested in turning points.

Example 2

Consider the function $f(x) = x^2 + 2$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = x^2 + 2$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\dfrac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes There are no vertical asymptotes, as there is no solution to the equation $f(x) = 0$.

Turning points Notice that the graph of $y = x^2 + 2$ has a minimum at $(0, 2)$. The reciprocal function therefore has a maximum at $(0, \frac{1}{2})$.

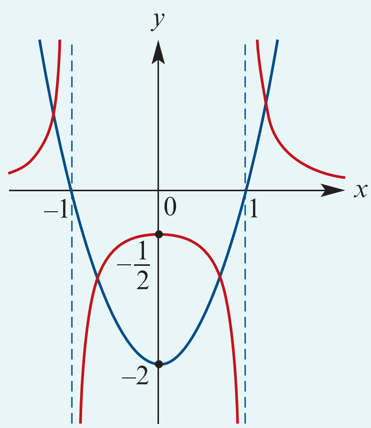
- If the graph of $y = f(x)$ has a local minimum at $x = a$, then the graph of $y = \dfrac{1}{f(x)}$ will have a local maximum at $x = a$.
- If the graph of $y = f(x)$ has a local maximum at $x = a$, then the graph of $y = \dfrac{1}{f(x)}$ will have a local minimum at $x = a$.

Example 3

Consider the function $f(x) = 2(x - 1)(x + 1)$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2(x - 1)(x + 1)$. This is shown in blue.



Horizontal asymptotes If $x \to \pm\infty$, then $\dfrac{1}{f(x)} \to 0$.

Therefore the line $y = 0$ is a horizontal asymptote of the reciprocal function.

Vertical asymptotes We have $f(x) = 0$ when $x = -1$ or $x = 1$. Therefore the lines $x = -1$ and $x = 1$ are vertical asymptotes of the reciprocal function.

Turning points The graph of $y = f(x)$ has a minimum at $(0, -2)$. Therefore the reciprocal has a local maximum at $(0, -\frac{1}{2})$.

ISBN 978-1-107-56765-8

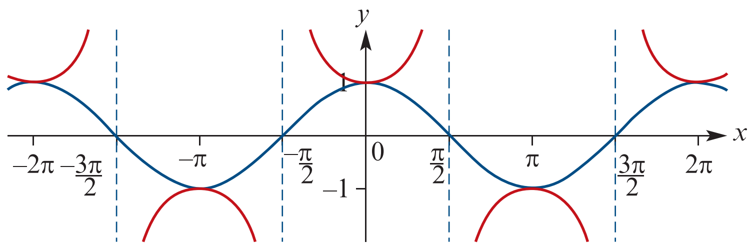
Cambridge University Press
Updated May 2021

Reciprocals of circular functions

We now briefly consider reciprocal circular functions, which were introduced in Chapter 14.

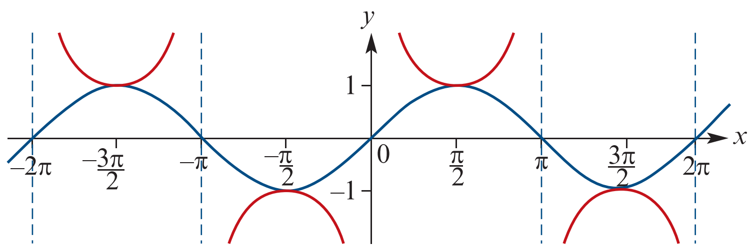
Secant

$$\sec x = \frac{1}{\cos x}$$



Cosecant

$$\operatorname{cosec} x = \frac{1}{\sin x}$$



The first graph shows $y = \cos x$ and $y = \sec x$; the second shows $y = \sin x$ and $y = \operatorname{cosec} x$.

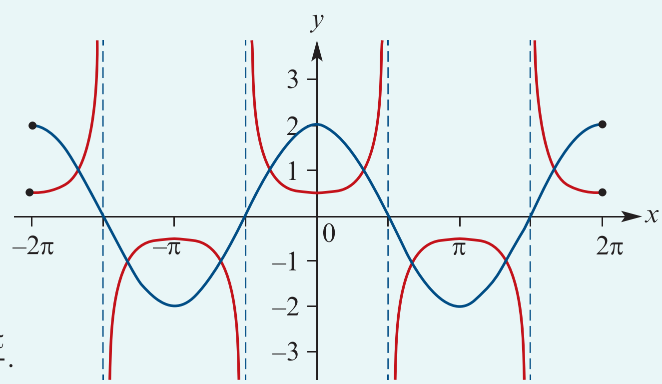
Note: The x-axis intercepts become vertical asymptotes.
Local maximums become local minimums, and vice versa.

Example 4

Let $f(x) = 2\cos x$ for $-2\pi \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 2\cos x$ for $x \in [-2\pi, 2\pi]$. This is shown in blue.



Vertical asymptotes

Vertical asymptotes of the reciprocal function will occur when $f(x) = 0$.

These are given by $x = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}$.

Turning points

The points $(0, 2)$ and $(\pm 2\pi, 2)$ are local maximums of $y = f(x)$. Therefore the points $(0, \frac{1}{2})$ and $(\pm 2\pi, \frac{1}{2})$ are local minimums of the reciprocal.

The points $(\pm\pi, -2)$ are local minimums of $y = f(x)$. Therefore the points $(\pm\pi, -\frac{1}{2})$ are local maximums of the reciprocal.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

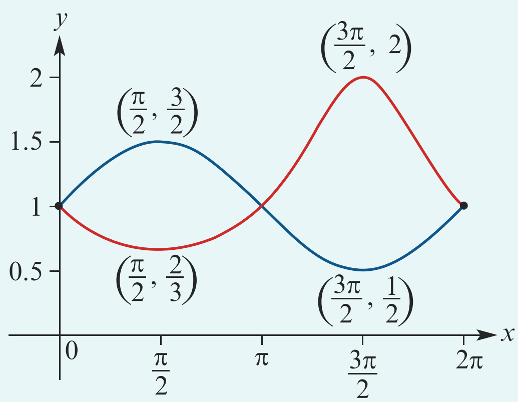
The graph of the next function has no x-axis intercepts, and so its reciprocal has no vertical asymptotes.

Example 5

Let $f(x) = 0.5\sin x + 1$ for $0 \le x \le 2\pi$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

Solution

We first sketch $y = 0.5\sin x + 1$ for $x \in [0, 2\pi]$. This is shown in blue.



Turning points

The point $(\frac{\pi}{2}, \frac{3}{2})$ is a local maximum of $y = f(x)$. Therefore the point $(\frac{\pi}{2}, \frac{2}{3})$ is a local minimum of the reciprocal.

The point $(\frac{3\pi}{2}, \frac{1}{2})$ is a local minimum of $y = f(x)$. Therefore the point $(\frac{3\pi}{2}, 2)$ is a local maximum of the reciprocal.

Section summary

Given the graph of a continuous function $y = f(x)$, we can sketch the graph of $y = \dfrac{1}{f(x)}$ with the help of the following observations:

Function $y = f(x)$	Reciprocal function $y = \dfrac{1}{f(x)}$
x-axis intercept at $x = a$	vertical asymptote $x = a$
local maximum at $x = a$	local minimum at $x = a$
local minimum at $x = a$	local maximum at $x = a$
above the x-axis	above the x-axis
below the x-axis	below the x-axis
increasing over an interval	decreasing over the interval
decreasing over an interval	increasing over the interval
values approach ∞	values approach 0 from above
values approach $-\infty$	values approach 0 from below
values approach 0 from above	values approach ∞
values approach 0 from below	values approach $-\infty$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Exercise 15A

Skillsheet

1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes:

Example 1, 2
Example 3

a $f(x) = x + 3$ **b** $f(x) = x^2$ **c** $f(x) = x^2 + 4$
d $f(x) = (x - 1)(x + 1)$ **e** $f(x) = 4 - x^2$ **f** $f(x) = (x - 1)^2 - 1$
g $f(x) = x^2 - 2x - 3$ **h** $f(x) = -x^2 - 2x + 3$ **i** $f(x) = x^3 + 1$

Example 4, 5

2 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes. Label asymptotes, turning points and endpoints.

a $f(x) = \sin x$ for $0 \le x \le 2\pi$ **b** $f(x) = \cos x$ for $0 \le x \le 2\pi$
c $f(x) = -2\cos x$ for $-\pi \le x \le \pi$ **d** $f(x) = \cos x + 1$ for $0 \le x \le 4\pi$
e $f(x) = -\sin x - 1$ for $-2\pi \le x \le 2\pi$ **f** $f(x) = \cos x - 2$ for $0 \le x \le 2\pi$
g $f(x) = -\sin x + 2$ for $0 \le x \le 2\pi$ **h** $f(x) = -2\cos x + 3$ for $-\pi \le x \le \pi$

3 Consider the quadratic function $f(x) = x^2 + 2x + 2$.

a By completing the square, find the turning point of the graph of $y = f(x)$.
b Hence sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

4 Consider the quadratic function $f(x) = 5x(1 - x)$.

a Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.
b Locate all points of intersection of the two graphs by solving $f(x) = 1$ and $f(x) = -1$.

5 Sketch the graphs of $y = 2\sin^2 x$ and $y = \dfrac{1}{2\sin^2 x}$ on the same set of axes, over the interval $0 \le x \le 2\pi$.

6 Consider the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 2^x - 1$. Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes.

7 Let $k \in \mathbb{R}$ and consider the function $f(x) = x^2 + 2kx + 1$.

a By completing the square, show that the graph of $y = f(x)$ has a minimum turning point at $(-k, 1 - k^2)$.
b For what values of k does the graph of $y = f(x)$ have:
i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts?
c Sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ when the graph of $y = f(x)$ has:
i no x-axis intercept **ii** one x-axis intercept **iii** two x-axis intercepts.

Hint: It helps to ignore the y-axis.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

15B Locus of points

Until now, all the curves we have studied have been described by an algebraic relationship between the x- and y-coordinates, such as $y = x^2 + 1$. In this section, we are interested in sets of points described by a geometric condition. A set described in this way is often called a **locus**. Many of these descriptions will give curves that are already familiar.

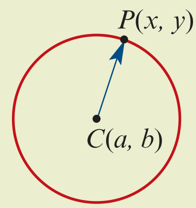
Circles

Circles have a very simple geometric description.

Locus definition of a circle

A **circle** is the locus of a point $P(x, y)$ that moves so that its distance from a fixed point $C(a, b)$ is constant.

Note: The constant distance is called the **radius** and the fixed point $C(a, b)$ is called the **centre** of the circle.



This definition can be used to find the equation of a circle.

Recall that the distance between points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Let r be the radius of the circle. Then

$$\begin{aligned} CP &= r \\ \sqrt{(x-a)^2 + (y-b)^2} &= r \\ (x-a)^2 + (y-b)^2 &= r^2 \end{aligned}$$

The circle with radius r and centre $C(a, b)$ has equation

$$(x-a)^2 + (y-b)^2 = r^2$$

Example 6

a Find the locus of points $P(x, y)$ whose distance from $C(2, -1)$ is 3.

b Find the centre and radius of the circle with equation $x^2 + 2x + y^2 - 4y = 1$.

Solution

a We know that the point $P(x, y)$ satisfies

$$\begin{aligned} CP &= 3 \\ \sqrt{(x-2)^2 + (y+1)^2} &= 3 \\ (x-2)^2 + (y+1)^2 &= 3^2 \end{aligned}$$

This is a circle with centre $(2, -1)$ and radius 3.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

b We must complete the square in both variables. This gives

$$\begin{aligned} x^2 + 2x + y^2 - 4y &= 1 \\ (x^2 + 2x + 1) - 1 + (y^2 - 4y + 4) - 4 &= 1 \\ (x+1)^2 + (y-2)^2 &= 6 \end{aligned}$$

Therefore the centre of the circle is $(-1, 2)$ and its radius is $\sqrt{6}$.

Straight lines

You have learned in previous years that a straight line is the set of points (x, y) satisfying

$$ax + by = c$$

for some constants a, b, c with $a \neq 0$ or $b \neq 0$.

Lines can also be described geometrically as follows.

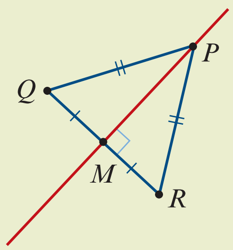
Locus definition of a straight line

Suppose that points Q and R are fixed.

A **straight line** is the locus of a point P that moves so that its distance from Q is the same as its distance from R. That is,

$$QP = RP$$

We can say that point P is **equidistant** from points Q and R.



Note: This straight line is the **perpendicular bisector** of line segment QR. To see this, we note that the midpoint M of QR is on the line. If P is any other point on the line, then

$$QP = RP, \quad QM = RM \quad \text{and} \quad MP = MP$$

and so $\triangle QMP$ is congruent to $\triangle RMP$. Therefore $\angle QMP = \angle RMP = 90°$.

