CHAPTER 5 Quadratic expressions and equations

Mathematics of flight

If you blow air across the top of a small piece of paper it will lift rather than be forced down. Daniel Bernoulli, an 18th-century Swiss mathematician and scientist, discovered the relationship between fluid pressure and fluid speed, simply represented by this quadratic equation:

$P = -\frac{1}{2}v^2 + c$, where P is internal pressure, v is speed and c is a constant.

As air is a fluid, Bernoulli's law shows us that with increased air speed there is decreased internal air pressure. This explains why, when a cyclonic wind blows across a house roof, the stronger air pressure from inside the house can push the roof off. The aerofoil shape of a bird or plane wing (i.e. concave-down on the top) causes air to flow at a higher speed over the wing's upper surface and hence air pressure is decreased. The air of higher pressure under the wing helps to lift the plane or bird. Bernoulli's quadratic equation shows us that wing lift is proportional to the square of the airspeed.

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Wing loading in kg/m^2 is calculated by dividing the fully loaded weight of a plane by the wing area. Planes with a large wing loading need large wings or a high airspeed or both to achieve lift. The gigantic Antonov cargo plane has a wing loading of around 660 kg/m^2 and can lift enormous weights due to its huge wing area of 905 m^2 and airspeed of 800 km/h. A hang glider has large wings and a low mass giving a wing loading of only 6 kg/m^2. Hence it can still obtain lift at very slow speeds.

Online resources

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In this chapter

Victorian Curriculum

NUMBER AND ALGEBRA
Patterns and algebra

Factorise algebraic expressions by taking out a common algebraic factor (VCMNA329)

Expand binomial products and factorise monic quadratic expressions using a variety of strategies (VCMNA332)

Substitute values into formulas to determine an unknown and re-arrange formulas to solve for a particular term (VCMNA333)

Linear and non-linear relationships

Solve simple quadratic equations using a range of strategies (VCMNA341)

(10A) Factorise monic and non-monic quadratic expressions and solve a wide range of quadratic equations derived from a variety of contexts (VCMNA362)

5A Expanding expressions CONSOLIDATING

Learning intentions

- To review how to apply the distributive law to expand brackets
- To be able to expand binomial products including perfect squares
- To be able to form a difference of perfect squares by expansion

You will recall that expressions that include numerals and pronumerals are central to the topic of algebra. Sound skills in algebra are essential for solving most mathematical problems and this includes the ability to expand expressions involving brackets. This includes binomial products, perfect squares and the difference of perfect squares. Exploring how projectiles fly subject to the Earth's gravity, for example, can be modelled with expressions with and without brackets.

Business analysts develop profit equations, which are quadratics, when sales and profit/item are linear relations of the selling price, e.g. $\$p$/ice-cream:

$$\begin{aligned}\text{Profit/week} &= \text{weekly sales} \times \text{profit/item}\\ &= 150(10 - p) \times (p - 2)\\ &= -150(p^2 - 12p + 20)\end{aligned}$$

LESSON STARTER Five key errors

Here are five expansion problems with incorrect answers. Discuss what error has been made and then give the correct expansion.

- $-2(x - 3) = -2x - 6$
- $(x + 3)^2 = x^2 + 9$
- $(x - 2)(x + 2) = x^2 + 4x - 4$
- $5 - 3(x - 1) = 2 - 3x$
- $(x + 3)(x + 5) = x^2 + 8x + 8$

KEY IDEAS

- **Like terms** have the same pronumeral part.
  - They can be collected (i.e. added and subtracted) to form a single term. For example: $7x - 11x = -4x$ and $4a^2b - 7ba^2 = -3a^2b$
- The **distributive law** is used to expand brackets.
  - $a(b + c) = ab + ac$ and $a(b - c) = ab - ac$
  - $(a + b)(c + d) = ac + ad + bc + bd$
  - $(a + b)(c + d)$ is called a binomial product because each expression in the brackets has two terms.

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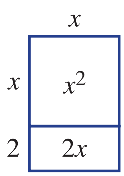
- **Perfect squares**
  - $(a+b)^2 = (a+b)(a+b)$
    $= a^2 + 2ab + b^2$
  - $(a-b)^2 = (a-b)(a-b)$
    $= a^2 - 2ab + b^2$
- **Difference of perfect squares (DOPS)**
  - $(a+b)(a-b) = a^2 - b^2$
- By definition, a perfect square is an integer that is the square of an integer; however, the rules above also apply for a wide range of values for a and b, including all real numbers.

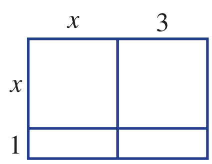
BUILDING UNDERSTANDING

1 Use each diagram to help expand the expressions.

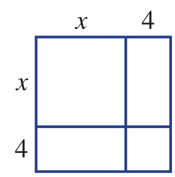
a $x(x+2)$



b $(x+3)(x+1)$



c $(x+4)^2$



2 Simplify these expressions.

a $2 \times 3x$ **b** $-4 \times 5x$ **c** $-x \times 4x$ **d** $5x \div 10$
e $-6x^2 \div (2x)$ **f** $3x - 21x$ **g** $-3x + 8x$ **h** $-5x - 8x$

Example 1 Expanding simple expressions

Expand and simplify where possible.

a $-3(x-5)$ **b** $2x(1-x)$ **c** $\frac{2}{7}(14x+3)$ **d** $x(2x-1) - x(3-x)$

SOLUTION	EXPLANATION
a $-3(x-5) = -3x + 15$	Use the distributive law: $a(b-c) = ab - ac$. $-3 \times x = -3x$ and $-3 \times (-5) = 15$
b $2x(1-x) = 2x - 2x^2$	Recall that $2x \times (-x) = -2x^2$.
c $\frac{2}{7}(14x+3) = \frac{2}{7} \times 14x + \frac{2}{7} \times 3$	When multiplying fractions cancel before multiplying numerators and denominators.
$= 4x + \frac{6}{7}$	Recall that $3 = \frac{3}{1}$.

Continued on next page

d $x(2x - 1) - x(3 - x) = 2x^2 - x - 3x + x^2$
$= 3x^2 - 4x$

Apply the distributive law to each set of brackets first, then simplify by collecting like terms. Recall that $-x \times (-x) = x^2$.

Now you try

Expand and simplify where possible.

a $-2(x - 4)$ **b** $5x(4 - x)$ **c** $\frac{3}{5}(10x + 1)$ **d** $x(5x - 1) - x(2 - 3x)$

Example 2 Expanding binomial products, perfect squares and difference of perfect squares

Expand the following.

a $(x + 5)(x + 4)$ **b** $(x - 4)^2$ **c** $(2x + 1)(2x - 1)$

SOLUTION

a $(x + 5)(x + 4) = x^2 + 4x + 5x + 20$
$= x^2 + 9x + 20$

EXPLANATION

For binomial products use $(a + b)(c + d) = ac + ad + bc + bd$. Simplify by collecting like terms.

b $(x - 4)^2 = (x - 4)(x - 4)$
$= x^2 - 4x - 4x + 16$
$= x^2 - 8x + 16$

Rewrite and expand using the distributive law.

Alternatively:
$(x - 4)^2 = x^2 - 2(x)(4) + 4^2$
$= x^2 - 8x + 16$

Alternatively for perfect squares $(a - b)^2 = a^2 - 2ab + b^2$. Here $a = x$ and $b = 4$.

c $(2x + 1)(2x - 1) = 4x^2 - 2x + 2x - 1$
$= 4x^2 - 1$

Expand, recalling that $2x \times 2x = 4x^2$. Cancel the $-2x$ and $+2x$ terms.

Alternatively:
$(2x + 1)(2x - 1) = (2x)^2 - (1)^2$
$= 4x^2 - 1$

Alternatively for difference of perfect squares $(a - b)(a + b) = a^2 - b^2$. Here $a = 2x$ and $b = 1$.

Now you try

Expand the following.

a $(x + 2)(x + 5)$ **b** $(x - 2)^2$ **c** $(3x + 2)(3x - 2)$

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Example 3 Expanding more binomial products

Expand and simplify.

a $(2x-1)(3x+5)$ b $2(x-3)(x-2)$ c $(x+2)(x+4)-(x-2)(x-5)$

SOLUTION

a $(2x-1)(3x+5) = 6x^2+10x-3x-5$
$= 6x^2+7x-5$

EXPLANATION

Expand using the distributive law and simplify.
Note: $2x \times 3x = 2 \times 3 \times x \times x = 6x^2$.

SOLUTION

b $2(x-3)(x-2) = 2(x^2-2x-3x+6)$
$= 2(x^2-5x+6)$
$= 2x^2-10x+12$

EXPLANATION

First expand the brackets using the distributive law, simplify and then multiply each term by 2.

