CHAPTER 3

Indices and surds

Environmental science

Environmental scientists investigate and measure the growth rates of bacteria and their effects on the waterways of countries around the world. The Yarra River in Melbourne, one of Victoria's most important rivers, has been in the spotlight for the past decade or so with high levels of pollutants and bacteria, such as E. coli, making it unsafe in certain locations for swimming, fishing and even rowing.

Scientists look at the rate at which bacteria doubles. This process follows an exponential growth pattern of $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, \ldots 2^n$ where n represents the number of generations. In a laboratory using favourable conditions, E. coli doubles approximately every 20 minutes. In summer where the temperature along the Yarra River can be over 30°C, the growth of this bacteria and others needs to be monitored. The Environmental Protection Agency (EPA) and Melbourne Water take samples along the waterway and analyse it for levels of bacteria and pollutants.

The decay rate of other pollutants is also examined by investigating their half-life, as this also follows an exponential pattern: $2^0, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, \ldots$.

ISBN 978-1-108-77290-7

Online resources

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In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Simplify algebraic products and quotients using index laws (VCMNA330)

Money and financial mathematics

Connect the compound interest formula to repeated applications of simple interest using appropriate digital technologies (VCMNA328)

Real numbers

(10A) Define rational and irrational numbers and perform operations with surds and fractional indices (VCMNA355)

Linear and non-linear relationships

(10A) Solve simple exponential equations (VCMNA360)

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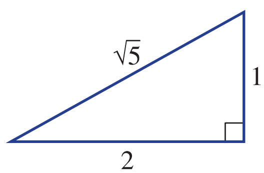
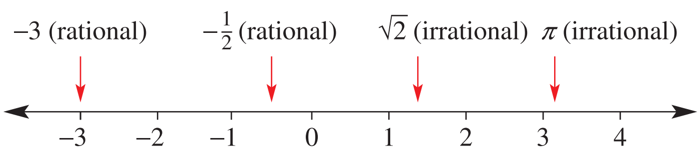
3A Irrational numbers including surds 10A

Learning intentions
- To know the meaning of the terms rational number, irrational number and surd
- To know how to identify a number as rational or irrational
- To know simple rules related to surds
- To be able to simplify surds using the highest square number factor

You will recall that when using Pythagoras' theorem to find unknown lengths in right-angled triangles, many answers expressed in exact form are surds. The length of the hypotenuse in this triangle, for example, is $\sqrt{5}$, which is a surd.



A surd is a number that uses a root sign ($\sqrt{\ }$), sometimes called a radical sign. They are irrational numbers, meaning that they cannot be expressed as a fraction in the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. Surds, together with other irrational numbers such as pi, and all rational numbers (fractions) make up the entire set of real numbers, which can be illustrated as points on a number line.

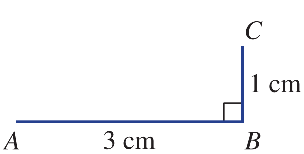


Many formulas contain numbers that are surds. The formulas for the speed of a rising weather balloon and the speed of its falling measuring device both include the surd $\sqrt{2}$.

LESSON STARTER Constructing surds

Someone asks you: 'How do you construct a line that is $\sqrt{10}$ cm long?' Use these steps to answer this question.

- First, draw a line segment AB that is 3 cm in length.
- Construct segment BC so that $BC = 1$ cm and $AB \perp BC$. You may wish to use a set square or pair of compasses.



- Now connect point A and point C and measure the length of the segment.
- Use Pythagoras' theorem to check the length of AC.

Use this idea to construct line segments with the following lengths. You may need more than one triangle for parts **d** to **f**.

a $\sqrt{2}$ **b** $\sqrt{17}$ **c** $\sqrt{20}$
d $\sqrt{3}$ **e** $\sqrt{6}$ **f** $\sqrt{22}$

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KEY IDEAS

- All **real** numbers can be located as a point on a number line. Real numbers include:
  - **rational numbers** (i.e. numbers that can be expressed as fractions)

    For example: $\frac{3}{7}, -\frac{4}{39}, -3, 1.6, 2.\dot{7}, 0.\overline{19}$

    The decimal representation of a rational number is either a **terminating** or **recurring decimal**.
  - **irrational numbers** (i.e. numbers that cannot be expressed as fractions)

    For example: $\sqrt{3}, -2\sqrt{7}, \sqrt{12} - 1, \pi, 2\pi - 3$

    The decimal representation of an irrational number is an **infinite non-recurring decimal**.
- **Surds** are irrational numbers that use a root sign ($\sqrt{\ }$).
  - For example: $\sqrt{2}, 5\sqrt{11}, -\sqrt{200}, 1 + \sqrt{5}$
  - These numbers are not surds: $\sqrt{4}(= 2), \sqrt[3]{125}(= 5), -\sqrt[4]{16}(= -2)$.
- The nth root of a number x is written $\sqrt[n]{x}$.
  - If $\sqrt[n]{x} = y$ then $y^n = x$. For example: $\sqrt[5]{32} = 2$ since $2^5 = 32$.
- The following rules apply to surds.
  - $(\sqrt{x})^2 = x$ and $\sqrt{x^2} = x$ when $x \geqslant 0$.
  - $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$ when $x \geqslant 0$ and $y \geqslant 0$.
  - $\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ when $x \geqslant 0$ and $y > 0$.
- When a factor of a number is a perfect square we call that factor a square factor. Examples of perfect squares are: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, …
- When simplifying surds, look for square factors of the number under the root sign and then use $\sqrt{a \times b} = \sqrt{a} \times \sqrt{b}$.

BUILDING UNDERSTANDING

1. Choose the correct word(s) from the words given in red to make the sentence true.
   a A number that cannot be expressed as a fraction is a rational/irrational number.
   b A surd is an irrational number that uses a root/square symbol.
   c The decimal representation of a surd is a terminating/recurring/non-recurring decimal.
   d $\sqrt{25}$ is a surd/rational number.
2. State the highest square factor of these numbers. For example, the highest square factor of 45 is 9.
   a 20 **b** 125 **c** 48 **d** 72

ISBN 978-1-108-77290-7

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Example 1 Defining and locating surds

Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{3}$ **b** 137% **c** $\dfrac{3}{7}$

SOLUTION

a $-\sqrt{3} = -1.732050807\ldots$
$-\sqrt{3}$ is irrational.

EXPLANATION

Use a calculator to express as a decimal.
The decimal does not terminate and there is no recurring pattern.

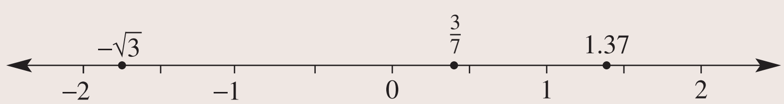
b $137\% = \dfrac{137}{100} = 1.37$
137% is rational.

137% is a fraction and can be expressed using a terminating decimal.

c $\dfrac{3}{7} = 0.\overline{428571}$
$\dfrac{3}{7}$ is rational.

$\dfrac{3}{7}$ is an infinitely recurring decimal.

Use the decimal equivalents to locate each number on the real number line.



Now you try

Express each number as a decimal and decide if they are rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{5}$ **b** -40% **c** $\dfrac{2}{7}$

Example 2 Simplifying surds

Simplify the following.

a $\sqrt{32}$ **b** $3\sqrt{200}$ **c** $\dfrac{5\sqrt{40}}{6}$ **d** $\sqrt{\dfrac{75}{9}}$

SOLUTION

a $\sqrt{32} = \sqrt{16 \times 2}$
$= \sqrt{16} \times \sqrt{2}$
$= 4\sqrt{2}$

EXPLANATION

When simplifying, choose the highest square factor of 32 (i.e. 16 rather than 4) as there is less work to do to arrive at the same answer.
Compare with
$\sqrt{32} = \sqrt{4 \times 8} = 2\sqrt{8} = 2\sqrt{4 \times 2} = 2 \times 2\sqrt{2} = 4\sqrt{2}$

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b $3\sqrt{200} = 3\sqrt{100 \times 2}$
$= 3 \times \sqrt{100} \times \sqrt{2}$
$= 3 \times 10 \times \sqrt{2}$
$= 30\sqrt{2}$

Select the apporopriate factors of 200 by finding its highest sqaure factor: 100.
Use $\sqrt{x \times y} = \sqrt{x} \times \sqrt{y}$ and simplify.

c $\dfrac{5\sqrt{40}}{6} = \dfrac{5\sqrt{4 \times 10}}{6}$
$= \dfrac{5 \times \sqrt{4} \times \sqrt{10}}{6}$
$= \dfrac{\cancel{10}^{5}\sqrt{10}}{\cancel{6}^{3}}$
$= \dfrac{5\sqrt{10}}{3}$

Select the appropriate factors of 40. The highest square factor is 4.

Cancel and simplify.

d $\sqrt{\dfrac{75}{9}} = \dfrac{\sqrt{75}}{\sqrt{9}}$
$= \dfrac{\sqrt{25 \times 3}}{\sqrt{9}} = \dfrac{5\sqrt{3}}{3}$

Use $\sqrt{\dfrac{x}{y}} = \dfrac{\sqrt{x}}{\sqrt{y}}$.

Then select the factors of 75 that include a square number and simplify.

Now you try

Simplify the following.

a $\sqrt{20}$ **b** $2\sqrt{300}$ **c** $\dfrac{2\sqrt{27}}{3}$ **d** $\sqrt{\dfrac{125}{16}}$

Example 3 Expressing as a single square root of a positive integer

Express these surds as a square root of a positive integer.

a $2\sqrt{5}$ **b** $7\sqrt{2}$

SOLUTION

a $2\sqrt{5} = \sqrt{4} \times \sqrt{5}$
$= \sqrt{20}$

b $7\sqrt{2} = \sqrt{49} \times \sqrt{2}$
$= \sqrt{98}$

EXPLANATION

Write 2 as $\sqrt{4}$ and then combine the two surds using $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

Write 7 as $\sqrt{49}$ and combine.

Now you try

Express these surds as a square root of a positive integer.

a $3\sqrt{2}$ **b** $5\sqrt{3}$

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Exercise 3A

FLUENCY	1, 2–6 (1/2)	2–7(1/3)	2–7(1/4)

Example 1

1 Express each number as a decimal and decide if they are rational or irrational. Then locate all the numbers on the same number line.

a $-\sqrt{2}$ b 150% c $\frac{6}{7}$

Example 1

2 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{5}$ b 18% c $\frac{2}{5}$ d -124%
e $1\frac{5}{7}$ f $-\sqrt{2}$ g $2\sqrt{3}$ h π

3 Decide if these numbers are surds.

a $\sqrt{7}$ b $2\sqrt{11}$ c $2\sqrt{25}$ d $-5\sqrt{144}$
e $\frac{3\sqrt{9}}{2}$ f $\frac{-5\sqrt{3}}{2}$ g $1-\sqrt{3}$ h $2\sqrt{1}+\sqrt{4}$

Example 2a

4 Simplify the following surds.

a $\sqrt{12}$ b $\sqrt{45}$ c $\sqrt{24}$ d $\sqrt{48}$
e $\sqrt{75}$ f $\sqrt{500}$ g $\sqrt{98}$ h $\sqrt{90}$
i $\sqrt{128}$ j $\sqrt{360}$ k $\sqrt{162}$ l $\sqrt{80}$

Example 2b,c

5 Simplify the following.

a $2\sqrt{18}$ b $3\sqrt{20}$ c $4\sqrt{48}$ d $2\sqrt{63}$
e $3\sqrt{98}$ f $4\sqrt{125}$ g $\frac{\sqrt{45}}{3}$ h $\frac{\sqrt{28}}{2}$
i $\frac{\sqrt{24}}{4}$ j $\frac{\sqrt{54}}{12}$ k $\frac{\sqrt{80}}{20}$ l $\frac{\sqrt{99}}{18}$
m $\frac{3\sqrt{44}}{2}$ n $\frac{5\sqrt{200}}{25}$ o $\frac{2\sqrt{98}}{7}$ p $\frac{3\sqrt{68}}{21}$
q $\frac{6\sqrt{75}}{20}$ r $\frac{4\sqrt{150}}{5}$ s $\frac{2\sqrt{108}}{18}$ t $\frac{3\sqrt{147}}{14}$

Example 2d

6 Simplify the following.

a $\sqrt{\frac{8}{9}}$ b $\sqrt{\frac{12}{49}}$ c $\sqrt{\frac{18}{25}}$ d $\sqrt{\frac{11}{25}}$
e $\sqrt{\frac{10}{9}}$ f $\sqrt{\frac{12}{144}}$ g $\sqrt{\frac{26}{32}}$ h $\sqrt{\frac{25}{50}}$
i $\sqrt{\frac{15}{27}}$ j $\sqrt{\frac{27}{4}}$ k $\sqrt{\frac{45}{72}}$ l $\sqrt{\frac{56}{76}}$

Example 3

7 Express these surds as a square root of a positive integer.

a $2\sqrt{3}$ b $4\sqrt{2}$ c $5\sqrt{2}$ d $3\sqrt{3}$
e $3\sqrt{5}$ f $6\sqrt{3}$ g $8\sqrt{2}$ h $10\sqrt{7}$
i $9\sqrt{10}$ j $5\sqrt{5}$ k $7\sqrt{5}$ l $11\sqrt{3}$

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PROBLEM-SOLVING	8, 9	8, 9, 11(1/2)	10, 11(1/2)

8 Simplify by searching for the highest square factor.

a $\sqrt{675}$

b $\sqrt{1183}$

c $\sqrt{1805}$

d $\sqrt{2883}$

9 Determine the exact side length, in simplest form, of a square with the given area.

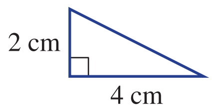
a 32 m^2 **b** 120 cm^2 **c** 240 mm^2

10 Determine the exact radius and diameter of a circle, in simplest form, with the given area.

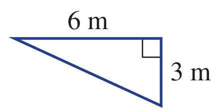
a $24\pi\text{ cm}^2$ **b** $54\pi\text{ m}^2$ **c** $128\pi\text{ m}^2$

11 Use Pythagoras' theorem to find the unknown length in these triangles, in simplest form.

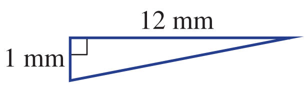
a



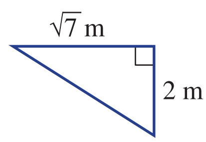
b



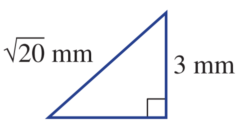
c



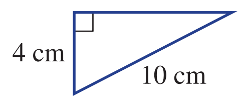
d



e



f



REASONING	12	12, 13	13, 14

12 Ricky uses the following working to simplify $\sqrt{72}$. Show how Ricky could have simplified $\sqrt{72}$ using fewer steps.

$$\begin{aligned}\sqrt{72} &= \sqrt{9 \times 8}\\ &= 3\sqrt{8}\\ &= 3\sqrt{4 \times 2}\\ &= 3 \times 2 \times \sqrt{2}\\ &= 6\sqrt{2}\end{aligned}$$

13 **a** List all the factors of 450 that are perfect squares.

b Now simplify $\sqrt{450}$ using the highest of these factors.

14 Use Pythagoras' theorem to construct a line segment with the given lengths. You can use only a ruler and a set square or compasses. Do not use a calculator.

a $\sqrt{10}$ cm

b $\sqrt{29}$ cm

c $\sqrt{6}$ cm

d $\sqrt{22}$ cm

ISBN 978-1-108-77290-7 Cambridge University Press

ENRICHMENT: Proving that $\sqrt{2}$ is irrational	–	–	15

15 We will prove that $\sqrt{2}$ is irrational by the method called 'proof by contradiction'. Your job is to follow and understand the proof, then copy it out and try explaining it to a friend or teacher.

a Before we start, we first need to show that if a perfect square a^2 is even then a is even. We do this by showing that if a is even then a^2 is even and if a is odd then a^2 is odd.

If a is even then $a = 2k$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k)^2 \\ &= 4k^2 \\ &= 2 \times 2k^2, \text{ which must be even.} \end{aligned}$$

If a is odd then $a = 2k + 1$, where k is an integer.

$$\begin{aligned} \text{So } a^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2 \times (2k^2 + 2k) + 1, \text{ which must be odd.} \end{aligned}$$

$\therefore$ If a^2 is even then a is even.

b Now, to prove $\sqrt{2}$ is irrational let's suppose that $\sqrt{2}$ is instead rational and can be written in the form $\frac{a}{b}$ in simplest form, where a and b are integers ($b \neq 0$) and at least one of a or b is odd.

$$\therefore \sqrt{2} = \frac{a}{b}$$

$$\text{So } 2 = \frac{a^2}{b^2} \text{ (squaring both sides)}$$

$$a^2 = 2b^2$$

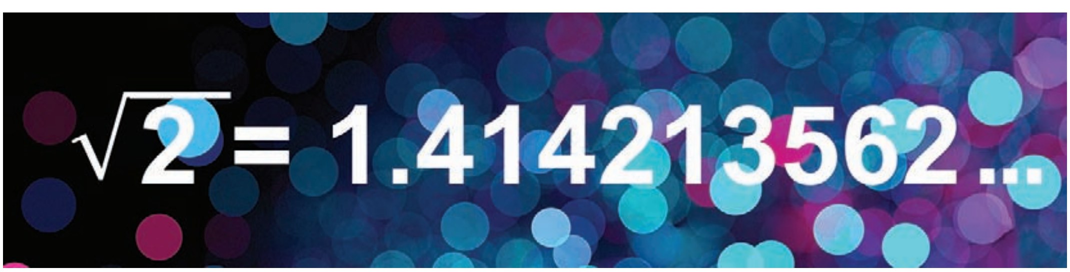
$\therefore a^2$ is even and, from part **a** above, a must be even.

If a is even, then $a = 2k$, where k is an integer.

$$\begin{aligned} \therefore \text{ If } a^2 &= 2b^2 \\ \text{Then } (2k)^2 &= 2b^2 \\ 4k^2 &= 2b^2 \\ 2k^2 &= b^2 \end{aligned}$$

$\therefore b^2$ is even and therefore b is even.

This is a contradiction because at least one of a or b must be odd. (Recall that $\frac{a}{b}$ in simplest form will have at least one of a or b being odd.) Therefore, the assumption that $\sqrt{2}$ can be written in the form $\frac{a}{b}$ must be incorrect and so $\sqrt{2}$ is irrational.



ISBN 978-1-108-77290-7

Cambridge University Press

3B Adding and subtracting surds 10A

Learning intentions

- To understand that only like surds can be combined under addition and subtraction
- To know how to add and subtract like surds
- To know that it is helpful to simplify all surds before determining if they can be added or subtracted

We can apply our knowledge of like terms in algebra to help simplify expressions involving the addition and subtraction of surds. Recall that $7x$ and $3x$ are like terms, so $7x + 3x = 10x$. The pronumeral x represents any number. When $x = 5$ then $7 \times 5 + 3 \times 5 = 10 \times 5$, and when $x = \sqrt{2}$ then $7\sqrt{2} + 3\sqrt{2} = 10\sqrt{2}$. Multiples of the same surd are called 'like surds' and can be collected (i.e. counted) in the same way as we collect like terms in algebra.

To design the Hearst Tower in New York, architects solved many equations, such a linear, quadratic and trigonometrical. Where possible, architects use surds in mathematical solutions to achieve precise results.

LESSON STARTER Can $3\sqrt{2} + \sqrt{8}$ be simplified?

To answer this question, first discuss these points.

- Are $3\sqrt{2}$ and $\sqrt{8}$ like surds?
- How can $\sqrt{8}$ be simplified?
- Now decide whether $3\sqrt{2} + \sqrt{8}$ can be simplified. Discuss why $3\sqrt{2} - \sqrt{7}$ cannot be simplified.

KEY IDEAS

- **Like surds** are multiples of the same surd.
  For example: $\sqrt{3}, -5\sqrt{3}, \sqrt{12} = 2\sqrt{3}, 2\sqrt{75} = 10\sqrt{3}$
- Like surds can be added and subtracted.
- Simplify all surds before attempting to add or subtract them.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 Decide if the following pairs of numbers are like surds.

a $\sqrt{3}, 2\sqrt{3}$ b $5, \sqrt{5}$ c $2\sqrt{2}, 2$ d $4\sqrt{6}, \sqrt{6}$

e $2\sqrt{3}, 5\sqrt{3}$ f $3\sqrt{7}, 3\sqrt{5}$ g $-2\sqrt{5}, 3\sqrt{5}$ h $-\sqrt{7}, -2\sqrt{7}$

2 Recall your basic skills in algebra to simplify these expressions.

a $11x - 5x$ b $2x - 7x$ c $-4a + 21a$ d $4t - 5t + 2t$

3 a Simplify the surd $\sqrt{48}$.

b Hence, simplify the following.

i $\sqrt{3} + \sqrt{48}$ ii $\sqrt{48} - 7\sqrt{3}$ iii $5\sqrt{48} - 3\sqrt{3}$

Example 4 Adding and subtracting surds

Simplify the following.

a $2\sqrt{3} + 4\sqrt{3}$

b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2}$

SOLUTION

a $2\sqrt{3} + 4\sqrt{3} = 6\sqrt{3}$

EXPLANATION

Collect the like surds by adding the coefficients: $2 + 4 = 6$.

SOLUTION

b $4\sqrt{6} + 3\sqrt{2} - 3\sqrt{6} + 2\sqrt{2} = \sqrt{6} + 5\sqrt{2}$

EXPLANATION

Collect like surds involving $\sqrt{6}$:
$4\sqrt{6} - 3\sqrt{6} = 1\sqrt{6} = \sqrt{6}$
Then collect those terms with $\sqrt{2}$.

Now you try

Simplify the following.

a $2\sqrt{5} + 3\sqrt{5}$

b $3\sqrt{7} + 2\sqrt{3} - 2\sqrt{7} + 5\sqrt{3}$

Example 5 Simplifying surds to add or subtract

Simplify these surds.

a $5\sqrt{2} - \sqrt{8}$

b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45}$

SOLUTION

a
$$\begin{aligned} 5\sqrt{2} - \sqrt{8} &= 5\sqrt{2} - \sqrt{4 \times 2} \\ &= 5\sqrt{2} - 2\sqrt{2} \\ &= 3\sqrt{2} \end{aligned}$$

EXPLANATION

First, look to simplify surds: $\sqrt{8}$ has a highest square factor of 4 and can be simplified to $2\sqrt{2}$.
Then subtract like surds.

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b $2\sqrt{5} - 3\sqrt{20} + 6\sqrt{45} = 2\sqrt{5} - 3\sqrt{4 \times 5} + 6\sqrt{9 \times 5}$
$= 2\sqrt{5} - 6\sqrt{5} + 18\sqrt{5}$
$= 14\sqrt{5}$

Simplify the surds and then collect like surds. Note that $3\sqrt{4 \times 5} = 3 \times \sqrt{4} \times \sqrt{5} = 6\sqrt{5}$.