Example 7

a Find the locus of points $P(x, y)$ that are equidistant from the points $Q(1, 1)$ and $R(3, 5)$.

b Show that this is the perpendicular bisector of line segment QR.

Solution

a We know that the point $P(x, y)$ satisfies

$$\begin{aligned} QP &= RP \\ \sqrt{(x-1)^2 + (y-1)^2} &= \sqrt{(x-3)^2 + (y-5)^2} \\ (x-1)^2 + (y-1)^2 &= (x-3)^2 + (y-5)^2 \\ x + 2y &= 8 \\ y &= -\tfrac{1}{2}x + 4 \end{aligned}$$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

b This line has gradient $-\frac{1}{2}$. The line through $Q(1, 1)$ and $R(3, 5)$ has gradient $\dfrac{5-1}{3-1} = 2$.

Because the product of the two gradients is -1, the two lines are perpendicular.

We also need to check that the line $y = -\frac{1}{2}x + 4$ passes through the midpoint of QR, which is $(2, 3)$. When $x = 2$, $y = -\frac{1}{2} \times 2 + 4 = 3$. Thus $(2, 3)$ is on the line.

Section summary

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.

Exercise 15B

Skillsheet

Example 6

1 Find the locus of points $P(x, y)$ whose distance from $Q(1, -2)$ is 4.

2 Find the locus of points $P(x, y)$ whose distance from $Q(-4, 3)$ is 5.

Example 7

3 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(-1, -1)$ and $R(1, 1)$.
b Show that this is the perpendicular bisector of line segment QR.

4 **a** Find the locus of points $P(x, y)$ that are equidistant from $Q(0, 2)$ and $R(1, 0)$.
b Show that this is the perpendicular bisector of line segment QR.

5 Point P is equidistant from points $Q(0, 1)$ and $R(2, 3)$. Moreover, its distance from point $S(3, 3)$ is 3. Find the possible coordinates of P.

6 Point P is equidistant from points $Q(0, 1)$ and $R(2, 0)$. Moreover, it is also equidistant from points $S(-1, 0)$ and $T(0, 2)$. Find the coordinates of P.

7 A valuable item is buried in a forest. It is 10 metres from a tree stump located at coordinates $T(0, 0)$ and 2 metres from a rock at coordinates $R(6, 10)$. Find the possible coordinates of the buried item.

8 Consider the three points $R(4, 5)$, $S(6, 1)$ and $T(1, -4)$.
a Find the locus of points $P(x, y)$ that are equidistant from the points R and S.
b Find the locus of points $P(x, y)$ that are equidistant from the points S and T.
c Hence find the point that is equidistant from the points R, S and T.
d Hence find the equation of the circle through the points R, S and T.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

9 Given two fixed points $A(0, 1)$ and $B(2, 5)$, find the locus of P if the gradient of AB equals that of BP.

10 A triangle OAP has vertices $O(0, 0)$, $A(4, 0)$ and $P(x, y)$, where $y > 0$. The triangle has area 12 square units. Find the locus of P.

11 **a** Determine the locus of a point $P(x, y)$ that moves so that its distance from the origin is equal to the sum of its x- and y-coordinates.

b Determine the locus of a point $P(x, y)$ that moves so that the *square* of its distance from the origin is equal to the sum of its x- and y-coordinates.

12 $A(0, 0)$ and $B(3, 0)$ are two vertices of a triangle ABP. The third vertex P is such that $AP : BP = 2$. Find the locus of P.

13 Find the locus of the point P that moves so that its distance from the line $y = 3$ is always 2 units.

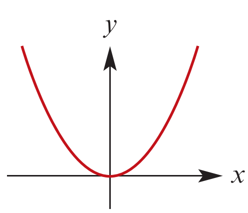
14 A steel pipe is too heavy to drag, but can be lifted at one end and rotated about its opposite end. How many moves are required to rotate the pipe into the parallel position indicated by the dotted line? The distance between the parallel lines is less than the length of the pipe.

15C Parabolas

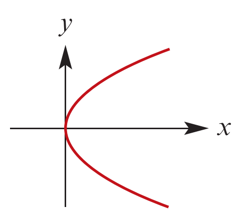
The parabola has been studied since antiquity and is admired for its range of applications, one of which we will explore at the end of this section.

The standard form of a parabola is $y = ax^2$.

Rotating the figure by 90° gives a parabola with equation $x = ay^2$.



$y = ax^2$



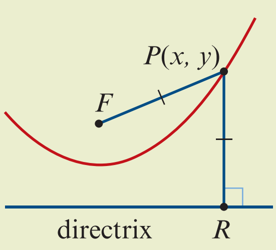
$x = ay^2$

The parabola can also be defined geometrically.

Locus definition of a parabola

A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.

Note: The fixed point is called the **focus** and the fixed line is called the **directrix**.



ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

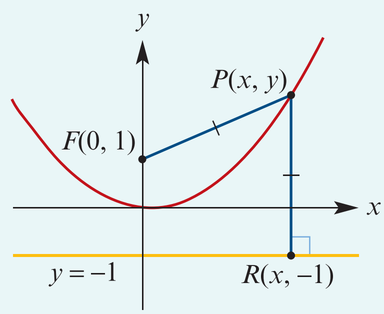
Example 8

Verify that the set of all points $P(x, y)$ that are equidistant from the point $F(0, 1)$ and the line $y = -1$ is a parabola.

Solution

We know that the point $P(x, y)$ satisfies

$$\begin{aligned} FP &= RP \\ \sqrt{x^2 + (y-1)^2} &= \sqrt{(y-(-1))^2} \\ x^2 + (y-1)^2 &= (y+1)^2 \\ x^2 + y^2 - 2y + 1 &= y^2 + 2y + 1 \\ x^2 - 2y &= 2y \\ x^2 &= 4y \\ y &= \frac{x^2}{4} \end{aligned}$$



Therefore the set of points is the parabola with equation $y = \dfrac{x^2}{4}$.

Example 9

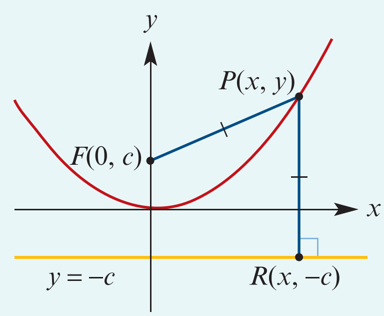
a Find the equation of the parabola with focus $F(0, c)$ and directrix $y = -c$.

b Hence find the focus of the parabola with equation $y = 2x^2$.

Solution

a A point $P(x, y)$ on the parabola satisfies

$$\begin{aligned} FP &= RP \\ \sqrt{x^2 + (y-c)^2} &= \sqrt{(y-(-c))^2} \\ x^2 + (y-c)^2 &= (y+c)^2 \\ x^2 + y^2 - 2cy + c^2 &= y^2 + 2cy + c^2 \\ x^2 - 2cy &= 2cy \\ x^2 &= 4cy \end{aligned}$$



The parabola has equation $4cy = x^2$.

b Since $\dfrac{y}{2} = x^2$, we solve $\dfrac{1}{2} = 4c$, giving $c = \dfrac{1}{8}$.

Hence the focus is $F\left(0, \dfrac{1}{8}\right)$.

In the previous example, we proved the following result:

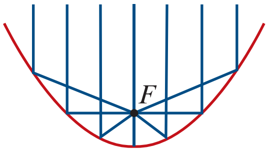
The parabola with focus $F(0, c)$ and directrix $y = -c$ has equation $4cy = x^2$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

A remarkable application

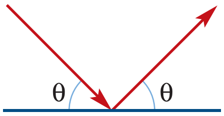
Parabolas have a remarkable property that makes them extremely useful. Light travelling parallel to the axis of symmetry of a reflective parabola is always reflected to its focus.



Parabolas can therefore be used to make reflective telescopes. Low intensity signals from outer space will reflect off the dish and converge at a receiver located at the focus.

To see how this works, we require a simple law of physics:

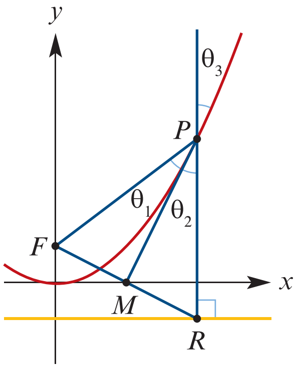
- When light is reflected off a surface, the angle between the ray and the tangent to the surface is preserved after reflection.



Reflective property of the parabola

Any ray of light parallel to the axis of symmetry of the parabola that reflects off the parabola at point P will pass through the focus at F.

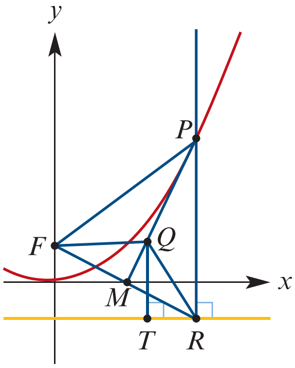
Proof Since point P is on the parabola, the distance to the focus F is the same as the distance to the directrix. Therefore $FP = RP$, and so $\triangle FPR$ is isosceles.



Let M be the midpoint of FR. Then $\triangle FMP$ is congruent to $\triangle RMP$ (by SSS). Therefore MP is the perpendicular bisector of FR and

$$\begin{aligned} \theta_1 &= \theta_2 && (\text{as } \triangle FMP \equiv \triangle RMP) \\ &= \theta_3 && (\text{vertically opposite angles}) \end{aligned}$$

However, we also need to ensure that line MP is tangent to the parabola. To see this, we will show that point P is the only point common to the parabola and line MP.



Take any other point Q on line MP. Suppose that point T is the point on the directrix closest to Q. Then

$$FQ = RQ > TQ$$

and so point Q is not on the parabola.

Section summary

- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- The fixed point is called the **focus** and the fixed line is called the **directrix**.
- The parabola with equation $4cy = x^2$ has focus $F(0, c)$ and directrix $y = -c$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Exercise 15C

Example 8 **1** Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, 3)$ is equal to the perpendicular distance to the line with equation $y = -3$.

2 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(0, -4)$ is equal to the perpendicular distance to the line with equation $y = 2$.

3 Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(2, 0)$ is equal to the perpendicular distance to the line with equation $x = -4$.

Example 9 **4** **a** Find the equation of the parabola with focus $F(c, 0)$ and directrix $x = -c$.

b Hence find the focus of the parabola with equation $x = 3y^2$.

5 **a** Find the equation of the locus of points $P(x, y)$ whose distance to the point $F(a, b)$ is equal to the perpendicular distance to the line with equation $y = c$.

b Hence find the equation of the parabola with focus $(1, 2)$ and directrix $y = 3$.

6 A parabola goes through the point $P(7, 9)$ and its focus is $F(1, 1)$. The axis of symmetry of the parabola is $x = 1$. Find the equation of its directrix.

Hint: The directrix will be a horizontal line, $y = c$. Expect to find two answers.

7 A parabola goes through the point $(1, 1)$, its axis of symmetry is the line $x = 2$ and its directrix is the line $y = 3$. Find the coordinates of its focus.

Hint: The focus must lie on the axis of symmetry.

15D Ellipses

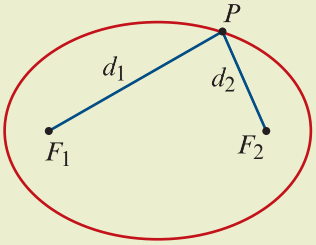
A ball casts a shadow that looks like a squashed circle. This figure – called an ellipse – is of considerable geometric significance. For instance, the planets in our solar system have elliptic orbits.

Locus definition of an ellipse

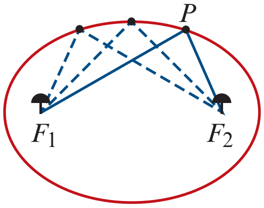
An **ellipse** is the locus of a point P that moves so that the sum of its distances from two fixed points F_1 and F_2 is a constant. That is,

$$F_1P + F_2P = k$$

Note: Points F_1 and F_2 are called the **foci** of the ellipse.



Drawing an ellipse An ellipse can be drawn by pushing two pins into paper. These will be the foci. A string of length k is tied to each of the two pins and the tip of a pen is used to pull the string taut and form a triangle. The pen will trace an ellipse if it is moved around the pins while keeping the string taut.

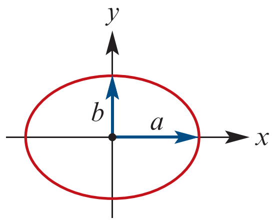


ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

▶ Cartesian equations of ellipses

The standard form of the Cartesian equation of an ellipse centred at the origin is



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

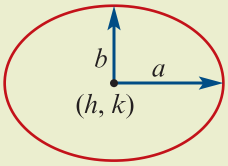
This ellipse has x-axis intercepts $\pm a$ and y-axis intercepts $\pm b$.

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is an ellipse centred at the point (h, k).