SOLUTION

c $(x+2)(x+4)-(x-2)(x-5)$
$= (x^2+4x+2x+8)-(x^2-5x-2x+10)$
$= (x^2+6x+8)-(x^2-7x+10)$
$= x^2+6x+8-x^2+7x-10$
$= 13x-2$

EXPLANATION

Expand each binomial product.
Remove brackets in the last step before simplifying.
$-(x^2-7x+10) = -1 \times x^2 + (-1) \times (-7x) + (-1) \times 10$
$= -x^2+7x-10$

Now you try

Expand and simplify.

a $(3x-1)(2x+7)$ b $3(x-1)(x-4)$ c $(x+3)(x+1)-(x-3)(x-4)$

Exercise 5A

FLUENCY	1, 2–6(1/3)	2–6(1/3)	2–6(1/4)

1 Expand and simplify where possible.

Example 1a
a i $-4(x-1)$ ii $-2(x-6)$

Example 1b
b i $3x(2-x)$ ii $7x(5-x)$

Example 1c
c i $\frac{4}{5}(15x+2)$ ii $\frac{7}{9}(18x-1)$

Example 1a–c
2 Expand and simplify where possible.

a $2(x+5)$ b $3(x-4)$ c $-5(x+3)$ d $-4(x-2)$
e $3(2x-1)$ f $4(3x+1)$ g $-2(5x-3)$ h $-5(4x+3)$
i $x(2x+5)$ j $x(3x-1)$ k $2x(1-x)$ l $3x(2-x)$
m $-2x(3x+2)$ n $-3x(6x-2)$ o $-5x(2-2x)$ p $-4x(1-4x)$
q $\frac{2}{5}(10x+4)$ r $\frac{3}{4}(8x-5)$ s $-\frac{1}{3}(6x+1)$ t $-\frac{1}{2}(4x-3)$
u $-\frac{3}{8}(24x-1)$ v $-\frac{2}{9}(9x+7)$ w $\frac{3x}{4}(3x+8)$ x $\frac{2x}{5}(7-3x)$

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Example 1d
3 Expand and simplify.
- **a** $x(3x - 1) + x(4 - x)$
- **b** $x(5x + 2) + x(x - 5)$
- **c** $x(4x - 3) - 2x(x - 5)$
- **d** $3x(2x + 4) - x(5 - 2x)$
- **e** $4x(2x - 1) + 2x(1 - 3x)$
- **f** $2x(2 - 3x) - 3x(2x - 7)$

Example 2a
4 Expand the following.
- **a** $(x + 2)(x + 8)$
- **b** $(x + 3)(x + 4)$
- **c** $(x + 7)(x + 5)$
- **d** $(x + 8)(x - 3)$
- **e** $(x + 6)(x - 5)$
- **f** $(x - 2)(x + 3)$
- **g** $(x - 7)(x + 3)$
- **h** $(x - 4)(x - 6)$
- **i** $(x - 8)(x - 5)$

Example 2b,c
5 Expand the following.
- **a** $(x + 5)^2$
- **b** $(x + 7)^2$
- **c** $(x + 6)^2$
- **d** $(x - 3)^2$
- **e** $(x - 8)^2$
- **f** $(x - 10)^2$
- **g** $(x + 4)(x - 4)$
- **h** $(x + 9)(x - 9)$
- **i** $(2x - 3)(2x + 3)$
- **j** $(3x + 4)(3x - 4)$
- **k** $(4x - 5)(4x + 5)$
- **l** $(8x - 7)(8x + 7)$

Example 3a
6 Expand the following using the distributive law.
- **a** $(2x + 1)(3x + 5)$
- **b** $(4x + 5)(3x + 2)$
- **c** $(5x + 3)(2x + 7)$
- **d** $(3x + 2)(3x - 5)$
- **e** $(5x + 3)(4x - 2)$
- **f** $(2x + 5)(3x - 5)$
- **g** $(4x - 5)(4x + 5)$
- **h** $(2x - 9)(2x + 9)$
- **i** $(5x - 7)(5x + 7)$
- **j** $(7x - 3)(2x - 4)$
- **k** $(5x - 3)(5x - 6)$
- **l** $(7x - 2)(8x - 2)$
- **m** $(2x + 5)^2$
- **n** $(5x + 6)^2$
- **o** $(7x - 1)^2$

PROBLEM-SOLVING	7–8(1/2)	7–9(1/3)	7–9(1/4), 10

7 Write the missing number.
- **a** $(x + ?)(x + 2) = x^2 + 5x + 6$
- **b** $(x + ?)(x + 5) = x^2 + 8x + 15$
- **c** $(x + 7)(x - ?) = x^2 + 4x - 21$
- **d** $(x + 4)(x - ?) = x^2 - 4x - 32$
- **e** $(x - 6)(x - ?) = x^2 - 7x + 6$
- **f** $(x - ?)(x - 8) = x^2 - 10x + 16$

Example 3b
8 Expand the following.
- **a** $2(x + 3)(x + 4)$
- **b** $3(x + 2)(x + 7)$
- **c** $-2(x + 8)(x + 2)$
- **d** $-4(x + 9)(x + 2)$
- **e** $5(x - 3)(x + 4)$
- **f** $3(x + 5)(x - 3)$
- **g** $-3(a + 2)(a - 7)$
- **h** $-5(a + 2)(a - 8)$
- **i** $4(a - 3)(a - 6)$
- **j** $3(y - 4)(y - 5)$
- **k** $-2(y - 3)(y - 8)$
- **l** $-6(y - 4)(y - 3)$
- **m** $3(2x + 3)(2x + 5)$
- **n** $6(3x - 4)(x + 2)$
- **o** $-2(x + 4)(3x - 7)$
- **p** $2(x + 3)^2$
- **q** $4(m + 5)^2$
- **r** $2(a - 7)^2$
- **s** $-3(y - 5)^2$
- **t** $3(2b - 1)^2$
- **u** $-3(2y - 6)^2$

Example 3c
9 Expand and simplify the following.
- **a** $(x + 1)(x + 3) + (x + 2)(x + 4)$
- **b** $(x + 8)(x + 3) + (x + 4)(x + 5)$
- **c** $(y + 3)(y - 1) + (y - 2)(y - 4)$
- **d** $(y - 7)(y + 4) + (y + 5)(y - 3)$
- **e** $(2a + 3)(a - 5) - (a + 6)(2a + 5)$
- **f** $(4b + 8)(b + 5) - (3b - 5)(b - 7)$
- **g** $(x + 5)^2 - 7$
- **h** $(x - 7)^2 - 9$
- **i** $3 - (2x - 9)^2$
- **j** $14 - (5x + 3)^2$

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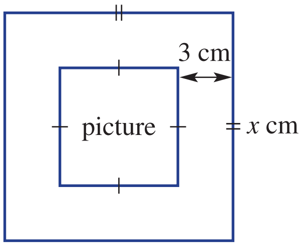
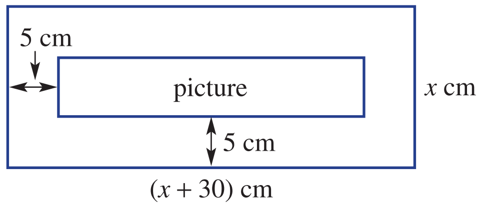
10 Find an expanded expression for the area of the pictures centred in these rectangular frames.

a



b



REASONING	12–13(1/2)	11–13(1/2)	11–14(1/2)

11 Prove the following by expanding the left-hand side.

a $(a+b)(a-b)=a^2-b^2$ **b** $(a+b)^2=a^2+2ab+b^2$

c $(a-b)^2=a^2-2ab+b^2$ **d** $(a+b)^2-(a-b)^2=4ab$

12 Use the distributive law to evaluate the following without the use of a calculator.
For example: $4\times 102=4\times 100+4\times 2=408$.

a 6×103 **b** 4×55 **c** 9×63 **d** 8×208

e 7×198 **f** 3×297 **g** 8×495 **h** 5×696

13 Each problem below has an incorrect answer. Find the error and give the correct answer.

a $-x(x-7)=-x^2-7x$ **b** $3a-7(4-a)=-4a-28$

c $(2x+3)^2=4x^2+9$ **d** $(x+2)^2-(x+2)(x-2)=0$

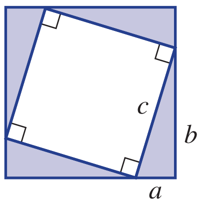
14 Expand these cubic expressions.

a $(x+2)(x+3)(x+1)$ **b** $(x+4)(x+2)(x+5)$ **c** $(x+3)(x-4)(x+3)$

d $(x-4)(2x+1)(x-3)$ **e** $(x+6)(2x-3)(x-5)$ **f** $(2x-3)(x-4)(3x-1)$

ENRICHMENT: Expanding to prove	–	–	15

15 One of the ways to prove Pythagoras' theorem is to arrange four congruent right-angled triangles around a square to form a larger square, as shown.



a Find an expression for the total area of the four shaded triangles by multiplying the area of one triangle by 4.

b Find an expression for the area of the four shaded triangles by subtracting the area of the inner square from the area of the outer square.

c By combining your results from parts **a** and **b**, expand and simplify to prove Pythagoras' theorem: $a^2+b^2=c^2$.

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5B Factorising expressions

Learning intentions

- To understand what it means to write an expression in factorised form
- To know to always look for a common factor before trying other factorising techniques
- To be able to recognise a difference of perfect squares including ones involving surds
- To be able to factorise using a common factor or a difference of perfect squares
- To be able to use the grouping technique to factorise

A common and key step in the simplification and solution of equations involves factorisation. Factorisation is the process of writing a number or expression as a product of its factors.

In this section we look at expressions in which all terms have a common factor, expressions that are a difference of perfect squares and four-term expressions, which can be factorised by grouping.