Now you try

Simplify these surds.

a $7\sqrt{2} - \sqrt{8}$
b $2\sqrt{3} - 2\sqrt{27} + 3\sqrt{12}$

Exercise 3B

FLUENCY	1, 2–4(½)	2–4(½)	2–4(⅓)

1 Simplify the following.

Example 4a
a **i** $3\sqrt{7} + 2\sqrt{7}$ **ii** $7\sqrt{3} + 2\sqrt{3}$

Example 4b
b **i** $7\sqrt{5} + 2\sqrt{3} - 5\sqrt{5} + 3\sqrt{3}$ **ii** $4\sqrt{5} + 6\sqrt{11} - 2\sqrt{5} + 2\sqrt{11}$

Example 4a
2 Simplify the following.

a $2\sqrt{5} + 4\sqrt{5}$
b $5\sqrt{3} - 2\sqrt{3}$
c $7\sqrt{2} - 3\sqrt{2}$
d $8\sqrt{2} - 5\sqrt{2}$
e $7\sqrt{5} + 4\sqrt{5}$
f $6\sqrt{3} - 5\sqrt{3}$
g $4\sqrt{10} + 3\sqrt{10} - \sqrt{10}$
h $6\sqrt{2} - 4\sqrt{2} + 3\sqrt{2}$
i $\sqrt{21} - 5\sqrt{21} + 2\sqrt{21}$
j $3\sqrt{11} - 8\sqrt{11} - \sqrt{11}$
k $-2\sqrt{13} + 5\sqrt{13} - 4\sqrt{13}$
l $10\sqrt{30} - 15\sqrt{30} - 2\sqrt{30}$

Example 4b
3 Simplify the following.

a $2\sqrt{3} + 3\sqrt{2} - \sqrt{3} + 2\sqrt{2}$
b $5\sqrt{6} + 4\sqrt{11} - 2\sqrt{6} + 3\sqrt{11}$
c $3\sqrt{5} - 4\sqrt{2} + \sqrt{5} - 3\sqrt{2}$
d $5\sqrt{2} + 2\sqrt{5} - 7\sqrt{2} - \sqrt{5}$
e $2\sqrt{3} + 2\sqrt{7} + 2\sqrt{3} - 2\sqrt{7}$
f $5\sqrt{11} + 3\sqrt{6} - 3\sqrt{6} - 5\sqrt{11}$
g $2\sqrt{2} - 4\sqrt{10} - 5\sqrt{2} + \sqrt{10}$
h $-4\sqrt{5} - 2\sqrt{15} + 5\sqrt{15} + 2\sqrt{5}$

Example 5a
4 Simplify the following.

a $\sqrt{8} - \sqrt{2}$
b $\sqrt{8} + 3\sqrt{2}$
c $\sqrt{27} + \sqrt{3}$
d $\sqrt{20} - \sqrt{5}$
e $4\sqrt{18} - 5\sqrt{2}$
f $2\sqrt{75} + 2\sqrt{3}$
g $3\sqrt{44} + 2\sqrt{11}$
h $3\sqrt{8} - \sqrt{18}$
i $\sqrt{24} + \sqrt{54}$
j $2\sqrt{125} - 3\sqrt{45}$
k $3\sqrt{72} + 2\sqrt{98}$
l $3\sqrt{800} - 4\sqrt{200}$

PROBLEM-SOLVING	5(½)	5(½), 7(½)	5–7(⅓)

Example 5b
5 Simplify the following.

a $\sqrt{2} + \sqrt{50} + \sqrt{98}$
b $\sqrt{6} - 2\sqrt{24} + 3\sqrt{96}$
c $5\sqrt{7} + 2\sqrt{5} - 3\sqrt{28}$
d $2\sqrt{80} - \sqrt{45} + 2\sqrt{63}$
e $\sqrt{150} - \sqrt{96} - \sqrt{162} + \sqrt{72}$
f $\sqrt{12} + \sqrt{125} - \sqrt{50} + \sqrt{180}$
g $7\sqrt{3} - 2\sqrt{8} + \sqrt{12} + 3\sqrt{8}$
h $\sqrt{36} - \sqrt{108} + \sqrt{25} - 3\sqrt{3}$
i $3\sqrt{49} + 2\sqrt{288} - \sqrt{144} - 2\sqrt{18}$
j $2\sqrt{200} + 3\sqrt{125} + \sqrt{32} - 3\sqrt{242}$

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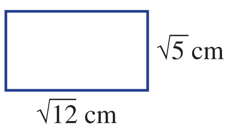
6 Simplify these surds that involve fractions. Remember to use the LCD (lowest common denominator).

a $\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{3}$ **b** $\frac{\sqrt{5}}{4} + \frac{\sqrt{5}}{3}$ **c** $\frac{\sqrt{2}}{5} - \frac{\sqrt{2}}{6}$

d $\frac{\sqrt{7}}{4} - \frac{\sqrt{7}}{12}$ **e** $\frac{2\sqrt{2}}{5} - \frac{\sqrt{2}}{2}$ **f** $\frac{3\sqrt{3}}{7} + \frac{\sqrt{3}}{2}$

g $\frac{7\sqrt{5}}{6} - \frac{4\sqrt{5}}{9}$ **h** $\frac{3\sqrt{3}}{10} - \frac{8\sqrt{3}}{15}$ **i** $\frac{-5\sqrt{10}}{6} + \frac{3\sqrt{10}}{8}$

7 Find the perimeter of these rectangles and triangles, in simplest form.

a


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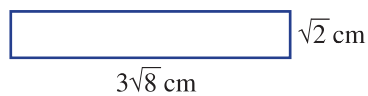

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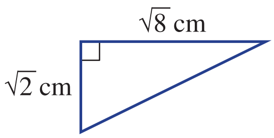

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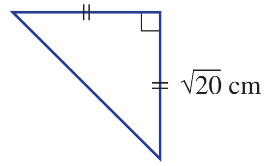
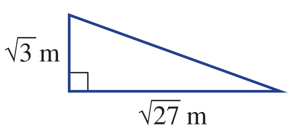
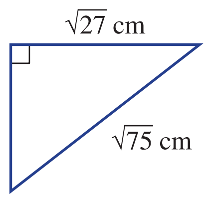


e


f


REASONING	8	8, 9(1/2)	9(1/2), 10

8 **a** Explain why $\sqrt{5}$ and $\sqrt{20}$ can be thought of as like surds.
b Explain why $3\sqrt{72}$ and $\sqrt{338}$ can be thought of as like surds.

9 Prove that each of the following simplifies to zero by showing all steps.

a $5\sqrt{3} - \sqrt{108} + \sqrt{3}$ **b** $\sqrt{6} + \sqrt{24} - 3\sqrt{6}$

c $6\sqrt{2} - 2\sqrt{32} + 2\sqrt{2}$ **d** $\sqrt{8} - \sqrt{18} + \sqrt{2}$

e $2\sqrt{20} - 7\sqrt{5} + \sqrt{45}$ **f** $3\sqrt{2} - 2\sqrt{27} - \sqrt{50} + 6\sqrt{3} + \sqrt{8}$

10 Prove that the surds in these expressions cannot be added or subtracted.

a $3\sqrt{12} - \sqrt{18}$ **b** $4\sqrt{8} + \sqrt{20}$ **c** $\sqrt{50} - 2\sqrt{45}$

d $5\sqrt{40} + 2\sqrt{75}$ **e** $2\sqrt{200} + 3\sqrt{300}$ **f** $\sqrt{80} - 2\sqrt{54}$

ENRICHMENT: Simplifying both surds and fractions	–	–	11(1/2)

11 To simplify the following, you will need to simplify surds and combine using a common denominator.

a $\frac{\sqrt{8}}{3} - \frac{\sqrt{2}}{5}$ **b** $\frac{\sqrt{12}}{4} + \frac{\sqrt{3}}{6}$ **c** $\frac{3\sqrt{5}}{4} - \frac{\sqrt{20}}{3}$ **d** $\frac{\sqrt{98}}{4} - \frac{5\sqrt{2}}{2}$

e $\frac{2\sqrt{75}}{5} - \frac{3\sqrt{3}}{2}$ **f** $\frac{\sqrt{63}}{9} - \frac{4\sqrt{7}}{5}$ **g** $\frac{2\sqrt{18}}{3} - \frac{\sqrt{72}}{2}$ **h** $\frac{\sqrt{54}}{4} + \frac{\sqrt{24}}{7}$

i $\frac{\sqrt{27}}{5} - \frac{\sqrt{108}}{10}$ **j** $\frac{5\sqrt{48}}{6} + \frac{2\sqrt{147}}{3}$ **k** $\frac{2\sqrt{96}}{5} - \frac{\sqrt{600}}{7}$ **l** $\frac{3\sqrt{125}}{14} - \frac{2\sqrt{80}}{21}$

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3C Multiplying and dividing surds 10A

Learning intentions

- To know how to multiply and divide surds
- To understand that, by definition, $\sqrt{x} \times \sqrt{x}$ is equal to x and that this can be helpful in simplifying multiplications
- To be able to apply the distributive law to brackets involving surds

When simplifying surds such as $\sqrt{18}$, we write $\sqrt{18} = \sqrt{9 \times 2} = \sqrt{9} \times \sqrt{2} = 3\sqrt{2}$, where we use the fact that $\sqrt{xy} = \sqrt{x} \times \sqrt{y}$. This can be used in reverse to simplify the product of two surds. A similar process is used for division.

A surd represents an accurate value until approximated with a decimal. Surveyor training includes solving problems where the trigonometry ratios are expressed as surds because these are exact values and give accurate results.

LESSON STARTER Exploring products and quotients

When adding and subtracting surds we can combine like surds only. Do you think this is true for multiplying and dividing surds?

- Use a calculator to find a decimal approximation for $\sqrt{5} \times \sqrt{3}$ and for $\sqrt{15}$.
- Use a calculator to find a decimal approximation for $2\sqrt{10} \div \sqrt{5}$ and for $2\sqrt{2}$.
- What do you notice about the results from above? Try other pairs of surds to see if your observations are consistent.

KEY IDEAS

- When multiplying surds, use the following result.
  - $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$
  - More generally: $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$
- When dividing surds, use the following result.
  - $\dfrac{\sqrt{x}}{\sqrt{y}} = \sqrt{\dfrac{x}{y}}$
  - More generally: $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$
- Use the distributive law to expand brackets.
  - $a(b + c) = ab + ac$

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BUILDING UNDERSTANDING

1 State the missing parts.

a $\sqrt{15} \div \sqrt{3} = \sqrt{\dfrac{15}{\text{---}}}$
$= \sqrt{\text{---}}$

b $\sqrt{42} \div \sqrt{7} = \sqrt{\dfrac{42}{\text{---}}}$
$= \sqrt{\text{---}}$

c $\sqrt{6} \times \sqrt{5} = \sqrt{6 \times \text{---}}$
$= \sqrt{\text{---}}$

d $\sqrt{11} \times \sqrt{2} = \sqrt{11 \times \text{---}}$
$= \sqrt{\text{---}}$

2 Use the definition of squares and square roots to simplify the following.

a $\sqrt{6} \times \sqrt{6}$ **b** $\sqrt{7^2}$ **c** $(\sqrt{5})^2$

3 Expand the brackets.

a $2(x + 3)$ **b** $5(2x - 1)$ **c** $6(5 - 4x)$

Example 6 Simplifying a product of two surds

Simplify the following.

a $\sqrt{2} \times \sqrt{3}$ **b** $2\sqrt{3} \times 3\sqrt{15}$ **c** $(2\sqrt{5})^2$

SOLUTION

a $\sqrt{2} \times \sqrt{3} = \sqrt{2 \times 3}$
$= \sqrt{6}$

EXPLANATION

Use $\sqrt{x} \times \sqrt{y} = \sqrt{xy}$.

SOLUTION

b $2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3 \times 15}$
$= 6\sqrt{45}$
$= 6\sqrt{9 \times 5}$
$= 6 \times \sqrt{9} \times \sqrt{5}$
$= 18\sqrt{5}$

EXPLANATION

Use $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$.
Then simplify the surd $\sqrt{45}$, which has a highest square factor of 9, using $\sqrt{9} = 3$.
Alternatively, using $\sqrt{15} = \sqrt{3} \times \sqrt{5}$:
$2\sqrt{3} \times 3\sqrt{15} = 2 \times 3 \times \sqrt{3} \times \sqrt{3} \times \sqrt{5}$
$= 2 \times 3 \times 3 \times \sqrt{5}$
$= 18\sqrt{5}$

SOLUTION

c $(2\sqrt{5})^2 = 2\sqrt{5} \times 2\sqrt{5}$
$= 4 \times 5$
$= 20$

EXPLANATION

Recall that $a^2 = a \times a$.
Combine the whole numbers and surd components by multiplying $2 \times 2 = 4$ and $\sqrt{5} \times \sqrt{5} = 5$.

Now you try

Simplify the following.

a $\sqrt{5} \times \sqrt{3}$ **b** $3\sqrt{2} \times 4\sqrt{6}$ **c** $(3\sqrt{7})^2$

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Example 7 Simplifying surds using division

Simplify these surds.

a $-\sqrt{10} \div \sqrt{2}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}}$

SOLUTION

a $-\sqrt{10} \div \sqrt{2} = -\sqrt{\dfrac{10}{2}}$
$= -\sqrt{5}$

b $\dfrac{12\sqrt{18}}{3\sqrt{3}} = \dfrac{12}{3}\sqrt{\dfrac{18}{3}}$
$= 4\sqrt{6}$

EXPLANATION

Use $\sqrt{x} \div \sqrt{y} = \sqrt{\dfrac{x}{y}}$.

Use $\dfrac{a\sqrt{x}}{b\sqrt{y}} = \dfrac{a}{b}\sqrt{\dfrac{x}{y}}$.

Now you try

Simplify these surds.

a $-\sqrt{15} \div \sqrt{5}$

b $\dfrac{14\sqrt{22}}{7\sqrt{11}}$

Example 8 Using the distributive law

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{3}(3\sqrt{5} - \sqrt{6})$

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6})$

SOLUTION

a $\sqrt{3}(3\sqrt{5} - \sqrt{6}) = 3\sqrt{15} - \sqrt{18}$
$= 3\sqrt{15} - \sqrt{9 \times 2}$
$= 3\sqrt{15} - 3\sqrt{2}$

b $3\sqrt{6}(2\sqrt{10} - 4\sqrt{6}) = 6\sqrt{60} - 12 \times 6$
$= 6\sqrt{4 \times 15} - 72$
$= 12\sqrt{15} - 72$

EXPLANATION

Expand the brackets $\sqrt{3} \times 3\sqrt{5} = 3\sqrt{15}$ and $\sqrt{3} \times \sqrt{6} = \sqrt{18}$.
Then simplify $\sqrt{18}$.

Expand the brackets and simplify the surds.
Recall that $\sqrt{6} \times \sqrt{6} = 6$ and $\sqrt{4 \times 15} = 2\sqrt{15}$.

Now you try

Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{2}(5\sqrt{3} - \sqrt{7})$

b $5\sqrt{3}(2\sqrt{6} - 3\sqrt{3})$

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Exercise 3C

FLUENCY	1, 2–8(1/2)	2–8(1/2)	2–8(1/3)

Example 6a, Example 6b, Example 6c

1 Simplify the following.

a i $\sqrt{7}\times\sqrt{6}$ ii $\sqrt{11}\times\sqrt{5}$

b i $2\sqrt{3}\times 5\sqrt{7}$ ii $3\sqrt{7}\times 2\sqrt{5}$

c i $(2\sqrt{3})^2$ ii $(3\sqrt{5})^2$

Example 6a

2 Simplify the following.

a $\sqrt{3}\times\sqrt{5}$ b $\sqrt{7}\times\sqrt{3}$ c $\sqrt{2}\times\sqrt{13}$

d $\sqrt{5}\times\sqrt{7}$ e $\sqrt{2}\times(-\sqrt{15})$ f $-\sqrt{6}\times\sqrt{5}$

g $-\sqrt{6}\times(-\sqrt{11})$ h $-\sqrt{3}\times(-\sqrt{2})$ i $\sqrt{10}\times\sqrt{7}$

Example 7a

3 Simplify the following.

a $\sqrt{20}\div\sqrt{2}$ b $\sqrt{18}\div\sqrt{3}$ c $\sqrt{33}\div(-\sqrt{11})$

d $-\sqrt{30}\div(-\sqrt{6})$ e $\dfrac{\sqrt{15}}{\sqrt{5}}$ f $\dfrac{\sqrt{30}}{\sqrt{3}}$

g $\dfrac{\sqrt{40}}{\sqrt{8}}$ h $-\dfrac{\sqrt{26}}{\sqrt{2}}$ i $-\dfrac{\sqrt{50}}{\sqrt{10}}$

4 Simplify the following, making use of $\sqrt{x}\times\sqrt{x}=x, x\geqslant 0$, in each part.

a $\sqrt{3}\times\sqrt{3}$ b $\sqrt{5}\times\sqrt{5}$ c $\sqrt{9}\times\sqrt{9}$

d $\sqrt{14}\times\sqrt{7}$ e $\sqrt{2}\times\sqrt{22}$ f $\sqrt{3}\times\sqrt{18}$

g $\sqrt{10}\times\sqrt{5}$ h $\sqrt{12}\times\sqrt{8}$ i $\sqrt{5}\times\sqrt{20}$

Example 6b

5 Simplify the following.

a $2\sqrt{5}\times\sqrt{15}$ b $3\sqrt{7}\times\sqrt{14}$ c $4\sqrt{6}\times\sqrt{21}$

d $-5\sqrt{10}\times\sqrt{30}$ e $3\sqrt{6}\times(-\sqrt{18})$ f $5\sqrt{3}\times\sqrt{15}$

g $3\sqrt{14}\times 2\sqrt{21}$ h $-4\sqrt{6}\times 5\sqrt{15}$ i $2\sqrt{10}\times(-2\sqrt{25})$

j $-2\sqrt{7}\times(-3\sqrt{14})$ k $4\sqrt{15}\times 2\sqrt{18}$ l $9\sqrt{12}\times 4\sqrt{21}$

Example 6c

6 Simplify the following.

a $(\sqrt{11})^2$ b $(\sqrt{13})^2$ c $(2\sqrt{3})^2$

d $(5\sqrt{5})^2$ e $(7\sqrt{3})^2$ f $(9\sqrt{2})^2$

Example 7b

7 Simplify the following.

a $\dfrac{6\sqrt{14}}{3\sqrt{7}}$ b $\dfrac{15\sqrt{12}}{5\sqrt{2}}$ c $\dfrac{4\sqrt{30}}{8\sqrt{6}}$

d $-\dfrac{8\sqrt{2}}{2\sqrt{26}}$ e $-\dfrac{3\sqrt{3}}{9\sqrt{21}}$ f $\dfrac{12\sqrt{70}}{18\sqrt{14}}$

Example 8

8 Use the distributive law to expand the following and then simplify the surds where necessary.

a $\sqrt{3}(\sqrt{2}+\sqrt{5})$ b $\sqrt{2}(\sqrt{7}-\sqrt{5})$ c $-\sqrt{5}(\sqrt{11}+\sqrt{13})$

d $-2\sqrt{3}(\sqrt{5}+\sqrt{7})$ e $3\sqrt{2}(2\sqrt{13}-\sqrt{11})$ f $4\sqrt{5}(\sqrt{5}-\sqrt{10})$

g $5\sqrt{3}(2\sqrt{6}+3\sqrt{10})$ h $-2\sqrt{6}(3\sqrt{2}-2\sqrt{3})$ i $3\sqrt{7}(2\sqrt{7}+3\sqrt{14})$

j $6\sqrt{5}(3\sqrt{15}-2\sqrt{8})$ k $-2\sqrt{8}(2\sqrt{2}-3\sqrt{20})$ l $2\sqrt{3}(7\sqrt{6}+5\sqrt{3})$

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PROBLEM-SOLVING	10	9(½), 10	9(½), 10, 11

9 Simplify the following.

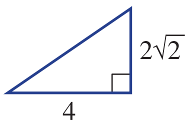
a $(2\sqrt{7})^2$ **b** $(-3\sqrt{2})^2$ **c** $-(5\sqrt{3})^2$

d $\sqrt{2}(3-\sqrt{3})-\sqrt{8}$ **e** $\sqrt{8}(\sqrt{6}+\sqrt{2})-\sqrt{3}$ **f** $\sqrt{5}(\sqrt{2}+1)-\sqrt{40}$

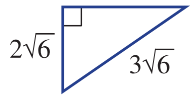
g $\sqrt{44}-2(\sqrt{11}-1)$ **h** $\sqrt{24}-2\sqrt{2}(\sqrt{3}-4)$ **i** $2\sqrt{3}(\sqrt{6}-\sqrt{3})-\sqrt{50}$

10 Determine the unknown side of the following right-angled triangles. Recall that $a^2+b^2=c^2$ for right-angled triangles.

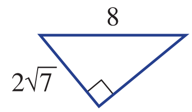
a



b



c



11 **a** The perimeter of a square is $2\sqrt{3}$ cm. Find its area.

b Find the length of a diagonal of a square that has an area of 12 cm^2.

REASONING	12	12, 13	13, 14

12 Use $\sqrt{x}\times\sqrt{y}=\sqrt{xy}$ to prove the following results.

a $\sqrt{6}\times\sqrt{6}=6$ **b** $-\sqrt{8}\times\sqrt{8}=-8$ **c** $-\sqrt{5}\times(-\sqrt{5})=5$

13 $\sqrt{8}\times\sqrt{27}$ could be simplified in two ways, as shown.

Method A

$$\begin{aligned}\sqrt{8}\times\sqrt{27}&=\sqrt{4\times2}\times\sqrt{9\times3}\\&=2\sqrt{2}\times3\sqrt{3}\\&=2\times3\times\sqrt{2\times3}\\&=6\sqrt{6}\end{aligned}$$

Method B

$$\begin{aligned}\sqrt{8}\times\sqrt{27}&=\sqrt{8\times27}\\&=\sqrt{216}\\&=\sqrt{36\times6}\\&=6\sqrt{6}\end{aligned}$$

a Describe the first step in method A.

b Why is it useful to simplify surds before multiplying, as in method A?

c Multiply by first simplifying each surd.

i $\sqrt{18}\times\sqrt{27}$ **ii** $\sqrt{24}\times\sqrt{20}$ **iii** $\sqrt{50}\times\sqrt{45}$

iv $\sqrt{54}\times\sqrt{75}$ **v** $2\sqrt{18}\times\sqrt{48}$ **vi** $\sqrt{108}\times(-2\sqrt{125})$

vii $-4\sqrt{27}\times(-\sqrt{28})$ **viii** $\sqrt{98}\times\sqrt{300}$ **ix** $2\sqrt{72}\times3\sqrt{80}$

14 $\dfrac{\sqrt{12}}{\sqrt{3}}$ could be simplified in two ways.

Method A

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}}&=\sqrt{\frac{12}{3}}\\&=\sqrt{4}\\&=2\end{aligned}$$

Method B

$$\begin{aligned}\frac{\sqrt{12}}{\sqrt{3}}&=\frac{2\cancel{\sqrt{3}}^{1}}{\cancel{\sqrt{3}}^{1}}\\&=2\end{aligned}$$

Choose a method to simplify these surds. Compare your method with that of another student.

a $\dfrac{\sqrt{27}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{20}}{\sqrt{5}}$ **c** $-\dfrac{\sqrt{162}}{\sqrt{2}}$

d $-\dfrac{2\sqrt{2}}{5\sqrt{8}}$ **e** $\dfrac{2\sqrt{45}}{15\sqrt{5}}$ **f** $\dfrac{5\sqrt{27}}{\sqrt{75}}$

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ENRICHMENT: Higher powers	–	–	15–16(1/2)

15 Look at this example before simplifying the following.

$$\begin{aligned}(2\sqrt{3})^3 &= 2^3(\sqrt{3})^3\\ &= 2\times 2\times 2\times \sqrt{3}\times\sqrt{3}\times\sqrt{3}\\ &= 8\times 3\times\sqrt{3}\\ &= 24\sqrt{3}\end{aligned}$$

a $(3\sqrt{2})^3$

b $(5\sqrt{3})^3$

c $2(3\sqrt{3})^3$

d $(\sqrt{5})^4$

e $(-\sqrt{3})^4$

f $(2\sqrt{2})^5$

g $-3(2\sqrt{5})^3$

h $2(-3\sqrt{2})^3$

i $5(2\sqrt{3})^4$

j $\dfrac{(2\sqrt{7})^3}{4}$

k $\dfrac{(3\sqrt{2})^3}{4}$

l $\dfrac{(3\sqrt{2})^4}{4}$

m $\dfrac{(5\sqrt{2})^2}{4}\times\dfrac{(2\sqrt{3})^3}{3}$

n $\dfrac{(2\sqrt{3})^2}{9}\times\dfrac{(-3\sqrt{2})^4}{3}$

o $\dfrac{(2\sqrt{5})^3}{5}\times\dfrac{(-2\sqrt{3})^5}{24}$

p $\dfrac{(3\sqrt{3})^3}{2}\div\dfrac{(5\sqrt{2})^2}{4}$

q $\dfrac{(2\sqrt{5})^4}{50}\div\dfrac{(2\sqrt{3})^3}{5}$

r $\dfrac{(2\sqrt{2})^3}{9}\div\dfrac{(2\sqrt{8})^2}{(\sqrt{27})^3}$

16 Fully expand and simplify these surds.

a $(2\sqrt{3}-\sqrt{2})^2+(\sqrt{3}+\sqrt{2})^2$

b $(\sqrt{5}-\sqrt{3})^2+(\sqrt{5}+\sqrt{3})^2$

c $(\sqrt{3}-4\sqrt{5})(\sqrt{3}+4\sqrt{5})-(\sqrt{3}-\sqrt{5})^2$

d $-10\sqrt{3}-(2\sqrt{3}-5)^2$

e $(\sqrt{3}-2\sqrt{6})^2+(1+\sqrt{2})^2$

f $(2\sqrt{7}-3)^2-(3-2\sqrt{7})^2$

g $(2\sqrt{3}-3\sqrt{2})(2\sqrt{3}+3\sqrt{2})-(\sqrt{6}-\sqrt{2})^2$

h $\sqrt{2}(2\sqrt{5}-3\sqrt{3})^2+(\sqrt{6}+\sqrt{5})^2$

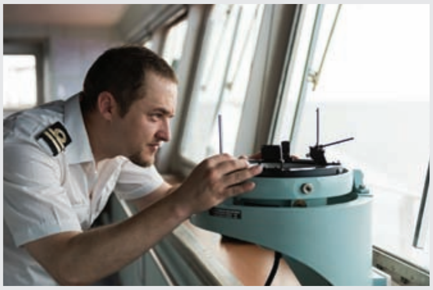
ISBN 978-1-108-77290-7

3D Rationalising the denominator 10A

Learning intentions

- To understand that a surd multiplied by itself gives a whole number
- To know that rationalising the denominator refers to converting an irrational denominator to one that is rational
- To be able to rationalise the denominator

As you know, it is easier to add or subtract fractions when the fractions are expressed with the same denominator. In a similar way, it is easier to work with surds such as $\frac{1}{\sqrt{2}}$ and $\frac{\sqrt{3}-1}{\sqrt{5}}$ when they are expressed using a whole number in the denominator. The process that removes a surd from the denominator is called 'rationalising the denominator' because the denominator is being converted from an irrational number to a rational number.