Example 10

For each of the following equations, sketch the graph of the corresponding ellipse. Give the coordinates of the centre and the axis intercepts.

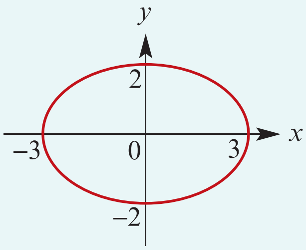
a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $4x^2 + 9y^2 = 1$ **c** $\dfrac{(x-1)^2}{4} + \dfrac{(y+2)^2}{9} = 1$

Solution

a The equation can be written as

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

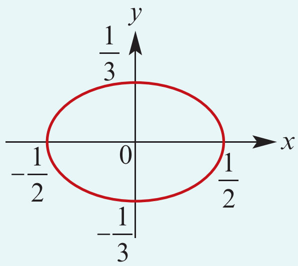
This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm 3$ and $y = \pm 2$.



b The equation can be written as

$$\frac{x^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{3})^2} = 1$$

This is an ellipse with centre $(0, 0)$ and axis intercepts at $x = \pm\frac{1}{2}$ and $y = \pm\frac{1}{3}$.



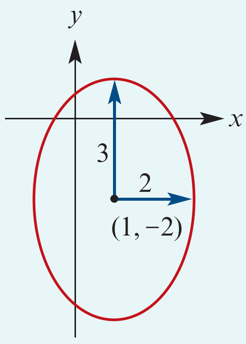
c This is an ellipse with centre $(1, -2)$.

To find the x-axis intercepts, let $y = 0$. Then solving for x gives

$$x = \frac{3 \pm 2\sqrt{5}}{3}$$

Likewise, to find the y-axis intercepts, let $x = 0$. This gives

$$y = \frac{-4 \pm 3\sqrt{3}}{2}$$



ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Using the locus definition

Example 11

Consider points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP + BP = 8$.

Solution

Let (x, y) be the coordinates of point P. If $AP + BP = 8$, then

$$\sqrt{(x+2)^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 8$$

and so

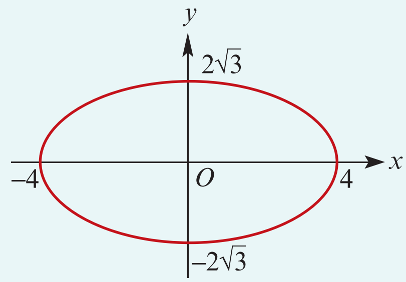
$$\sqrt{(x+2)^2 + y^2} = 8 - \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$\begin{aligned}(x+2)^2 + y^2 &= 64 - 16\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2\\ x^2 + 4x + 4 + y^2 &= 64 - 16\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2\\ x - 8 &= -2\sqrt{(x-2)^2 + y^2}\end{aligned}$$

Square both sides again:

$$x^2 - 16x + 64 = 4(x^2 - 4x + 4 + y^2)$$



Simplifying yields

$$3x^2 + 4y^2 = 48$$

i.e. $$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

This is an ellipse with centre the origin and axis intercepts at $x = \pm 4$ and $y = \pm 2\sqrt{3}$.

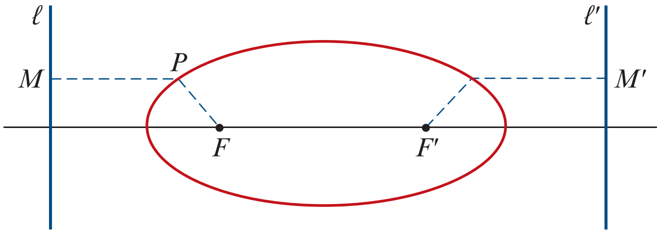
Every point P on the ellipse satisfies $AP + BP = 8$.

Note: You might like to consider the general version of this example with $A(-c, 0)$, $B(c, 0)$ and $AP + BP = 2a$, where $a > c > 0$.

It can also be shown that an ellipse is the locus of points $P(x, y)$ satisfying

$$FP = eMP$$

where F is a fixed point, $0 < e < 1$ and MP is the perpendicular distance from P to a fixed line ℓ. From the symmetry of the ellipse, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eM'P$ defines the same locus, where $M'P$ is the perpendicular distance from P to ℓ'.



ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Example 12

Find the equation of the locus of points $P(x, y)$ if the distance from P to the point $F(1, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = \frac{1}{2}MP$.

Solution

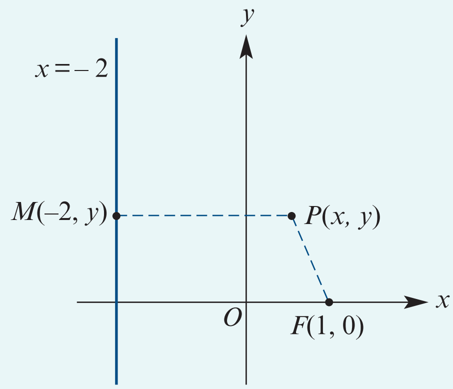
Let (x, y) be the coordinates of point P.

If $FP = \frac{1}{2}MP$, then

$$\sqrt{(x-1)^2 + y^2} = \frac{1}{2}\sqrt{(x+2)^2}$$

Square both sides:

$$(x-1)^2 + y^2 = \frac{1}{4}(x+2)^2$$
$$4(x^2 - 2x + 1) + 4y^2 = x^2 + 4x + 4$$
$$3x^2 - 12x + 4y^2 = 0$$



Complete the square:

$$3(x^2 - 4x + 4) + 4y^2 = 12$$
$$3(x-2)^2 + 4y^2 = 12 \quad \text{or equivalently} \quad \frac{(x-2)^2}{4} + \frac{y^2}{3} = 1$$

This is an ellipse with centre $(2, 0)$.

Section summary

- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 (called the **foci**) is equal to a fixed positive constant.
- The graph of
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
is an ellipse centred at the point (h, k).

Exercise 15D

Skillsheet

Example 10

1 Sketch the graph of each ellipse, labelling the axis intercepts:

a $\frac{x^2}{9} + \frac{y^2}{64} = 1$ **b** $\frac{x^2}{100} + \frac{y^2}{25} = 1$ **c** $\frac{y^2}{9} + \frac{x^2}{64} = 1$ **d** $25x^2 + 9y^2 = 225$

2 Sketch the graph of each ellipse, labelling the centre and the axis intercepts:

a $\frac{(x-3)^2}{9} + \frac{(y-4)^2}{16} = 1$ **b** $\frac{(x+3)^2}{9} + \frac{(y+4)^2}{25} = 1$

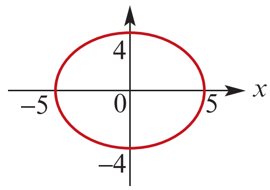
c $\frac{(y-3)^2}{16} + \frac{(x-2)^2}{4} = 1$ **d** $25(x-5)^2 + 9y^2 = 225$

ISBN 978-1-107-56765-8

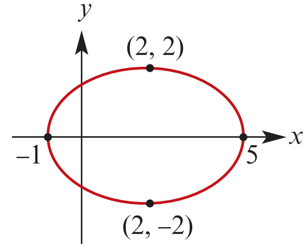
Cambridge University Press Updated May 2021

3 Find the Cartesian equations of the following ellipses:

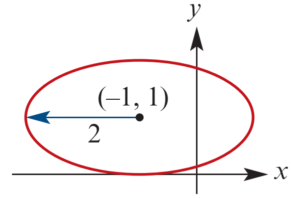
a



b



c



Example 11 **4** Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(1, 0)$ and $B(-1, 0)$ is 4 units.

5 Find the locus of the point P as it moves such that the sum of its distances from two fixed points $A(0, 2)$ and $B(0, -2)$ is 6 units.

Example 12 **6** Find the equation of the locus of points $P(x, y)$ such that the distance from P to the point $F(2, 0)$ is half the distance MP, the perpendicular distance from P to the line with equation $x = -4$. That is, $FP = \frac{1}{2}MP$.

7 A circle has equation $x^2 + y^2 = 1$. It is then dilated by a factor of 3 from the x-axis and by a factor of 5 from the y-axis. Find the equation of the image and sketch its graph.

15E Hyperbolas

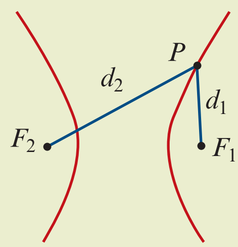
Hyperbolas are defined analogously to ellipses, but using the difference instead of the sum.

Locus definition of a hyperbola

A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant. That is,

$$|F_2P - F_1P| = k$$

Note: Points F_1 and F_2 are called the **foci** of the hyperbola.



The standard form of the Cartesian equation of a hyperbola centred at the origin is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we can see the following result:

The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k).

Note: Interchanging x and y in this equation produces another hyperbola (rotated by 90°).

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Asymptotes of the hyperbola

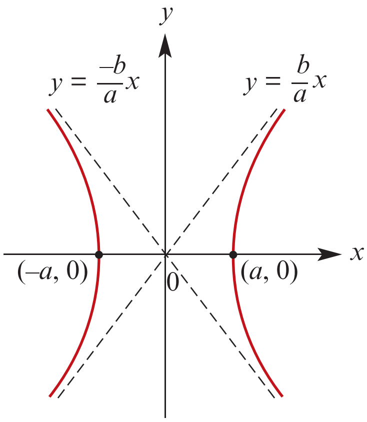
We now investigate the behaviour of the hyperbola as $x \to \pm\infty$. We first show that the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

has asymptotes

$$y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x$$

To see why this should be the case, we rearrange the equation of the hyperbola as follows:



$$\begin{aligned}\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\ \frac{y^2}{b^2} &= \frac{x^2}{a^2} - 1 \\ y^2 &= \frac{b^2x^2}{a^2} - b^2 \\ &= \frac{b^2x^2}{a^2}\left(1 - \frac{a^2}{x^2}\right)\end{aligned}$$

If $x \to \pm\infty$, then $\frac{a^2}{x^2} \to 0$. Therefore $y^2 \to \frac{b^2x^2}{a^2}$ as $x \to \pm\infty$. That is,

$$y \to \pm\frac{bx}{a} \quad \text{as} \quad x \to \pm\infty$$

Applying the translation defined by $(x, y) \to (x + h, y + k)$, we obtain the following result:

The hyperbola with equation

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

has asymptotes given by

$$y - k = \pm\frac{b}{a}(x - h)$$

Example 13

For each of the following equations, sketch the graph of the corresponding hyperbola. Give the coordinates of the centre, the axis intercepts and the equations of the asymptotes.

a $\frac{x^2}{9} - \frac{y^2}{4} = 1$

b $\frac{y^2}{9} - \frac{x^2}{4} = 1$

c $(x-1)^2 - (y+2)^2 = 1$

d $\frac{(y-1)^2}{4} - \frac{(x+2)^2}{9} = 1$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

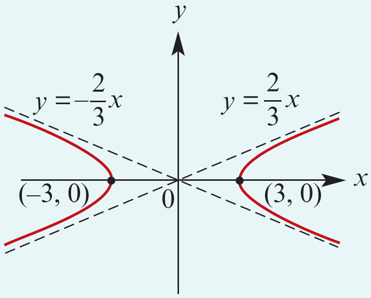
Solution

a Since $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$, we have

$$y^2 = \frac{4x^2}{9}\left(1 - \frac{9}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{2}{3}x$.

If $y = 0$, then $x^2 = 9$ and so $x = \pm 3$. The x-axis intercepts are $(3, 0)$ and $(-3, 0)$. The centre is $(0, 0)$.



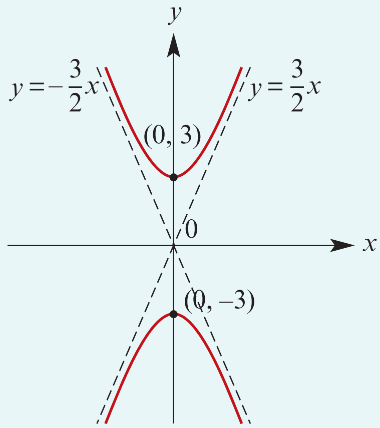
b Since $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$, we have

$$y^2 = \frac{9x^2}{4}\left(1 + \frac{4}{x^2}\right)$$

Thus the equations of the asymptotes are $y = \pm\frac{3}{2}x$.

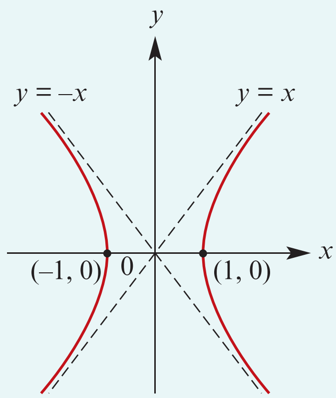
The y-axis intercepts are $(0, 3)$ and $(0, -3)$.

The centre is $(0, 0)$.



c First sketch the graph of $x^2 - y^2 = 1$. The asymptotes are $y = x$ and $y = -x$. The centre is $(0, 0)$ and the axis intercepts are $(1, 0)$ and $(-1, 0)$.