After a car accident, crash investigators use the length of tyre skid marks to determine a vehicle's speed before braking. The quadratic equation $u^2 + 2as = 0$ relates to speed, u, to a known braking distance, s, and deceleration $a = -10$ m/s^2 on a dry, flat bitumen road.

LESSON STARTER But there are no common factors!

An expression such as $xy + 4x + 3y + 12$ has no common factors across all four terms, but it can still be factorised. The method of grouping can be used.

- Complete this working to show how to factorise the expression.

$$xy + 4x + 3y + 12 = x(\_\_\_\_\_\_) + 3(\_\_\_\_\_\_)$$
$$= (\_\_\_\_\_\_)(x + 3)$$

- Now repeat with the expression rearranged.

$$xy + 3y + 4x + 12 = y(\_\_\_\_\_\_) + 4(\_\_\_\_\_\_)$$
$$= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$$

- Are the two results equivalent?

KEY IDEAS

- **Factorise** expressions with **common factors** by 'taking out' the common factors.
  For example: $-5x - 20 = -5(x + 4)$ and $4x^2 - 8x = 4x(x - 2)$.
- Factorise a **difference of perfect squares** (DOPS) using $a^2 - b^2 = (a + b)(a - b)$.
  - We use surds when a^2 or b^2 is not a perfect square, such as 1, 4, 9, …
    For example: $x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$ using $(\sqrt{5})^2 = 5$.
- Factorise four-term expressions if possible by **grouping** terms and factorising each pair.
  For example: $x^2 + 5x - 2x - 10 = x(x + 5) - 2(x + 5)$
  $= (x + 5)(x - 2)$

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BUILDING UNDERSTANDING

1. Determine the highest common factor of these pairs of terms.
   a $7x$ and 14 b $-5y$ and -25 c $12a^2$ and $9a$ d $-3x^2y$ and $-6xy$
2. State the missing parts.
   a If $x(x - 1) = x^2 - x$, then $x^2 - x = x(\_\_\_\_\_\_)$
   b If $2(1 - x) = 2 - 2x$, then $2 - 2x = \_\_\_\_\_\_$
   c If $(x + 2)(x - 2) = \_\_\_\_\_\_$, then $x^2 - 4 = \_\_\_\_\_\_$
   d If $(3x - 7)(\_\_\_\_\_\_) = 9x^2 - 49$, then $9x^2 - 49 = \_\_\_\_\_\_$

Example 4 Taking out common factors

Factorise by taking out common factors.

a $-3x - 12$ b $20a^2 + 30a$ c $2(x + 1) - a(x + 1)$

SOLUTION

a $-3x - 12 = -3(x + 4)$

EXPLANATION

-3 is common to both $-3x$ and -12. Divide each term by -3 to determine the terms in the brackets. Expand to check.

b $20a^2 + 30a = 10a(2a + 3)$

The HCF of $20a^2$ and $30a$ is $10a$.

c $2(x + 1) - a(x + 1) = (x + 1)(2 - a)$

$(x + 1)$ is a common factor to both parts of the expression.

Now you try

Factorise by taking out common factors.

a $-2x - 8$ b $15a^2 + 20a$ c $3(x + 2) - a(x + 2)$

Example 5 Factorising difference of perfect squares

Factorise the following difference of perfect squares. You may need to look for a common factor first.

a $x^2 - 16$ b $9a^2 - 4b^2$ c $12y^2 - 1200$ d $(x + 3)^2 - 4$

SOLUTION

a $x^2 - 16 = (x)^2 - (4)^2$
$= (x + 4)(x - 4)$

EXPLANATION

Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x$ and $b = 4$.

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b $9a^2 - 4b^2 = (3a)^2 - (2b)^2$ $= (3a + 2b)(3a - 2b)$	$9a^2 = (3a)^2$ and $4b^2 = (2b)^2$.
c $12y^2 - 1200 = 12(y^2 - 100)$ $= 12(y + 10)(y - 10)$	First, take out the common factor of 12. $100 = 10^2$, use $a^2 - b^2 = (a + b)(a - b)$.
d $(x + 3)^2 - 4 = (x + 3)^2 - (2)^2$ $= (x + 3 + 2)(x + 3 - 2)$ $= (x + 5)(x + 1)$	Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x + 3$ and $b = 2$. Simplify each bracket.

Now you try

Factorise the following difference of perfect squares. You may need to look for a common factor first.

a $x^2 - 25$ **b** $16a^2 - 9b^2$ **c** $2y^2 - 98$ **d** $(x + 2)^2 - 36$

Example 6 Factorising DOPS using surds

Factorise these DOPS using surds.

a $x^2 - 10$ **b** $x^2 - 24$ **c** $(x - 1)^2 - 5$

SOLUTION	EXPLANATION
a $x^2 - 10 = x^2 - (\sqrt{10})^2$ $= (x + \sqrt{10})(x - \sqrt{10})$	Recall that $(\sqrt{10})^2 = 10$.
b $x^2 - 24 = x^2 - (\sqrt{24})^2$ $= (x + \sqrt{24})(x - \sqrt{24})$ $= (x + 2\sqrt{6})(x - 2\sqrt{6})$	Use $(\sqrt{24})^2 = 24$. Simplify: $\sqrt{24} = \sqrt{4 \times 6} = \sqrt{4} \times \sqrt{6} = 2\sqrt{6}$
c $(x - 1)^2 - 5 = (x - 1)^2 - (\sqrt{5})^2$ $= (x - 1 + \sqrt{5})(x - 1 - \sqrt{5})$	Use $a^2 - b^2 = (a + b)(a - b)$, where $a = x - 1$ and $b = \sqrt{5}$.

Now you try

Factorise these DOPS using surds.

a $x^2 - 7$ **b** $x^2 - 32$ **c** $(x - 5)^2 - 2$

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Example 7 Factorisation by grouping

Factorise by grouping $x^2 - x + ax - a$.

SOLUTION

$$x^2 - x + ax - a = x(x - 1) + a(x - 1)$$
$$= (x - 1)(x + a)$$

EXPLANATION

Factorise two pairs of terms, then take out the common binomial factor $(x - 1)$.

Now you try

Factorise by grouping $x^2 - 2x + ax - 2a$.

Exercise 5B

FLUENCY	1, 2–6(1/2)	2–7(1/3)	2–7(1/4)

Example 4a, Example 4b, Example 4c

1 Factorise by taking out common factors.

a **i** $-4x - 12$ **ii** $-9x - 36$

b **i** $10a^2 + 40a$ **ii** $17a^2 + 34a$

c **i** $4(x + 2) - a(x + 2)$ **ii** $11(x + 5) - a(x + 5)$

Example 4a,b

2 Factorise by taking out the common factors.

a $3x - 18$ **b** $4x + 20$ **c** $7a + 7b$ **d** $9a - 15$
e $-5x - 30$ **f** $-4y - 2$ **g** $-12a - 3$ **h** $-2ab - bc$
i $4x^2 + x$ **j** $5x^2 - 2x$ **k** $6b^2 - 18b$ **l** $14a^2 - 21a$
m $10a - 5a^2$ **n** $12x - 30x^2$ **o** $-2x - x^2$ **p** $-4y - 8y^2$
q $ab^2 - a^2b$ **r** $2x^2yz - 4xy$ **s** $-12m^2n - 12mn^2$ **t** $6xyz^2 - 3z^2$

Example 4c

3 Factorise, noting the common binomial factor. (*Hint*: For parts **g**-**i**, insert a 1 where appropriate.)

a $5(x - 1) - a(x - 1)$ **b** $b(x + 2) + 3(x + 2)$ **c** $a(x + 5) - 4(x + 5)$
d $x(x + 2) + 5(x + 2)$ **e** $x(x - 4) - 2(x - 4)$ **f** $3(x + 1) - x(x + 1)$
g $a(x + 3) + (x + 3)$ **h** $x(x - 2) - (x - 2)$ **i** $(x - 6) - x(x - 6)$

Example 5a,b

4 Factorise the following difference of perfect squares.

a $x^2 - 9$ **b** $x^2 - 25$ **c** $y^2 - 49$ **d** $y^2 - 1$
e $4x^2 - 9$ **f** $36a^2 - 25$ **g** $1 - 81y^2$ **h** $100 - 9x^2$
i $25x^2 - 4y^2$ **j** $64x^2 - 25y^2$ **k** $9a^2 - 49b^2$ **l** $144a^2 - 49b^2$

Example 5c,d

5 Factorise the following.

a $2x^2 - 32$ **b** $5x^2 - 45$ **c** $6y^2 - 24$ **d** $3y^2 - 48$
e $3x^2 - 75y^2$ **f** $3a^2 - 300b^2$ **g** $12x^2 - 27y^2$ **h** $63a^2 - 112b^2$
i $(x + 5)^2 - 16$ **j** $(x - 4)^2 - 9$ **k** $(a - 3)^2 - 64$ **l** $(a - 7)^2 - 1$
m $(3x + 5)^2 - x^2$ **n** $(2y + 7)^2 - y^2$ **o** $(5x + 11)^2 - 4x^2$ **p** $(3x - 5y)^2 - 25y^2$