Working through a problem using surds provides exact value solutions. Navigation training uses surd manipulation to solve problems of speed and direction, applying Pythagoras' theorem and trigonometry.

LESSON STARTER What do I multiply by?

When trying to rationalise the denominator in a surd like $\frac{1}{\sqrt{2}}$, you must multiply the surd by a chosen number so that the denominator is converted to a whole number.

- First, decide what each of the following is equivalent to.
  a $\frac{\sqrt{3}}{\sqrt{3}}$ **b** $\frac{\sqrt{2}}{\sqrt{2}}$ **c** $\frac{\sqrt{21}}{\sqrt{21}}$
- Recall that $\sqrt{x} \times \sqrt{x} = x$ and simplify the following.
  a $\sqrt{5} \times \sqrt{5}$ **b** $2\sqrt{3} \times \sqrt{3}$ **c** $4\sqrt{7} \times \sqrt{7}$
- Now, decide what you can multiply $\frac{1}{\sqrt{2}}$ by so that:
  - the value of $\frac{1}{\sqrt{2}}$ does not change, and
  - the denominator becomes a whole number.
- Repeat this for:
  a $\frac{1}{\sqrt{5}}$ **b** $\frac{3}{2\sqrt{3}}$

KEY IDEAS

- **Rationalising a denominator** involves multiplying by a number equivalent to 1, which changes the denominator to a whole number.
  $$\frac{x}{\sqrt{y}} = \frac{x}{\sqrt{y}} \times \frac{\sqrt{y}}{\sqrt{y}} = \frac{x\sqrt{y}}{y}$$

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 Simplify.

a $\frac{\sqrt{6}}{\sqrt{6}}$ b $\frac{2\sqrt{5}}{4\sqrt{5}}$ c $-\frac{\sqrt{8}}{\sqrt{2}}$ d $\frac{\sqrt{72}}{\sqrt{2}}$

2 State the missing number.

a $\sqrt{3} \times \_\_ = 3$ b $\sqrt{10} \times \sqrt{10} = \_\_$ c $2\sqrt{5} \times \_\_ = 10$ d $\_\_ \times 3\sqrt{7} = 21$

3 Use a calculator to find a decimal approximation to each number in the following pairs of numbers. What do you notice?

a $\frac{1}{\sqrt{7}}, \frac{\sqrt{7}}{7}$ b $\frac{5}{\sqrt{3}}, \frac{5\sqrt{3}}{3}$ c $\frac{11\sqrt{11}}{\sqrt{5}}, \frac{11\sqrt{55}}{5}$

Example 9 Rationalising the denominator

Rationalise the denominator in the following.

a $\frac{2}{\sqrt{3}}$ b $\frac{3\sqrt{2}}{\sqrt{5}}$ c $\frac{2\sqrt{7}}{5\sqrt{2}}$ d $\frac{1-\sqrt{3}}{\sqrt{3}}$

SOLUTION

a
$$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{2\sqrt{3}}{3}$$

EXPLANATION

Choose the appropriate fraction equivalent to 1 to multiply by. In this case, choose $\frac{\sqrt{3}}{\sqrt{3}}$ since $\sqrt{3} \times \sqrt{3} = 3$.

b
$$\frac{3\sqrt{2}}{\sqrt{5}} = \frac{3\sqrt{2}}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$$
$$= \frac{3\sqrt{10}}{5}$$

Choose the appropriate fraction. In this case, use $\frac{\sqrt{5}}{\sqrt{5}}$ since $\sqrt{5} \times \sqrt{5} = 5$. Recall $\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5} = \sqrt{10}$.

c
$$\frac{2\sqrt{7}}{5\sqrt{2}} = \frac{2\sqrt{7}}{5\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$
$$= \frac{\cancel{2}^{1}\sqrt{14}}{\cancel{10}^{5}}$$
$$= \frac{\sqrt{14}}{5}$$

Choose the appropriate fraction; i.e. $\frac{\sqrt{2}}{\sqrt{2}}$.
$5 \times \sqrt{2} \times \sqrt{2} = 5 \times 2 = 10$
Cancel the common factor of 2.

d
$$\frac{1-\sqrt{3}}{\sqrt{3}} = \frac{1-\sqrt{3}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{\sqrt{3}-3}{3}$$

Expand using the distributive law:
$(1-\sqrt{3}) \times \sqrt{3} = 1 \times \sqrt{3} - \sqrt{3} \times \sqrt{3} = \sqrt{3} - 3$

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Now you try

Rationalise the denominator in the following.

a $\frac{3}{\sqrt{2}}$ b $\frac{4\sqrt{3}}{\sqrt{7}}$ c $\frac{2\sqrt{5}}{3\sqrt{2}}$ d $\frac{2-\sqrt{7}}{\sqrt{7}}$

Exercise 3D

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

1 Rationalise the denominator in the following.

Example 9a a i $\frac{3}{\sqrt{5}}$ ii $\frac{7}{\sqrt{6}}$

Example 9b b i $\frac{2\sqrt{3}}{\sqrt{5}}$ ii $\frac{3\sqrt{5}}{\sqrt{2}}$

Example 9c c i $\frac{3\sqrt{7}}{5\sqrt{3}}$ ii $\frac{2\sqrt{5}}{7\sqrt{6}}$

Example 9d d i $\frac{1-\sqrt{2}}{\sqrt{2}}$ ii $\frac{4-\sqrt{5}}{\sqrt{5}}$

Example 9a

2 Rationalise the denominators.

a $\frac{1}{\sqrt{2}}$ b $\frac{1}{\sqrt{7}}$ c $\frac{3}{\sqrt{11}}$ d $\frac{4}{\sqrt{5}}$

e $\frac{5}{\sqrt{3}}$ f $\frac{8}{\sqrt{2}}$ g $\frac{\sqrt{5}}{\sqrt{3}}$ h $\frac{\sqrt{2}}{\sqrt{7}}$

3 Rewrite each of the following in the form $\frac{\sqrt{a}}{\sqrt{b}}$ and then rationalise the denominators.

a $\sqrt{\frac{2}{3}}$ b $\sqrt{\frac{5}{7}}$ c $\sqrt{\frac{6}{11}}$ d $\sqrt{\frac{2}{5}}$

e $\sqrt{\frac{7}{3}}$ f $\sqrt{\frac{6}{7}}$ g $\sqrt{\frac{10}{3}}$ h $\sqrt{\frac{17}{2}}$

Example 9b

4 Rationalise the denominators.

a $\frac{4\sqrt{2}}{\sqrt{7}}$ b $\frac{5\sqrt{2}}{\sqrt{3}}$ c $\frac{3\sqrt{5}}{\sqrt{2}}$

d $\frac{3\sqrt{6}}{\sqrt{7}}$ e $\frac{7\sqrt{3}}{\sqrt{10}}$ f $\frac{2\sqrt{7}}{\sqrt{15}}$

Example 9c

5 Rationalise the denominators.

a $\frac{4\sqrt{7}}{5\sqrt{3}}$ b $\frac{2\sqrt{3}}{3\sqrt{2}}$ c $\frac{5\sqrt{7}}{3\sqrt{5}}$ d $\frac{4\sqrt{5}}{5\sqrt{10}}$

e $\frac{2\sqrt{7}}{3\sqrt{35}}$ f $\frac{5\sqrt{12}}{3\sqrt{27}}$ g $\frac{9\sqrt{6}}{2\sqrt{3}}$ h $\frac{7\sqrt{90}}{2\sqrt{70}}$

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PROBLEM-SOLVING	7	6(1/2), 7	6–8(1/3)

Example 9d

6 Rationalise the denominators.

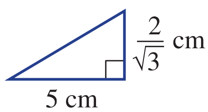
a $\frac{1 + \sqrt{2}}{\sqrt{3}}$ **b** $\frac{3 + \sqrt{5}}{\sqrt{7}}$ **c** $\frac{2 - \sqrt{3}}{\sqrt{5}}$

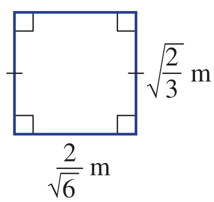
d $\frac{\sqrt{3} - \sqrt{5}}{\sqrt{2}}$ **e** $\frac{\sqrt{5} + \sqrt{2}}{\sqrt{7}}$ **f** $\frac{\sqrt{10} - \sqrt{7}}{\sqrt{3}}$

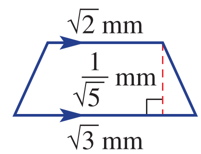
g $\frac{\sqrt{2} + \sqrt{7}}{\sqrt{6}}$ **h** $\frac{\sqrt{5} + \sqrt{2}}{\sqrt{10}}$ **i** $\frac{\sqrt{6} - \sqrt{10}}{\sqrt{5}}$

j $\frac{4\sqrt{2} - 5\sqrt{3}}{\sqrt{6}}$ **k** $\frac{3\sqrt{5} + 5\sqrt{2}}{\sqrt{10}}$ **l** $\frac{3\sqrt{10} + 5\sqrt{3}}{\sqrt{2}}$

7 Determine the exact value of the area of the following shapes. Express your answers using a rational denominator.

a


b


c


8 Simplify the following by first rationalising denominators and then using a common denominator.

a $\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{2}}$ **b** $\frac{3}{\sqrt{5}} + \frac{1}{\sqrt{2}}$ **c** $\frac{3}{\sqrt{7}} - \frac{2}{\sqrt{3}}$

d $\frac{5}{2\sqrt{3}} - \frac{2}{3\sqrt{2}}$ **e** $\frac{1}{3\sqrt{2}} + \frac{5}{4\sqrt{3}}$ **f** $\frac{3}{2\sqrt{5}} + \frac{2}{5\sqrt{3}}$

g $\frac{7\sqrt{2}}{5\sqrt{7}} - \frac{2\sqrt{7}}{3\sqrt{2}}$ **h** $\frac{10\sqrt{6}}{3\sqrt{5}} + \frac{4\sqrt{2}}{3\sqrt{3}}$ **i** $\frac{5\sqrt{2}}{3\sqrt{5}} - \frac{4\sqrt{7}}{3\sqrt{6}}$

REASONING	9	9, 10(1/2)	10(1/3), 11

9 Explain why multiplying a number by $\frac{\sqrt{x}}{\sqrt{x}}$ does not change its value.

10 Rationalise the denominators and simplify the following.

a $\frac{\sqrt{3} + a}{\sqrt{7}}$ **b** $\frac{\sqrt{6} + a}{\sqrt{5}}$ **c** $\frac{\sqrt{2} + a}{\sqrt{6}}$

d $\frac{\sqrt{3} - 3a}{\sqrt{3}}$ **e** $\frac{\sqrt{5} - 5a}{\sqrt{5}}$ **f** $\frac{\sqrt{7} - 7a}{\sqrt{7}}$

g $\frac{4a + \sqrt{5}}{\sqrt{10}}$ **h** $\frac{3a + \sqrt{3}}{\sqrt{6}}$ **i** $\frac{2a + \sqrt{7}}{\sqrt{14}}$

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11 To explore how to simplify a number such as $\frac{3}{4-\sqrt{2}}$, first answer these questions.

a Simplify.

i $(4-\sqrt{2})(4+\sqrt{2})$ **ii** $(3-\sqrt{7})(3+\sqrt{7})$ **iii** $(5\sqrt{2}-\sqrt{3})(5\sqrt{2}+\sqrt{3})$

b What do you notice about each question and answer in part **a** above?

c Now decide what to multiply $\frac{3}{4-\sqrt{2}}$ by to rationalise the denominator.

d Rationalise the denominator in these expressions.

i $\frac{3}{4-\sqrt{2}}$ **ii** $\frac{-3}{\sqrt{3}-1}$ **iii** $\frac{\sqrt{2}}{\sqrt{4}-\sqrt{3}}$ **iv** $\frac{2\sqrt{6}}{\sqrt{6}-2\sqrt{5}}$

ENRICHMENT: Binomial denominators	–	–	12(1/2)

12 Rationalise the denominators in the following by forming a 'difference of two perfect squares'.

For example:
$$\begin{aligned}\frac{2}{\sqrt{2}+1} &= \frac{2}{\sqrt{2}+1}\times\frac{\sqrt{2}-1}{\sqrt{2}-1}\\ &= \frac{2(\sqrt{2}-1)}{(\sqrt{2}+1)(\sqrt{2}-1)}\\ &= \frac{2\sqrt{2}-2}{2-1}\\ &= 2\sqrt{2}-2\end{aligned}$$

a $\frac{5}{\sqrt{3}+1}$ **b** $\frac{4}{\sqrt{3}-1}$ **c** $\frac{3}{\sqrt{5}-2}$ **d** $\frac{4}{1-\sqrt{2}}$

e $\frac{3}{1-\sqrt{3}}$ **f** $\frac{7}{6-\sqrt{7}}$ **g** $\frac{4}{3-\sqrt{10}}$ **h** $\frac{7}{2-\sqrt{5}}$

i $\frac{2}{\sqrt{11}-\sqrt{2}}$ **j** $\frac{6}{\sqrt{2}+\sqrt{5}}$ **k** $\frac{4}{\sqrt{3}+\sqrt{7}}$ **l** $\frac{\sqrt{2}}{\sqrt{7}+1}$

m $\frac{\sqrt{6}}{\sqrt{6}-1}$ **n** $\frac{3\sqrt{2}}{\sqrt{7}-2}$ **o** $\frac{2\sqrt{5}}{\sqrt{5}+2}$ **p** $\frac{b}{\sqrt{a}+\sqrt{b}}$

q $\frac{a}{\sqrt{a}-\sqrt{b}}$ **r** $\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$ **s** $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{b}}$ **t** $\frac{\sqrt{ab}}{\sqrt{a}-\sqrt{b}}$

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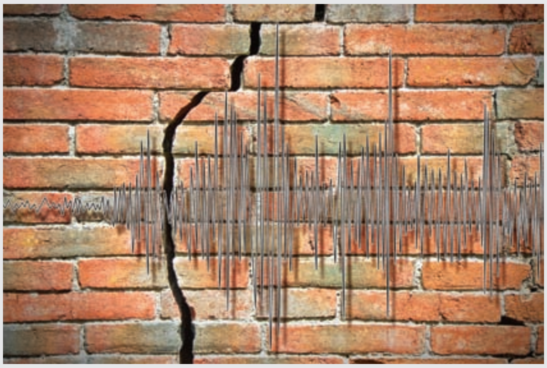
3E Review of index laws CONSOLIDATING

Learning intentions

- To know that powers are used as a short hand way of writing repeated multiplications
- To understand that index laws for multiplication and division apply only to common bases
- To know how to combine powers with the same base under multiplication and division
- To know how to apply powers where brackets are involved
- To know that any number (except 0) to the power of zero is equal to 1
- To be able to combine a number of index laws to simplify an expression

From your work in Year 9 you will recall that powers (i.e numbers with indices) can be used to represent repeated multiplication of the same factor. For example, $2 \times 2 \times 2 = 2^3$ and $5 \times x \times x \times x \times x = 5x^4$. The five basic index laws and the zero power will be revised in this section.



Index laws efficiently simplify powers of a base. Powers of 2 calculate the size of digital data and bacterial populations, and powers of 10 are used when calculating earthquake and sound level intensities.

LESSON STARTER Recall the laws

Try to recall how to simplify each expression and use words to describe the index law used.

- $5^3 \times 5^7$
- $x^4 \div x^2$
- $(a^7)^2$
- $(2a)^3$
- $\left(\dfrac{x}{3}\right)^4$
- $(4x^2)^0$

KEY IDEAS

- Recall that $a = a^1$ and $5a = 5^1 \times a^1$.
- The index laws
  - Law 1: $a^m \times a^n = a^{m+n}$ Retain the base and add the indices.
  - Law 2: $a^m \div a^n = \dfrac{a^m}{a^n} = a^{m-n}$ Retain the base and subtract the indices.
  - Law 3: $(a^m)^n = a^{m \times n}$ Retain the base and multiply the indices.
  - Law 4: $(a \times b)^m = a^m \times b^m$ Distribute the index number across the bases.
  - Law 5: $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{b^m}$ Distribute the index number across the bases.
- The zero power: $a^0 = 1$ Any number (except 0) to the power of zero is equal to 1.

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BUILDING UNDERSTANDING

1 Simplify, using index form.

a $3 \times 3 \times 3 \times 3$

b $7 \times 7 \times 7 \times 7 \times 7 \times 7$

c $2 \times x \times x \times 3 \times x$

d $2 \times b \times a \times 4 \times b \times a \times a$

2 State the missing parts to this table.

x	4	3	2	1	0
2^x		$2^3 = 8$			

3 State the missing components.

a $2^2 \times 2^3 = 2 \times 2 \times$ ______
$= 2^{--}$

b $\frac{x^5}{x^3} = \frac{x \times x \times x \times \_\_\_\_\_\_\_\_\_\_}{\_\_\_\_\_\_\_\_\_\_}$
$= x^{--}$

c $(a^2)^3 = a \times a$______ $\times$ ______
$= a^{--}$

d $(2x)^0 \times 2x^0 =$ ____ $\times$ ____
$= 2$

Example 10 Using index law 1

Simplify the following using the first index law.

a $x^5 \times x^4$

b $3a^2b \times 4ab^3$

SOLUTION

a $x^5 \times x^4 = x^9$

b $3a^2b \times 4ab^3 = 12a^3b^4$

EXPLANATION

There is a common base of x, so add the indices.

Multiply coefficients and add indices for each base a and b. Recall that $a = a^1$.

Now you try

Simplify the following using the first index law.

a $x^3 \times x^4$

b $2ab^2 \times 7a^2b^3$

Example 11 Using index law 2

Simplify the following using the second index law.

a $m^7 \div m^5$

b $4x^2y^5 \div (8xy^2)$

SOLUTION

a $m^7 \div m^5 = m^2$

EXPLANATION

Subtract the indices when dividing terms with the same base.

Continued on next page

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b $4x^2y^5 \div (8xy^2) = \dfrac{4x^2y^5}{8xy^2}$

$= \dfrac{xy^3}{2}$

$= \dfrac{1}{2}xy^3$

First, express as a fraction.
Divide the coefficients and subtract the indices of x and y (i.e. $x^{2-1}\,y^{5-2}$).

Now you try

Simplify the following using the second index law.

a $m^5 \div m^3$

b $5x^2y^4 \div (10xy^2)$

Example 12 Combining index laws

Simplify the following using the index laws.

a $(a^3)^4$

b $(2y^5)^3$

c $\left(\dfrac{3x^2}{5y^2z}\right)^3$

d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y}$

SOLUTION

a $(a^3)^4 = a^{12}$

b $(2y^5)^3 = 2^3y^{15}$
$= 8y^{15}$

c $\left(\dfrac{3x^2}{5y^2z}\right)^3 = \dfrac{3^3x^6}{5^3y^6z^3}$
$= \dfrac{27x^6}{125y^6z^3}$

d $\dfrac{3(xy^2)^3 \times 4x^4y^2}{8x^2y} = \dfrac{3x^3y^6 \times 4x^4y^2}{8x^2y}$
$= \dfrac{12x^7y^8}{8x^2y}$
$= \dfrac{3x^5y^7}{2}$

EXPLANATION

a Use index law 3 and multiply the indices.

b Use index law 4 and multiply the indices for each base 2 and y. Note: $2 = 2^1$.

c Raise the coefficients to the power 3 and multiply the indices of each base by 3.

d Remove brackets first by multiplying the indices for each base.
Simplify the numerator using index law 1.

Simplify the fraction using index law 2, subtracting indices of the same base.

Now you try

Simplify the following using the index laws.

a $(a^2)^3$

b $(3y^3)^3$

c $\left(\dfrac{2x^3}{7yz^2}\right)^2$

d $\dfrac{4(x^2y)^3 \times 2xy^2}{2x^2y}$

ISBN 978-1-108-77290-7

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Example 13 Using the zero power

Evaluate, using the zero power.

a $4a^0$

b $2p^0 + (3p)^0$

SOLUTION

a $4a^0 = 4 \times 1$
$= 4$

b $2p^0 + (3p)^0 = 2 \times 1 + 1$
$= 3$

EXPLANATION

Any number to the power of zero is equal to 1.

Note: $(3p)^0$ is not the same as $3p^0$.

Now you try

Evaluate, using the zero power.

a $2a^0$

b $5p^0 + (7p)^0$

Exercise 3E

FLUENCY	1, 2–5(1/2)	2–5(1/3)	2–5(1/4)

1 Simplify the following using the first index law.

Example 10a

a **i** $x^2 \times x^5$ **ii** $x^6 \times x^2$

Example 10b

b **i** $5a^2b \times 2ab^2$ **ii** $4a^3b \times 3ab^4$

Example 10

2 Simplify, using the first index law.

a $a^5 \times a^4$ **b** $x^3 \times x^2$ **c** $b \times b^5$

d $7m^2 \times 2m^3$ **e** $2s^4 \times 3s^3$ **f** $t^8 \times 2t^8$

g $\frac{1}{5}p^2 \times p$ **h** $\frac{1}{4}c^4 \times \frac{2}{3}c^3$ **i** $\frac{3}{5}s \times \frac{3s}{5}$

j $2x^2y \times 3xy^2$ **k** $3a^2b \times 5ab^5$ **l** $3v^7w \times 6v^2w$

m $3x^4 \times 5xy^2 \times 10y^4$ **n** $2rs^3 \times 3r^4s \times 2r^2s^2$ **o** $4m^6n^7 \times mn \times 5mn^2$

Example 11

3 Simplify, using the second index law.

a $x^5 \div x^2$ **b** $a^7 \div a^6$ **c** $q^9 \div q^6$

d $b^5 \div b$ **e** $\frac{y^8}{y^3}$ **f** $\frac{d^8}{d^3}$

g $\frac{j^7}{j^6}$ **h** $\frac{m^{15}}{m^9}$ **i** $2x^2y^3 \div x$

j $3r^5s^2 \div (r^3s)$ **k** $6p^4q^2 \div (3q^2p^2)$ **l** $16m^7x^5 \div (8m^3x^4)$

m $\frac{5a^2b^4}{a^2b}$ **n** $\frac{8st^4}{2t^3}$ **o** $\frac{2v^5}{8v^3}$

p $\frac{7a^2b}{14ab}$ **q** $\frac{-3x^4y}{9x^3y}$ **r** $\frac{-8x^2y^3}{16x^2y}$

Example 12a–c

4 Simplify using the third, fourth and fifth index laws.

a $(x^5)^2$ **b** $(t^3)^2$ **c** $4(a^2)^3$ **d** $5(y^5)^3$

e $(4t^2)^3$ **f** $(2u^2)^2$ **g** $(3r^3)^3$ **h** $(3p^4)^4$

i $\left(\dfrac{a^2}{b^3}\right)^2$ **j** $\left(\dfrac{x^3}{y^4}\right)^3$ **k** $\left(\dfrac{x^2y^3}{z^4}\right)^2$ **l** $\left(\dfrac{u^4w^2}{v^2}\right)^4$

m $\left(\dfrac{3f^2}{5g}\right)^3$ **n** $\left(\dfrac{3a^2b}{2pq^3}\right)^2$ **o** $\left(\dfrac{at^3}{3g^4}\right)^3$ **p** $\left(\dfrac{4p^2q^3}{3r}\right)^4$

Example 13

5 Evaluate the following using the zero power.

a $8x^0$ **b** $3t^0$ **c** $(5z)^0$ **d** $(10ab^2)^0$

e $5(g^3h^3)^0$ **f** $8x^0 - 5$ **g** $4b^0 - 9$ **h** $7x^0 - 4(2y)^0$

PROBLEM-SOLVING	6(1/2), 7	6(1/2), 7	6(1/3), 7, 8

Example 12d

6 Use appropriate index laws to simplify the following.

a $x^6 \times x^5 \div x^3$ **b** $x^2y \div (xy) \times xy^2$

c $x^4n^7 \times x^3n^2 \div (xn)$ **d** $\dfrac{x^2y^3 \times x^2y^4}{x^3y^5}$

e $\dfrac{m^2w \times m^3w^2}{m^4w^3}$ **f** $\dfrac{r^4s^7 \times r^4s^7}{r^4s^7}$

g $\dfrac{9x^2y^3 \times 6x^7y^5}{12xy^6}$ **h** $\dfrac{4x^2y^3 \times 12x^2y^2}{24x^4y}$

i $\dfrac{16a^8b \times 4ab^7}{32a^7b^6}$ **j** $(3m^2n^4)^3 \times mn^2$

k $-5(a^2b)^3 \times (3ab)^2$ **l** $(4f^2g)^2 \times f^2g^4 \div \left(3(fg^2)^3\right)$

m $\dfrac{4m^2n \times 3(m^2n)^3}{6m^2n}$ **n** $\dfrac{(7y^2z)^2 \times 3yz^2}{7(yz)^2}$

o $\dfrac{2(ab)^2 \times (2a^2b)^3}{4ab^2 \times 4a^7b^3}$ **p** $\dfrac{(2m^3)^2}{3(mn^4)^0} \times \dfrac{(6n^5)^2}{(-2n)^3m^4}$

7 Simplify.

a $(-3)^3$ **b** $-(3)^3$ **c** $(-3)^4$ **d** -3^4

8 Simplify.

a $\left((x^2)^3\right)^2$ **b** $\left((a^5)^3\right)^7$ **c** $\left(\left(\dfrac{a^2}{b}\right)^3\right)^5$

REASONING	9(1/2)	9(1/2), 10	9(1/3), 10, 11

9 Evaluate without the use of a calculator.

a $\dfrac{13^3}{13^2}$ **b** $\dfrac{18^7}{18^6}$ **c** $\dfrac{9^8}{9^6}$ **d** $\dfrac{4^{10}}{4^7}$

e $\dfrac{25^2}{5^4}$ **f** $\dfrac{36^2}{6^4}$ **g** $\dfrac{27^2}{3^4}$ **h** $\dfrac{32^2}{2^7}$

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10 When Billy uses a calculator to raise -2 to the power 4 he gets -16 when the answer is actually 16. What has he done wrong?