Note: This hyperbola is called a **rectangular hyperbola**, as its asymptotes are perpendicular.



Now to sketch the graph of

$$(x - 1)^2 - (y + 2)^2 = 1$$

we apply the translation $(x, y) \to (x + 1, y - 2)$.

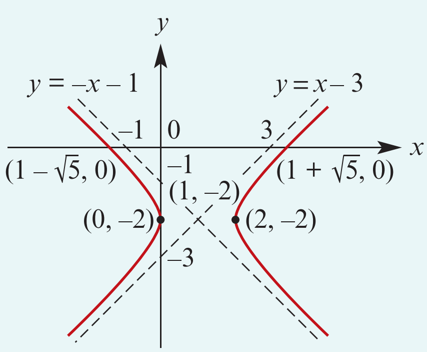
The new centre is $(1, -2)$ and the asymptotes have equations $y + 2 = \pm(x - 1)$. That is, $y = x - 3$ and $y = -x - 1$.

Axis intercepts

If $x = 0$, then $y = -2$.

If $y = 0$, then $(x - 1)^2 = 5$ and so $x = 1 \pm \sqrt{5}$.

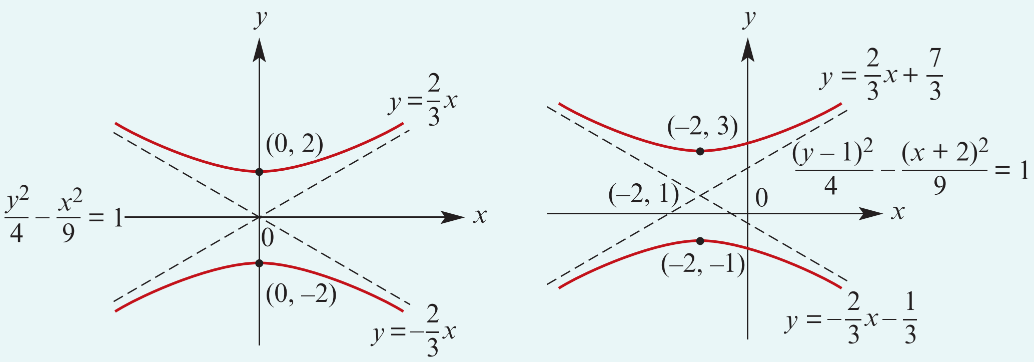
Therefore the axis intercepts are $(0, -2)$ and $(1 \pm \sqrt{5}, 0)$.



ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

d The graph of $\dfrac{(y-1)^2}{4} - \dfrac{(x+2)^2}{9} = 1$ is obtained from the hyperbola $\dfrac{y^2}{4} - \dfrac{x^2}{9} = 1$ through the translation $(x, y) \to (x-2, y+1)$. Its centre will be $(-2, 1)$.



Using the locus definition

Example 14

Consider the points $A(-2, 0)$ and $B(2, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 3$.

Solution

Let (x, y) be the coordinates of point P.

If $AP - BP = 3$, then

$$\sqrt{(x+2)^2 + y^2} - \sqrt{(x-2)^2 + y^2} = 3$$

and so

$$\sqrt{(x+2)^2 + y^2} = 3 + \sqrt{(x-2)^2 + y^2}$$

Square both sides, then expand and simplify:

$$\begin{aligned}
(x+2)^2 + y^2 &= 9 + 6\sqrt{(x-2)^2 + y^2} + (x-2)^2 + y^2 \\
x^2 + 4x + 4 + y^2 &= 9 + 6\sqrt{(x-2)^2 + y^2} + x^2 - 4x + 4 + y^2 \\
8x - 9 &= 6\sqrt{(x-2)^2 + y^2}
\end{aligned}$$

Note that this only holds if $x \geq \dfrac{9}{8}$. Squaring both sides again gives

$$\begin{aligned}
64x^2 - 144x + 81 &= 36(x^2 - 4x + 4 + y^2) \\
28x^2 - 36y^2 &= 63 \\
\frac{4x^2}{9} - \frac{4y^2}{7} &= 1 \qquad \text{for } x \geq \frac{3}{2}
\end{aligned}$$

This is the right branch of a hyperbola with centre the origin and x-axis intercept $\dfrac{3}{2}$.

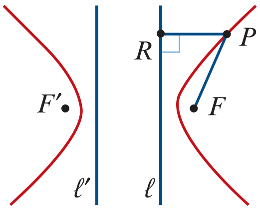
ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

It can also be shown that a hyperbola is the locus of points $P(x, y)$ satisfying

$$FP = eRP$$

where F is a fixed point, $e > 1$ and RP is the perpendicular distance from P to a fixed line ℓ.



From the symmetry of the hyperbola, it is clear that there is a second point F' and a second line ℓ' such that $F'P = eR'P$ defines the same locus, where $R'P$ is the perpendicular distance from P to ℓ'.

Example 15

Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance from P to the point $F(1, 0)$ is twice the distance MP, the perpendicular distance from P to the line with equation $x = -2$. That is, $FP = 2MP$.

Solution

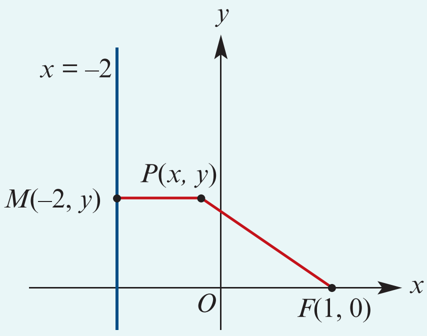
Let (x, y) be the coordinates of point P.

If $FP = 2MP$, then

$$\sqrt{(x-1)^2 + y^2} = 2\sqrt{(x+2)^2}$$

Squaring both sides gives

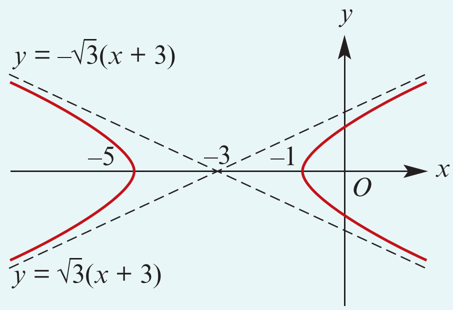
$$\begin{aligned}(x-1)^2 + y^2 &= 4(x+2)^2 \\ x^2 - 2x + 1 + y^2 &= 4(x^2 + 4x + 4) \\ 3x^2 + 18x - y^2 + 15 &= 0\end{aligned}$$



By completing the square, we obtain

$$\begin{aligned}3(x^2 + 6x + 9) - 27 - y^2 + 15 &= 0 \\ 3(x+3)^2 - y^2 &= 12 \\ \frac{(x+3)^2}{4} - \frac{y^2}{12} &= 1\end{aligned}$$

This is a hyperbola with centre $(-3, 0)$.



Section summary

- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 (called the **foci**) is a constant. That is, $|F_2P - F_1P| = k$.
- The graph of

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Exercise 15E

Skillsheet

Example 13

1 Sketch the graph of each of the following hyperbolas. Label axis intercepts and give the equations of the asymptotes.

a $\dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$ **b** $x^2 - \dfrac{y^2}{4} = 1$

c $\dfrac{y^2}{25} - \dfrac{x^2}{100} = 1$ **d** $25x^2 - 9y^2 = 225$

2 Sketch the graph of each of the following hyperbolas. State the centre and label axis intercepts and asymptotes.

a $(x-1)^2 - (y+2)^2 = 1$ **b** $\dfrac{(x+1)^2}{4} - \dfrac{(y-2)^2}{16} = 1$

c $\dfrac{(y-3)^2}{9} - (x-2)^2 = 1$ **d** $25(x-4)^2 - 9y^2 = 225$

e $x^2 - 4y^2 - 4x - 8y - 16 = 0$ **f** $9x^2 - 25y^2 - 90x + 150y = 225$

Example 14

3 Consider the points $A(4, 0)$ and $B(-4, 0)$. Find the equation of the locus of points P satisfying $AP - BP = 6$.

4 Find the equation of the locus of points $P(x, y)$ satisfying $AP - BP = 4$, given coordinates $A(-3, 0)$ and $B(3, 0)$.

Example 15

5 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(5, 0)$ is twice the distance MP, the perpendicular distance to P from the line with equation $x = -1$. That is, $FP = 2MP$.

6 Find the equation of the locus of points $P(x, y)$ that satisfy the property that the distance to P from the point $F(0, -1)$ is twice the distance MP, the perpendicular distance to P from the line with equation $y = -4$. That is, $FP = 2MP$.

15F Parametric equations

A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

The variable t is called the **parameter**, and for each choice of t we get a point in the plane $(f(t), g(t))$. The set of all such points will be a curve in the plane.

It is sometimes useful to think of t as being *time*, so that the equations $x = f(t)$ and $y = g(t)$ give the position of an object at time t. Points on the curve can be plotted by substituting various values of t into the two equations.

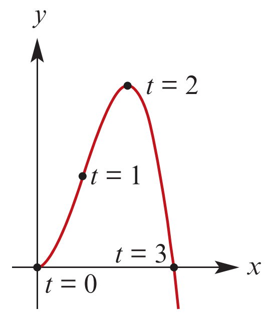
ISBN 978-1-107-56765-8

For instance, we can plot points on the curve defined by the parametric equations

$$x = t \quad \text{and} \quad y = 3t^2 - t^3$$

by letting $t = 0, 1, 2, 3$.

t	0	1	2	3
x	0	1	2	3
y	0	2	4	0



In this instance, it is possible to eliminate the parameter t to obtain a Cartesian equation in x and y alone. Substituting $t = x$ into the second equation gives $y = 3x^2 - x^3$.

Lines

Example 16

a Find the Cartesian equation for the curve defined by the parametric equations

$$x = t + 2 \quad \text{and} \quad y = 2t - 3$$

b Find parametric equations for the line through the points $A(2, 3)$ and $B(4, 7)$.

Solution

a Substitute $t = x - 2$ into the second equation to give

$$\begin{aligned} y &= 2(x - 2) - 3 \\ &= 2x - 7 \end{aligned}$$

Thus every point lies on the straight line with equation $y = 2x - 7$.

b The gradient of the straight line through points $A(2, 3)$ and $B(4, 7)$ is

$$m = \frac{7 - 3}{4 - 2} = 2$$

Therefore the line has equation

$$\begin{aligned} y - 3 &= 2(x - 2) \\ y &= 2x - 1 \end{aligned}$$

We can simply let $x = t$ and so $y = 2t - 1$.

Note: There are infinitely many pairs of parametric equations that describe the same curve. In part **b**, we could also let $x = 2t$ and $y = 4t - 1$. These parametric equations describe exactly the same set of points. As t increases, the point moves along the same line twice as fast.

Parabolas

Example 17

Find the Cartesian equation of the parabola defined by the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 + 1$$

Solution

Substitute $t = x + 1$ into the second equation to give $y = (x + 1)^2 + 1$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Circles

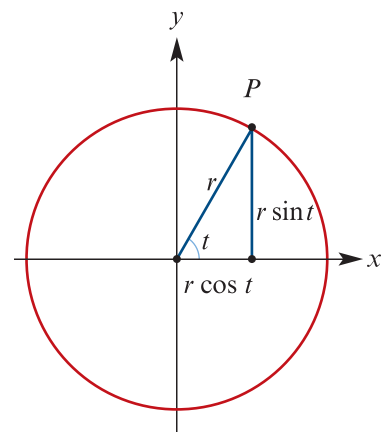
We have seen that the circle with radius r and centre at the origin can be written in Cartesian form as

$$x^2 + y^2 = r^2$$

We now introduce the parameter t and let

$$x = r\cos t \quad \text{and} \quad y = r\sin t$$

As t increases from 0 to 2π, the point $P(x, y)$ travels from $(r, 0)$ anticlockwise around the circle and returns to its original position.



To demonstrate that this parameterises the circle, we evaluate

$$\begin{aligned} x^2 + y^2 &= r^2\cos^2 t + r^2\sin^2 t \\ &= r^2(\cos^2 t + \sin^2 t) \\ &= r^2 \end{aligned}$$

where we have used the Pythagorean identity $\cos^2 t + \sin^2 t = 1$.

Example 18

a Find the Cartesian equation of the circle defined by the parametric equations

$$x = \cos t + 1 \quad \text{and} \quad y = \sin t - 2$$

b Find parametric equations for the circle with Cartesian equation

$$(x + 1)^2 + (y + 3)^2 = 4$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$x - 1 = \cos t \quad \text{and} \quad y + 2 = \sin t$$

Using the Pythagorean identity:

$$(x - 1)^2 + (y + 2)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the circle with equation $(x - 1)^2 + (y + 2)^2 = 1$.

b We let

$$\cos t = \frac{x + 1}{2} \quad \text{and} \quad \sin t = \frac{y + 3}{2}$$

giving

$$x = 2\cos t - 1 \quad \text{and} \quad y = 2\sin t - 3$$

We can easily check that these equations parameterise the given circle.