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Example 6

6 Factorise using surds and remember to simplify surds where possible.

a $x^2 - 7$
b $x^2 - 5$
c $x^2 - 19$
d $x^2 - 21$
e $x^2 - 14$
f $x^2 - 30$
g $x^2 - 15$
h $x^2 - 11$
i $x^2 - 8$
j $x^2 - 18$
k $x^2 - 45$
l $x^2 - 20$
m $x^2 - 32$
n $x^2 - 48$
o $x^2 - 50$
p $x^2 - 200$
q $(x + 2)^2 - 6$
r $(x + 5)^2 - 10$
s $(x - 3)^2 - 11$
t $(x - 1)^2 - 7$
u $(x - 6)^2 - 15$
v $(x + 4)^2 - 21$
w $(x + 1)^2 - 19$
x $(x - 7)^2 - 26$

Example 7

7 Factorise by grouping.

a $x^2 + 4x + ax + 4a$
b $x^2 + 7x + bx + 7b$
c $x^2 - 3x + ax - 3a$
d $x^2 + 2x - ax - 2a$
e $x^2 + 5x - bx - 5b$
f $x^2 + 3x - 4ax - 12a$
g $x^2 - ax - 4x + 4a$
h $x^2 - 2bx - 5x + 10b$
i $3x^2 - 6ax - 7x + 14a$

PROBLEM-SOLVING

PROBLEM-SOLVING			
	8(1/2)	8–9(1/2)	8–10(1/3)

8 Factorise fully and simplify surds.

a $x^2 - \frac{2}{9}$
b $x^2 - \frac{3}{4}$
c $x^2 - \frac{7}{16}$
d $x^2 - \frac{5}{36}$
e $(x - 2)^2 - 20$
f $(x + 4)^2 - 27$
g $(x + 1)^2 - 75$
h $(x - 7)^2 - 40$
i $3x^2 - 4$
j $5x^2 - 9$
k $7x^2 - 5$
l $6x^2 - 11$
m $-9 + 2x^2$
n $-16 + 5x^2$
o $-10 + 3x^2$
p $-7 + 13x^2$

9 Factorise by first rearranging.

a $xy - 6 - 3x + 2y$
b $ax - 12 + 3a - 4x$
c $ax - 10 + 5x - 2a$
d $xy + 12 - 3y - 4x$
e $2ax + 3 - a - 6x$
f $2ax - 20 + 8a - 5x$

10 Factorise fully.

a $5x^2 - 120$
b $3x^2 - 162$
c $7x^2 - 126$
d $2x^2 - 96$
e $2(x + 3)^2 - 10$
f $3(x - 1)^2 - 21$
g $4(x - 4)^2 - 48$
h $5(x + 6)^2 - 90$

REASONING

REASONING			
	11(1/2)	11(1/2), 12	11(1/2), 13, 14

11 Evaluate the following, without the use of a calculator, by first factorising.

a $16^2 - 14^2$
b $18^2 - 17^2$
c $13^2 - 10^2$
d $15^2 - 11^2$
e $17^2 - 15^2$
f $11^2 - 9^2$
g $27^2 - 24^2$
h $52^2 - 38^2$

12 a Show that $4 - (x + 2)^2 = -x(x + 4)$ by factorising the left-hand side.

b Now factorise the following.

i $9 - (x + 3)^2$
ii $16 - (x + 4)^2$
iii $25 - (x - 5)^2$
iv $25 - (x + 2)^2$
v $49 - (x - 1)^2$
vi $100 - (x + 4)^2$

13 a Prove that, in general, $(x + a)^2 \neq x^2 + a^2$.

b Are there any values of x for which $(x + a)^2 = x^2 + a^2$? If so, what are they?

14 Show that $x^2 - \frac{4}{9} = \frac{1}{9}(3x + 2)(3x - 2)$ using two different methods.

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ENRICHMENT: Hidden DOPS	–	–	15(1/2), 16

15 Factorise and simplify the following without initially expanding the brackets.

- **a** $(x + 2)^2 - (x + 3)^2$
- **b** $(y - 7)^2 - (y + 4)^2$
- **c** $(a + 3)^2 - (a - 5)^2$
- **d** $(b + 5)^2 - (b - 5)^2$
- **e** $(s - 3)^2 - (s + 3)^2$
- **f** $(y - 7)^2 - (y + 7)^2$
- **g** $(2w + 3x)^2 - (3w + 4x)^2$
- **h** $(d + 5e)^2 - (3d - 2e)^2$
- **i** $(4f + 3j)^2 - (2f - 3j)^2$
- **j** $(3r - 2p)^2 - (2p - 3r)^2$

16 **a** Is it possible to factorise $x^2 + 5y - y^2 + 5x$? Can you show how?

b Also try factorising:

- **i** $x^2 + 7x + 7y - y^2$
- **ii** $x^2 - 2x - 2y - y^2$
- **iii** $4x^2 + 4x + 6y - 9y^2$
- **iv** $25y^2 + 15y - 4x^2 + 6x$

Factorising is a key component of the proof of Fermat's last theorem, which states that there are no solutions to $x^n + y^n = z^n$ for $n \geqslant 3$. Although it looks simple, it took the best mathematicians on Earth 358 years to find a proof of this theorem. It was finally proved in 1994 by Andrew Wiles, and his proof is almost 130 pages long!

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Using calculators to expand and factorise

1 Expand and simplify $3(2x - 3)(5x + 1) - (4x - 1)^2$.

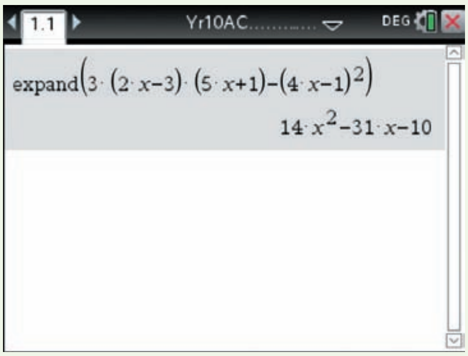
2 Factorise.

a $ax^2 + x - ax - 1$

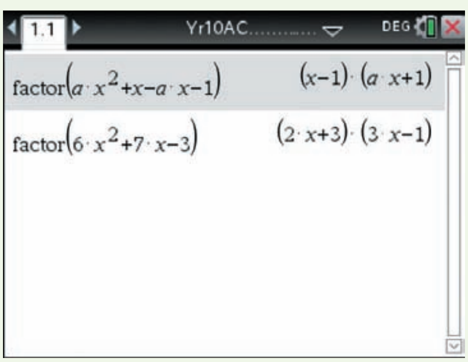
b $6x^2 + 7x - 3$

Using the TI-Nspire:

1 In a **calculator** page use (menu) **>Algebra>Expand** and type as shown.



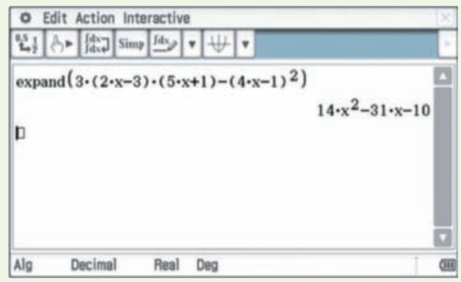
2 In a **Calculator** page use (menu) **>Algebra>Factor** and type as shown.



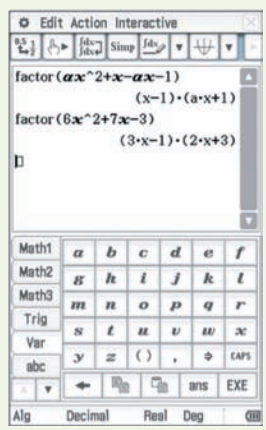
Note: Use a multiplication sign between the *a* and *x*.

Using the ClassPad:

1 In the **Main** application, type and highlight the expression, then tap **Interactive**, **Transformation**, **expand** and type in as shown below.



2 Use the **VAR** keyboard to type the expression as shown. Highlight the expression and tap **Interactive**, **Transformation**, **factor**.



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5C Factorising monic quadratic trinomials

Learning intentions

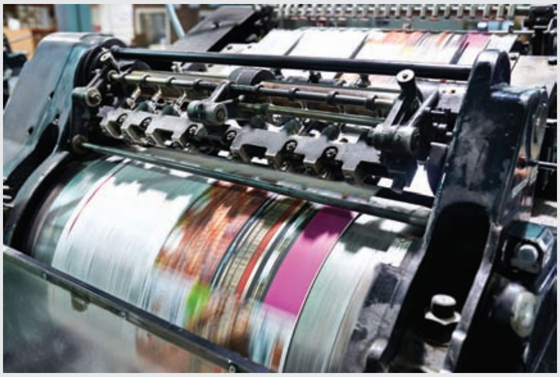
- To be able to identify a monic quadratic trinomial
- To understand the relationship between expanding brackets to form a trinomial and factorising a monic trinomial
- To know how to factorise a monic quadratic trinomial
- To be able to simplify algebraic fractions by first factorising and cancelling common factors

A quadratic trinomial of the form $x^2 + bx + c$ is called a monic quadratic because the coefficient of x^2 is 1.

Now consider:

$$\begin{aligned}(x+m)(x+n) &= x^2 + xn + mx + mn \\ &= x^2 + (m+n)x + mn\end{aligned}$$

We can see from this expansion that mn gives the constant term (c) and $m + n$ is the coefficient of x. This tells us that to factorise a monic quadratic trinomial we should look for factors of the constant term (c) that add to give the coefficient of the middle term (b).