11 Find the value of a in these equations in which the index is unknown.

a $2^a = 8$ **b** $3^a = 81$ **c** $2^{a+1} = 4$

d $(-3)^a = -27$ **e** $(-5)^a = 625$ **f** $(-4)^{a-1} = 1$

ENRICHMENT: Indices in equations	–	–	12–14(1/2)

12 If $x^4 = 3$, find the value of:

a x^8 **b** $x^4 - 1$ **c** $2x^{16}$ **d** $3x^4 - 3x^8$

13 Find the value(s) of x.

a $x^4 = 16$ **b** $2^{x-1} = 16$ **c** $2^{2x} = 16$ **d** $2^{2x-3} = 16$

14 Find the possible pairs of positive integers for x and y when:

a $x^y = 16$ **b** $x^y = 64$ **c** $x^y = 81$ **d** $x^y = 1$

ISBN 978-1-108-77290-7

3F Negative indices

Learning intentions
- To understand how a negative power relates to division
- To know how to rewrite expressions involving negative indices with positive indices
- To be able to apply index laws to expressions involving negative indices

The study of indices can be extended to include negative powers. Using the second index law and the fact that $a^0 = 1$, we can establish rules for negative powers.

$$a^0 \div a^n = a^{0-n} \text{ (index law 2)} \qquad \text{also} \qquad a^0 \div a^n = 1 \div a^n \text{ (as } a^0 = 1)$$
$$= a^{-n} \qquad\qquad\qquad\qquad = \frac{1}{a^n}$$

Therefore: $a^{-n} = \dfrac{1}{a^n}$

Also: $\dfrac{1}{a^{-n}} = 1 \div a^{-n}$
$$= 1 \div \frac{1}{a^n}$$
$$= 1 \times \frac{a^n}{1}$$
$$= a^n$$

Therefore: $\dfrac{1}{a^{-n}} = a^n$.

A half-life is the time taken for radioactive material to halve in size. Calculations of the quantity remaining after multiple halving use negative powers of 2. Applications include radioactive waste management and diagnostic medicine.

LESSON STARTER The disappearing bank balance

Due to fees, an initial bank balance of $64 is halved every month.

Balance ($)	64	32	16	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
Positive indices only	2^6	2^5							$\frac{1}{2^2}$	
Positive and negative indices	2^6		2^4					2^{-1}		

- Copy and complete the table and continue each pattern.
- Discuss the differences in the way indices are used at the end of the rows.
- What would be a way of writing $\frac{1}{16}$ using positive indices?
- What would be a way of writing $\frac{1}{16}$ using negative indices?

KEY IDEAS

- $a^{-m} = \dfrac{1}{a^m}$ For example, $2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}$.
- $\dfrac{1}{a^{-m}} = a^m$ For example, $\dfrac{1}{2^{-3}} = 2^3 = 8$.

BUILDING UNDERSTANDING

1. State the next three terms in these patterns.
   a $2^3, 2^2, 2^1, 2^0, 2^{-1}$, ___, ___, ___
   b x^2, x^1, x^0, ___, ___, ___
2. Recall that $\dfrac{1}{4} = \dfrac{1}{2^2}$. Similarly, state these fractions using positive indices.
   a $\dfrac{1}{9}$ b $\dfrac{1}{25}$ c $\dfrac{5}{16}$ d $-\dfrac{2}{27}$
3. State each rule for negative indices by completing each statement.
   a $a^{-b} =$ _______ b $\dfrac{1}{a^{-b}} =$ _______
4. Express the following with positive indices using $a^{-m} = \dfrac{1}{a^m}$ and evaluate.
   a 5^{-2} b 3^{-3} c 4×7^{-2}

Example 14 Writing expressions using positive indices

Express each of the following using positive indices.

a b^{-4} b $3x^{-4}y^2$ c $\dfrac{5}{x^{-3}}$

SOLUTION	EXPLANATION
a $b^{-4} = \dfrac{1}{b^4}$	Use $a^{-n} = \dfrac{1}{a^n}$.
b $3x^{-4}y^2 = \dfrac{3y^2}{x^4}$	x is the only base with a negative power. $\dfrac{3}{1} \times \dfrac{1}{x^4} \times \dfrac{y^2}{1} = \dfrac{3y^2}{x^4}$.
c $\dfrac{5}{x^{-3}} = 5 \times x^3$ $= 5x^3$	Use $\dfrac{1}{a^{-n}} = a^n$ and note that $\dfrac{5}{x^{-3}} = 5 \times \dfrac{1}{x^{-3}}$.

Now you try

Express each of the following using positive indices.

a b^{-3} b $2x^{-2}y^3$ c $\dfrac{2}{x^{-4}}$

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Example 15 Using index laws with negative indices

Simplify the following expressing answers using positive indices.

a $\dfrac{2a^3b^2}{a^5b^3}$ **b** $\dfrac{4m^{-2}n^3}{8m^5n^{-4}}$

SOLUTION

a $\dfrac{2a^3b^2}{a^5b^3} = 2a^{-2}b^{-1}$

$= \dfrac{2}{a^2b}$

b $\dfrac{4m^{-2}n^3}{8m^5n^{-4}} = \dfrac{{}^{1}\cancel{4}m^{-7}n^7}{\cancel{8}^{2}}$

$= \dfrac{n^7}{2m^7}$

EXPLANATION

Use index law 2 to subtract powers with common base a^{3-5} and b^{2-3}.

Express with positive powers $\dfrac{2}{1} \times \dfrac{1}{a^2} \times \dfrac{1}{b}$.

Cancel common factor of 4 and subtract powers m^{-2-5} and $n^{3-(-4)}$.

Express with positive powers $\dfrac{1}{2} \times \dfrac{1}{m^7} \times \dfrac{n^7}{1}$.

Now you try

Simplify the following expressing answers using positive indices.

a $\dfrac{5a^2b^3}{a^4b^7}$ **b** $\dfrac{3m^{-3}n^6}{9m^4n^{-2}}$

Example 16 Simplifying more complex expressions

Simplify the following and express your answers using positive indices.

a $\dfrac{(p^{-2}q)^4}{5p^{-1}q^3} \times \left(\dfrac{p^{-2}}{q^3}\right)^{-3}$ **b** $\left(\dfrac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\dfrac{5m^{-2}n^3}{r}\right)^2$

SOLUTION

a $\dfrac{(p^{-2}q)^4}{5p^{-1}q^3} \times \left(\dfrac{p^{-2}}{q^3}\right)^{-3} = \dfrac{p^{-8}q^4}{5p^{-1}q^3} \times \dfrac{p^6}{q^{-9}}$

$= \dfrac{p^{-2}q^4}{5p^{-1}q^{-6}}$

$= \dfrac{p^{-1}q^{10}}{5}$

$= \dfrac{q^{10}}{5p}$

EXPLANATION

Deal with brackets first by multiplying the power to each of the indices within the brackets.

Use index laws 1 and 2 to combine indices of like bases. Simplify each numerator and denominator first: $p^{-8+6} = p^{-2}$ and $q^{3+(-9)} = q^{-6}$.

Then $p^{-2-(-1)}q^{4-(-6)} = p^{-1}q^{10}$.

Use $a^{-n} = \dfrac{1}{a^n}$ to express p^{-1} with a positive index.

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b $\left(\dfrac{2m^3}{r^2n^{-4}}\right)^3 \div \left(\dfrac{5m^{-2}n^3}{r}\right)^2 = \dfrac{2^3m^9}{r^6n^{-12}} \div \dfrac{5^2m^{-4}n^6}{r^2}$

Multiply the bracket power to each of the indices within the bracket.

$= \dfrac{8m^9}{r^6n^{-12}} \times \dfrac{r^2}{25m^{-4}n^6}$

Multiply by the reciprocal of the divisor.

$= \dfrac{8m^{13}r^{-4}}{25n^{-6}}$

Use index laws 1 and 2 to combine indices of like bases.

$= \dfrac{8m^{13}n^6}{25r^4}$

Write the answer with positive powers.

Now you try

Simplify the following and express your answers using positive indices.

a $\dfrac{(p^{-1}q)^3}{2p^{-2}q^2} \times \left(\dfrac{p^{-2}}{q^2}\right)^{-1}$ **b** $\left(\dfrac{2m^2}{r^4n^{-3}}\right)^2 \div \left(\dfrac{4m^{-1}n^2}{r}\right)^3$

Exercise 3F

FLUENCY	1, 2–5(1/2)	2–6(1/3)	2–6(1/4)

1 Express each of the following using positive indices.

Example 14a — **a** **i** b^{-2} **ii** a^{-5}

Example 14b — **b** **i** $4x^{-1}y^3$ **ii** $7x^{-3}y^2$

Example 14c — **c** **i** $\dfrac{2}{x^{-2}}$ **ii** $\dfrac{6}{x^{-7}}$

Example 14a,b

2 Express the following using positive indices.

a x^{-5} **b** a^{-4} **c** $2m^{-4}$ **d** $3y^{-7}$

e $3a^2b^{-3}$ **f** $4m^3n^{-3}$ **g** $10x^{-2}y^5z$ **h** $3x^{-4}y^{-2}z^3$

i $\dfrac{1}{3}p^{-2}q^3r$ **j** $\dfrac{1}{5}d^2e^{-4}f^5$ **k** $\dfrac{3}{8}u^2v^{-6}w^7$ **l** $\dfrac{2}{5}b^3c^{-5}d^{-2}$

Example 14c

3 Express the following using positive indices.

a $\dfrac{1}{x^{-2}}$ **b** $\dfrac{2}{y^{-3}}$ **c** $\dfrac{4}{m^{-7}}$ **d** $\dfrac{3}{b^{-5}}$

e $\dfrac{2b^4}{d^{-3}}$ **f** $\dfrac{3m^2}{n^{-4}}$ **g** $\dfrac{4b^4}{3a^{-3}}$ **h** $\dfrac{5h^3}{2g^{-3}}$

Example 15a

4 Use index laws 1 and 2 to simplify the following. Write your answers using positive indices.

a $x^3 \times x^{-2}$ **b** $a^7 \times a^{-4}$ **c** $2b^5 \times b^{-9}$ **d** $3y^{-6} \times y^3$

e $x^2y^3 \times x^{-3}y^{-4}$ **f** $4a^{-6}y^4 \times a^3y^{-2}$ **g** $2a^{-3}b \times 3a^{-2}b^{-3}$ **h** $6a^4b^3 \times 3a^{-6}b$

i $\dfrac{a^4b^3}{a^2b^5}$ **j** $\dfrac{m^3n^2}{mn^3}$ **k** $\dfrac{3x^2y}{6xy^2}$ **l** $\dfrac{4m^3n^4}{7m^2n^7}$

m $a^3b^4 \div (a^2b^7)$ **n** $p^2q^3 \div (p^7q^2)$ **o** $\dfrac{p^2q^2r^4}{pq^4r^5}$ **p** $\dfrac{12r^4s^6}{9rs^8}$

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Example 15b

5 Express the following in simplest form with positive indices.

a $\dfrac{2x^{-2}}{3x^{-3}}$ **b** $\dfrac{7d^{-3}}{10d^{-5}}$ **c** $\dfrac{5s^{-2}}{3s}$ **d** $\dfrac{4f^{-5}}{3f^{-3}}$

e $\dfrac{f^3g^{-2}}{f^{-2}g^3}$ **f** $\dfrac{r^{-3}s^{-4}}{r^3s^{-2}}$ **g** $\dfrac{3w^{-2}x^3}{6w^{-3}x^{-2}}$ **h** $\dfrac{15c^3d}{12c^{-2}d^{-3}}$

6 Express the following with positive indices.

a $\left(\dfrac{2x^2}{x^3}\right)^4$ **b** $\left(\dfrac{m^3}{4m^5}\right)^3$ **c** $2(x^{-7})^3$ **d** $4(d^{-2})^3$

e $(3t^{-4})^2$ **f** $5(x^2)^{-2}$ **g** $(3x^{-5})^4$ **h** $-8(x^5)^{-3}$

i $(4y^{-2})^{-2}$ **j** $(3h^{-3})^{-4}$ **k** $7(j^{-2})^{-4}$ **l** $2(t^{-3})^{-2}$

PROBLEM-SOLVING	7(1/2)	7–8(1/2)	7–8(1/3), 9

Example 16

7 Simplify the following and express your answers with positive indices.

a $(a^3b^2)^3 \times (a^2b^4)^{-1}$ **b** $(2p^2)^4 \times (3p^2q)^{-2}$ **c** $2(x^2y^{-1})^2 \times (3xy^4)^3$

d $\dfrac{2a^3b^2}{a^{-3}} \times \dfrac{2a^2b^5}{b^4}$ **e** $\dfrac{(3rs^2)^4}{r^{-3}s^4} \times \dfrac{(2r^2s)^2}{s^7}$ **f** $\dfrac{4(x^{-2}y^4)^2}{x^2y^{-3}} \times \dfrac{xy^4}{2x^{-2}y}$

g $\left(\dfrac{a^2b^3}{b^{-2}}\right)^2 \div \left(\dfrac{ab^4}{a^2}\right)^{-2}$ **h** $\left(\dfrac{m^4n^{-2}}{r^3}\right)^2 \div \left(\dfrac{m^{-3}n^2}{r^3}\right)^2$ **i** $\dfrac{3(x^2y^{-4})^2}{2(xy^2)^2} \div \dfrac{(xy)^{-3}}{(3x^{-2}y^4)^2}$

8 Evaluate without the use of a calculator.

a 5^{-2} **b** 4^{-3} **c** 2×7^{-2} **d** $5 \times (-3^{-4})$

e $3^{10} \times (3^2)^{-6}$ **f** $(4^2)^{-5} \times 4(4^{-3})^{-3}$ **g** $\dfrac{2}{7^{-2}}$ **h** $\dfrac{-3}{4^{-2}}$

i $\left(\dfrac{2}{3}\right)^{-2}$ **j** $\left(\dfrac{-5}{4}\right)^{-3}$ **k** $\dfrac{(4^{-2})^3}{4^{-4}}$ **l** $\dfrac{(10^{-4})^{-2}}{(10^{-2})^{-3}}$

9 The width of a hair on a spider is approximately 3^{-5} cm. How many centimetres is this, correct to four decimal places?

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REASONING	11	10, 11	10, 12, 13

10 **a** Simplify these numbers.

i $\left(\frac{2}{3}\right)^{-1}$ **ii** $\left(\frac{5}{7}\right)^{-1}$ **iii** $\left(\frac{2x}{y}\right)^{-1}$

b What is $\left(\frac{a}{b}\right)^{-1}$ when expressed in simplest form?

11 A student simplifies $2x^{-2}$ and writes $2x^{-2} = \frac{1}{2x^2}$. Explain the error made.

12 Evaluate the following by combining fractions.

a $2^{-1} + 3^{-1}$ **b** $3^{-2} + 6^{-1}$ **c** $\left(\frac{3}{4}\right)^{-1} - \left(\frac{1}{2}\right)^{0}$

d $\left(\frac{3}{2}\right)^{-1} - 5(2^{-2})$ **e** $\left(\frac{4}{5}\right)^{-2} - \left(\frac{2^{-2}}{3}\right)$ **f** $\left(\frac{3}{2^{-2}}\right) - \left(\frac{2^{-1}}{3^{-2}}\right)^{-1}$

13 Prove that $\left(\frac{1}{2}\right)^{x} = 2^{-x}$ giving reasons.

ENRICHMENT: Simple equations with negative indices	–	–	14(1/2)

14 Find the value of x.

a $2^x = \frac{1}{4}$ **b** $2^x = \frac{1}{32}$ **c** $3^x = \frac{1}{27}$ **d** $\left(\frac{3}{4}\right)^x = \frac{4}{3}$

e $\left(\frac{2}{3}\right)^x = \frac{9}{4}$ **f** $\left(\frac{2}{5}\right)^x = \frac{125}{8}$ **g** $\frac{1}{2^x} = 8$ **h** $\frac{1}{3^x} = 81$

i $\frac{1}{2^x} = 1$ **j** $5^{x-2} = \frac{1}{25}$ **k** $3^{x-3} = \frac{1}{9}$ **l** $10^{x-5} = \frac{1}{1000}$

m $\left(\frac{3}{4}\right)^{2x+1} = \frac{64}{27}$ **n** $\left(\frac{2}{5}\right)^{3x-5} = \frac{25}{4}$ **o** $\left(\frac{3}{2}\right)^{3x+2} = \frac{16}{81}$ **p** $\left(\frac{7}{4}\right)^{1-x} = \frac{4}{7}$

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Progress quiz

3A 10A

1 Express each number as a decimal and decide if it is rational or irrational. Then locate all the numbers on the same number line.

a $\sqrt{10}$ **b** $\frac{22}{7}$ **c** π **d** 315%

3A 10A

2 Simplify the following.

a $\sqrt{98}$ **b** $2\sqrt{75}$ **c** $\frac{5\sqrt{32}}{8}$ **d** $\sqrt{\frac{125}{16}}$

3A 10A

3 Express $8\sqrt{3}$ as the square root of a positive integer.

3B 10A

4 Simplify the following.

a $7\sqrt{3} - 5\sqrt{3} + \sqrt{3}$ **b** $4\sqrt{2} - 3\sqrt{5} + 2\sqrt{2} + 5\sqrt{5}$

c $5\sqrt{48} - 2\sqrt{12}$ **d** $7\sqrt{45} - \sqrt{243} - 2\sqrt{20} + \sqrt{27}$

3C 10A

5 Simplify the following.

a $-\sqrt{3} \times \sqrt{5}$ **b** $-5\sqrt{21} \times (-\sqrt{14})$ **c** $14\sqrt{7} \div (21\sqrt{35})$

d $\sqrt{6} \times \sqrt{6}$ **e** $(2\sqrt{13})^2$

3C 10A

6 Use the distributive law to expand $2\sqrt{3}(\sqrt{6} + 5\sqrt{24})$ and simplify the surds where necessary.

3D 10A

7 Rationalise the denominators.

a $\frac{3}{\sqrt{7}}$ **b** $\frac{2\sqrt{3}}{\sqrt{5}}$ **c** $\frac{\sqrt{6} - 3\sqrt{5}}{\sqrt{2}}$

3E

8 Simplify, using index laws.

a $a^3 \times a^2$ **b** $4x^2y \times 3xy^3$ **c** $h^6 \div h^2$ **d** $5m^9n^4 \div (10m^3n)$

e $(a^2)^3$ **f** $(3m^5)^2$ **g** $\left(\frac{2p^4q^3}{7rt^2}\right)^2$ **h** $(2ab)^0 + 5m^0$

3F

9 Simplify the following where possible and express your answers using positive indices.

a x^{-3} **b** $2a^{-2}b^4c^{-3}$ **c** $\frac{7}{m^{-2}}$ **d** $\frac{4d^{-7}}{5d^{-5}}$

e $\left(\frac{4k^3}{k^7}\right)^2$ **f** $(2a^{-2})^{-3}$ **g** $6a^{-3}m^4 \times 2a^{-2}m^{-3}$ **h** $\frac{20c^{-3}d^2}{15c^{-1}d^{-3}}$

3F

10 Simplify the following and express your answers using positive indices.

a $\frac{(a^3b)^5}{5a^3b^{-2}} \times \left(\frac{a^{-4}}{b^6}\right)^2$ **b** $\left(\frac{3x^2}{c^2d^{-3}}\right)^2 \div \left(\frac{2x^{-2}c^3}{d}\right)^3$

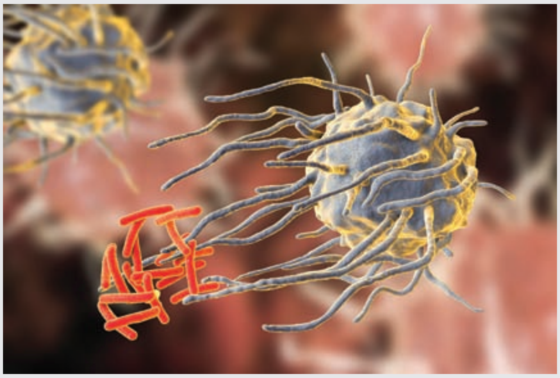
ISBN 978-1-108-77290-7

3G Scientific notation CONSOLIDATING

Learning intentions

- To understand that very large and very small numbers can be written in a shorthand form
- To know the general form of a number in scientific notation
- To be able to convert between scientific notation and basic numerals
- To know the meaning of the term significant figure
- To be able to round a number to a desired number of significant figures
- To know how to use technology in working with scientific notation

Scientific notation is useful when working with very large or very small numbers. Combined with the use of significant figures, numbers can be written down with an appropriate degree of accuracy and without the need to write all the zeros that define the position of the decimal place. The approximate distance between the Earth and the Sun is 150 million kilometres or 1.5×10^8 km when written in scientific notation using two significant figures. Negative indices can be used for very small numbers, such as 0.0000382 g $= 3.82 \times 10^{-5}$ g.



Everyday users of scientific notation include astronomers, space scientists, chemists, engineers, environmental scientists, physicists, biologists, lab technicians and medical researchers. This image shows white blood cells engulfing tuberculosis bacteria.

LESSON STARTER Amazing facts large and small

Think of an object, place or living thing that is associated with a very large or small number.

- Give three examples using very large numbers.
- Give three examples using very small numbers.
- Can you remember how to write these numbers using scientific notation?
- How are significant figures used when writing numbers with scientific notation?

KEY IDEAS

- A number written using **scientific notation** is of the form $a \times 10^m$, where $1 \leqslant a < 10$ or $-10 < a \leqslant -1$ and m is an integer.
  - Large numbers: $24\,800\,000 = 2.48 \times 10^7$
    $9\,020\,000\,000 = 9.02 \times 10^9$
  - Small numbers: $0.00307 = 3.07 \times 10^{-3}$
    $-0.0000012 = -1.2 \times 10^{-6}$

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- **Significant figures** are counted from left to right, starting at the first non-zero digit.
  - When using scientific notation the digit to the left of the decimal point is the first significant figure.
    For example: $20\,190\,000 = 2.019 \times 10^7$ shows four significant figures.
  - The $\boxed{\times 10^n}$, $\boxed{\text{EE}}$ or $\boxed{\text{Exp}}$ keys can be used on calculators to enter numbers using scientific notation; e.g. 2.3E–4 means 2.3×10^{-4}.

BUILDING UNDERSTANDING

1. How many significant figures are showing in these numbers?
   a 2.12×10^7 b 1.81×10^{-3} c 461 d 0.0000403
2. State these numbers as powers of 10.
   a 1000 b 10 000 000 c 0.000001 d $\frac{1}{1000}$
3. Convert to numbers using scientific notation by stating the missing power.
   a $43\,000 = 4.3 \times 10^{\square}$ b $712\,000 = 7.12 \times 10^{\square}$ c $9012 = 9.012 \times 10^{\square}$
   d $0.00078 = 7.8 \times 10^{\square}$ e $0.00101 = 1.01 \times 10^{\square}$ f $0.00003 = 3 \times 10^{\square}$

Example 17 Converting from scientific notation to a basic numeral

Write these numbers as a basic numeral.
a 5.016×10^5 b 3.2×10^{-7}

SOLUTION	EXPLANATION
a $5.016 \times 10^5 = 501\,600$	Move the decimal point 5 places to the right.
b $3.2 \times 10^{-7} = 0.00000032$	Move the decimal point 7 places to the left.

Now you try

Write these numbers as a basic numeral.
a 2.048×10^4 b 4.7×10^{-5}

Example 18 Converting to scientific notation using significant figures

Write these numbers in scientific notation, using three significant figures.
a 5 218 300 b 0.0042031

SOLUTION	EXPLANATION
a $5\,218\,300 = 5.22 \times 10^6$	Place the decimal point after the first non-zero digit. The digit following the third digit is at least 5, so round up.