ISBN 978-1-107-56765-8

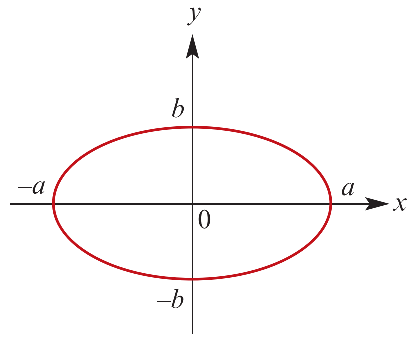
Cambridge University Press
Updated May 2021

Ellipses

An ellipse can be thought of as a squashed circle.

This is made apparent from the parametric equations for an ellipse:

$$x = a\cos t \quad \text{and} \quad y = b\sin t$$



As with the circle, we see the sine and cosine functions, but these are now scaled by different constants, giving different dilations from the x- and y-axes.

We can turn this pair of parametric equations into one Cartesian equation as follows:

$$\frac{x}{a} = \cos t \quad \text{and} \quad \frac{y}{b} = \sin t$$

giving

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2 t + \sin^2 t = 1$$

which is the standard form of an ellipse centred at the origin with axis intercepts at $x = \pm a$ and $y = \pm b$.

Example 19

a Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t + 1 \quad \text{and} \quad y = 2\sin t - 1$$

b Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\cos t$ and $\sin t$ respectively. This gives

$$\frac{x-1}{3} = \cos t \quad \text{and} \quad \frac{y+1}{2} = \sin t$$

Using the Pythagorean identity:

$$\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 = \cos^2 t + \sin^2 t = 1$$

So every point on the graph lies on the ellipse with equation $\dfrac{(x-1)^2}{3^2} + \dfrac{(y+1)^2}{2^2} = 1$.

b We let

$$\cos t = \frac{x-1}{2} \quad \text{and} \quad \sin t = \frac{y+2}{4}$$

giving

$$x = 2\cos t + 1 \quad \text{and} \quad y = 4\sin t - 2$$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Hyperbolas

We can parameterise a hyperbola using the equations

$$x = a \sec t \quad \text{and} \quad y = b \tan t$$

From these two equations, we can find the more familiar Cartesian equation:

$$\frac{x}{a} = \sec t \quad \text{and} \quad \frac{y}{b} = \tan t$$

giving

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \sec^2 t - \tan^2 t = 1$$

which is the standard form of a hyperbola centred at the origin.

Example 20

a Find the Cartesian equation of the hyperbola defined by the parametric equations

$$x = 3 \sec t - 1 \quad \text{and} \quad y = 2 \tan t + 2$$

b Find parametric equations for the hyperbola with Cartesian equation

$$\frac{(x+2)^2}{4} - \frac{(y-3)^2}{16} = 1$$

Solution

a We rearrange each equation to isolate $\sec t$ and $\tan t$ respectively. This gives

$$\frac{x+1}{3} = \sec t \quad \text{and} \quad \frac{y-2}{2} = \tan t$$

and therefore

$$\left(\frac{x+1}{3}\right)^2 - \left(\frac{y-2}{2}\right)^2 = \sec^2 t - \tan^2 t = 1$$

So each point on the graph lies on the hyperbola with equation $\dfrac{(x+1)^2}{3^2} - \dfrac{(y-2)^2}{2^2} = 1$.

b We let

$$\sec t = \frac{x+2}{2} \quad \text{and} \quad \tan t = \frac{y-3}{4}$$

giving

$$x = 2 \sec t - 2 \quad \text{and} \quad y = 4 \tan t + 3$$

Parametric equations with restricted domains

Example 21

Eliminate the parameter to determine the graph of the parameterised curve

$$x = t - 1, \quad y = t^2 - 2t + 1 \qquad \text{for } 0 \le t \le 2$$

ISBN 978-1-107-56765-8

Cambridge University Press Updated May 2021

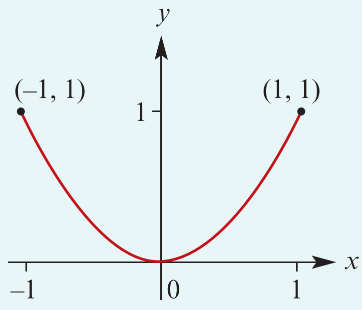
Solution

Substitute $t = x + 1$ from the first equation into the second equation, giving

$$\begin{aligned} y &= (x+1)^2 - 2(x+1) + 1 \\ &= x^2 + 2x + 1 - 2x - 2 + 1 \\ &= x^2 \end{aligned}$$

Since $0 \le t \le 2$, it follows that $-1 \le x \le 1$.

Therefore, as t increases from 0 to 2, the point travels along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$.



▶ Intersections of curves defined parametrically

It is often difficult to find the intersection of two curves defined parametrically. This is because, although the curves may intersect, they might do so for different values of the parameter t.

In many instances, it is easiest to find the points of intersection using the Cartesian equations for the two curves.

Example 22

Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = 5\cos t \quad$ and $\quad y = 5\sin t$

line $\quad x = t - 3 \quad$ and $\quad y = 2t - 8$

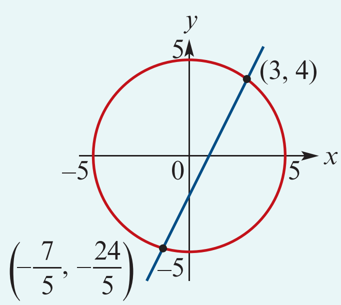
Solution

The Cartesian equation of the circle is $x^2 + y^2 = 25$.

The Cartesian equation of the line is $y = 2x - 2$.

Substituting the second equation into the first gives

$$\begin{aligned} x^2 + (2x-2)^2 &= 25 \\ x^2 + 4x^2 - 8x + 4 &= 25 \\ 5x^2 - 8x - 21 &= 0 \\ (x-3)(5x+7) &= 0 \end{aligned}$$



This gives solutions $x = 3$ and $x = -\frac{7}{5}$.

Substituting these into the equation $y = 2x - 2$ gives $y = 4$ and $y = -\frac{24}{5}$ respectively.

The points of intersection are $(3, 4)$ and $\left(-\frac{7}{5}, -\frac{24}{5}\right)$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

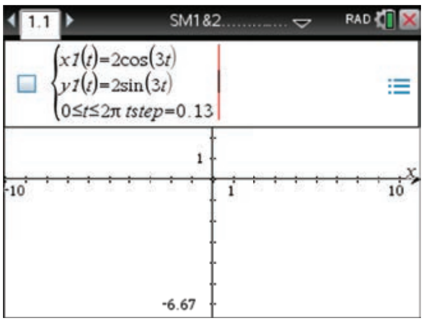
▶ Using a CAS calculator with parametric equations

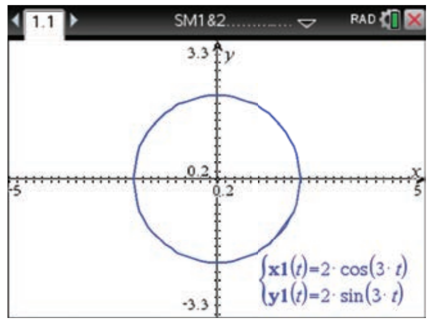
Example 23

Plot the graph of the parametric curve given by $x = 2\cos(3t)$ and $y = 2\sin(3t)$.

Using the TI-Nspire

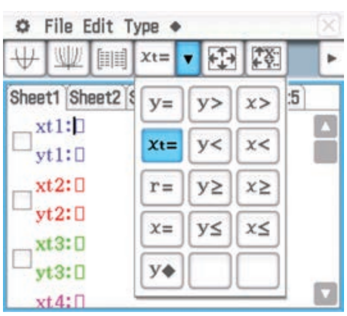
- Open a **Graphs** application (on) > **New Document** > **Add Graphs**).
- Use (menu) > **Graph Entry/Edit** > **Parametric** to show the entry line for parametric equations.
- Enter $x1(t) = 2\cos(3t)$ and $y1(t) = 2\sin(3t)$ as shown.



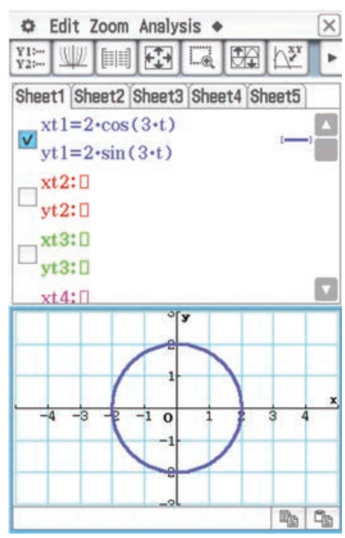


Using the Casio ClassPad

- Open the **Graph & Table** application.
- Clear all equations and graphs.
- Tap on $\boxed{y=}$ in the toolbar and select $\boxed{xt=}$.



- Enter the equations in $xt1$ and $yt1$ as shown.
- Tick the box and tap .



ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Section summary

- A **parametric curve** in the plane is a pair of functions

$$x = f(t) \quad \text{and} \quad y = g(t)$$

where t is called the **parameter** of the curve. For example:

	Cartesian equation	Parametric equations
Circle	$x^2 + y^2 = r^2$	$x = r\cos t$ and $y = r\sin t$
Ellipse	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$x = a\cos t$ and $y = b\sin t$
Hyperbola	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$x = a\sec t$ and $y = b\tan t$

- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

Exercise 15F

Skillsheet

1 Consider the parametric equations

$$x = t - 1 \quad \text{and} \quad y = t^2 - 1$$

Example 16

a Find the Cartesian equation of the curve described by these equations.

b Sketch the curve and label the points on the curve corresponding to $t = 0, 1, 2$.

Example 17

2 For each of the following pairs of parametric equations, find the Cartesian equation and sketch the curve:

a $x = t + 1$ and $y = 2t + 1$

b $x = t - 1$ and $y = 2t^2 + 1$

c $x = t^2$ and $y = t^6$

d $x = t + 2$ and $y = \dfrac{1}{t+1}$

Example 18

3 **a** Find the Cartesian equation of the circle defined by the parametric equations

$$x = 2\cos t \quad \text{and} \quad y = 2\sin t$$

Example 19

b Find the Cartesian equation of the ellipse defined by the parametric equations

$$x = 3\cos t - 1 \quad \text{and} \quad y = 2\sin t + 2$$

c Find parametric equations for the circle with Cartesian equation

$$(x + 3)^2 + (y - 2)^2 = 9$$

d Find parametric equations for the ellipse with Cartesian equation

$$\frac{(x + 2)^2}{9} + \frac{(y - 1)^2}{4} = 1$$

4 Find parametric equations for the line through the points $A(-1, -2)$ and $B(1, 4)$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Example 21

5 **a** Eliminate the parameter t to determine the equation of the parameterised curve

$$x = t - 1 \quad \text{and} \quad y = -2t^2 + 4t - 2 \qquad \text{for } 0 \le t \le 2$$

b Sketch the graph of this curve over an appropriate domain.

Example 22

6 Find the points of intersection of the circle and line defined by the parametric equations:

circle $\quad x = \cos t \quad$ and $\quad y = \sin t$

line $\quad x = 3t + 6 \quad$ and $\quad y = 4t + 8$

7 A curve is parameterised by the equations

$$x = \sin t \quad \text{and} \quad y = 2\sin^2 t + 1 \qquad \text{for } 0 \le t \le 2\pi$$

a Find the curve's Cartesian equation.

b What is the domain of the curve?

c What is the range of the curve?

d Sketch the graph of the curve.

8 A curve is parameterised by the equations

$$x = 2^t \quad \text{and} \quad y = 2^{2t} + 1 \qquad \text{for } t \in \mathbb{R}$$

a Find the curve's Cartesian equation.

b What is the domain of the curve?

c What is the range of the curve?

d Sketch the graph of the curve.

9 Eliminate the parameter to determine the graph of the parameterised curve

$$x = \cos t \quad \text{and} \quad y = 1 - 2\sin^2 t \qquad \text{for } 0 \le t \le 2\pi$$

10 Consider the parametric equations

$$x = 2^t + 2^{-t} \quad \text{and} \quad y = 2^t - 2^{-t}$$

a Show that the Cartesian equation of the curve is $\dfrac{x^2}{4} - \dfrac{y^2}{4} = 1$ for $x \ge 2$.

b Sketch the graph of the curve.

11 Consider the circle with Cartesian equation $x^2 + (y - 1)^2 = 1$.

a Sketch the graph of the circle.

b Show that the parametric equations $x = \cos t$ and $y = \sin t + 1$ define the same circle.

c A different parameterisation of the circle can be found without the use of the cosine and sine functions. Suppose that t is any real number and let $P(x, y)$ be the point of intersection of the line $y = 2 - tx$ with the circle. Solve for x and y in terms of t, assuming that $x \ne 0$.

d Verify that the equations found in part **c** parameterise the same circle.