Trinomial quadratics can model the revenue and profits from book publishing. Market research and past sales are used to develop unique quadratic models which find the book's selling price that predicts maximum revenue.

LESSON STARTER Factorising $x^2 - 6x - 72$

Discuss what is wrong with each of these statements when trying to factorise $x^2 - 6x - 72$.

- Find factors of 72 that add to 6.
- Find factors of 72 that add to −6.
- Find factors of −72 that add to 6.
- $-18 \times 4 = -72$ so $x^2 - 6x - 72 = (x - 18)(x + 4)$
- $-9 \times 8 = -72$ so $x^2 - 6x - 72 = (x - 9)(x + 8)$

Can you write a correct statement that correctly factorises $x^2 - 6x - 72$?

KEY IDEAS

- **Monic quadratics** have a coefficient of x^2 equal to 1.
- Monic quadratics of the form $x^2 + bx + c$ can be factorised by finding the two numbers that multiply to give the constant term (c) and add to give the coefficient of x (i.e. b).

$$x^2 + \underbrace{(m+n)}_{b}x + \underbrace{mn}_{c} = (x+m)(x+n)$$

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BUILDING UNDERSTANDING

1 Find two integers that multiply to give the first number and add to give the second number.

a 18, 11 b 20, 12 c −15, 2 d −12, 1
e −24, −5 f −30, −7 g 10, −7 h 36, −15

2 A number (except zero) divided by itself always equals 1.

For example: $\frac{a^1}{a^1} = 1$, $\frac{2(x-3)^1}{(x-3)^1} = 2$, $\frac{(a+5)^1}{2(a+5)^1} = \frac{1}{2}$

Invent some algebraic fractions that are equal to:

a 1 b 3 c −5 d $\frac{1}{3}$

3 Simplify by cancelling common factors. For parts f to g, first factorise the numerator.

a $\frac{2x}{4}$ b $\frac{6a}{2a}$ c $\frac{3(x+1)}{9(x+1)}$ d $\frac{2(x-2)}{8(x-2)}$

e $\frac{8(x+4)}{12(x+4)}$ f $\frac{x^2+x}{x}$ g $\frac{x^2-2x}{x}$ h $\frac{x^2-3x}{2x}$

Example 8 Factorising trinomials of the form $x^2 + bx + c$

Factorise.

a $x^2 + 8x + 15$ b $x^2 - 5x + 6$ c $2x^2 - 10x - 28$ d $x^2 - 8x + 16$

SOLUTION	EXPLANATION
a $x^2 + 8x + 15 = (x+3)(x+5)$	$3 \times 5 = 15$ and $3 + 5 = 8$ Check: $(x+3)(x+5) = x^2 + 5x + 3x + 15 = x^2 + 8x + 15$
b $x^2 - 5x + 6 = (x-3)(x-2)$	$-3 \times (-2) = 6$ and $-3 + (-2) = -5$ Check: $(x-3)(x-2) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$
c $2x^2 - 10x - 28 = 2(x^2 - 5x - 14) = 2(x-7)(x+2)$	First, take out the common factor of 2. $-7 \times 2 = -14$ and $-7 + 2 = -5$
d $x^2 - 8x + 16 = (x-4)(x-4) = (x-4)^2$	$-4 \times (-4) = 16$ and $-4 + (-4) = -8$ $(x-4)(x-4) = (x-4)^2$ is a perfect square.

Now you try

Factorise:

a $x^2 + 7x + 12$ b $x^2 - 10x + 24$ c $2x^2 - 2x - 12$ d $x^2 - 6x + 9$

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Example 9 Simplifying algebraic fractions

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - x - 6}{x + 2}$ **b** $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

SOLUTION

a $\dfrac{x^2 - x - 6}{x + 2} = \dfrac{(x - 3)\cancel{(x + 2)}^1}{\cancel{(x + 2)}^1}$

$= x - 3$

EXPLANATION

First, factorise $x^2 - x - 6$ and then cancel $(x + 2)$.

SOLUTION

b $\dfrac{x^2 - 9}{x^2 - 2x - 15} \times \dfrac{x^2 - 4x - 5}{2x - 6}$

$= \dfrac{\cancel{(x + 3)}^1\cancel{(x - 3)}^1}{\cancel{(x - 5)}^1\cancel{(x + 3)}^1} \times \dfrac{\cancel{(x - 5)}^1(x + 1)}{2\cancel{(x - 3)}^1}$

$= \dfrac{x + 1}{2}$

EXPLANATION

First, factorise all expressions in the numerators and denominators. Cancel to simplify where possible.

Now you try

Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 - 2x - 8}{x + 2}$ **b** $\dfrac{x^2 - 4}{x^2 + x - 2} \times \dfrac{x^2 + 3x - 4}{2x - 4}$

Exercise 5C

FLUENCY	1, 2–4(1/4)	2–5(1/3)	2–5(1/4)

1 Factorise.

Example 8a **a** i $x^2 + 3x + 2$ ii $x^2 + 6x + 5$

Example 8b **b** i $x^2 - 4x + 3$ ii $x^2 - 11x + 30$

Example 8c **c** i $2x^2 - 8x - 10$ ii $3x^2 - 9x - 30$

Example 8d **d** i $x^2 - 4x + 4$ ii $x^2 - 10x + 25$

Example 8a,b

2 Factorise these quadratic trinomials.

a $x^2 + 7x + 6$ **b** $x^2 + 5x + 6$ **c** $x^2 + 6x + 9$ **d** $x^2 + 7x + 10$

e $x^2 + 7x + 12$ **f** $x^2 + 11x + 18$ **g** $x^2 + 5x - 6$ **h** $x^2 + x - 6$

i $x^2 + 2x - 8$ **j** $x^2 + 3x - 4$ **k** $x^2 + 7x - 30$ **l** $x^2 + 9x - 22$

m $x^2 - 7x + 10$ **n** $x^2 - 6x + 8$ **o** $x^2 - 7x + 12$ **p** $x^2 - 2x + 1$

q $x^2 - 9x + 18$ **r** $x^2 - 11x + 18$ **s** $x^2 - 4x - 12$ **t** $x^2 - x - 20$

u $x^2 - 5x - 14$ **v** $x^2 - x - 12$ **w** $x^2 + 4x - 32$ **x** $x^2 - 3x - 10$

Example 8c

3 Factorise by first taking out the common factor.

a $2x^2 + 14x + 20$ **b** $3x^2 + 21x + 36$ **c** $2x^2 + 22x + 36$

d $5x^2 - 5x - 10$ **e** $4x^2 - 16x - 20$ **f** $3x^2 - 9x - 30$

g $-2x^2 - 14x - 24$ **h** $-3x^2 + 9x - 6$ **i** $-2x^2 + 10x + 28$

j $-4x^2 + 4x + 8$ **k** $-5x^2 - 20x - 15$ **l** $-7x^2 + 49x - 42$

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Example 8d

4 Factorise these perfect squares.

a $x^2 - 4x + 4$ **b** $x^2 + 6x + 9$ **c** $x^2 + 12x + 36$ **d** $x^2 - 14x + 49$
e $x^2 - 18x + 81$ **f** $x^2 - 20x + 100$ **g** $2x^2 + 44x + 242$ **h** $3x^2 - 24x + 48$
i $5x^2 - 50x + 125$ **j** $-3x^2 + 36x - 108$ **k** $-2x^2 + 28x - 98$ **l** $-4x^2 - 72x - 324$

Example 9a

5 Use factorisation to simplify these algebraic fractions. In some cases, you may need to remove a common factor first.

a $\frac{x^2 - 3x - 54}{x - 9}$ **b** $\frac{x^2 + x - 12}{x + 4}$ **c** $\frac{x^2 - 6x + 9}{x - 3}$
d $\frac{x + 2}{x^2 + 9x + 14}$ **e** $\frac{x - 3}{x^2 - 8x + 15}$ **f** $\frac{x + 1}{x^2 - 5x - 6}$
g $\frac{2(x + 12)}{x^2 + 4x - 96}$ **h** $\frac{x^2 - 5x - 36}{3(x - 9)}$ **i** $\frac{x^2 - 15x + 56}{5(x - 8)}$

PROBLEM-SOLVING	6(1/2)	6–7(1/3)	6–8(1/3)

Example 9b

6 Simplify by first factorising.

a $\frac{x^2 - 4}{x^2 - x - 6} \times \frac{5x - 15}{x^2 + 4x - 12}$ **b** $\frac{x^2 + 3x + 2}{x^2 + 4x + 3} \times \frac{x^2 - 9}{3x + 6}$
c $\frac{x^2 + 2x - 3}{x^2 - 25} \times \frac{2x - 10}{x + 3}$ **d** $\frac{x^2 - 9}{x^2 - 5x + 6} \times \frac{4x - 8}{x^2 + 8x + 15}$
e $\frac{x^2 - 4x + 3}{x^2 + 4x - 21} \times \frac{4x + 4}{x^2 - 1}$ **f** $\frac{x^2 + 6x + 8}{x^2 - 4} \times \frac{6x - 24}{x^2 - 16}$
g $\frac{x^2 - x - 6}{x^2 + x - 12} \times \frac{x^2 + 5x + 4}{x^2 - 1}$ **h** $\frac{x^2 - 4x - 12}{x^2 - 4} \times \frac{x^2 - 6x + 8}{x^2 - 36}$