ISBN 978-1-108-77290-7

Cambridge University Press

b $0.0042031 = 4.20 \times 10^{-3}$

Round down in this case, but retain the zero to show the value of the third significant figure.

Now you try

Write these numbers in scientific notation, using three significant figures.

a 7 937 200   b 0.00027103

Exercise 3G

FLUENCY	1, 2–5(½)	2–5(½)	2–6(⅓)

1 Write these numbers as a basic numeral.

Example 17a
a i 3.126×10^3 ii 5.04×10^6

Example 17b
b i 2.8×10^{-3} ii 5.91×10^{-6}

Example 17a

2 Write these numbers as a basic numeral.

a 3.12×10^3 b 5.4293×10^4 c 7.105×10^5
d 8.213×10^6 e 5.95×10^4 f -8.002×10^5
g -1.012×10^4 h 9.99×10^6 i 2.105×10^8
j -5.5×10^4 k 2.35×10^9 l 1.237×10^{12}

Example 17b

3 Write these numbers as a basic numeral.

a 4.5×10^{-3} b 2.72×10^{-2} c 3.085×10^{-4}
d 7.83×10^{-3} e -9.2×10^{-5} f 2.65×10^{-1}
g 1.002×10^{-4} h -6.235×10^{-6} i 9.8×10^{-1}
j -5.45×10^{-10} k 3.285×10^{-12} l 8.75×10^{-7}

Example 18a

4 Write these numbers in scientific notation, using three significant figures.

a 6241   b −572 644   c 30 248
d 423 578   e −10 089   f 34 971 863
g 72 477   h 356 088   i 110 438 523
j 909 325   k −4 555 678   l 9 826 100 005

Example 18b

5 Write these numbers in scientific notation, using three significant figures.

a 0.002423   b −0.018754   c 0.000125
d −0.0078663   e 0.0007082   f 0.11396
g 0.000006403   h 0.00007892   i 0.000129983
j 0.00000070084   k 0.000000009886   l −0.0004998

6 Write in scientific notation, using the number of significant figures given in the brackets.

a −23 900 (2)   b 5 707 159 (3)   c 703 780 030 (2)
d 4875 (3)   e 0.00192 (2)   f −0.00070507 (3)
g 0.000009782 (2)   h −0.35708 (4)   i 0.000050034 (3)

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PROBLEM-SOLVING	7–8(½)	7–8(½)	7–8(⅓), 9

7 Write the following numerical facts using scientific notation.
  a The area of Australia is about 7 700 000 km^2.
  b The number of stones used to build the Pyramid of Khufu is about 2 500 000.
  c The greatest distance of Pluto from the Sun is about 7 400 000 000 km.
  d A human hair is about 0.01 cm wide.
  e The mass of a neutron is about 0.000000000000000000000000001675 kg.
  f The mass of a bacteria cell is about 0.00000000000095 g.

8 Use a calculator to evaluate the following, giving the answers in scientific notation using three significant figures.
  a $(2.31)^{-7}$
  b $(5.04)^{-4}$
  c $(2.83 \times 10^2)^{-3}$
  d $5.1 \div (8 \times 10^2)$
  e $(9.3 \times 10^{-2}) \times (8.6 \times 10^8)$
  f $(3.27 \times 10^4) \div (9 \times 10^{-5})$
  g $\sqrt{3.23 \times 10^{-6}}$
  h $\pi(3.3 \times 10^7)^2$
  i $\sqrt[3]{5.73 \times 10^{-4}}$

9 The speed of light is approximately 3×10^5 km/s and the average distance between Pluto and the Sun is about 5.9×10^9 km. How long does it take for light from the Sun to reach Pluto? Answer correct to the nearest minute.

REASONING	10	10, 11(½)	11–12(⅓), 13

10 Explain why 38×10^7 is not written using scientific notation.

11 Write the following using scientific notation.
  a 21×10^3
  b 394×10^7
  c 6004×10^{-2}
  d 179×10^{-6}
  e 0.2×10^4
  f 0.007×10^2
  g 0.01×10^9
  h 0.06×10^8
  i 0.4×10^{-2}
  j 0.0031×10^{-11}
  k 210.3×10^{-6}
  l 9164×10^{-24}

12 Combine your knowledge of index laws with scientific notation to evaluate the following and express using scientific notation.
  a $(3 \times 10^2)^2$
  b $(2 \times 10^3)^3$
  c $(8 \times 10^4)^2$
  d $(12 \times 10^{-5})^2$
  e $(5 \times 10^{-3})^{-2}$
  f $(4 \times 10^5)^{-2}$
  g $(1.5 \times 10^{-3})^2$
  h $(8 \times 10^{-8})^{-1}$
  i $(5 \times 10^{-2}) \times (2 \times 10^{-4})$
  j $(3 \times 10^{-7}) \times (4.25 \times 10^2)$
  k $(15 \times 10^8) \times (12 \times 10^{-11})$
  l $(18 \times 10^5) \div (9 \times 10^3)$
  m $(240 \times 10^{-4}) \div (3 \times 10^{-2})$
  n $(2 \times 10^{-8}) \div (50 \times 10^4)$
  o $(5 \times 10^2) \div (20 \times 10^{-3})$

13 Rewrite 3×10^{-4} with a positive index and use this to explain why, when expressing 3×10^{-4} as a basic numeral, the decimal point is moved four places to the left.

ENRICHMENT: $E = mc^2$	–	–	14

14 $E = mc^2$ is a formula derived by Albert Einstein (1879–1955). The formula relates the energy (E joules) of an object to its mass (m kg), where c is the speed of light (approximately 3×10^8 m/s).

Use $E = mc^2$ to answer these questions, using scientific notation.
  a Find the energy, in joules, contained inside an object with these given masses.
    i 10 kg
    ii 26 000 kg
    iii 0.03 kg
    iv 0.00001 kg
  b Find the mass, in kilograms, of an object that contains these given amounts of energy. Give your answer using three significant figures.
    i 1×10^{25} J
    ii 3.8×10^{16} J
    iii 8.72×10^4 J
    iv 1.7×10^{-2} J
  c The mass of the Earth is about 6×10^{24} kg. How much energy does this convert to?

ISBN 978-1-108-77290-7

3H Rational indices 10A

Learning intentions

- To understand how a rational index relates to the root of a number
- To know how to convert between bases with rational indices and surd form
- To be able to evaluate some numbers with rational indices without a calculator
- To be able to apply index laws to expressions involving rational indices

The square and cube roots of numbers, such as $\sqrt{81} = 9$ and $\sqrt[3]{64} = 4$, can be written using fractional powers.

The following shows that $\sqrt{9} = 9^{\frac{1}{2}}$ and $\sqrt[3]{8} = 8^{\frac{1}{3}}$.

Consider:

$$\begin{aligned} \sqrt{9} \times \sqrt{9} &= 3 \times 3 \\ &= 9 \end{aligned} \quad \text{and} \quad \begin{aligned} 9^{\frac{1}{2}} \times 9^{\frac{1}{2}} &= 9^{\frac{1}{2}+\frac{1}{2}} \\ &= 9 \end{aligned}$$

$$\therefore \sqrt{9} = 9^{\frac{1}{2}}$$

Also:

$$\begin{aligned} \sqrt[3]{8} \times \sqrt[3]{8} \times \sqrt[3]{8} &= 2 \times 2 \times 2 \\ &= 8 \end{aligned} \quad \text{and} \quad \begin{aligned} 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} \times 8^{\frac{1}{3}} &= 8^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} \\ &= 8 \end{aligned}$$

$$\therefore \sqrt[3]{8} = 8^{\frac{1}{3}}$$

Fractional indices are used in finance, electrical engineering, architecture, carpentry and for solving packing problems. Volume to the power of one-third (i.e. the cube root) finds a cube's side length and helps find a sphere's radius.

A rational index is an index that can be expressed as a fraction.

LESSON STARTER Making the connection

For each part below use your knowledge of index laws and basic surds to simplify the numbers. Then discuss the connection that can be made between numbers that have a $\sqrt{\ }$ sign and numbers that have fractional powers.

- $\sqrt{5} \times \sqrt{5}$ and $5^{\frac{1}{2}} \times 5^{\frac{1}{2}}$
- $\sqrt[3]{27} \times \sqrt[3]{27} \times \sqrt[3]{27}$ and $27^{\frac{1}{3}} \times 27^{\frac{1}{3}} \times 27^{\frac{1}{3}}$
- $(\sqrt{5})^2$ and $\left(5^{\frac{1}{2}}\right)^2$
- $(\sqrt[3]{64})^3$ and $\left(64^{\frac{1}{3}}\right)^3$

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KEY IDEAS

- $a^{\frac{1}{n}} = \sqrt[n]{a}$
  - $\sqrt[n]{a}$ is the nth root of a.

  For example: $3^{\frac{1}{2}} = \sqrt{3}$ or $\sqrt{3}$, $5^{\frac{1}{3}} = \sqrt[3]{5}$, $7^{\frac{1}{10}} = \sqrt[10]{7}$
- $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = (\sqrt[n]{a})^m$ or $a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$

  For example: $8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = (\sqrt[3]{8})^2 = 2^2 = 4$ or $8^{\frac{2}{3}} = (8^2)^{\frac{1}{3}} = (64)^{\frac{1}{3}} = \sqrt[3]{64} = 4$
- In most cases, the index laws apply to **rational indices** (i.e. fractional indices) just as they do for indices that are integers.

BUILDING UNDERSTANDING

1 State the missing components for each statement.

a $2^{\square} = 8$ and $\sqrt[3]{8} =$ ____

b $2^{\square} = 32$ and $\sqrt[\square]{32} = 2$

c $3^{\square} = 81$ and $\sqrt[\square]{81} = 3$

d $10^{\square} = 100000$ and $\sqrt[\square]{100000} = 10$

2 Evaluate:

a $\sqrt{9}$

b $\sqrt{121}$

c $\sqrt[3]{27}$

d $\sqrt[3]{64}$

e $\sqrt[4]{16}$

f $\sqrt[4]{81}$

g $\sqrt[5]{32}$

h $\sqrt[5]{100000}$

3 Using a calculator, enter and evaluate each pair of numbers in their given form. Round your answer to two decimal places.

a $\sqrt[3]{7}$, $7^{\frac{1}{3}}$

b $\sqrt[5]{10}$, $10^{\frac{1}{5}}$

c $\sqrt{100}$, $100^{\frac{1}{2}}$

Example 19 Writing in index form

Express the following in index form.

a $\sqrt{15}$

b $\sqrt{7x^5}$

c $3\sqrt[4]{x^7}$

d $10\sqrt{10}$

SOLUTION	EXPLANATION
a $\sqrt{15} = 15^{\frac{1}{2}}$	$\sqrt{\ }$ means the square root or $\sqrt[2]{\ }$. Note: $\sqrt[n]{a} = a^{\frac{1}{n}}$.
b $\sqrt{7x^5} = (7x^5)^{\frac{1}{2}} = 7^{\frac{1}{2}}x^{\frac{5}{2}}$	Rewrite $\sqrt{\ }$ as power $\frac{1}{2}$, then apply index laws to simplify: $5 \times \frac{1}{2} = \frac{5}{2}$.

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c $3\sqrt[4]{x^7} = 3(x^7)^{\frac{1}{4}}$
$= 3x^{\frac{7}{4}}$

$\sqrt[4]{}$ means to the power of $\frac{1}{4}$.
Apply index law 3 to multiply indices.

d $10\sqrt{10} = 10 \times 10^{\frac{1}{2}}$
$= 10^{\frac{3}{2}}$

Rewrite the square root as power $\frac{1}{2}$ and then add indices for the common base 10. Recall $10 = 10^1$, so $1 + \frac{1}{2} = \frac{3}{2}$.

An alternative answer is $\sqrt{100} \times \sqrt{10} = 1000^{\frac{1}{2}}$.

Now you try

Express the following in index form.

a $\sqrt{11}$ b $\sqrt{3x^7}$ c $2\sqrt[4]{x^9}$ d $7\sqrt{7}$

Example 20 Writing in surd form

Express the following in surd form.

a $3^{\frac{1}{5}}$ b $5^{\frac{2}{3}}$

SOLUTION

a $3^{\frac{1}{5}} = \sqrt[5]{3}$

b $5^{\frac{2}{3}} = (5^{\frac{1}{3}})^2$
$= (\sqrt[3]{5})^2$

Alternatively:
$5^{\frac{2}{3}} = (5^2)^{\frac{1}{3}}$
$= \sqrt[3]{25}$

EXPLANATION

$a^{\frac{1}{n}} = \sqrt[n]{a}$

Use index law 3 whereby $\frac{2}{3} = \frac{1}{3} \times 2$.
$5^{\frac{1}{3}} = \sqrt[3]{5}$

$\frac{1}{3} \times 2$ is the same as $2 \times \frac{1}{3}$.

Now you try

Express the following in surd form.

a $5^{\frac{1}{3}}$ b $11^{\frac{2}{3}}$

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Example 21 Evaluating numbers with fractional indices

Evaluate the following without a calculator.

a $16^{\frac{1}{2}}$ b $16^{\frac{1}{4}}$ c $27^{-\frac{1}{3}}$

SOLUTION

a $16^{\frac{1}{2}} = \sqrt{16}$
$= 4$

b $16^{\frac{1}{4}} = \sqrt[4]{16}$
$= 2$

c $27^{-\frac{1}{3}} = \dfrac{1}{27^{\frac{1}{3}}}$
$= \dfrac{1}{\sqrt[3]{27}}$
$= \dfrac{1}{3}$

EXPLANATION

$16^{\frac{1}{2}}$ means $\sqrt{16}$.

$16^{\frac{1}{4}}$ means $\sqrt[4]{16}$ and $2^4 = 16$.

Rewrite, using positive indices. Recall that $a^{-m} = \dfrac{1}{a^m}$.

$27^{\frac{1}{3}}$ means $\sqrt[3]{27}$ and $3^3 = 27$.

Now you try

Evaluate the following without a calculator.

a $25^{\frac{1}{2}}$ b $27^{\frac{1}{3}}$ c $16^{-\frac{1}{4}}$

Exercise 3H

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/3)

1 Express the following in index form.

Example 19a — a i $\sqrt{13}$ ii $\sqrt{19}$

Example 19b — b i $\sqrt{6x^3}$ ii $\sqrt{11x^7}$

Example 19c — c i $4\sqrt[4]{x^5}$ ii $7\sqrt[3]{x^2}$

Example 19d — d i $6\sqrt{6}$ ii $13\sqrt{13}$

Example 19a,b

2 Express the following in index form.

a $\sqrt{29}$ b $\sqrt[3]{35}$ c $\sqrt[5]{x^2}$ d $\sqrt[4]{b^3}$
e $\sqrt{2a}$ f $\sqrt[3]{4t^7}$ g $\sqrt[5]{10t^2}$ h $\sqrt[8]{8m^4}$

Example 19c,d

3 Express the following in index form.

a $7\sqrt{x^5}$ b $6\sqrt[3]{n^7}$ c $3\sqrt[4]{y^{12}}$
d $5\sqrt[3]{p^2r}$ e $2\sqrt[3]{a^4b^2}$ f $2\sqrt[4]{g^3h^5}$
g $5\sqrt{5}$ h $7\sqrt{7}$ i $4\sqrt[3]{4}$

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Example 20

4 Express the following in surd form.

a $2^{\frac{1}{5}}$ **b** $8^{\frac{1}{7}}$ **c** $6^{\frac{1}{3}}$ **d** $11^{\frac{1}{10}}$

e $3^{\frac{2}{3}}$ **f** $7^{\frac{2}{3}}$ **g** $2^{\frac{3}{5}}$ **h** $3^{\frac{4}{7}}$

Example 21

5 Evaluate without using a calculator.

a $36^{\frac{1}{2}}$ **b** $27^{\frac{1}{3}}$ **c** $64^{\frac{1}{3}}$

d $49^{\frac{1}{2}}$ **e** $16^{\frac{1}{4}}$ **f** $125^{\frac{1}{3}}$

g $9^{-\frac{1}{2}}$ **h** $32^{-\frac{1}{5}}$ **i** $81^{-\frac{1}{4}}$

j $1000^{-\frac{1}{3}}$ **k** $400^{-\frac{1}{2}}$ **l** $10\,000^{-\frac{1}{4}}$

PROBLEM-SOLVING	6(½)	6–7(½)	6–8(⅓)

6 Evaluate without using a calculator.

a $8^{\frac{2}{3}}$ **b** $32^{\frac{3}{5}}$ **c** $36^{\frac{3}{2}}$

d $16^{\frac{5}{4}}$ **e** $16^{-\frac{3}{4}}$ **f** $27^{-\frac{2}{3}}$

g $64^{-\frac{2}{3}}$ **h** $25^{-\frac{3}{2}}$ **i** $\dfrac{1}{25^{-\frac{3}{2}}}$

j $\dfrac{2}{4^{\frac{5}{2}}}$ **k** $\dfrac{3}{9^{\frac{5}{2}}}$ **l** $\dfrac{10}{100^{\frac{3}{2}}}$

7 Use index laws to simplify the following.

a $a^{\frac{1}{2}} \times a^{\frac{3}{2}}$ **b** $m^{\frac{3}{2}} \times m^{\frac{3}{2}}$ **c** $x^{\frac{7}{3}} \div x^{\frac{4}{3}}$ **d** $b^{\frac{5}{4}} \div b^{\frac{3}{4}}$

e $\left(s^{\frac{3}{2}}\right)^{\frac{4}{7}}$ **f** $\left(y^{\frac{1}{3}}\right)^{\frac{1}{3}}$ **g** $\left(t^{\frac{2}{11}}\right)^{0}$ **h** $\left(\dfrac{a^{\frac{2}{3}}}{b^{\frac{4}{3}}}\right)^{\frac{3}{4}}$

8 Simplify the following.

a $\sqrt{25s^4}$ **b** $\sqrt[3]{27t^6}$ **c** $\sqrt[4]{16t^8}$ **d** $\sqrt[3]{125t^{12}}$

e $(x^3)^{\frac{1}{3}}$ **f** $(b^{12})^{\frac{1}{3}}$ **g** $\left(t^{\frac{1}{4}}\right)^{12}$ **h** $\left(m^{\frac{1}{5}}\right)^{10}$

i $(16a^2b^8)^{\frac{1}{2}}$ **j** $(216m^6n^3)^{\frac{1}{3}}$ **k** $(32x^{10}y^{15})^{\frac{1}{5}}$ **l** $(343r^9t^6)^{\frac{1}{3}}$

m $\sqrt{\dfrac{25}{49}}$ **n** $\sqrt[3]{\dfrac{8x^3}{27}}$ **o** $\left(\dfrac{32}{x^{10}}\right)^{\frac{1}{5}}$ **p** $\left(\dfrac{10^2x^4}{0.01}\right)^{\frac{1}{4}}$

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REASONING	9	9	9, 10

9 As shown below, $16^{\frac{5}{4}}$ can be evaluated in two ways.

Method A

$$16^{\frac{5}{4}} = (16^5)^{\frac{1}{4}}$$
$$= (1\,048\,576)^{\frac{1}{4}}$$
$$= \sqrt[4]{1\,048\,576}$$
$$= 32$$

Method B

$$16^{\frac{5}{4}} = \left(16^{\frac{1}{4}}\right)^5$$
$$= (\sqrt[4]{16})^5$$
$$= 2^5$$
$$= 32$$

a If $16^{\frac{5}{4}}$ is to be evaluated without a calculator, which method above would be preferable?

b Use your preferred method to evaluate the following without a calculator.

i $8^{\frac{5}{3}}$ **ii** $36^{\frac{3}{2}}$ **iii** $16^{\frac{7}{4}}$ **iv** $27^{\frac{4}{3}}$

v $125^{\frac{4}{3}}$ **vi** $\left(\frac{1}{9}\right)^{\frac{3}{2}}$ **vii** $\left(\frac{4}{25}\right)^{\frac{5}{2}}$ **viii** $\left(\frac{27}{1000}\right)^{\frac{4}{3}}$

10 Explain why $\sqrt[6]{64}$ is not a surd.

ENRICHMENT: Does it exist?	–	–	11

11 We know that when $y = \sqrt{x}$, where $x < 0$, y is not a real number. This is because the square of y cannot be negative; i.e. $y^2 \neq x$ since y^2 is positive and x is negative.

But we know that $(-2)^3 = -8$ so $\sqrt[3]{-8} = -2$.

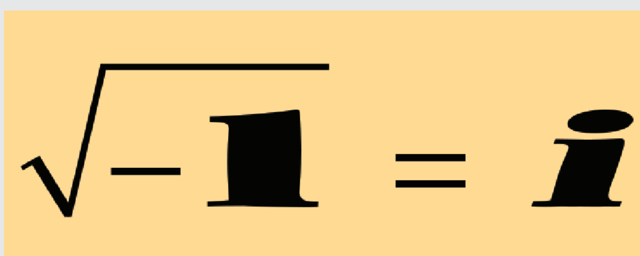
a Evaluate:

i $\sqrt[3]{-27}$ **ii** $\sqrt[3]{-1000}$ **iii** $\sqrt[5]{-32}$ **iv** $\sqrt[7]{-2187}$

b Decide if these are real numbers.

i $\sqrt{-5}$ **ii** $\sqrt[3]{-7}$ **iii** $\sqrt[5]{-16}$ **iv** $\sqrt[4]{-12}$

c If $y = \sqrt[n]{x}$ and $x < 0$, for what values of n is y a real number?



The square root of a negative value is not a real number. $\sqrt{-1} = i$ and is found in a special set of numbers called complex numbers.

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3I Exponential equations 10A

Learning intentions

- To know the form of an exponential equation
- To be able to rewrite an expression using its lowest base
- To be able to solve simple exponential equations using a common base

Equations can take many forms. For example, $2x - 1 = 5$ and $5(a - 3) = -3(3a + 7)$ are both linear equations; $x^2 = 9$ and $3x^2 - 4x - 9 = 0$ are quadratic equations; and $2^x = 8$ and $3^{2x} - 3^x - 6 = 0$ are exponential equations. Exponential equations contain a pronumeral within the index or indices of the terms in the equation. To solve for the unknown in exponential equations we use our knowledge of indices and surds and try to equate powers where possible.

Solving exponential equations can predict the timing of future outcomes. When will my new apartment double in value? When will Australia's population reach 30 million? How long before my coffee goes cold?

LESSON STARTER 2 to the power of what number is 5?

We know that 2 to the power of 2 is 4 and 2 to the power of 3 is 8, but 2 to the power of what number is 5? That is, what is x when $2^x = 5$?

- Use a calculator and trial and error to estimate the value of x when $2^x = 5$ by completing this table.

x	2	3	2.5	2.1	
2^x	4	8	5.65 …		
Result	too small	too big	too big		

- Continue trying values until you find the answer, correct to three decimal places.

KEY IDEAS

- A simple **exponential equation** is of the form $a^x = b$, where $a > 0$, $b > 0$ and $a \neq 1$.
  - There is only one solution to exponential equations of this form.
- Many exponential equations can be solved by expressing both sides of the equation using the same base.
  - We use this fact: if $a^x = a^y$ then $x = y$.

ISBN 978-1-108-77290-7

BUILDING UNDERSTANDING

1 **a** Evaluate the following.

i 2^2 **ii** 2^3 **iii** 2^4 **iv** 2^5

b Hence, state the value of x when:

i $2^x = 8$ **ii** $2^x = 32$ **iii** $2^x = 64$

2 Complete these patterns, which involve powers.

a 2, 4, 8, ___, ___, ___, ___, ___, ___

b 3, 9, 27, ___, ___, ___, ___, ___

c 4, 16, ___, ___, ___, ___

d 5, 25, ___, ___, ___

e 6, 36, ___, ___

3 State these numbers in index form. For example, $32 = 2^5$.

a 9 **b** 125 **c** 243 **d** 128 **e** 729

Example 22 Solving exponential equations

Solve for x in each of the following.

a $2^x = 16$ **b** $3^x = \frac{1}{9}$ **c** $25^x = 125$

SOLUTION

EXPLANATION

a $2^x = 16$

$2^x = 2^4$

$\therefore x = 4$

Rewrite 16 as a power, using the base 2.

Equate powers using the result: if $a^x = a^y$ then $x = y$.

b $3^x = \frac{1}{9}$

$3^x = \frac{1}{3^2}$

$3^x = 3^{-2}$

$\therefore x = -2$

Rewrite 9 as a power of 3, then write using a negative index.

Equate powers with the same base.

c $25^x = 125$

$(5^2)^x = 5^3$

$5^{2x} = 5^3$

$\therefore 2x = 3$

$x = \frac{3}{2}$

Since 25 and 125 are both powers of 5, rewrite both with a base of 5.

Apply index law 3 to remove brackets and multiply indices, then equate powers and solve for x.

Now you try

Solve for x in each of the following.

a $3^x = 27$ **b** $2^x = \frac{1}{8}$ **c** $16^x = 64$

Example 23 Solving exponential equations with a variable on both sides

Solve $3^{2x-1} = 27^x$.