12 The curve with parametric equations $x = \dfrac{t}{2\pi}\cos t$ and $y = \dfrac{t}{2\pi}\sin t$ is called an **Archimedean spiral**.

a With the help of your calculator, sketch the curve over the interval $0 \le t \le 6\pi$.

b Label the points on the curve corresponding to $t = 0, 1, 2, 3, 4, 5, 6$.

ISBN 978-1-107-56765-8

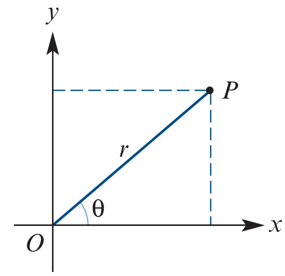
Cambridge University Press
Updated May 2021

15G Polar coordinates

Until now, we have described each point in the plane by a pair of numbers (x, y). These are called Cartesian coordinates, and take their name from the French intellectual René Descartes (1596–1650) who introduced them. However, they are not the only way to describe points in the plane. In fact, for many situations it is more convenient to use **polar coordinates**.

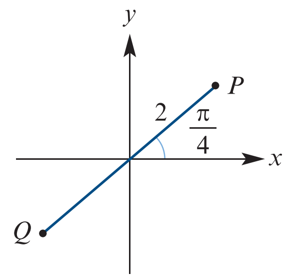
Using polar coordinates, every point P in the plane is described by a pair of numbers (r, θ).

- The number r is the distance from the origin O to P.
- The number θ measures the angle between the positive direction of the x-axis and the ray OP, as shown.



For example, the diagram on the right shows the point P with polar coordinates $\left(2, \frac{\pi}{4}\right)$.

We can even make sense of polar coordinates such as $Q\left(-2, \frac{\pi}{4}\right)$: go to the direction $\frac{\pi}{4}$ and then move a distance of 2 in the opposite direction.



Converting between the two coordinate systems requires little more than basic trigonometry.

- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

- If a point P has Cartesian coordinates (x, y), then its polar coordinates (r, θ) satisfy

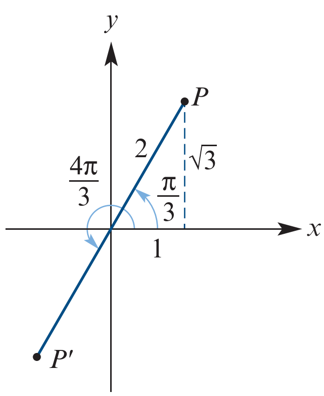
$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

Non-uniqueness of polar coordinates

Polar coordinates differ from Cartesian coordinates in that each point in the plane has more than one representation in polar coordinates.

For example, the following polar coordinates all represent the same point:

$$\left(2, \frac{\pi}{3}\right), \quad \left(-2, \frac{4\pi}{3}\right) \quad \text{and} \quad \left(2, \frac{7\pi}{3}\right)$$



The point $P(r, \theta)$ can be described in infinitely many ways:

$$(r, \theta + 2n\pi) \quad \text{and} \quad (-r, \theta + (2n+1)\pi) \qquad \text{for all } n \in \mathbb{Z}$$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Example 24

Convert polar coordinates $\left(2, \frac{5\pi}{6}\right)$ into Cartesian coordinates.

Solution

$$x = r\cos\theta \qquad y = r\sin\theta$$
$$= 2\cos\left(\frac{5\pi}{6}\right) \qquad = 2\sin\left(\frac{5\pi}{6}\right)$$
$$= -\sqrt{3} \qquad = 1$$

The Cartesian coordinates are $(-\sqrt{3}, 1)$.

Example 25

For each pair of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$.

a $(3, 3)$ **b** $(1, -\sqrt{3})$ **c** $(-5, 0)$ **d** $(0, 3)$

Solution

a $r = \sqrt{3^2 + 3^2} = 3\sqrt{2}$

$\theta = \tan^{-1}(1) = \frac{\pi}{4}$

The point has polar coordinates $\left(3\sqrt{2}, \frac{\pi}{4}\right)$.

We could also let $r = -3\sqrt{2}$ and add π to the angle, giving $\left(-3\sqrt{2}, \frac{5\pi}{4}\right)$.

b $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$

$\theta = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

The point has polar coordinates $\left(2, -\frac{\pi}{3}\right)$.

We could also let $r = -2$ and add π to the angle, giving $\left(-2, \frac{2\pi}{3}\right)$.

c $r = 5$ and $\theta = \pi$

The point has polar coordinates $(5, \pi)$.

We could also let $r = -5$ and subtract π from the angle, giving $(-5, 0)$.

d $r = 3$ and $\theta = \frac{\pi}{2}$

The point has polar coordinates $\left(3, \frac{\pi}{2}\right)$.

We could also let $r = -3$ and subtract π from the angle, giving $\left(-3, -\frac{\pi}{2}\right)$.

Section summary

- Each point P in the plane can be represented using polar coordinates (r, θ), where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.
- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\cos\theta$$

- If a point P has Cartesian coordinates (x, y), then all its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \qquad (\text{if } x \neq 0)$$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

- Each point in the plane has more than one representation in polar coordinates. For example, the coordinates $\left(2, \frac{\pi}{4}\right)$, $\left(2, \frac{9\pi}{4}\right)$ and $\left(-2, \frac{5\pi}{4}\right)$ all represent the same point.

Exercise 15G

Example 24

1 Plot the points with the following polar coordinates and then find their Cartesian coordinates:

a $A\left(1, \frac{\pi}{2}\right)$ **b** $B\left(2, \frac{3\pi}{4}\right)$ **c** $C\left(3, \frac{-\pi}{2}\right)$ **d** $D\left(-2, \frac{\pi}{4}\right)$ **e** $E(-1, \pi)$

f $F\left(0, \frac{\pi}{4}\right)$ **g** $G\left(4, -\frac{5\pi}{6}\right)$ **h** $H\left(-2, \frac{2\pi}{3}\right)$ **i** $I\left(-2, -\frac{\pi}{4}\right)$

Example 25

2 For each of the following pairs of Cartesian coordinates, find two representations using polar coordinates, one with $r > 0$ and the other with $r < 0$:

a $(1, -1)$ **b** $(1, \sqrt{3})$ **c** $(2, -2)$

d $(-\sqrt{2}, -\sqrt{2})$ **e** $(3, 0)$ **f** $(0, -2)$

3 Two points have polar coordinates $P\left(2, \frac{\pi}{6}\right)$ and $Q\left(3, \frac{\pi}{2}\right)$ respectively. Find the exact length of line segment PQ.

4 Two points have polar coordinates $P(r_1, \theta_1)$ and $Q(r_2, \theta_2)$. Find a formula for the length of PQ.

15H Graphing using polar coordinates

Polar coordinates are useful for describing and sketching curves in the plane, especially in situations that involve symmetry with respect to the origin. Suppose that f is a function. The graph of f in polar coordinates is simply the set of all points (r, θ) such that $r = f(\theta)$.

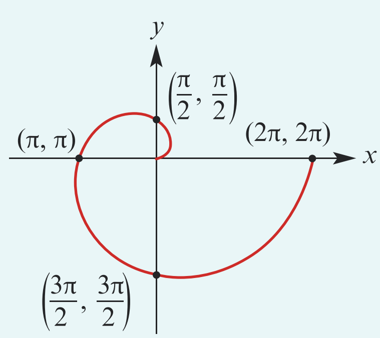
Example 26

Sketch the spiral with polar equation $r = \theta$, for $0 \le \theta \le 2\pi$.

Solution

The distance r from the origin exactly matches the angle θ. So as the angle increases, so too does the distance from the origin.

Note that the coordinates on the graph are in polar form.

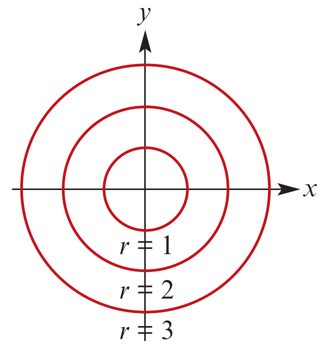


ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Circles

If a circle is centred at the origin, then its polar equation could not be simpler.



A circle of radius a centred at the origin has polar equation

$$r = a$$

That is, the distance r from the origin is constant, having no dependence on the angle θ. This illustrates rather forcefully the utility of polar coordinates: they simplify situations that involve symmetry with respect to the origin.

For circles not centred at the origin, the polar equations are less obvious.

Example 27

A curve has polar equation $r = 2\sin\theta$. Show that its Cartesian equation is $x^2 + (y-1)^2 = 1$.

Solution

The trick here is to first multiply both sides of the polar equation by r to get

$$r^2 = 2r\sin\theta$$

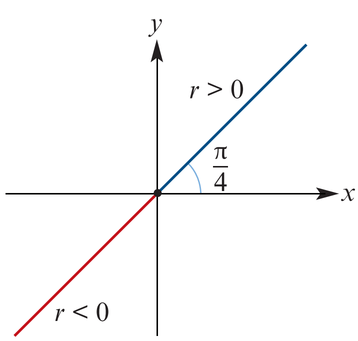
Since $r^2 = x^2 + y^2$ and $r\sin\theta = y$, this equation becomes

$$\begin{aligned} x^2 + y^2 &= 2y \\ x^2 + y^2 - 2y &= 0 \\ x^2 + (y^2 - 2y + 1) - 1 &= 0 \qquad \text{(completing the square)} \\ x^2 + (y-1)^2 &= 1 \end{aligned}$$

This is a circle with centre (0, 1) and radius 1.

Lines

For a straight line through the origin, the angle θ is fixed and the distance r varies. Because we have allowed negative values of r, the straight line goes in both directions.



The straight line shown has equation

$$\theta = \frac{\pi}{4}$$

For a straight line that does not go through the origin, the equation is more complicated. A line in Cartesian form $ax + by = c$ can be converted into polar form by substituting $x = r\cos\theta$ and $y = r\sin\theta$.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Example 28

a Express $x + y = 1$ in polar form.

b Express $r = \dfrac{2}{3\cos\theta - 4\sin\theta}$ in Cartesian form.

Solution

a Since $x = r\cos\theta$ and $y = r\sin\theta$, the equation $x + y = 1$ becomes

$$r\cos\theta + r\sin\theta = 1$$
$$r(\cos\theta + \sin\theta) = 1$$

Therefore the straight line has polar equation

$$r = \frac{1}{\cos\theta + \sin\theta}$$

b Since $\dfrac{x}{r} = \cos\theta$ and $\dfrac{y}{r} = \sin\theta$, the equation becomes

$$r = \frac{2}{\dfrac{3x}{r} - \dfrac{4y}{r}}$$
$$r = \frac{2r}{3x - 4y}$$
$$1 = \frac{2}{3x - 4y}$$

Therefore the Cartesian equation is

$$3x - 4y = 2$$

Section summary

- For a function f, the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.
- Polar coordinates are useful for describing graphs that are symmetric about the origin. For example:
  - The circle centred at the origin with radius a has equation $r = a$.
  - The line through the origin at angle α to the positive x-axis has equation $\theta = \alpha$.
- To convert between the polar form and the Cartesian form of an equation, substitute $x = r\cos\theta$ and $y = r\sin\theta$.

Exercise 15H

Skillsheet

1 Sketch the spiral with polar equation $r = \dfrac{\theta}{2\pi}$, for $0 \le \theta \le 4\pi$.

Example 26

2 Express each of the following Cartesian equations in polar form:

a $x = 4$ **b** $xy = 1$ **c** $y = x^2$

d $x^2 + y^2 = 9$ **e** $x^2 - y^2 = 1$ **f** $2x - 3y = 5$

3 Express each of the following polar equations in Cartesian form:

a $r = \dfrac{2}{\cos\theta}$ **b** $r = 2$ **c** $\theta = \dfrac{\pi}{4}$ **d** $r = \dfrac{4}{3\cos\theta - 2\sin\theta}$

Example 27

4 By finding the Cartesian equation, show that each of the following polar equations describes a circle:

a $r = 6\cos\theta$ **b** $r = 4\sin\theta$ **c** $r = -6\cos\theta$ **d** $r = -8\sin\theta$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

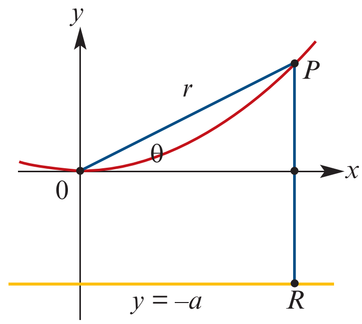
5 Show that the graph of $r = 2a\cos\theta$ is a circle of radius a centred at $(a, 0)$.

Example 28

6 **a** Show that the graph of $r = \dfrac{a}{\cos\theta}$ is a vertical line.

b Find the polar form of the horizontal line $y = a$.