7 Simplify these expressions that involve surds.

a $\frac{x^2 - 7}{x + \sqrt{7}}$ **b** $\frac{x^2 - 10}{x - \sqrt{10}}$ **c** $\frac{x^2 - 12}{x + 2\sqrt{3}}$
d $\frac{\sqrt{5}x + 3}{5x^2 - 9}$ **e** $\frac{\sqrt{3}x - 4}{3x^2 - 16}$ **f** $\frac{7x^2 - 5}{\sqrt{7}x + \sqrt{5}}$
g $\frac{(x + 1)^2 - 2}{x + 1 + \sqrt{2}}$ **h** $\frac{(x - 3)^2 - 5}{x - 3 - \sqrt{5}}$ **i** $\frac{(x - 6)^2 - 6}{x - 6 + \sqrt{6}}$

8 Simplify using factorisation.

a $\frac{x^2 + 2x - 3}{x^2 - 25} \div \frac{3x - 3}{2x + 10}$ **b** $\frac{x^2 + 3x + 2}{x^2 + 4x + 3} \div \frac{4x + 8}{x^2 - 9}$ **c** $\frac{x^2 - x - 12}{x^2 - 9} \div \frac{x^2 - 16}{3x + 12}$
d $\frac{x^2 - 49}{x^2 - 3x - 28} \div \frac{4x + 28}{6x + 24}$ **e** $\frac{x^2 + 5x - 14}{x^2 + 2x - 3} \div \frac{x^2 + 9x + 14}{x^2 + x - 2}$ **f** $\frac{x^2 + 8x + 15}{x^2 + 5x - 6} \div \frac{x^2 + 6x + 5}{x^2 + 7x + 6}$

REASONING	9	9, 10(1/2)	10(1/2), 11, 12

9 A businessman is showing off his new formula to determine the company's profit, in millions of dollars, after t years.

$$\text{Profit} = \frac{t^2 - 49}{5t - 40} \times \frac{t^2 - 5t - 24}{2t^2 - 8t - 42}$$

Show that this is really the same as

$$\text{Profit} = \frac{t + 7}{10}.$$

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10 Note that an expression with a perfect square can be simplified as shown.

$$\frac{(x+3)^2}{x+3} = \frac{(x+3)\cancel{(x+3)}^1}{\cancel{x+3}^1}$$
$$= x+3$$

Use this idea to simplify the following.

a $\frac{x^2-6x+9}{x-3}$ **b** $\frac{x^2+2x+1}{x+1}$ **c** $\frac{x^2-16x+64}{x-8}$

d $\frac{6x-12}{x^2-4x+4}$ **e** $\frac{4x+20}{x^2+10x+25}$ **f** $\frac{x^2-14x+49}{5x-35}$

11 **a** Prove that $\frac{a^2+2ab+b^2}{a^2+ab} \div \frac{a^2-b^2}{a^2-ab} = 1$.

b Make up your own expressions, like the one in part **a**, which equal 1. Ask a classmate to check them.

12 Simplify.

a $\frac{a^2+2ab+b^2}{a(a+b)} \div \frac{a^2-b^2}{a^2-2ab+b^2}$ **b** $\frac{a^2-2ab+b^2}{a^2-b^2} \div \frac{a^2-b^2}{a^2+2ab+b^2}$

c $\frac{a^2-b^2}{a^2-2ab+b^2} \div \frac{a^2-b^2}{a^2+2ab+b^2}$ **d** $\frac{a^2+2ab+b^2}{a(a+b)} \div \frac{a(a-b)}{a^2-2ab+b^2}$

ENRICHMENT: Addition and subtraction with factorisation	–	–	13(1/2)

13 Factorisation can be used to help add and subtract algebraic fractions. Here is an example.

$$\begin{aligned}\frac{3}{x-2} + \frac{x}{x^2-6x+8} &= \frac{3}{x-2} + \frac{x}{(x-2)(x-4)}\\ &= \frac{3(x-4)}{(x-2)(x-4)} + \frac{x}{(x-2)(x-4)}\\ &= \frac{3x-12+x}{(x-2)(x-4)}\\ &= \frac{4x-12}{(x-2)(x-4)}\\ &= \frac{4(x-3)}{(x-2)(x-4)}\end{aligned}$$

Now simplify the following.

a $\frac{2}{x+3} + \frac{x}{x^2-x-12}$ **b** $\frac{4}{x+2} + \frac{3x}{x^2-7x-18}$

c $\frac{3}{x+4} - \frac{2x}{x^2-16}$ **d** $\frac{4}{x^2-9} - \frac{1}{x^2-8x+15}$

e $\frac{x+4}{x^2-x-6} - \frac{x-5}{x^2-9x+18}$ **f** $\frac{x+3}{x^2-4x-32} - \frac{x}{x^2+7x+12}$

g $\frac{x+1}{x^2-25} - \frac{x-2}{x^2-6x+5}$ **h** $\frac{x+2}{x^2-2x+1} - \frac{x+3}{x^2+3x-4}$

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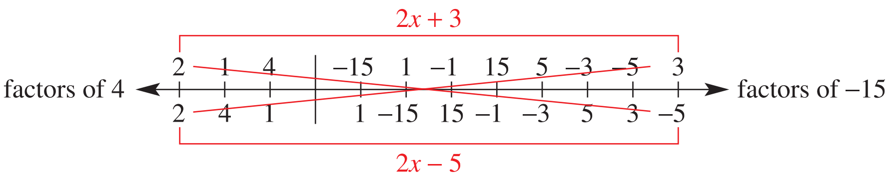
5D Factorising non-monic quadratic trinomials 10A

Learning intentions
- To understand the relationship between expansion and factorisation for binomial products
- To know and be able to apply the process for factorising non-monic quadratic trinomials

There are a number of ways of factorising non-monic quadratic trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. The cross method, for example, uses lists of factors of a and c so that a correct combination can be found. For example, to factorise $4x^2 - 4x - 15$:



$2 \times (-5) + 2 \times 3 = -4$, so choose $(2x + 3)$ and $(2x - 5)$.
$\therefore\ 4x^2 - 4x - 15 = (2x + 3)(2x - 5)$

The method outlined in this section, however, uses grouping.

LESSON STARTER Does the order matter?

To factorise the non-monic quadratic $4x^2 - 4x - 15$ using grouping, we multiply a by c, which is $4 \times (-15) = -60$. Then we look for numbers that multiply to give -60 and add to give -4 (the coefficient of x).

- What are the two numbers that multiply to give -60 and add to give -4?
- Complete the following using grouping.

  $4x^2 - 4x - 15 = 4x^2 - 10x + 6x - 15$ $\quad -10 \times 6 = -60, \quad -10 + 6 = -4$
  $= 2x(\_\_\_\_\_\_) + 3(\_\_\_\_\_\_)$
  $= (2x - 5)(\_\_\_\_\_\_)$
- If we changed the order of the $-10x$ and $+6x$ do you think the result would change? Copy and complete to find out.

  $4x^2 - 4x - 15 = 4x^2 + 6x - 10x - 15$ $\quad 6 \times (-10) = -60, \quad 6 + (-10) = -4$
  $= 2x(\_\_\_\_\_\_) - 5(\_\_\_\_\_\_)$
  $= (\_\_\_\_\_\_)(\_\_\_\_\_\_)$

KEY IDEAS

- To factorise a **non-monic** trinomial of the form $ax^2 + bx + c$, follow these steps:
  - Find two numbers that multiply to give $a \times c$ and add to give b.
    For $15x^2 - x - 6$, $a \times c = 15 \times (-6) = -90$.
    The factors of -90 that add to -1 (b) are -10 and 9.

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- Use the two numbers shown in the example above to split bx, then factorise by grouping.
  $15x^2 - x - 6 = 15x^2 - 10x + 9x - 6$
  $= 5x(3x - 2) + 3(3x - 2) = (3x - 2)(5x + 3)$

- There are other valid methods that can be used to factorise non-monic trinomials. The cross method is illustrated in the introduction.

BUILDING UNDERSTANDING

1 State the missing numbers in this table.

$ax^2 + bx + c$	$a \times c$	Two numbers that multiply to give $a \times c$ and add to give b
$6x^2 + 13x + 6$	36	9 and ______
$8x^2 + 18x + 4$	32	
$12x^2 + x - 6$		-8 and ______
$10x^2 - 11x - 6$		
$21x^2 - 20x + 4$		-6 and ______
$15x^2 - 13x + 2$		

2 Factorise by grouping pairs.

a $x^2 + 2x + 5x + 10$
b $x^2 - 7x - 2x + 14$
c $6x^2 - 8x + 3x - 4$
d $8x^2 - 4x + 6x - 3$
e $5x^2 + 20x - 2x - 8$
f $12x^2 - 6x - 10x + 5$

Example 10 Factorising non-monic quadratics

Factorise.

a $6x^2 + 19x + 10$
b $9x^2 + 6x - 8$

SOLUTION

a $6x^2 + 19x + 10 = 6x^2 + 15x + 4x + 10$
$= 3x(2x + 5) + 2(2x + 5)$
$= (2x + 5)(3x + 2)$

EXPLANATION

$a \times c = 6 \times 10 = 60$; choose 15 and 4 since $15 \times 4 = 60$ and $15 + 4 = 19$ (b).
Factorise by grouping.