SOLUTION	EXPLANATION
$3^{2x-1} = 27^x$	Rewrite 27 as a power of 3.
$3^{2x-1} = (3^3)^x$	Apply index law 3 to remove brackets and then equate powers.
$3^{2x-1} = 3^{3x}$	
$\therefore 2x - 1 = 3x$	Subtract $2x$ from both sides and answer with x as the subject.
$-1 = x$	
$\therefore x = -1$	

Now you try

Solve $5^{3x-1} = 25^x$.

Exercise 3I

FLUENCY	1, 2–4(1/2)	2–4(1/2)	2–4(1/3)

1 Solve for x in each of the following.

Example 22a
a **i** $5^x = 25$ **ii** $2^x = 8$

Example 22b
b **i** $2^x = \frac{1}{4}$ **ii** $5^x = \frac{1}{125}$

Example 22c
c **i** $9^x = 27$ **ii** $4^x = 8$

Example 22a
2 Solve for x in each of the following.

a $3^x = 27$ **b** $2^x = 8$ **c** $6^x = 36$
d $9^x = 81$ **e** $5^x = 125$ **f** $4^x = 64$
g $3^x = 81$ **h** $6^x = 216$ **i** $5^x = 625$
j $2^x = 32$ **k** $10^x = 10000$ **l** $7^x = 343$

Example 22b
3 Solve for x in each of the following.

a $7^x = \frac{1}{49}$ **b** $9^x = \frac{1}{81}$ **c** $11^x = \frac{1}{121}$
d $4^x = \frac{1}{256}$ **e** $3^x = \frac{1}{243}$ **f** $5^{-x} = \frac{1}{125}$
g $3^{-x} = \frac{1}{9}$ **h** $2^{-x} = \frac{1}{64}$ **i** $7^{-x} = \frac{1}{343}$

Example 22c
4 Solve for x in each of the following.

a $9^x = 27$ **b** $8^x = 16$ **c** $25^x = 125$
d $16^x = 64$ **e** $81^x = 9$ **f** $216^x = 6$
g $32^x = 2$ **h** $10000^x = 10$ **i** $7^{-x} = 49$
j $4^{-x} = 256$ **k** $16^{-x} = 64$ **l** $25^{-x} = 125$

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PROBLEM-SOLVING	5	5, 6(1/2)	6(1/3), 7

5 The population of bacteria in a dish is given by the rule $P = 2^t$, where P is the bacteria population and t is the time in minutes.

a What is the initial population of bacteria; i.e. when $t = 0$?

b What is the population of bacteria after:

i 1 minute? ii 5 minutes? iii 1 hour? iv 1 day?

c How long does it take for the population to reach:

i 8? ii 256? iii more than 1000?

Example 23

6 Solve for x in each of the following.

a $2^{x+1} = 8^x$ b $3^{2x+1} = 27^x$ c $7^{x+9} = 49^{2x}$

d $5^{x+3} = 25^{2x}$ e $6^{2x+3} = 216^{2x}$ f $9^{x+12} = 81^{x+5}$

g $27^{x+3} = 9^{2x}$ h $25^{x+3} = 125^{3x}$ i $32^{2x+3} = 128^{2x}$

j $27^{2x+3} = 9^{2x-1}$ k $9^{x-1} = 27^{2x-6}$ l $49^{2x-3} = 343^{2x-1}$

7 Would you prefer \$1 million now or 1 cent doubled every second for 30 seconds? Give reasons for your preference.

REASONING	8	8, 9(1/2)	9(1/2), 10, 11(1/2)

8 Consider a^x, where $a = 1$.

a Evaluate 1^x when:

i $x = 1$ ii $x = 3$ iii $x = 10\,000\,000$

b Are there any solutions to the equation $a^x = 2$ when $a = 1$? Give a reason.

9 Recall that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x} = x^{\frac{1}{3}}$. Now solve the following.

a $3^x = \sqrt{81}$ b $5^x = \sqrt{25}$ c $6^x = \sqrt[3]{36}$ d $4^x = \sqrt[4]{64}$

e $2^x = \sqrt[4]{32}$ f $3^x = \sqrt[9]{27}$ g $25^x = \sqrt[5]{125}$ h $9^x = \dfrac{1}{\sqrt[3]{27}}$

10 a Write these numbers as decimals.

i $\dfrac{1}{2^2}$ ii 2^{-3} iii 10^{-3} iv $\left(\dfrac{1}{5}\right)^4$

b Write these decimal numbers as powers of prime numbers.

i 0.04 ii 0.0625 iii 0.5 iv 0.0016

11 Show how you can use techniques from this section to solve these equations involving decimals.

a $10^x = 0.0001$ b $2^x = 0.015625$ c $5^x = 0.00032$

d $(0.25)^x = 0.5$ e $(0.04)^x = 125$ f $(0.0625)^{x+1} = \dfrac{1}{2}$

ENRICHMENT: Mixing index laws with equations	–	–	12(1/2)

12 Solve for n in the following.

a $3^n \times 9^n = 27$ b $5^{3n} \times 25^{-2n+1} = 125$ c $2^{-3n} \times 4^{2n-2} = 16$

d $3^{2n-1} = \dfrac{1}{81}$ e $7^{2n+3} = \dfrac{1}{49}$ f $5^{3n+2} = \dfrac{1}{625}$

g $6^{2n-6} = 1$ h $11^{3n-1} = 11$ i $8^{5n-1} = 1$

j $\dfrac{3^{n-2}}{9^{1-n}} = 9$ k $\dfrac{5^{3n-3}}{25^{n-3}} = 125$ l $\dfrac{36^{3+2n}}{6^n} = 1$

ISBN 978-1-108-77290-7 Cambridge University Press

3J Graphs of exponentials

Learning intentions
- To know what defines an exponential relation
- To know the meaning of the term asymptote
- To know the basic features of an exponential graph
- To be able to sketch simple exponential graphs including those involving reflections
- To know how to find the point of intersection of an exponential graph and a horizontal line

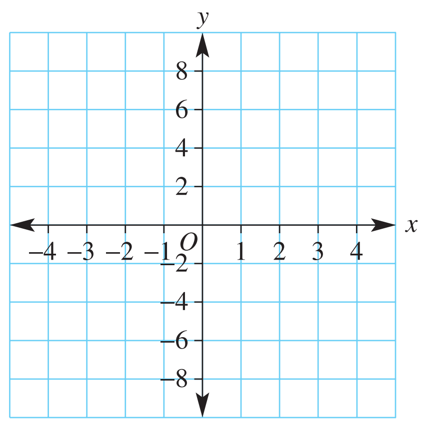
We saw earlier that indices can be used to describe some special relations. The population of the world, for example, or the balance of an investment account can be described using exponential rules that include indices. The rule $A = 100\,000(1.05)^t$ describes the account balance of \$100 000 invested at 5% p.a. compound interest for t years.

When a patient receives medication, the blood concentration decays exponentially as the body breaks it down. Exponential rules can determine the safe time between doses, from the highest safe level to the lowest effective level.

LESSON STARTER What do $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ all have in common?

Complete this table and graph all three relations on the same set of axes before discussing the points below.

x	–3	–2	–1	0	1	2	3
$y_1 = 2^x$	$\frac{1}{8}$			1		4	
$y_2 = -2^x$							
$y_3 = 2^{-x}$							



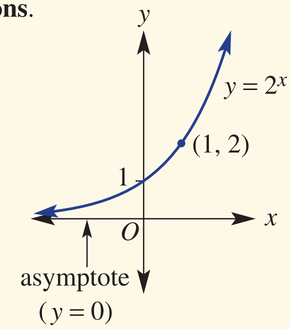
- Discuss the shape of each graph.
- Where does each graph cut the y-axis?
- Do the graphs have x-intercepts? Why not?
- What is the one feature they all have in common?

ISBN 978-1-108-77290-7

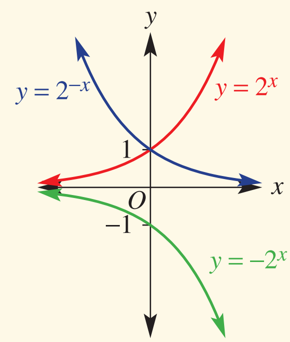
Cambridge University Press

KEY IDEAS

- $y = 2^x$, $y = (0.4)^x$, $y = 3 \times (1.1)^x$ are examples of **exponential relations**.
- An **asymptote** is a line that a curve approaches, by getting closer and closer to it, but never reaching.



- A simple **exponential** rule is of the form $y = a^x$, where $a > 0$ and $a \neq 1$.
  - y-intercept is 1.
  - $y = 0$ is the equation of the asymptote.
- The graph of $y = -a^x$ is the reflection of the graph of $y = a^x$ in the x-axis. (Note: $y = -a^x$ means $y = -1 \times a^x$.)
- The graph of $y = a^{-x}$ is the reflection of the graph of $y = a^x$ in the y-axis.



- To find the intersection points of a simple exponential and a horizontal line, use the method of substitution and equate powers after expressing both sides of the equation using the same base.
For example, for $y = 2^x$ and $y = 16$, solve
$$2^x = 16$$
$$2^x = 2^4$$
$$\therefore x = 4$$

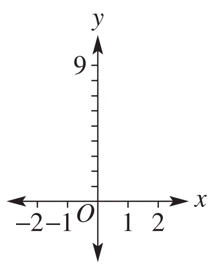
BUILDING UNDERSTANDING

1 Consider the exponential rule $y = 3^x$.

a Complete this table.

x	−2	−1	0	1	2
y		$\frac{1}{3}$	1		

b Plot the points in the table to form the graph of $y = 3^x$.



2 Complete the following.

a Graphs of the form $y = a^x$, $a > 0$ have an ______________ with equation $y = 0$ (the x-axis).

b The y-intercept of the graph $y = a^x$, $a > 0$ is ___________.

c The graph of $y = 4^{-x}$ is a reflection of the graph of $y = 4^x$ in the ______________.

d The graph of $y = -5^x$ is a reflection of the graph of $y = 5^x$ in the ______________.

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3 **a** Explain the difference between a^{-2} and $-a^2$. **b** True or false: $-3^2 = \frac{1}{3^2}$? Explain why.

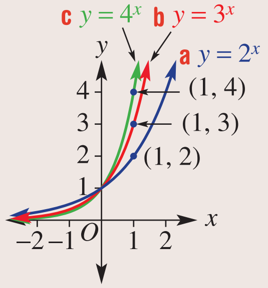
c Express with negative indices: $\frac{1}{5^3}, \frac{1}{3^2}, \frac{1}{2}$. **d** Simplify: $-3^2, -5^3, -2^{-2}$.

Example 24 Sketching graphs of exponentials

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 3^x$ **c** $y = 4^x$

SOLUTION



EXPLANATION

$a^0 = 1$, so all y-intercepts are at 1.

$y = 4^x$ is steeper than $y = 3^x$, which is steeper than $y = 2^x$.

Substitute $x = 1$ into each rule to obtain a second point to indicate the steepness of each curve.

Now you try

Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

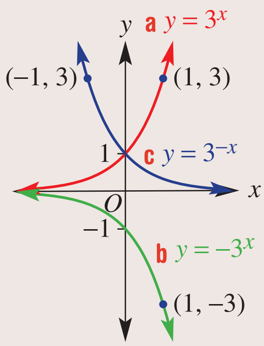
a $y = 2^x$ **b** $y = 5^x$

Example 25 Sketching with reflections

Sketch the graphs of these exponentials on the same set of axes.

a $y = 3^x$ **b** $y = -3^x$ **c** $y = 3^{-x}$

SOLUTION



EXPLANATION

The graph of $y = -3^x$ is a reflection of the graph of $y = 3^x$ in the x-axis.

Check: $x = 1, y = -3^1 = -3$

The graph of $y = 3^{-x}$ is a reflection of the graph of $y = 3^x$ in the y-axis.

Check: $x = 1, y = 3^{-1} = \frac{1}{3}$
$x = -1, y = 3^1 = 3$

Continued on next page

ISBN 978-1-108-77290-7

Now you try

Sketch the graphs of these exponentials on the same set of axes.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 26 Solving exponential equations

Find the intersection of the graphs of $y = 2^x$ and $y = 8$.

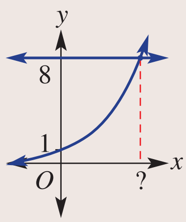
SOLUTION

$y = 2^x$
$8 = 2^x$
$2^3 = 2^x$
$x = 3$
$\therefore$ Intersection point is (3, 8).

EXPLANATION

Set $y = 8$ and write 8 with base 2.

Since the bases are the same, equate the powers.



Now you try

Find the intersection of the graphs of $y = 3^x$ and $y = 27$.

Exercise 3J

FLUENCY	1–5	2–6	2–6

Example 24
1 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 6^x$

Example 24
2 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = 4^x$ **c** $y = 5^x$

Example 25
3 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^x$ **b** $y = -2^x$ **c** $y = 2^{-x}$

Example 25b
4 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = -2^x$ **b** $y = -5^x$ **c** $y = -3^x$

Example 25c

5 Sketch the graph of the following on the same set of axes, labelling the y-intercept and the point where $x = 1$.

a $y = 2^{-x}$ **b** $y = 3^{-x}$ **c** $y = 6^{-x}$

6 **a** Find the coordinates on the graph of $y = 3^x$, where:

i $x = 0$ **ii** $x = -1$ **iii** $y = 1$ **iv** $y = 9$

b Find the coordinates on the graph of $y = -2^x$, where:

i $x = 4$ **ii** $x = -1$ **iii** $y = -1$ **iv** $y = -4$

c Find the coordinates on the graph of $y = 4^{-x}$, where:

i $x = 1$ **ii** $x = -3$ **iii** $y = 1$ **iv** $y = \frac{1}{4}$

PROBLEM-SOLVING	7, 8	7(½), 8, 9	7(½), 8–10

Example 26

7 **a** Find the intersection of the graphs of $y = 2^x$ and $y = 4$.

b Find the intersection of the graphs of $y = 3^x$ and $y = 9$.

c Find the intersection of the graphs of $y = -4^x$ and $y = -4$.

d Find the intersection of the graphs of $y = 2^{-x}$ and $y = 8$.

8 A study shows that the population of a town is modelled by the rule $P = 2^t$, where t is in years and P is in thousands of people.

a State the number of people in the town at the start of the study.

b State the number of people in the town after:

i 1 year **ii** 3 years

c When is the town's population expected to reach:

i 4000 people? **ii** 16 000 people?

9 A single bacterium divides into two every second, so one cell becomes 2 in the first second and in the next second two cells become 4 and so on.

a Write a rule for the number of bacteria, N, after t seconds.

b How many bacteria will there be after 10 seconds?

c How long does it take for the population to exceed 10 000? Round to the nearest second.

10 Use trial and error to find x when $2^x = 5$. Give the answer correct to three decimal places.

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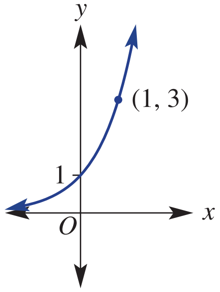
REASONING	11	11, 12	11–14

11 Match equations **a**–**f** with graphs **A**–**F** below.

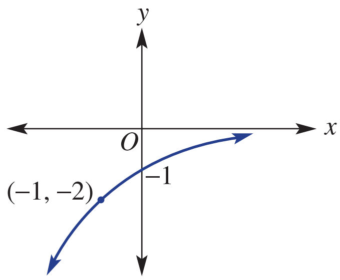
a $y = -x - 2$ **b** $y = 3^x$ **c** $y = 3^{-x}$

d $y = -2^x$ **e** $y = x$ **f** $y = -2^{-x}$

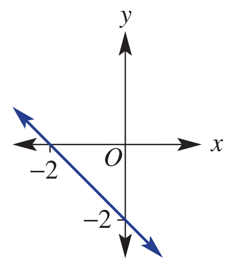
A



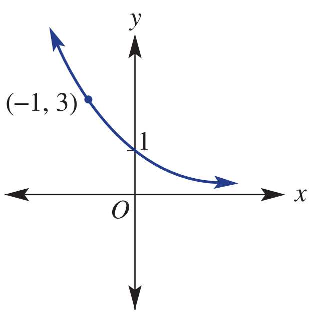
B



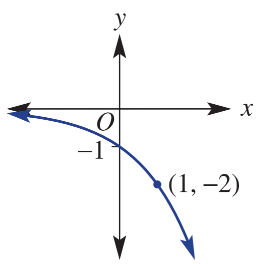
C



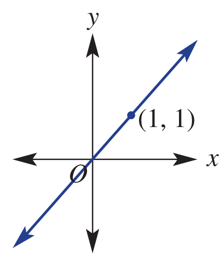
D



E



F



12 Explain why the point $(2, 5)$ does not lie on the curve with equation $y = 2^x$.

13 Describe and draw the graph of the line with equation $y = a^x$ when $a = 1$.

14 Explain why $2^x = 0$ is never true for any value of x.

ENRICHMENT: $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$	–	–	15

15 Consider the exponential rules $y = 2^{-x}$ and $y = \left(\frac{1}{2}\right)^x$.

a Using $-3 \leqslant x \leqslant 3$, sketch graphs of the rules on the same set of axes. What do you notice?

b Write the following rules in the form $y = a^x$, where $0 < a < 1$.

i $y = 3^{-x}$ **ii** $y = 5^{-x}$ **iii** $y = 10^{-x}$

c Write the following rules in the form $y = a^{-x}$, where $a > 1$.

i $y = \left(\frac{1}{4}\right)^x$ **ii** $y = \left(\frac{1}{7}\right)^x$ **iii** $y = \left(\frac{1}{11}\right)^x$

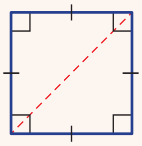
d Prove that $\left(\frac{1}{a}\right)^x = a^{-x}$, for $a > 0$.

ISBN 978-1-108-77290-7 Cambridge University Press

Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

Square diagonals



1 Square sand boxes produced by a company for playgrounds are labelled on the packaging with their diagonal length.

A landscaper is interested in the relationship between this diagonal length and other properties of the sand box including perimeter and area.

- **a** A square sand box has a diagonal length of $\sqrt{3}$ m. Give the area and perimeter of this sand box in simplified form.
- **b** A second square sand box has diagonal length $(2 + 2\sqrt{2})$ m.
  - **i** Find the exact area occupied by this sand box in m^2, using $(a + b)(c + d) = ac + ad + bc + bd$ to expand.
  - **ii** Express the side length of the sand box in metres in the form $\sqrt{a + b\sqrt{c}}$ where a, b and c are integers.
- **c** To determine the side length of the sand box in part **b** in simplified form, consider the following.
  - **i** Use expansion to show that $(\sqrt{x} + \sqrt{y})^2 = x + y + 2\sqrt{xy}$ where x and y are positive integers.
  - **ii** Make use of the result in part **i** to simplify $\sqrt{7 + 2\sqrt{10}}$ and $\sqrt{7 + 4\sqrt{3}}$.
  - **iii** Hence, simplify your answer to part **b ii** and give the perimeter of the sand box.

Air conditioner thermostat

2 An air conditioning unit inside a room has a thermostat that controls the temperature of the room. The temperature of the room, T°C, n hours after the air conditioning unit switches on is given by $T = 17 + \dfrac{8}{2^n}$.

The air conditioning unit is set to turn on when the room temperature reaches 25°C.

A technician wishes to investigate how exponential equations can model the change in air temperature and how thermostats can be used to control the use of air conditioners.

- **a** If the air conditioning unit remains on for 1 hour after it switches on, what will be the temperature in the room?
- **b** After how many hours of the unit being on would the temperature in the room reach 19°C?

The unit is programmed to switch off when the temperature in the room reaches 20°C.

- **c** Find the longest consecutive period of time that the unit could be on for, correct to one decimal place.
- **d** Sketch a graph of the temperature in the room, T, from when the unit switches on until when it switches off.
- **e** Express the rule for the temperature T in the form $T = 17 + 2^{k-n}$ where k is an integer.

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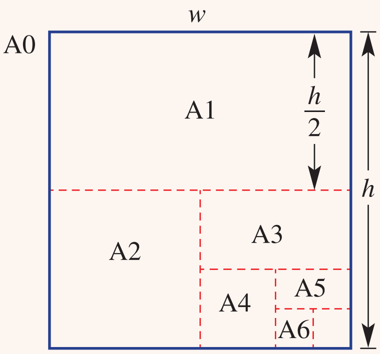
The thermostat is adjusted so that it turns on at 24°C and so that the fan strength is decreased. This unit switches off when the room is cooled to 21°C, which occurs after it has been on for 2 hours.

f Find the values of a and k, where a and k are integers, if the rule for the temperature, T°C, of the room n hours after this unit is turned on is given by $T = a + 2^{k-n}$.

International paper sizes

3 The A series of paper sizes, e.g. A4, are based on international standards. The paper sizes are such that the ratio between the height and width of each paper size is the same. The height is taken to be the longer side length of each rectangle.

Let an A0 piece of paper have width w mm and height h mm.



A paper company wants to explore the A series paper sizes and use ratios to connect the lengths and widths of successive sizes. It wishes to use these ratios to then determine various widths and heights and the rules that link these dimensions.

a Complete the table below for the corresponding height and width of the A series paper in terms of h and w.

An	Width	Height
A0	w	h
A1	$\frac{h}{2}$	w
A2		
A3		
A4		
A5		
A6		

b Determine the ratio of the height to the width of A series paper if it is the same for each paper size An.

c From your result in part **b**, write a rule for the height, h, of A series paper in terms of its width, w.

d A0 paper has an area of 1 square metre (1000 mm × 1000 mm). Determine the dimensions, w and h, of A0 paper in exact form in mm.

e Use your values from part **d** and your table from part **a** to determine the dimensions of an A4 sheet to the nearest millimetre. Measure a sheet of A4 paper to compare.

f Consider the table in part **a** and paper sizes An.

- **i** Describe the changes to the values of the width and height as n increases when n is even and when n is odd.
- **ii** Use your table and dimensions from part **d** to come up with rules for w and h when n is even and when n is odd.
- **iii** Use your rule from part **ii** to find the length and width for A3 and A4 paper and check by measuring the paper.

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3K Exponential growth and decay

Learning intentions
- To understand how percentage increase and decrease relate to exponential growth and decay
- To know the general form of the exponential growth and decay model
- To be able to write an exponential rule from a word problem and apply it

The population of a country increasing by 5% per year and an investment increasing, on average, by 12% per year are examples of exponential growth. When an investment grows exponentially, the increase per year is not constant. The annual increase is calculated on the value of the investment at that time, and this changes from year to year because of the added investment returns. The more money you have invested, the more interest you will make in a year.

Population growth can be modelled using exponential equations. Governments use projected population numbers when planning for future infrastructure, land use, and water, energy and food security.

In the same way, a population can grow exponentially. A growth of 5% in a large population represents many more babies born in a year than 5% of a small population.

Here we will focus on exponential growth and decay in general and compound interest will be studied in the next section.

LESSON STARTER A compound rule

Imagine you have an antique car valued at \$100 000 and you hope that it will increase in value at 10% p.a. The 10% increase is to be added to the value of the car each year.

- Discuss how to calculate the value of the car after 1 year.
- Discuss how to calculate the value of the car after 2 years.
- Complete this table.

Year	0	1	2	3
Value (\$)	100 000	$100\,000 \times 1.1$ = ______	$100\,000 \times 1.1 \times$ ____ = ______	________ = ______

- Discuss how indices can be used to calculate the value of the car after the second year.
- Discuss how indices can be used to calculate the value of the car after the tenth year.
- What might be the rule connecting the value of the car (\$$A$) and the time n years?
- Repeat the steps above if the value of the car decreases by 10% p.a.

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KEY IDEAS

- **Per annum** (p.a.) means 'per year'.
- Exponential growth and decay can be modelled by the rule $A = ka^t$, where A is the amount, k is the initial amount and t is the time.
  - When $a > 1$, exponential growth occurs.
  - When $0 < a < 1$, exponential decay occurs.
- For a **growth** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 + \frac{r}{100}$.
- For a **decay** rate of $r\%$ p.a., the base 'a' is calculated using $a = 1 - \frac{r}{100}$.
- The basic **exponential formula** can be summarised as $A = A_0\left(1 \pm \frac{r}{100}\right)^n$.
  - The subscript zero is often used to indicate the initial value of a quantity (e.g. P_0 is initial population).

BUILDING UNDERSTANDING

1. An antique ring is purchased for \$1000 and is expected to grow in value by 5% per year. Round your answers to the nearest cent.
   a Find the increase in value in the first year.
   b Find the value of the ring at the end of the first year.
   c Find the increase in value in the second year.
   d Find the increase in value in the third year.
   e Find the value of the ring at the end of the fifth year.
2. The mass of a limestone 5 kg rock exposed to the weather is decreasing at a rate of 2% per annum.
   a Find the mass of the rock at the end of the first year.
   b State the missing numbers for the mass of the rock (M kg) after t years.
   $M = 5(1 - \_\_\_\_)^t$
   $= 5 \times \_\_\_\_^t$
   c Use your rule to calculate the mass of the rock after 5 years, correct to two decimal places.
3. Decide if the following represent exponential *growth* or exponential *decay*.
   a $A = 1000 \times 1.3^t$
   b $A = 350 \times 0.9^t$
   c $P = P_0\left(1 + \frac{3}{100}\right)^t$
   d $T = T_0\left(1 - \frac{7}{100}\right)^t$

Example 27 Writing exponential rules

Form exponential rules for the following situations.

a John has a painting that is valued at \$100 000 and it is expected to increase in value by 14% per annum.

b A city's initial population of 50 000 is decreasing by 12% per year.