7 A set of points $P(x, y)$ is such that the distance from P to the origin O is equal to the perpendicular distance from P to the line $y = -a$, where $a > 0$. This set of points is a parabola. Suppose that point P has polar coordinates (r, θ).



a Show that the distance from P to the line is $a + r\sin\theta$.

b Conclude that the equation for the parabola can be written as $r = \dfrac{a}{1 - \sin\theta}$.

15I Further graphing using polar coordinates

Various geometrically significant and beautiful figures are best described using polar coordinates.

Cardioids

The name **cardioid** comes from the Greek word for heart. A cardioid is the curve traced by a point on the perimeter of a circle that is rolling around a fixed circle of the same radius.

Example 29

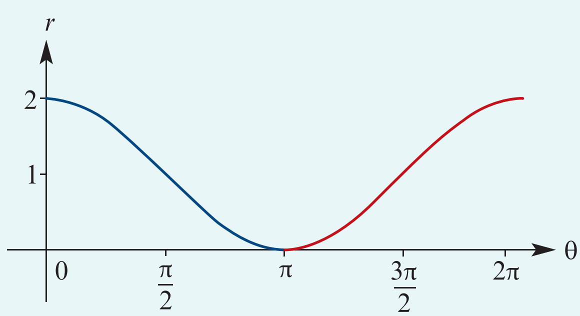
Graph the cardioid with equation $r = 1 + \cos\theta$, for $\theta \in [0, 2\pi]$.

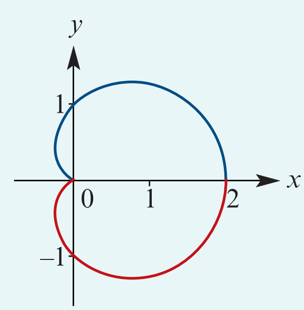
Solution

To help sketch this curve, we first graph the function $r = 1 + \cos\theta$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases.

- As the angle θ increases from 0 to π, the distance r decreases from 2 to 0.
- As the angle θ increases from π to 2π, the distance r increases from 0 to 2.

This gives the graph of the cardioid shown on the right.

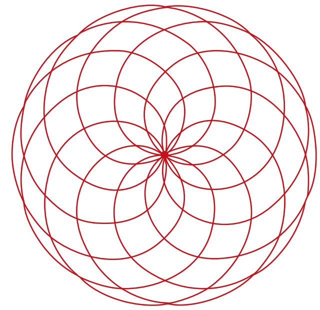




ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

▶ Roses



This impressive curve is fittingly called a **rose**. It belongs to the family of curves with polar equations of the form

$$r = \cos(n\theta)$$

For the example shown, $n = \frac{5}{8}$.

Example 30

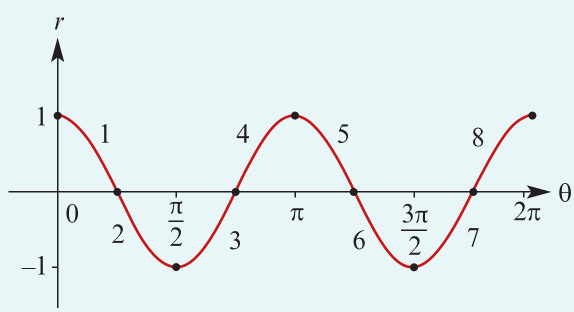
A curve has polar equation $r = \cos(2\theta)$.

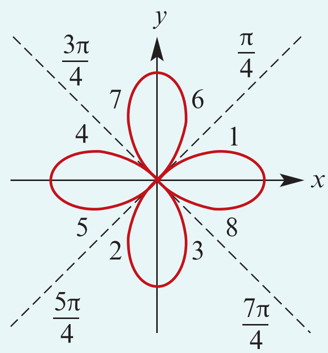
a Sketch the graph of the curve.

b Show that its Cartesian equation is $(x^2 + y^2)^3 = (x^2 - y^2)^2$.

Solution

a To help sketch this curve, we first graph the function $r = \cos(2\theta)$ using Cartesian coordinates, as shown on the left. This allows us to see how r changes as θ increases. Using numbers, we have labelled how each section of this graph corresponds to a section of the rose shown on the right.





b Using the double angle formula $\cos(2\theta) = \cos^2\theta - \sin^2\theta$, we have

$$\begin{aligned}
r &= \cos(2\theta) \\
r &= \cos^2\theta - \sin^2\theta \\
r &= \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \\
r^3 &= x^2 - y^2 \\
\left(\sqrt{x^2 + y^2}\right)^3 &= x^2 - y^2 \\
(x^2 + y^2)^3 &= (x^2 - y^2)^2
\end{aligned}$$

Note: This example further illustrates how polar coordinates can give more pleasing equations than their Cartesian counterparts.

The curve in this example is a **four-leaf rose**. More generally, the equations $r = \cos(n\theta)$ and $r = \sin(n\theta)$ give $2n$-leaf roses if n is even, and give n-leaf roses if n is odd.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

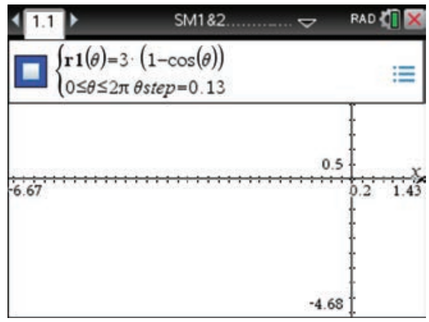
► Using a CAS calculator with polar coordinates

Example 31

Plot the graph of $r = 3(1 - \cos\theta)$.

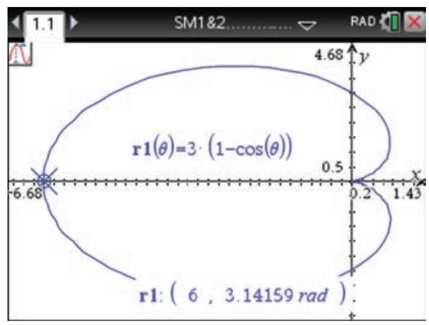
Using the TI-Nspire

- Open a **Graphs** application (on > **New Document** > **Add Graphs**) and set to polar using menu > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = 3(1 - \cos(\theta))$ as shown. The variable θ is entered using π▸ or the Symbols palette (ctrl).



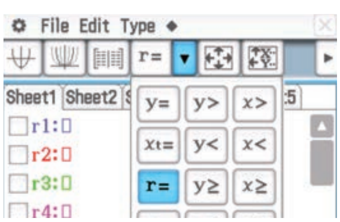
Note: The domain and the step size can be adjusted in this window.

- Set the scale using menu > **Window/Zoom** > **Zoom – Fit**.
- You can see the polar coordinates (r, θ) of points on the graph using menu > **Trace** > **Graph Trace**.
- To go to the point where $\theta = \pi$, simply type π and then press enter. The cursor will move to the point $(r, \theta) = (6, \pi)$ as shown.

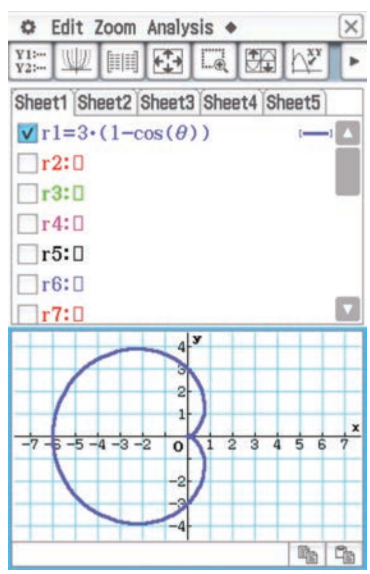


Using the Casio ClassPad

- Open the **Graph & Table** application .
- Clear all equations and graphs.
- Tap on y= in the toolbar and select r=.



- Enter $3(1 - \cos(\theta))$ in $r1$.
- Tick the box and tap .
- Select **Zoom** > **Initialize** to adjust the window.



Note: The variable θ is found in the Trig keyboard.

ISBN 978-1-107-56765-8

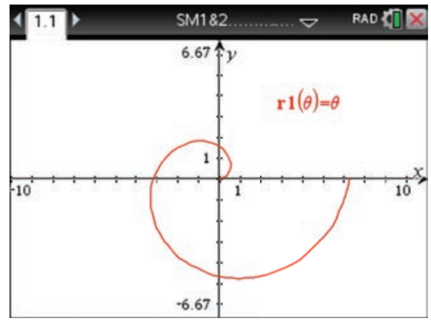
Cambridge University Press
Updated May 2021

Example 32

Plot the graph of $r = \theta$ for $0 \le \theta \le 6\pi$.

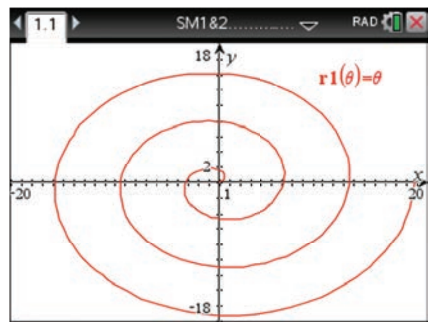
Using the TI-Nspire

- Open a **Graphs** application (⌂ on > **New Document** > **Add Graphs**) and set to polar using menu > **Graph Entry/Edit** > **Polar**.
- Enter $r1(\theta) = \theta$.
- The graph is shown.



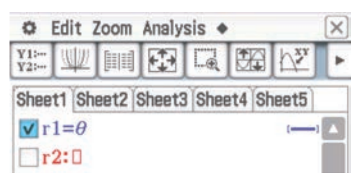
- If the domain is extended, the graph continues to spiral out. This can be observed by extending the domain to $0 \le \theta \le 6\pi$.

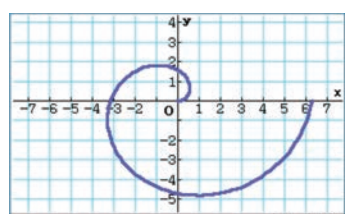
Note: Change the domain in the graph entry line; it is not the window setting.



Using the Casio ClassPad

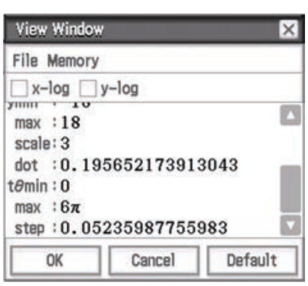
- Open the **Graph & Table** application .
- Clear all equations and graphs.
- Tap y= in the toolbar and select r=.
- Enter θ in $r1$. Tick the box and tap the graph icon.

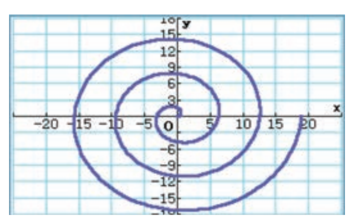




To extend the domain of the graph:

- Tap the View Window icon in the toolbar.
- Scroll down to the bottom of the list of settings.
- Set $t\theta$ max to 6π as shown below.





ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Section summary

- To sketch the curve $r = f(\theta)$ in polar coordinates, it helps to first sketch the graph in Cartesian coordinates.

Exercise 15I

1 Consider the polar equation

$$r = \frac{e}{1 + e\sin\theta}$$

When $0 < e < 1$, it can be shown that this equation describes an ellipse.

a With the help of your calculator, sketch the graphs of this polar equation when $e = 0.7, 0.8, 0.9$ on the same set of axes.

b The number e is called the **eccentricity** of the ellipse. What happens to the shape of the ellipse as the value of e is increased?

Example 29

2 The curve with polar equation $r = 1 - \sin\theta$ is a cardioid.

a Sketch the graph of the cardioid.

b Show that its Cartesian equation is $(x^2 + y^2 + y)^2 = x^2 + y^2$.

Example 30

3 Sketch the graphs of the roses with the following polar equations:

a $r = \cos(3\theta)$

b $r = \sin(3\theta)$

4 The polar equation $r = \sin(2\theta)$ defines a four-leaf rose.

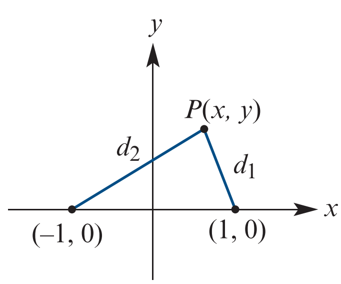
a Sketch the graph of the rose.

b Using a suitable double angle formula, show that its Cartesian equation is $(x^2 + y^2)^3 = 4x^2y^2$.

5 **a** With the help of your calculator, sketch the graph of the **lemniscate**, which has polar equation $r = \sqrt{2\cos(2\theta)}$.

b Using an appropriate double angle formula, find its equation in Cartesian coordinates.

c Show that the lemniscate can also be described as the set of all points $P(x, y)$ in the diagram that satisfy $d_1 d_2 = 1$.