SOLUTION

b $9x^2 + 6x - 8 = 9x^2 + 12x - 6x - 8$
$= 3x(3x + 4) - 2(3x + 4)$
$= (3x + 4)(3x - 2)$

EXPLANATION

$a \times c = 9 \times (-8) = -72$; choose 12 and -6 since $12 \times (-6) = -72$ and $12 + (-6) = 6$ (b).

Now you try

Factorise.

a $6x^2 + 11x + 3$
b $8x^2 + 10x - 3$

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Example 11 Simplifying algebraic fractions involving quadratic expressions

Simplify $\dfrac{4x^2 - 9}{10x^2 + 13x - 3} \times \dfrac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$.

SOLUTION

$$\dfrac{4x^2 - 9}{10x^2 + 13x - 3} \times \dfrac{25x^2 - 10x + 1}{10x^2 - 17x + 3}$$

$$= \dfrac{(2x+3)^1(2x-3)^1}{(2x+3)^1(5x-1)^1} \times \dfrac{(5x-1)^1(5x-1)^1}{(2x-3)^1(5x-1)^1}$$

$$= 1$$

EXPLANATION

First, use the range of factorising techniques to factorise all quadratics.

Cancel to simplify.

Now you try

Simplify $\dfrac{9x^2 - 4}{12x^2 - 17x + 6} \times \dfrac{16x^2 - 24x + 9}{12x^2 - x - 6}$.

Exercise 5D

FLUENCY	1, 2(1/2)	2–3(1/3)	2–3(1/4)

1 Factorise.

Example 10a: **a** **i** $8x^2 + 14x + 3$ **ii** $10x^2 + 19x + 6$

Example 10b: **b** **i** $6x^2 + 13x - 5$ **ii** $8x^2 + 2x - 3$

Example 10: **2** Factorise the following.

a $3x^2 + 10x + 3$ **b** $2x^2 + 3x + 1$ **c** $3x^2 + 8x + 4$ **d** $3x^2 - 5x + 2$

e $2x^2 - 11x + 5$ **f** $5x^2 + 2x - 3$ **g** $3x^2 - 11x - 4$ **h** $3x^2 - 2x - 1$

i $7x^2 + 2x - 5$ **j** $2x^2 - 9x + 7$ **k** $3x^2 + 2x - 8$ **l** $2x^2 + 5x - 12$

m $2x^2 - 9x - 5$ **n** $13x^2 - 7x - 6$ **o** $5x^2 - 22x + 8$ **p** $8x^2 - 14x + 5$

q $6x^2 + x - 12$ **r** $10x^2 + 11x - 6$ **s** $6x^2 + 13x + 6$ **t** $4x^2 - 5x + 1$

u $8x^2 - 14x + 5$ **v** $8x^2 - 26x + 15$ **w** $6x^2 - 13x + 6$ **x** $9x^2 + 9x - 10$

3 Factorise the following.

a $18x^2 + 27x + 10$ **b** $20x^2 + 39x + 18$ **c** $21x^2 + 22x - 8$

d $30x^2 + 13x - 10$ **e** $40x^2 - x - 6$ **f** $28x^2 - 13x - 6$

g $24x^2 - 38x + 15$ **h** $45x^2 - 46x + 8$ **i** $25x^2 - 50x + 16$

PROBLEM-SOLVING	4(1/2), 6	4–5(1/3), 6	4–5(1/3), 6

4 Factorise by first taking out the common factor.

a $6x^2 + 38x + 40$ **b** $6x^2 - 15x - 36$ **c** $48x^2 - 18x - 3$

d $32x^2 - 88x + 60$ **e** $16x^2 - 24x + 8$ **f** $90x^2 + 90x - 100$

g $-50x^2 - 115x - 60$ **h** $12x^2 - 36x + 27$ **i** $20x^2 - 25x + 5$

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5 Simplify by first factorising.

a $\frac{6x^2 - x - 35}{3x + 7}$
b $\frac{8x^2 + 10x - 3}{2x + 3}$
c $\frac{9x^2 - 21x + 10}{3x - 5}$
d $\frac{10x - 2}{15x^2 + 7x - 2}$
e $\frac{4x + 6}{14x^2 + 17x - 6}$
f $\frac{20x - 12}{10x^2 - 21x + 9}$
g $\frac{2x^2 + 11x + 12}{6x^2 + 11x + 3}$
h $\frac{12x^2 - x - 1}{8x^2 + 14x + 3}$
i $\frac{10x^2 + 3x - 4}{14x^2 - 11x + 2}$
j $\frac{9x^2 - 4}{15x^2 + 4x - 4}$
k $\frac{14x^2 + 19x - 3}{49x^2 - 1}$
l $\frac{8x^2 - 2x - 15}{16x^2 - 25}$

6 A cable is suspended across a farm channel. The height (h), in metres, of the cable above the water surface is modelled by the equation $h = 3x^2 - 19x + 20$, where x metres is the distance from one side of the channel.

a Factorise the right-hand side of the equation.
b Determine the height of the cable when $x = 3$. Interpret this result.
c Determine where the cable is at the level of the water surface.

REASONING	7(1/2)	7–8(1/2)	7–8(1/3), 9

Example 11

7 Combine all your knowledge of factorising to simplify the following.

a $\frac{9x^2 - 16}{x^2 - 6x + 9} \times \frac{x^2 + x - 12}{3x^2 + 8x - 16}$
b $\frac{4x^2 - 1}{6x^2 - x - 2} \times \frac{9x^2 - 4}{8x - 4}$
c $\frac{1 - x^2}{15x + 9} \times \frac{25x^2 + 30x + 9}{5x^2 + 8x + 3}$
d $\frac{20x^2 + 21x - 5}{16x^2 + 8x - 15} \times \frac{16x^2 - 24x + 9}{25x^2 - 1}$
e $\frac{100x^2 - 25}{2x^2 - 9x - 5} \div \frac{2x^2 - 7x + 3}{5x^2 - 40x + 75}$
f $\frac{3x^2 - 12}{30x + 15} \div \frac{2x^2 - 3x - 2}{4x^2 + 4x + 1}$
g $\frac{9x^2 - 6x + 1}{6x^2 - 11x + 3} \div \frac{9x^2 - 1}{6x^2 - 7x - 3}$
h $\frac{16x^2 - 25}{4x^2 - 7x - 15} \div \frac{4x^2 - 17x + 15}{16x^2 - 40x + 25}$

8 Find a method to show how $-12x^2 - 5x + 3$ factorises to $(1 - 3x)(4x + 3)$. Then factorise the following.

a $-8x^2 + 2x + 15$
b $-6x^2 + 11x + 10$
c $-12x^2 + 13x + 4$
d $-8x^2 + 18x - 9$
e $-14x^2 + 39x - 10$
f $-15x^2 - x + 6$

9 Make up your own complex expression like those in Question 7, which simplifies to 1. Check your expression with your teacher or a classmate.

ENRICHMENT: Non-monics with addition and subtraction	–	–	10(1/2)

10 Factorise the quadratics in the expressions and then simplify using a common denominator.

a $\frac{2}{2x - 3} + \frac{x}{8x^2 - 10x - 3}$
b $\frac{3}{3x - 1} - \frac{x}{6x^2 + 13x - 5}$
c $\frac{4x}{2x - 5} + \frac{x}{8x^2 - 18x - 5}$
d $\frac{4x}{12x^2 - 11x + 2} - \frac{3x}{3x - 2}$
e $\frac{2}{4x^2 - 1} + \frac{1}{6x^2 - x - 2}$
f $\frac{2}{9x^2 - 25} - \frac{3}{9x^2 + 9x - 10}$
g $\frac{4}{8x^2 - 18x - 5} - \frac{2}{12x^2 - 5x - 2}$
h $\frac{1}{10x^2 - 19x + 6} + \frac{2}{4x^2 + 8x - 21}$

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5E Factorising by completing the square

Learning intentions

- To know the expanded form of a perfect square
- To be able to carry out the process of completing the square
- To know how to factorise by first completing the square
- To understand that not all quadratic expressions can be factorised and to be able to identify those that can't

Consider the quadratic expression $x^2 + 6x + 1$. We cannot factorise this using the methods we have established in the previous exercises because there are no factors of 1 that add to 6.

We can, however, use our knowledge of perfect squares and the difference of perfect squares to help find factors using surds.

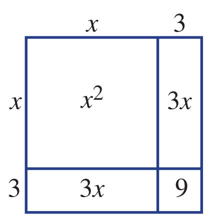
The statistical analysis of agricultural research data has found that quadratic equations model harvest yields (kg/ha) versus the quantity of nitrogen fertiliser (kg/ha) used. The CSIRO provides Australian farmers with numerous mathematical models.

LESSON STARTER Make a perfect square

This diagram is a square. Its sides are $x + 3$ and its area is given by $x^2 + 6x + 9 = (x + 3)^2$.

Use a similar diagram to help make a perfect square for the following and determine the missing number for each.

- $x^2 + 8x + ?$
- $x^2 + 12x + ?$



Can you describe a method for finding the missing number without drawing a diagram?