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SOLUTION

a Let A = the value in \$ of the painting at any time
n = the number of years the painting is kept
$r = 14$
$A_0 = 100\,000$

$$A = 100\,000\left(1 + \frac{14}{100}\right)^n$$
$$\therefore A = 100\,000(1.14)^n$$

EXPLANATION

Define your variables.

$$A = A_0\left(1 \pm \frac{r}{100}\right)^n$$

Substitute $r = 14$ and $A_0 = 100\,000$ and use '+' since we have growth.

b Let P = the population at any time
n = the number of years the population decreases
$r = 12$
$P_0 = 50\,000$

$$P = 50\,000\left(1 - \frac{12}{100}\right)^n$$
$$\therefore P = 50\,000(0.88)^n$$

Define your variables.

$$P = P_0\left(1 \pm \frac{r}{100}\right)^n$$

Substitute $r = 12$ and $P_0 = 50\,000$ and use '–' since we have decay.

Now you try

Form exponential rules for the following situations.

a Caz has a vase that is valued at \$50 000 and it is expected to increase in value by 16% per annum.

b A town's initial population of 10 000 is decreasing by 9% per year.

Example 28 Applying exponential rules

House prices are rising at 9% per year and Zoe's flat is currently valued at \$600 000.

a Determine a rule for the value of Zoe's flat (\$$V$) in n years' time.

b What will be the value of her flat:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Zoe's flat will be valued at \$900 000, to one decimal place.

SOLUTION

a Let V = value of Zoe's flat at any time
V_0 = starting value \$600 000
n = number of years from now
$r = 9$

$$V = V_0(1.09)^n$$
$$\therefore V = 600\,000(1.09)^n$$

EXPLANATION

Define your variables.

$$V = V_0\left(1 \pm \frac{r}{100}\right)^n$$

Use '+' since we have growth.

Continued on next page

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b **i** When $n = 1$, $V = 600\,000(1.09)^1$
$= 654\,000$
Zoe's flat would be valued at \$654 000 next year.

Substitute $n = 1$ for next year.

ii When $n = 3$, $V = 600\,000(1.09)^3$
$= 777\,017.40$
In 3 years' time Zoe's flat will be valued at about \$777 017.

For 3 years, substitute $n = 3$.

c

n	4	5	4.6	4.8	4.7
V	846 949	923 174	891 894	907 399	899 613

Zoe's flat will be valued at \$900 000 in about 4.7 years' time.

Try a value of n in the rule. If V is too low, increase your n value. If V is too high, decrease your n value. Continue this process until you get close to 900 000.

Now you try

House prices are rising at 7% per year and Andrew's apartment is currently valued at \$400 000.

a Determine a rule for the value of Andrew's apartment (\$$V$) in n years' time.

b What will be the value of his apartment:

i next year? **ii** in 3 years' time?

c Use trial and error to find when Andrew's apartment will be valued at \$500 000, to one decimal place.

Exercise 3K

FLUENCY	1, 2–4	2(½), 3–5	2(½), 3, 5, 6

1 Form exponential rules for the following situations.

Example 27a **a** Lara has a necklace that is valued at \$6000 and it is expected to increase in value by 12% per annum.

Example 27b **b** A village's initial population of 2000 is decreasing by 8% per year.

Example 27 **2** Define variables and form exponential rules for the following situations.

a A flat is purchased for \$200 000 and is expected to grow in value by 17% per annum.

b A house initially valued at \$530 000 is losing value at 5% per annum.

c The value of a car, bought for \$14 200, is decreasing at 3% per annum.

d An oil spill, initially covering an area of 2 square metres, is increasing at 5% per minute.

e A tank with 1200 litres of water is leaking at a rate of 10% of the water in the tank every hour.

f A human cell of area 0.01 cm^2 doubles its area every minute.

g A population, which is initially 172 500, is increasing at 15% per year.

h A substance of mass 30 g is decaying at a rate of 8% per hour.

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Example 28

3 The value of a house purchased for \$500 000 is expected to grow by 10% per year. Let \$$A$ be the value of the house after t years.

a Write the missing number in the rule connecting A and t.

$A = 500\,000 \times \_\_\_\_\_^{t}$

b Use your rule to find the expected value of the house after the following number of years. Round your answer to the nearest cent.

i 3 years **ii** 10 years **iii** 20 years

c Use trial and error to estimate when the house will be worth \$1 million. Round your answer to one decimal place.

4 A share portfolio, initially worth \$300 000, is reduced by 15% p.a. over a number of years. Let \$$A$ be the share portfolio value after t years.

a Write the missing number in the rule connecting A and t.

$A = \_\_\_\_\_ \times 0.85^{t}$

b Use your rule to find the value of the shares after the following number of years. Round your answer to the nearest cent.

i 2 years **ii** 7 years **iii** 12 years

c Use trial and error to estimate when the share portfolio will be valued at \$180 000. Round your answer to one decimal place.

5 A water tank containing 15 000 L has a small hole that reduces the amount of water by 6% per hour.

a Determine a rule for the volume of water (V) left after t hours.

b Calculate (to the nearest litre) the amount of water left in the tank after:

i 3 hours **ii** 7 hours

c How much water is left after two days? Round your answer to two decimal places.

d Using trial and error, determine when the tank holds less than 500 L of water, to one decimal place.

6 Megan invests \$50 000 in a superannuation scheme that has an annual return of 11%.

a Determine the rule for the value of her investment (V) after n years.

b How much will Megan's investment be worth in:

i 4 years? **ii** 20 years?

c Find the approximate time before her investment is worth \$100 000. Round your answer to two decimal places.

PROBLEM-SOLVING	7, 8	7, 8	8, 9

7 A certain type of bacteria grows according to the equation $N = 3000(2.6)^{t}$, where N is the number of cells present after t hours.

a How many bacteria are there at the start?

b Determine the number of cells (round to the whole number) present after:

i 1 hour **ii** 2 hours **iii** 4.6 hours

c If 50 000 000 bacteria are needed to make a drop of serum, determine how long you will have to wait to make a drop (to the nearest minute).

8 A car tyre has 10 mm of tread when new. It is considered unroadworthy when there is only 3 mm left. The rubber wears at 12.5% every 10 000 km.

a Write an equation relating the depth of tread (D) for every 10 000 km travelled.

b Using trial and error, determine when the tyre becomes unroadworthy, to the nearest 10 000 km.

c If a tyre lasts 80 000 km, it is considered to be of good quality. Is this a good quality tyre?

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9 A cup of coffee has an initial temperature of 90°C and the surrounding temperature is 0°C.

a If the temperature relative to surroundings reduces by 8% every minute, determine a rule for the temperature of the coffee (T°C) after t minutes.

b What is the temperature of the coffee (to one decimal place) after:

i 90 seconds? **ii** 2 minutes?

c When is the coffee suitable to drink if it is best consumed at a temperature of 68.8°C? Give your answer to the nearest second.

REASONING	10	10	10, 11

10 The monetary value of things can be calculated using different time periods. Consider a \$1000 collector's item that is expected to grow in value by 10% p.a. over 5 years.

- If the increase in value is added annually then $r = 10$ and $t = 5$, so $A = 1000(1.1)^5$.
- If the increase in value is added monthly then $r = \frac{10}{12}$ and $t = 5 \times 12 = 60$, so $A = 1000\left(1 + \frac{10}{1200}\right)^{60}$.

a If the increase in value is added annually, find the value of the collectors' item, to the nearest cent, after:

i 5 years **ii** 8 years **iii** 15 years

b If the increase in value is added monthly, find the value of the collectors' item, to the nearest cent, after:

i 5 years **ii** 8 years **iii** 15 years

11 You inherit a \$2000 necklace that is expected to grow in value by 7% p.a. What will the necklace be worth, to the nearest cent, after 5 years if the increase in value is added:

a annually? **b** monthly? **c** weekly (assume 52 weeks in the year)?

ENRICHMENT: Half-life	–	–	12–14

Half-life is the period of time it takes for an object to decay by half. It is often used to compare the rate of decay for radioactive materials.

12 A 100 g mass of a radioactive material decays at a rate of 10% every 10 years.

a Find the mass of the material after the following time periods. Round your answer to one decimal place, where necessary.

i 10 years **ii** 30 years **iii** 60 years

b Estimate the half-life of the radioactive material (i.e. find how long it takes for the material to decay to 50 g). Use trial and error and round your answer to the nearest year.

13 An ice sculpture, initially containing 150 L of water, melts at a rate of 3% per minute.

a What will be the volume of the ice sculpture after half an hour? Round your answer to the nearest litre.

b Estimate the half-life of the ice sculpture. Give your answer in minutes, correct to one decimal place.

14 The half-life of a substance is 100 years. Find the rate of decay per annum, expressed as a percentage correct to one decimal place.

ISBN 978-1-108-77290-7

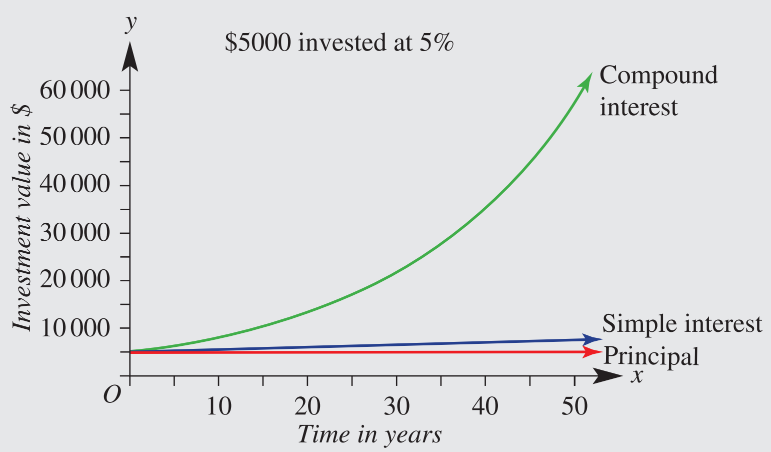
3L Compound interest

Learning intentions

- To know the meaning of the term compound interest
- To know how to apply the compound interest formula
- To know how compound interest is calculated for different time periods
- To be able to determine the total amount and the interest in a compound interest scenario

For simple interest, the interest is always calculated on the principal amount. Sometimes, however, interest is calculated on the actual amount present in an account at each time period that interest is calculated. This means that the interest is added to the amount, then the next lot of interest is calculated again using this new amount.

This process is called compound interest.



The 'magic' growth of compound interest comes from interest paid on previous interest. Retirement savings are especially suited to benefit from compound interest, as this type of investment grows at an increasingly faster rate over time, as you can see in the graph above.

Compound interest can be calculated using updated applications of the simple interest formula or by using the compound interest formula. It is a common example of exponential growth.

LESSON STARTER Investing using updated simple interest

Consider investing $400 at 12% per annum.

- Copy and complete the table below.

Time (n)	Amount (A)	Interest (I)	New amount
1st year	$400	$48	$448
2nd year	$448	$53.76	$501.76
3rd year	$501.76		
4th year			

- What is the balance at the end of 4 years if interest is added to the amount at the end of each year?
- Thinking about this as exponential growth, write a rule linking A with n.

ISBN 978-1-108-77290-7

KEY IDEAS

- **Compound interest** is calculated using updated applications of the simple interest formula.
  For example, \$100 compounded at 10% p.a. for 2 years.
  Year 1: $100 + 10\%$ of $100 = \$110$
  Year 2: $110 + 10\%$ of $110 = \$121$, so compound interest $= \$21$
- The total amount in an account using compound interest for a given number of time periods is given by:
  $A = P\left(1 + \frac{r}{100}\right)^n$, where:
  - Principal (P) = the amount of money borrowed or invested.
  - Rate of interest (r) = the percentage applied to the principal per period of investment.
  - Periods (n) = the number of periods the principal is invested.
  - Amount (A) = the total amount of your investment.
- Interest = amount (A) − principal (P)

BUILDING UNDERSTANDING

1. Consider \$500 invested at 10% p.a., compounded annually.
   a. How much interest is earned in the first year?
   b. What is the balance of the account once the first year's interest is added?
   c. How much interest is earned in the second year?
   d. What is the balance of the account at the end of the second year?
   e. Use your calculator to work out $500(1.1)^2$.
2. By considering an investment of \$4000 at 5% p.a., compounded annually, calculate the missing values in the table below.

Year	Amount (\$)	Interest (\$)	New amount (\$)
1	4000	200	4200
2	4200		
3			
4			
5			

3. Find the value of the following, correct to two decimal places.
   a. $\$1000 \times 1.05 \times 1.05$
   b. $\$1000 \times 1.05^2$
   c. $\$1000 \times 1.05 \times 1.05 \times 1.05$
   d. $\$1000 \times 1.05^3$
4. State the missing numbers.
   a. \$700 invested at 8% p.a., compounded annually for 2 years.
      $A = \square(1.08)^{\square}$
   b. \$1000 invested at 15% p.a., compounded annually for 6 years.
      $A = 1000(\square)^6$
   c. \$850 invested at 6% p.a., compounded annually for 4 years.
      $A = 850(\square)^{\square}$

ISBN 978-1-108-77290-7 Cambridge University Press

Example 29 Using the compound interest formula

Determine the amount after 5 years if \$4000 is compounded annually at 8%. Round to the nearest cent.

SOLUTION	EXPLANATION
$P = 4000, n = 5, r = 8$	List the values for the terms
$A = P\left(1 + \frac{r}{100}\right)^n$ $= 4000\left(1 + \frac{8}{100}\right)^5$	Write the formula and then substitute the known values.
$= 4000(1.08)^5$	Simplify and evaluate.
$= \$5877.31$	Write your answer to two decimal places (the nearest cent).

Now you try

Determine the amount after 6 years if \$3000 is compounded annually at 7%. Round to the nearest cent.

Example 30 Converting rates and time periods

Calculate the number of periods and the rates of interest offered per period for the following.

a 6% p.a. over 4 years, paid monthly

b 18% p.a. over 3 years, paid quarterly

SOLUTION	EXPLANATION
a $n = 4 \times 12$ $\quad r = 6 \div 12$ $= 48$ $\quad = 0.5$	4 years is the same as 48 months, as 12 months = 1 year. 6% p.a. = 6% in 1 year. Divide by 12 to find the monthly rate.
b $n = 3 \times 4$ $\quad r = 18 \div 4$ $= 12$ $\quad = 4.5$	There are 4 quarters in 1 year.

Now you try

Calculate the number of periods and the rates of interest offered per period for the following.

a 5% p.a. over 5 years, paid monthly

b 14% p.a. over 3 years, paid quarterly

ISBN 978-1-108-77290-7

Example 31 Finding compounded amounts using months

Anthony's investment of \$4000 is compounded at 8.4% p.a. over 5 years. Determine the amount he will have after 5 years if the interest is paid monthly. Round to the nearest cent.

SOLUTION	EXPLANATION
$P = 4000$	List the values of the terms you know.
$n = 5 \times 12$ $= 60$	Convert the time in years to the number of periods (in this case, months); 60 months = 5 years.
$r = 8.4 \div 12$ $= 0.7$	Convert the rate per year to the rate per period (months) by dividing by 12.
$A = P\left(1 + \frac{r}{100}\right)^n$	Write the formula.
$= 4000(1 + 0.007)^{60}$ $= 4000(1.007)^{60}$	Substitute the values, $0.7 \div 100 = 0.007$.
$= \$6078.95$	Simplify and evaluate, rounding to the nearest cent.

Now you try

Wendy's investment of \$7000 is compounded at 6.2% p.a. over 4 years. Determine the amount she will have after 4 years if the interest is paid monthly. Round to the nearest cent.

Exercise 3L

FLUENCY	1, 2–5(1/2)	2–5(1/2)	2–5(1/2)

Example 29

1 Determine the amount after 4 years if \$5000 is compounded annually at 6%. Round to the nearest cent.

Example 29

2 Determine the amount after 5 years if:
- a \$4000 is compounded annually at 5%
- b \$8000 is compounded annually at 8.35%
- c \$6500 is compounded annually at 16%
- d \$6500 is compounded annually at 8%.

3 Determine the amount if \$100 000 is compounded annually at 6% for:
- a 1 year
- b 2 years
- c 3 years
- d 5 years
- e 10 years
- f 15 years.

Example 30

4 Calculate the number of periods (n) and the rates of interest (r) offered per period for the following. (Round the interest rate to three decimal places where necessary.)
- a 6% p.a. over 3 years, paid bi-annually
- b 12% p.a. over 5 years, paid monthly
- c 4.5% p.a. over 2 years, paid fortnightly
- d 10.5% p.a. over 3.5 years, paid quarterly
- e 15% p.a. over 8 years, paid quarterly
- f 9.6% p.a. over 10 years, paid monthly

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Example 31

5 Calculate the value of the following investments if interest is compounded monthly.

a \$2000 at 6% p.a. for 2 years
b \$34 000 at 24% p.a. for 4 years
c \$350 at 18% p.a. for 8 years
d \$670 at 6.6% p.a. for $2\frac{1}{2}$ years
e \$250 at 7.2% p.a. for 12 years
f \$1200 at 4.8% p.a. for $3\frac{1}{3}$ years

PROBLEM-SOLVING	6	6, 7	6, 8

6 Darinia invests \$5000 compounded monthly at 18% p.a. Determine the value of the investment after:
a 1 month **b** 3 months **c** 5 months.

7 An investment of \$8000 is compounded at 12.6% over 3 years. Determine the amount the investor will have after 3 years if the interest is compounded monthly.

8 **a** For each rate below, calculate the amount of compound interest paid on \$8000 at the end of 3 years.
i 12% compounded annually
ii 12% compounded bi-annually (i.e. twice a year)
iii 12% compounded monthly
iv 12% compounded weekly
v 12% compounded daily
b What is the interest difference between annual and daily compounding in this case?

REASONING	9	9	9(½), 10

9 The following are expressions relating to compound interest calculations. Determine the principal (P), number of periods (n), rate of interest per period (r%), annual rate of interest (R%) and the overall time (t).

a $300(1.07)^{12}$, bi-annually
b $5000(1.025)^{24}$, monthly
c $1000(1.00036)^{65}$, fortnightly
d $3500(1.000053)^{30}$, daily
e $10\,000(1.078)^{10}$, annually
f $6000(1.0022)^{91}$, fortnightly

10 Paula must decide whether to invest her \$13 500 for 6 years at 4.2% p.a. compounded monthly or 5.3% compounded bi-annually. Decide which investment would be the best choice for Paula.

ENRICHMENT: Double your money	–	–	11

11 You have \$100 000 to invest and wish to double that amount. Use trial and error in the following.
a Determine, to the nearest whole number of years, the length of time it will take to do this using the compound interest formula at rates of:
i 12% p.a. **ii** 6% p.a. **iii** 8% p.a.
iv 16% p.a. **v** 10% p.a. **vi** 20% p.a.
b If the amount of investment is \$200 000 and you wish to double it, determine the time it will take using the same interest rates as above.
c Are the lengths of time to double your investment the same in part **a** and part **b**?

ISBN 978-1-108-77290-7

3M Comparing simple and compound interest

Learning intentions

- To know the formulas for simple and compound interest
- To know how to use technology and the formulas to compare simple and compound interest

In the following exercise, we compare compound and simple interest and look at their applications to the banking world. You are expected to use technology to its best advantage when solving the problems in this section.

Finance industries employ highly trained mathematicians to model investment outcomes and analyse risk. In 2018, Australia's total pension funds (superannuation) invested in local and global markets exceeded $2600 billion.

LESSON STARTER Who earns the most?

- Ceanna invests $500 at 8% p.a., compounded monthly over 3 years.
- Huxley invests $500 at 10% p.a., compounded annually over 3 years.
- Loreli invests $500 at 15% p.a. simple interest over 3 years.
  - How much does each person have at the end of the 3 years?
  - Who earned the most?

KEY IDEAS

For either form of interest, you can calculate the total amount of your investment using technology.

- **CAS or graphics calculator**
  To create programs for the two types of interest, enter the following data. This will allow you to calculate both types of interest for a given time period. If you invest $100 000 at 8% p.a. paid monthly for 2 years, you will be asked for P, r, t or n and the calculator will do the work for you. (Note: Some modifications may be needed for other calculators or languages.)

```
Define simple()=
Prgm
Request "Enter Principal:",p
Request "Enter interest rate:",r
Request "Enter time:",t
(p·r·t)/100 → i
Disp "Interest is",i
Disp "Amount is",p+i
EndPrgm
```

```
Define compound()=
Prgm
Request "Enter Principal:",p
Request "Enter interest rate:",r
Request "Enter time:",t
p·(1+r/100)^t → a
Disp "Interest is",round(a−p,2)
Disp "Amount is",round(a,2)
EndPrgm
```

ISBN 978-1-108-77290-7

- **Spreadsheet**

Copy and complete the spreadsheets as shown, to compile a simple interest and compound interest sheet.

Book1.xls

	A	B	C	D	E
1	Interest calculator				
2		Principal		Rate	
3		4000		=5.4/100/12	
4					
5		Simple interest		Compound interest	
6	Time (months)	Interest	Amount	Interest	Amount
7	0	0	=B$3+B7	0	=B$3*(1+D$3)^A7
8	=A7+1	=B$3*D$3	=C7+B8	=E8-E7	=B$3*(1+D$3)^A8
9	=A8+1	=B$3*D$3	=C8+B9	=E9-E8	=B$3*(1+D$3)^A9
10	=A9+1	=B$3*D$3	=C9+B10	=E10-E9	=B$3*(1+D$3)^A10
11	=A10+1	=B$3*D$3	=C10+B11	=E11-E10	=B$3*(1+D$3)^A11
12	=A11+1	=B$3*D$3	=C11+B12	=E12-E11	=B$3*(1+D$3)^A12
13	=A12+1	=B$3*D$3	=C12+B13	=E13-E12	=B$3*(1+D$3)^A13
14					
15					
16					
17					

Sheet 1 Sheet 2 Sheet 3

Fill in the principal in B3 and the rate per period in D3. For example, for \$4000 invested at 5.4% p.a. paid monthly, B3 will be 4000 and D3 will be $\frac{0.054}{12}$.

- Recall the simple interest formula from previous years: $I = \frac{Prt}{100}$ where I is the total amount of interest, P is the initial amount or principal, r is percentage interest rate and t is the time.

BUILDING UNDERSTANDING

1. Which is better on an investment of \$100 for 2 years:
   A simple interest calculated at 20% p.a. or
   B compound interest calculated at 20% p.a. and paid annually?
2. State the values of P, r and n for an investment of \$750 at 7.5% p.a., compounded annually for 5 years.
3. State the values of I, P, r and t for an investment of \$300 at 3% p.a. simple interest over 300 months.
4. Use the simple interest formula $I = \frac{Prt}{100}$ to find the simple interest on an investment of \$2000 at 4% p.a. over 3 years.

ISBN 978-1-108-77290-7

Example 32 Comparing simple and compound interest using technology

Find the total amount of the following investments, using technology.

a $5000 at 5% p.a., compounded annually for 3 years

b $5000 at 5% p.a., simple interest for 3 years

SOLUTION	EXPLANATION
a $5788.13	$A = P\left(1 + \frac{r}{100}\right)^n$ using $P = 5000$, $r = 5$ and $n = 3$. Alternatively, use a spreadsheet or computer program. Refer to the Key ideas.
b $5750	Total $= P + \frac{Prt}{100}$ using $P = 5000$, $r = 5$ and $t = 3$.

Now you try

Find the total amount of the following investments, using technology.

a $4000 at 6% p.a., compounded annually for 3 years

b $4000 at 6% p.a., simple interest for 3 years

Exercise 3M

FLUENCY	1–3	2, 3	2, 3

1 Find the total amount of the following investments, using technology.

Example 32a **a** $7000 at 4% p.a., compounded annually for 5 years

Example 32b **b** $7000 at 4% p.a., simple interest for 5 years

Example 32

2 **a** Find the total amount of the following investments, using technology.

- **i** $6000 at 6% p.a., compounded annually for 3 years
- **ii** $6000 at 3% p.a., compounded annually for 5 years
- **iii** $6000 at 3.4% p.a., compounded annually for 4 years
- **iv** $6000 at 10% p.a., compounded annually for 2 years
- **v** $6000 at 5.7% p.a., compounded annually for 5 years

b Which of the above yields the most interest?

3 **a** Find the total amount of the following investments, using technology where possible.