ISBN 978-1-107-56765-8

Chapter summary

AS

Nrich

Reciprocal functions

- If $y = f(x)$ is a function, then the **reciprocal function** is defined by the rule $y = \frac{1}{f(x)}$.
- To sketch the graph of $y = \frac{1}{f(x)}$, we first sketch the graph of $y = f(x)$.
- The x-axis intercepts of $y = f(x)$ will become vertical asymptotes of $y = \frac{1}{f(x)}$.
- Local maximums of $y = f(x)$ will become local minimums of $y = \frac{1}{f(x)}$, and vice versa.

Parabolas, ellipses and hyperbolas

- A **locus** is the set of points described by a geometric condition.
- A **circle** is the locus of a point P that moves so that its distance from a fixed point C is constant.
- A **straight line** is the locus of a point P that moves so that it is equidistant from two fixed points Q and R.
- A **parabola** is the locus of a point P that moves so that its distance from a fixed point F is equal to its perpendicular distance from a fixed line.
- An **ellipse** is the locus of a point P that moves so that the sum of its distances d_1 and d_2 from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$
is an ellipse centred at the point (h, k).
- A **hyperbola** is the locus of a point P that moves so that the difference between its distances from two fixed points F_1 and F_2 is a constant.
- The graph of
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$
is a hyperbola centred at the point (h, k). The asymptotes are $y - k = \pm\frac{b}{a}(x - h)$.

Parametric curves

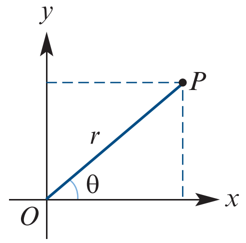
- A **parametric curve** in the plane is a pair of functions
$$x = f(t) \quad \text{and} \quad y = g(t)$$
where t is called the **parameter** of the curve.
- It can be helpful to think of the parameter t as describing time. Parametric curves are then useful for describing the motion of an object.
- We can sometimes find the Cartesian equation of a parametric curve by eliminating t and solving for y in terms of x.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Review

Polar coordinates



- Each point P in the plane can be represented using polar coordinates (r, θ), where:
  - r is the distance from the origin O to P
  - θ is the angle between the positive direction of the x-axis and the ray OP.
- If a point P has polar coordinates (r, θ), then its Cartesian coordinates (x, y) satisfy

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \qquad (1)$$

- If a point P has Cartesian coordinates (x, y), then all its polar coordinates (r, θ) satisfy

$$r^2 = x^2 + y^2 \quad \text{and} \quad \tan\theta = \frac{y}{x} \quad (\text{if } x \neq 0) \qquad (2)$$

- If f is a function, then the graph of f in polar coordinates is the set of all points (r, θ) such that $r = f(\theta)$.
- To convert between the polar form and the Cartesian form of an equation, use formulas (1) and (2) above.

Technology-free questions

1 For each of the following functions, sketch the graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ on the same set of axes:

a $f(x) = \frac{1}{2}(x^2 - 4)$ **b** $f(x) = (x + 1)^2 + 1$

c $f(x) = \cos(x) + 1,\ x \in [0, 2\pi]$ **d** $f(x) = \sin(x) + 2,\ x \in [0, 2\pi]$

2 The point $P(x, y)$ moves so that it is equidistant from $Q(2, -1)$ and $R(1, 2)$. Find the locus of the point P.

3 Find the locus of the point $P(x, y)$ such that $AP = 6$, given point $A(3, 2)$.

4 A circle has equation $x^2 + 4x + y^2 - 8y = 0$. Find the coordinates of the centre and the radius of the circle.

5 Sketch the graph of each ellipse and find the coordinates of its axis intercepts:

a $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ **b** $\dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{9} = 1$

6 An ellipse has equation $x^2 + 4x + 2y^2 = 0$. Find the coordinates of the centre and the axis intercepts of the ellipse.

7 Sketch the graph of each hyperbola and write down the equations of its asymptotes:

a $x^2 - \dfrac{y^2}{4} = 1$ **b** $\dfrac{(y-1)^2}{16} - \dfrac{(x+2)^2}{4} = 1$

ISBN 978-1-107-56765-8

8 A point $P(x, y)$ moves so that its distance from the point $K(-2, 5)$ is twice its distance from the line $x = 1$. Find its locus.

9 For each of the following pairs of parametric equations, find the corresponding Cartesian equation:

a $x = 2t - 1$ and $y = 6 - 4t$

b $x = 2\cos t$ and $y = 2\sin t$

c $x = 3\cos t + 1$ and $y = 5\sin t - 1$

d $x = \cos t$ and $y = 3\sin^2 t - 2$

10 A curve has parametric equations $x = t - 1$ and $y = 2t^2 - 1$ for $0 \le t \le 2$.

a Find the curve's Cartesian equation.

b What is the domain of the curve?

c What is the range of the curve?

d Sketch the graph of the curve.

11 Convert the polar coordinates $\left(2, \dfrac{3\pi}{4}\right)$ into Cartesian coordinates.

12 The point P has Cartesian coordinates $(2, -2\sqrt{3})$. Find two representations of P using polar coordinates, one with $r > 0$ and the other with $r < 0$.

13 Find an equation for the straight line $2x + 3y = 5$ using polar coordinates.

14 Show that the graph of $r = 6\sin\theta$ is a circle of radius 3 centred at $(0, 3)$.
Hint: First multiply both sides of the equation by r.

Multiple-choice questions

1 The graph of $y = \dfrac{1}{x^2 + 8x + k}$ will have two vertical asymptotes provided

A $k = 16$
B $k < 16$
C $k > 16$
D $k < -4$ or $k > 4$
E $-4 < k < 4$

2 The locus of points $P(x, y)$ which satisfy the property that $AP = BP$, given points $A(2, -5)$ and $B(-4, 1)$, is described by the equation

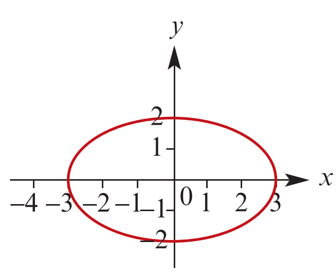
A $y = x - 1$
B $y = x - 6$
C $y = -x - 3$
D $y = x + 1$
E $y = 3 - x$

3 A parabola has focus $(0, 2)$ and directrix $y = -2$. Which of the following is not true about the parabola?

A Its axis of symmetry is the line $x = 0$.
B It passes through the origin.
C It contains no point below the x-axis.
D The point $(2, 1)$ lies on the parabola.
E The point $(4, 2)$ lies on the parabola.

4 The equation of the graph shown is

A $\dfrac{x^2}{3} + \dfrac{y^2}{2} = 1$
B $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1$
C $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
D $\dfrac{x^2}{3} - \dfrac{y^2}{2} = 1$
E $3x^2 + 2y^2 = 1$



ISBN 978-1-107-56765-8

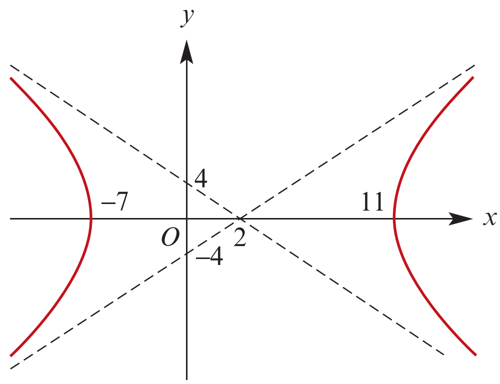
Cambridge University Press Updated May 2021

Review

5 The graph of the ellipse with equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$ has x-axis intercepts at

A $(-5, 0)$ and $(-3, 0)$ **B** $(-3, 0)$ and $(3, 0)$ **C** $(0, -5)$ and $(0, 5)$
D $(-5, 0)$ and $(5, 0)$ **E** $(5, 0)$ and $(3, 0)$

6 The equation of the graph shown is



A $\frac{(x+2)^2}{27} - \frac{y^2}{108} = 1$

B $\frac{(x-2)^2}{9} - \frac{y^2}{34} = 1$

C $\frac{(x+2)^2}{81} - \frac{y^2}{324} = 1$

D $\frac{(x-2)^2}{81} - \frac{y^2}{324} = 1$

E $\frac{(x+2)^2}{9} - \frac{y^2}{36} = 1$

7 The asymptotes of the hyperbola with equation $\frac{(y-2)^2}{9} - \frac{(x+3)^2}{4} = 1$ intersect at the point

A $(3, 2)$ **B** $(3, -2)$ **C** $(-3, 2)$ **D** $(2, -3)$ **E** $(-2, 3)$

8 An ellipse is parameterised by the equations $x = 4\cos t + 1$ and $y = 2\sin t - 1$. The coordinates of its x-axis intercepts are

A $(1 - 3\sqrt{2}, 0),\ (1 + 3\sqrt{2}, 0)$ **B** $(-3, 0),\ (5, 0)$
C $(1 - 2\sqrt{3}, 0),\ (1 + 2\sqrt{3}, 0)$ **D** $(0, -3),\ (0, 5)$
E $(0, 1 - 2\sqrt{3}),\ (0, 1 + 2\sqrt{3})$

9 Which of the following pairs of polar coordinates represent the same point?

A $\left(2, \frac{\pi}{4}\right)$ and $\left(2, \frac{3\pi}{4}\right)$ **B** $\left(3, \frac{\pi}{2}\right)$ and $\left(-3, \frac{\pi}{2}\right)$ **C** $\left(2, \frac{\pi}{3}\right)$ and $\left(-2, \frac{2\pi}{3}\right)$
D $\left(3, \frac{\pi}{4}\right)$ and $\left(3, \frac{5\pi}{4}\right)$ **E** $\left(1, \frac{\pi}{6}\right)$ and $\left(-1, \frac{7\pi}{6}\right)$

10 A curve has polar equation $r = 1 + \cos\theta$. Its equation in Cartesian coordinates is

A $xy = x^2 + y^2$ **B** $(x^2 + y^2 - x)^2 = x^2 + y^2$ **C** $x = x^2 + y^2$
D $(x^2 + y^2 - y)^2 = x^2 + y^2$ **E** $y = x^2 + y^2$

Extended-response questions

1 Consider points $A(0, 3)$ and $B(6, 0)$. Find the locus of the point $P(x, y)$ given that:

a $AP = BP$
b $AP = 2BP$

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021

Review

2 Find the equation of the locus of points $P(x, y)$ which satisfy the property that the distance to P from the point $F(0, 4)$ is equal to:

a MP, the perpendicular distance from the line with equation $y = -2$

b half the distance MP, the perpendicular distance from the line $y = -2$

c twice the distance MP, the perpendicular distance from the line $y = -2$.

3 A ball is thrown into the air. The position of the ball at time $t \geq 0$ is given by the parametric equations $x = 10t$ and $y = 20t - 5t^2$.

a Find the Cartesian equation of the ball's flight.

b Sketch the graph of the ball's path.

c What is the maximum height reached by the ball?

A second ball is thrown into the air. Its position at time $t \geq 0$ is given by the parametric equations $x = 60 - 10t$ and $y = 20t - 5t^2$.

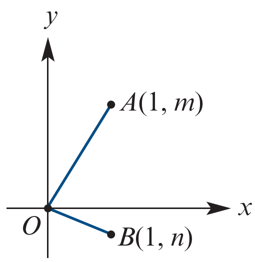
d Find the Cartesian equation of the second ball's flight.

e Sketch the graph of the second ball's path on the same set of axes.

f Find the points of intersection of the two paths.

g Do the balls collide?

4 Consider the lines $y = mx$ and $y = nx$ shown in the diagram.

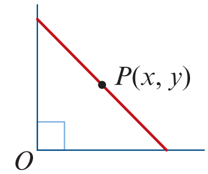


a Use the diagram and Pythagoras' theorem to prove that if $\angle AOB = 90°$, then $mn = -1$.

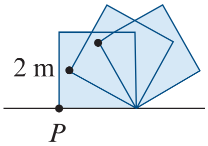
b Use the diagram and the cosine rule to prove that if $mn = -1$, then $\angle AOB = 90°$.

c Consider points $A(0, 4)$ and $B(8, 10)$. Find the equation of the locus of points $P(x, y)$ where $AP \perp BP$.

5 A ladder of length 6 metres stands against a vertical wall. The ladder then slides along on the floor until it lies flat. Show that the midpoint $P(x, y)$ of the ladder moves along a circular path.



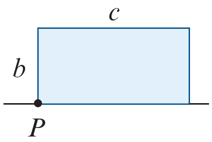
6 A square box of side length 2 metres is too heavy to lift, but can be rolled along the flat ground, using each edge as a pivot. The box is rolled one full revolution.



a Sketch the full path of the point P.

b Find the total distance travelled by the point P.

c A second rectangular box has width b metres and length c metres. Sketch the path taken by the point P when the box is rolled one full revolution, and find the total distance travelled by this point.



d For the second box, find the area between the path taken by P and the ground.

ISBN 978-1-107-56765-8

Cambridge University Press
Updated May 2021