- **i** $6000 at 6% p.a. simple interest for 3 years
- **ii** $6000 at 3% p.a. simple interest for 6 years
- **iii** $6000 at 3.4% p.a. simple interest for 7 years
- **iv** $6000 at 10% p.a. simple interest for 2 years
- **v** $6000 at 5.7% p.a. simple interest for 5 years

b Which of the above yields the most interest?

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PROBLEM-SOLVING | 4 | 4 | 4

4 a Determine the total simple and compound interest accumulated in the following cases.
   i $4000 at 6% p.a. payable annually for:
      I 1 year II 2 years III 5 years IV 10 years.
   ii $4000 at 6% p.a. payable bi-annually for:
      I 1 year II 2 years III 5 years IV 10 years.
   iii $4000 at 6% p.a. payable monthly for:
      I 1 year II 2 years III 5 years IV 10 years.
b Would you prefer the same rate of compound interest or simple interest if you were investing money?
c Would you prefer the same rate of compound interest or simple interest if you were borrowing money and paying off the loan in instalments?

REASONING | 5 | 5 | 6

5 a Copy and complete the following table when simple interest is applied.

Principal	Rate	Overall time	Interest	Amount
$7000		5 years		$8750
$7000		5 years		$10 500
	10%	3 years	$990	
	10%	3 years	$2400	
$9000	8%	2 years		
$18 000	8%	2 years		

b Explain the effect on the interest when we double the:
   i rate ii period iii overall time.

6 Copy and complete the following table when compound interest is applied. You may need to use a calculator and trial and error to find some of the missing numbers. Write your answers correct to two decimal places where necessary.

Principal	Rate	Period	Overall time	Interest	Amount
$7000		annually	5 years		$8750
$7000		annually	5 years		$10 500
$9000	8%	fortnightly	2 years		
$18 000	8%	fortnightly	2 years		

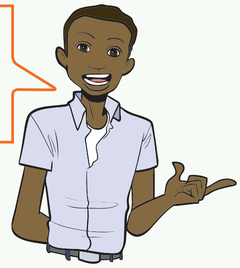
ENRICHMENT: Changing the parameters | – | – | 7, 8

7 If you invest $5000, determine the interest rate per annum (to two decimal places) if the total amount is approximately $7500 after 5 years and if:
a interest is compounded annually
b interest is compounded quarterly
c interest is compounded weekly
Comment on the effect of changing the period for each payment on the rate needed to achieve the same total amount in a given time.

8 a Determine, to one decimal place, the equivalent simple interest rate for the following investments over 3 years.
   i $8000 at 4%, compounded annually ii $8000 at 8%, compounded annually
b If you double or triple the compound interest rate, how is the simple interest rate affected?

Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.



1 Write $3^{n-1} + 3^{n-1} + 3^{n-1}$ as a single term with base 3.

2 Simplify.

a $\dfrac{25^6 \times 5^4}{125^5}$

b $\dfrac{8^x \times 3^x}{6^x \times 9^x}$

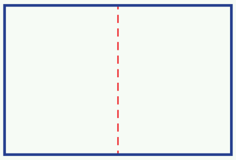
3 Solve $3^{2x} \times 27^{x+1} = 81$.

4 Simplify.

a $\dfrac{2^{n+1} - 2^{n+2}}{2^{n-1} - 2^{n-2}}$

b $\dfrac{2^{a+3} - 4 \times 2^a}{2^{2a+1} - 4^a}$

5 A rectangular piece of paper has an area of $100\sqrt{2}$ cm^2. The piece of paper is such that, when it is folded in half along the dashed line as shown, the new rectangle is similar (i.e. of the same shape) to the original rectangle. What are the dimensions of the piece of paper?



6 Simplify the following, leaving your answer with a rational denominator.

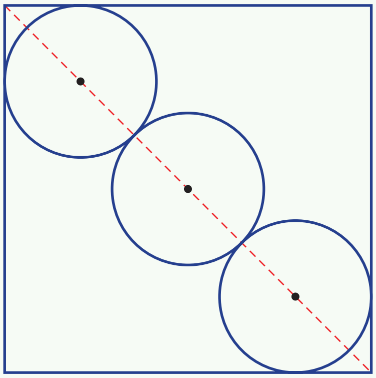
$$\frac{\sqrt{2}}{2\sqrt{2}+1} + \frac{2}{\sqrt{3}+1}$$

7 Simplify.

a $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\sqrt{xy}}$

b $\dfrac{x^{\frac{1}{2}}y^{-\frac{1}{2}} - x^{-\frac{1}{2}}y^{\frac{1}{2}}}{x^{-1}y^{-1}}$

8 Three circles, each of radius 1 unit, fit inside a square such that the two outer circles touch the middle circle and the sides of the square, as shown. Given the centres of the circle lie on the diagonal of the square, find the exact area of the square.



9 Given that $5^{x+1} - 5^{x-2} = 620\sqrt{5}$, find the value of x.

ISBN 978-1-108-77290-7

Negative indices

$a^{-m} = \frac{1}{a^m}$

$\frac{1}{a^{-m}} = a^m$

Scientific notation

Numbers (large and small) in scientific notation are of the form $a \times 10^m$, where $1 \le a < 10$ or $-10 < a \le -1$ and m is an integer.

e.g. $3.45 \times 10^5 = 345\,000$

$2.1 \times 10^{-3} = 0.0021$

e.g 1475 231 in scientific notation, using three significant figures is 1.48×10^6.

Index laws

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $a^m \div a^n = a^{m-n}$

Law 3: $(a^m)^n = a^{mn}$

Law 4: $(ab)^m = a^m b^m$

Law 5: $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$

Zero power : $a^0 = 1,\ a \ne 0$

Rational Indices (10A)

$\sqrt{a} = a^{\frac{1}{2}}$

$\sqrt[n]{a} = a^{\frac{1}{n}}$

$\sqrt[n]{a^m} = a^{\frac{m}{n}}$

e.g. $27^{\frac{1}{3}} = \sqrt[3]{27}$

$= 3$

Indices and surds

Simplifying surds (10A)

A surd uses symbol $\sqrt{\ }$ and as a decimal is infinite and non-recurring

Simplify surds: use the highest square factor.

e.g. **1** $\sqrt{20} = \sqrt{4 \times 5}$

$= \sqrt{4} \times \sqrt{5}$

$= 2\sqrt{5}$

2 $4\sqrt{27} = 4 \times \sqrt{9} \times \sqrt{3}$

$= 12\sqrt{3}$

Exponential equations (10A)

If $a^x = a^y$, then $x = y$.

e.g. **1** $2^x = 16$

$2^x = 2^4$

$\therefore x = 4$

2 $27^{x-4} = 9^x$

$(3^3)^{x-4} = (3^2)^x$

$3^{3x-12} = 3^{2x}$

$\therefore 3x - 12 = 2x$

$x = 12$

Multiply/Divide (10A)

$\sqrt{x} \times \sqrt{y} = \sqrt{xy}$

$a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$

$\sqrt{x} \div \sqrt{y} = \sqrt{\frac{x}{y}}$

$a\sqrt{x} \div (b\sqrt{y}) = \frac{a}{b}\sqrt{\frac{x}{y}}$

Add/Subtract (10A)

Like surds only

e.g. **1** $3\sqrt{2} + 4\sqrt{2} = 7\sqrt{2}$

$4\sqrt{2} + \sqrt{5} - \sqrt{2} = 3\sqrt{2} + \sqrt{5}$

Simplify first

e.g. **2** $3\sqrt{8} - 4\sqrt{2}$

$= 3 \times \sqrt{4} \times \sqrt{2} - 4\sqrt{2}$

$= 6\sqrt{2} - 4\sqrt{2}$

$= 2\sqrt{2}$

Exponential growth/decay

Rule: amount $A = A_0\left(1 \pm \frac{r}{100}\right)^n$

A_0 = initial amount

r = rate of growth/decay, as a %

n = time period

Use + for growth.

Use – for decay.

Rationalise the denominator (10A)

Express with a whole number in the denominato r

$\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}$

$= \frac{2\sqrt{5}}{5}$

Compound interest

$A = P\left(1 + \frac{r}{100}\right)^n$

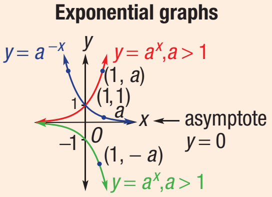
A (amount): total value

P (principal): initial amount

r (interest rate): per period

n (period): number of periods

Exponential graphs



ISBN 978-1-108-77290-7

Cambridge University Press

Chapter checklist

Chapter checklist: Success criteria

			✔
3A	1. **I can decide if a number is rational or irrational.** e.g. Express $\sqrt{6}$ as a decimal and decide if it is rational or irrational.	10A	
3A	2. **I can simplify a surd using the highest square number factor.** e.g. Simplify $\sqrt{75}$.	10A	
3A	3. **I can express a surd as a square root of an integer.** e.g. Express $3\sqrt{7}$ in the form $\sqrt{a}$ where a is an integer.	10A	
3B	4. **I can add and subtract expressions involving like surds.** e.g. Simplify $\sqrt{3} + 5\sqrt{2} + 3\sqrt{3} - 2\sqrt{2}$.	10A	
3B	5. **I can simplify surds to add or subtract.** e.g. Simplify $3\sqrt{12} + 2\sqrt{27}$.	10A	
3C	6. **I can multiply surds.** e.g. Simplify $3\sqrt{5} \times 2\sqrt{10}$.	10A	
3C	7. **I can simplify surds using division.** e.g. Simplify $4\sqrt{15} \div \sqrt{3}$.	10A	
3C	8. **I can apply the distributive law to expressions involving surds.** e.g. Expand and simplify $2\sqrt{5}\left(3\sqrt{3} - \sqrt{5}\right)$.	10A	
3D	9. **I can rationalise a denominator.** e.g. Rationalise $\frac{4\sqrt{2}}{\sqrt{7}}$.	10A	
3E	10. **I can apply index laws to multiply common bases.** e.g. Simplify $3x^2y^3 \times 2xy^4$.		
3E	11. **I can apply index laws in division.** e.g. Simplify $3x^5y^3 \div (12x^2y)$.		
3E	12. **I can simplify using a number of index laws.** e.g. Simplify $3(x^2y)^3 \times \left(\frac{2}{x}\right)^2$ using index laws.		
3E	13. **I can use the zero power.** e.g. Evaluate $(3a)^0 - 5a^0$.		
3F	14. **I can rewrite an expression using positive indices.** e.g. Express $3x^2y^{-3}$ using positive indices.		
3F	15. **I can rewrite an expression with a negative power in the denominator using positive indices.** e.g. Express $\frac{3}{y^{-4}}$ using positive indices.		
3F	16. **I can simplify expressions and apply index laws to negative indices.** e.g. Simplify $\frac{(x^{-1}y)^{-3}}{4x^{-2}y^3}$ and express using positive indices.		

ISBN 978-1-108-77290-7

Section	Checklist item		✔
3G	**17. I can convert from scientific notation to a basic numeral.** e.g. Write 3.07×10^4 and 4.1×10^{-3} as basic numerals.		
3G	**18. I can convert to scientific notation using significant figures.** e.g. Write 0.0035892 in scientific notation using three significant figures.		
3G	**19. I can use technology to perform calculations in scientific notation.** e.g. Evaluate $\sqrt{3.02 \times 10^{24}}$, answering in scientific notation using three significant figures.		
3H	**20. I can write roots in index form.** e.g. Express $\sqrt[3]{x^6}$ in index form.	10A	
3H	**21. I can write rational indices in surd form.** e.g. Express $4^{\frac{3}{2}}$ in surd form.	10A	
3H	**22. I can evaluate numbers with rational indices.** e.g. Evaluate $25^{-\frac{1}{2}}$ without a calculator.	10A	
3I	**23. I can solve exponential equations using a common base.** e.g. Solve $3^x = 27$ for x.	10A	
3I	**24. I can solve exponential equations that require a common base to be found.** e.g. Solve $25^x = 125^{x-2}$.	10A	
3J	**25. I can sketch a graph of an exponential equation.** e.g. Sketch $y = 3^x$ labelling the y-intercept and one other point.		
3J	**26. I can sketch exponential graphs involving reflections.** e.g. Sketch $y = 2^x$, $y = -2^x$ and $y = 2^{-x}$ on the same axes.		
3J	**27. I can find the intersection of horizontal lines and exponential graphs.** e.g. Find the intersection of the graphs of $y = 3^x$ and $y = 27$.		
3K	**28. I can form an exponential rule for a situation.** e.g. Write an exponential rule for the value of Scott's car purchased for $35 000 and decreasing in value by 15% per year.		
3K	**29. I can apply exponential rules.** e.g. The value of a house in n years' time is given by $V = 590\,000(1.06)^n$. What will be the value in 4 years' time and find when it will be valued at $1 000 000.		
3L	**30. I can calculate compound interest using the formula.** e.g. Determine the amount after 3 years if $6000 is compounded annually at 4%. Round to the nearest cent.		
3L	**31. I can find compounded amounts using different time periods.** e.g. An investment of $2000 is compounded at 4.8% over 3 years. Determine the amount after 3 years if the interest is paid monthly.		
3M	**32. I can compare interest using technology.** e.g. Use technology to compare the total amount of the investment of $6000 for 4 years at 5% p.a. using simple interest and compound interest.		

ISBN 978-1-108-77290-7

Cambridge University Press

Chapter review

Short-answer questions

3A 10A

1 Simplify the following surds.

a $\sqrt{24}$ b $\sqrt{72}$ c $3\sqrt{200}$ d $4\sqrt{54}$

e $\sqrt{\frac{4}{49}}$ f $\sqrt{\frac{8}{9}}$ g $\frac{5\sqrt{28}}{2}$ h $\frac{2\sqrt{45}}{15}$

3B/C 10A

2 Simplify the following.

a $2\sqrt{3}+4+5\sqrt{3}$ b $6\sqrt{5}-\sqrt{7}-4\sqrt{5}+3\sqrt{7}$ c $\sqrt{8}+3\sqrt{2}$

d $4\sqrt{3}+2\sqrt{18}-4\sqrt{2}$ e $2\sqrt{5}\times\sqrt{6}$ f $-3\sqrt{2}\times 2\sqrt{10}$

g $\frac{2\sqrt{15}}{\sqrt{3}}$ h $\frac{5\sqrt{14}}{15\sqrt{2}}$ i $\frac{\sqrt{27}}{3}-\sqrt{3}$

3C 10A

3 Expand and simplify.

a $\sqrt{2}(2\sqrt{3}+4)$ b $2\sqrt{3}(2\sqrt{15}-\sqrt{3})$ c $(\sqrt{11})^2$ d $(4\sqrt{3})^2$

3D 10A

4 Rationalise the denominator.

a $\frac{1}{\sqrt{6}}$ b $\frac{10}{\sqrt{2}}$ c $\frac{6\sqrt{3}}{\sqrt{2}}$ d $\frac{4\sqrt{7}}{\sqrt{2}}$

e $\frac{3\sqrt{3}}{2\sqrt{6}}$ f $\frac{5\sqrt{5}}{4\sqrt{10}}$ g $\frac{\sqrt{5}+2}{\sqrt{2}}$ h $\frac{4\sqrt{2}-\sqrt{3}}{\sqrt{3}}$

3E/F

5 Simplify the following, expressing all answers with positive indices when required.

a $(5y^3)^2$ b $7m^0-(5n)^0$ c $4x^2y^3\times 5x^5y^7$

d $3x^2y^{-4}$ e $\left(\frac{3x}{y^{-3}}\right)^2\times\frac{x^{-5}}{6y^2}$ f $\frac{3(a^2b^{-4})^2}{(2ab^2)^2}\div\frac{(ab)^{-2}}{(3a^{-2}b)^2}$

3H 10A

6 Express in index form.

a $\sqrt{21}$ b $\sqrt[3]{x}$ c $\sqrt[3]{m^5}$ d $\sqrt[5]{a^2}$

e $\sqrt{10x^3}$ f $\sqrt[3]{2a^9b}$ g $7\sqrt{7}$ h $4\sqrt[3]{4}$

3H 10A

7 Evaluate these without using a calculator.

a $25^{\frac{1}{2}}$ b $64^{\frac{1}{3}}$ c $\left(\frac{1}{8}\right)^{\frac{1}{3}}$

d $49^{-\frac{1}{2}}$ e $100^{-\frac{1}{2}}$ f $125^{-\frac{1}{3}}$

3G

8 a Write the following numbers as a basic numeral.

i 3.21×10^3 ii 4.024×10^6 iii 7.59×10^{-3} iv 9.81×10^{-5}

b Write the following numbers in scientific notation, using three significant figures.

i 0.0003084 ii 0.0000071753 iii 5678200 iv 119830000

3K

9 Form exponential rules for the following situations.

a An antique bought for \$800 is expected to grow in value by 7% per year.

b A balloon with volume 3000 cm^3 is leaking air at a rate of 18% per minute.

ISBN 978-1-108-77290-7 Cambridge University Press

3L/M

10 Determine the final amount after 4 years if:
- **a** \$1000 is compounded annually at 5%
- **b** \$3000 is compounded monthly at 4%
- **c** \$5000 is compounded daily at 3%.

3I 10A

11 Solve the following exponential equations for x.
- **a** $3^x = 27$
- **b** $7^x = 49$
- **c** $4^{2x+1} = 64$
- **d** $2^{x-2} = 16$
- **e** $9^x = \frac{1}{81}$
- **f** $5^x = \frac{1}{125}$
- **g** $36^x = 216$
- **h** $8^{x+1} = 32$
- **i** $7^{3x-4} = 49^x$
- **j** $11^{x-5} = \frac{1}{121}$
- **k** $100^{x-2} = 1000^x$
- **l** $9^{3-2x} = 27^{x+2}$

3J

12 Sketch the following graphs, labelling the y-intercept and the point where $x = 1$.
- **a** $y = 4^x$
- **b** $y = -3^x$
- **c** $y = 5^{-x}$

Multiple-choice questions

3A 10A

1 Which of the following is a surd?

A $\sqrt{36}$ **B** π **C** $\sqrt{7}$ **D** $\sqrt[3]{8}$ **E** $1.\dot{6}$

3A 10A

2 A square has an area of 75 square units. Its side length, in simplified form, is:

A $3\sqrt{5}$ **B** 8.5 **C** $25\sqrt{3}$ **D** $5\sqrt{3}$ **E** $6\sqrt{15}$

3A 10A

3 $4\sqrt{5}$ is equivalent to:

A $\sqrt{100}$ **B** $\sqrt{80}$ **C** $2\sqrt{10}$ **D** $\sqrt{20}$ **E** $\sqrt{40}$

3B 10A

4 $3\sqrt{12} + 7 - 4\sqrt{3}$ simplifies to:

A $7 + \sqrt{3}$ **B** $8\sqrt{3} + 7$ **C** $6\sqrt{6} + 3\sqrt{3}$ **D** $3\sqrt{3}$ **E** $2\sqrt{3} + 7$

3C 10A

5 The expanded form of $2\sqrt{5}(5 - 3\sqrt{3})$ is:

A $10\sqrt{5} - 6\sqrt{15}$ **B** $7\sqrt{5} - 5\sqrt{15}$ **C** $10\sqrt{5} - 12\sqrt{2}$ **D** $10 - 5\sqrt{15}$ **E** $7\sqrt{5} - 5\sqrt{3}$

3D 10A

6 $\frac{2\sqrt{5}}{\sqrt{6}}$ is equivalent to:

A $\frac{2\sqrt{30}}{\sqrt{6}}$ **B** $\frac{5\sqrt{6}}{3}$ **C** $2\sqrt{5}$ **D** $\frac{\sqrt{30}}{3}$ **E** $\frac{\sqrt{30}}{10}$

3E

7 The simplified form of $\frac{(6xy^3)^2}{3x^3y^2 \times 4x^4y^0}$ is:

A $\frac{y^4}{2x^6}$ **B** $\frac{3y^3}{x^{10}}$ **C** $\frac{y^6}{x^5}$ **D** $\frac{3y^4}{x^5}$ **E** $\frac{y^6}{2x^6}$

3F

8 $\frac{8a^{-1}b^{-2}}{12a^3b^{-5}}$ expressed with positive indices is:

A $\frac{2a^2}{3b^3}$ **B** $\frac{a^2b^3}{96}$ **C** $\frac{2b^3}{3a^4}$ **D** $-\frac{2b^7}{3a^2}$ **E** $\frac{3}{2}a^4b^7$

ISBN 978-1-108-77290-7

3G **9** The radius of the Earth is approximately 6 378 137 m. In scientific notation, using three significant figures, this is:

A 6.378×10^6 m **B** 6.38×10^6 m **C** 6.4×10^5 m
D 6.37×10^6 m **E** 6.36×10^6 m

3H 10A **10** $\sqrt{8x^6}$ in index form is:

A $8x^3$ **B** $8x^2$ **C** $4x^3$ **D** $8^{\frac{1}{2}}x^3$ **E** $8^{\frac{1}{2}}x^4$

3I 10A **11** The solution to $3^{2x-1} = 9^2$ is:

A $x = \frac{3}{2}$ **B** $x = 2$ **C** $x = \frac{5}{2}$ **D** $x = 6$ **E** $x = 3$

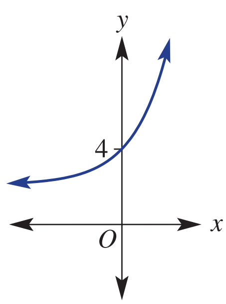
3K **12** A rule for the amount $\$A$ in an account after n years for an initial investment of \$5000 that is increasing at 7% per annum is:

A $A = 5000(1.7)^n$ **B** $A = 5000(0.93)^n$ **C** $A = 5000(0.3)^n$
D $A = 5000(1.07)^n$ **E** $A = 5000(0.7)^n$

3J **13** The graph of $y = 3^x$ intersects the y-axis at:

A $(0, 3)$ **B** $(3, 0)$ **C** $(0, 1)$ **D** $(1, 3)$ **E** $\left(0, \frac{1}{3}\right)$

3J **14** The graph of $y = 4^{-x}$ is:

A


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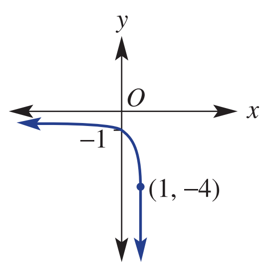

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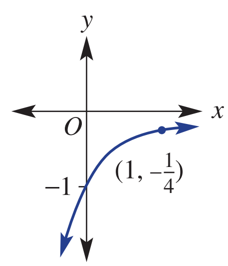

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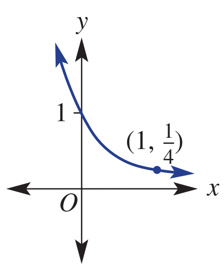

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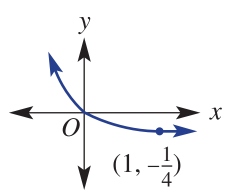

3J **15** The graph of $y = 3^x$ and $y = \frac{1}{3}$ intersect at the point:

A $(1, 3)$ **B** $\left(-1, \frac{1}{3}\right)$ **C** $(-1, 3)$ **D** $\left(\frac{1}{9}, \frac{1}{3}\right)$ **E** $\left(1, \frac{1}{3}\right)$

ISBN 978-1-108-77290-7

Extended-response questions

10A **1** A small rectangular jewellery box has a base with dimensions $3\sqrt{15}$ cm by $(12 + \sqrt{3})$ cm and a height of $(2\sqrt{5} + 4)$ cm.

- **a** Determine the exact area of the base of the box, in expanded and simplified form.
- **b** What is the exact volume of the box?
- **c** Julie's earring boxes occupy an area of $9\sqrt{5}$ cm^2. What is the exact number that would fit across the base of the jewellery box? Give your answer with a rational denominator.
- **d** The surface of Julie's rectangular dressing table has dimensions $(\sqrt{2} - 1)$ m by $(\sqrt{2} + 1)$ m.
  - **i** Find the area of the dressing table, in square centimetres.
  - **ii** What percentage of the area of the dressing table does the jewellery box occupy? Give your answer to one decimal place.

2 Georgia invests $10000 in shares in a new company. She has been told that their value is expected to increase at 6.5% per year.

- **a** Write a rule for Georgia's expected value, V dollars, in shares after n years.
- **b** Use your rule to find the value she expects the shares to be after:
  - **i** 2 years
  - **ii** 5 years
- **c** When her shares are valued at $20000 Georgia plans to cash them in. According to this rule, how many years will it take to reach this amount? Give your answer to one decimal place.
- **d** After 6 years there is a downturn in the market and the shares start to drop, losing value at 3% per year.
  - **i** What is the value of Georgia's shares prior to the downturn in the market? Give your answer to the nearest dollar.
  - **ii** Using your answer from part **d i**, write a rule for the expected value, V dollars, of Georgia's shares t years after the market downturn.
  - **iii** Ten years after Georgia initially invested in the shares the market is still falling at this rate. She decides it's time to sell her shares. What is their value, to the nearest dollar? How does this compare with the original amount of $10000 she invested?

ISBN 978-1-108-77290-7