KEY IDEAS

- Recall for a perfect square $(x + a)^2 = x^2 + 2ax + a^2$ and $(x - a)^2 = x^2 - 2ax + a^2$.
- To **complete the square** for $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$.
  - $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
- To factorise by completing the square:
  - Add $\left(\frac{b}{2}\right)^2$ and balance by subtracting $\left(\frac{b}{2}\right)^2$.
  - Factorise the perfect square and simplify.
  - Factorise using DOPS: $a^2 - b^2 = (a + b)(a - b)$; surds can be used.

$$\begin{aligned} x^2 + 6x + 1 &= x^2 + 6x + \left(\frac{6}{2}\right)^2 - \left(\frac{6}{2}\right)^2 + 1 \\ &= \left(x + \frac{6}{2}\right)^2 - 8 \\ &= (x + 3)^2 - (\sqrt{8})^2 \\ &= (x + 3 + \sqrt{8})(x + 3 - \sqrt{8}) \\ &= (x + 3 + 2\sqrt{2})(x + 3 - 2\sqrt{2}) \end{aligned}$$

- Not all quadratic expressions factorise. This will be seen when you end up with expressions such as $(x + 3)^2 + 6$, which is *not* a difference of two perfect squares.

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BUILDING UNDERSTANDING

1. These expressions are of the form $x^2 + bx$. Evaluate $\left(\frac{b}{2}\right)^2$ for each one.
   a $x^2 + 6x$
   b $x^2 + 2x$
   c $x^2 - 4x$
   d $x^2 - 8x$
   e $x^2 + 5x$
   f $x^2 - 9x$
2. Factorise these perfect squares.
   a $x^2 + 4x + 4$
   b $x^2 + 8x + 16$
   c $x^2 + 10x + 25$
   d $x^2 - 12x + 36$
   e $x^2 - 6x + 9$
   f $x^2 - 18x + 81$
3. Factorise using surds. Recall that $a^2 - b^2 = (a + b)(a - b)$.
   a $(x + 1)^2 - 5$
   b $(x + 4)^2 - 10$
   c $(x - 3)^2 - 11$

Example 12 Completing the square

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 10x$

b $x^2 - 7x$

SOLUTION	EXPLANATION
a $\left(\frac{10}{2}\right)^2 = 5^2 = 25$	For $x^2 + bx$, add $\left(\frac{b}{2}\right)^2$. Here $b = 10$, and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 + 10x + 25 = (x + 5)^2$	$x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$
b $\left(\frac{-7}{2}\right)^2 = \frac{49}{4}$	In $x^2 - 7x$, $b = -7$ and evaluate $\left(\frac{b}{2}\right)^2$.
$x^2 - 7x + \frac{49}{4} = \left(x - \frac{7}{2}\right)^2$	Factorise the perfect square.

Now you try

Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 12x$

b $x^2 - 9x$

Example 13 Factorising by completing the square

Factorise the following by completing the square if possible.

a $x^2 + 8x - 3$

b $x^2 - 2x + 8$

Continued on next page

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SOLUTION

a $x^2 + 8x - 3 = \left(x^2 + 8x + \left(\frac{8}{2}\right)^2\right) - \left(\frac{8}{2}\right)^2 - 3$

$= \left(x + \frac{8}{2}\right)^2 - 16 - 3$

$= (x + 4)^2 - 19$

$= (x + 4)^2 - (\sqrt{19})^2$

$= (x + 4 - \sqrt{19})(x + 4 + \sqrt{19})$

EXPLANATION

Add $\left(\frac{b}{2}\right)^2$ to complete the square and balance by subtracting $\left(\frac{b}{2}\right)^2$ also.

Factorise the resulting perfect square and simplify.

Express 19 as $(\sqrt{19})^2$ to set up a DOPS.
Apply $a^2 - b^2 = (a + b)(a - b)$ using surds.

b $x^2 - 2x + 8 = \left(x^2 - 2x + \left(\frac{2}{2}\right)^2\right) - \left(\frac{2}{2}\right)^2 + 8$

$= \left(x - \frac{2}{2}\right)^2 + 7$

$= (x - 1)^2 + 7$

$\therefore x^2 - 2x + 8$ cannot be factorised.

Add $\left(\frac{2}{2}\right)^2 = (1)^2$ to complete the square and balance by subtracting $(1)^2$ also.
Factorise the perfect square and simplify.
$(x - 1)^2 + 7$ is not a *difference* of perfect squares.

Now you try

Factorise the following by completing the square if possible.

a $x^2 + 6x - 1$

b $x^2 - 4x + 7$

Example 14 Factorising with fractions

Factorise $x^2 + 3x + \frac{1}{2}$.

SOLUTION

$x^2 + 3x + \frac{1}{2} = \left(x^2 + 3x + \left(\frac{3}{2}\right)^2\right) - \left(\frac{3}{2}\right)^2 + \frac{1}{2}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{9}{4} + \frac{1}{2}$

$= \left(x + \frac{3}{2}\right)^2 - \frac{7}{4}$

$= \left(x + \frac{3}{2}\right)^2 - \left(\sqrt{\frac{7}{4}}\right)^2$

$= \left(x + \frac{3}{2} - \frac{\sqrt{7}}{2}\right)\left(x + \frac{3}{2} + \frac{\sqrt{7}}{2}\right)$

$= \left(x + \frac{3 - \sqrt{7}}{2}\right)\left(x + \frac{3 + \sqrt{7}}{2}\right)$

EXPLANATION

Add $\left(\frac{3}{2}\right)^2$ to complete the square and balance by subtracting $\left(\frac{3}{2}\right)^2$. Leave in fraction form.

Factorise the perfect square and simplify.

$-\frac{9}{4} + \frac{1}{2} = -\frac{9}{4} + \frac{2}{4} = -\frac{7}{4}$

Recall that $\sqrt{\frac{7}{4}} = \frac{\sqrt{7}}{\sqrt{4}} = \frac{\sqrt{7}}{2}$ and use DOPS.

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Now you try

Factorise $x^2 + 5x + \frac{1}{2}$.

Exercise 5E

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

1 Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

Example 12a **a** i $x^2 + 8x$ ii $x^2 + 14x$

Example 12b **b** i $x^2 - 5x$ ii $x^2 - 11x$

Example 12

2 Decide what number must be added to these expressions to complete the square. Then factorise the resulting perfect square.

a $x^2 + 6x$ **b** $x^2 + 12x$ **c** $x^2 + 4x$ **d** $x^2 + 8x$
e $x^2 - 10x$ **f** $x^2 - 2x$ **g** $x^2 - 8x$ **h** $x^2 - 12x$
i $x^2 + 5x$ **j** $x^2 + 9x$ **k** $x^2 + 7x$ **l** $x^2 + 11x$
m $x^2 - 3x$ **n** $x^2 - 7x$ **o** $x^2 - x$ **p** $x^2 - 9x$

Example 13a

3 Factorise by completing the square.

a $x^2 + 4x + 1$ **b** $x^2 + 6x + 2$ **c** $x^2 + 2x - 4$
d $x^2 + 10x - 4$ **e** $x^2 - 8x + 13$ **f** $x^2 - 12x + 10$
g $x^2 - 4x - 3$ **h** $x^2 - 8x - 5$ **i** $x^2 + 14x + 6$

Example 13b

4 Factorise, if possible.

a $x^2 + 6x + 11$ **b** $x^2 + 4x + 7$ **c** $x^2 + 8x + 1$
d $x^2 + 4x + 2$ **e** $x^2 + 10x + 3$ **f** $x^2 + 4x - 6$
g $x^2 - 10x + 30$ **h** $x^2 - 6x + 6$ **i** $x^2 - 12x + 2$
j $x^2 - 2x + 2$ **k** $x^2 - 8x - 1$ **l** $x^2 - 4x + 6$

PROBLEM-SOLVING	5(1/2)	5–6(1/2)	5–7(1/3)

Example 14

5 Factorise the following.

a $x^2 + 3x + 1$ **b** $x^2 + 7x + 2$ **c** $x^2 + 5x - 2$ **d** $x^2 + 9x - 3$
e $x^2 - 3x + \frac{1}{2}$ **f** $x^2 - 5x + \frac{1}{2}$ **g** $x^2 - 5x - \frac{3}{2}$ **h** $x^2 - 9x - \frac{5}{2}$

6 Factorise by first taking out the common factor.

a $2x^2 + 12x + 8$ **b** $3x^2 + 12x - 3$ **c** $4x^2 - 8x - 16$
d $3x^2 - 24x + 6$ **e** $-2x^2 - 4x + 10$ **f** $-3x^2 - 30x - 3$
g $-4x^2 - 16x + 12$ **h** $-2x^2 + 16x + 4$ **i** $-3x^2 + 24x - 15$

7 Factorise the following.

a $3x^2 + 9x + 3$ **b** $5x^2 + 15x - 35$ **c** $2x^2 - 10x + 4$
d $4x^2 - 28x + 12$ **e** $-3x^2 - 21x + 6$ **f** $-2x^2 - 14x + 8$
g $-4x^2 + 12x + 20$ **h** $-3x^2 + 9x + 6$ **i** $-2x^2 + 10x + 8$

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REASONING	8	8	8, 9

8 A student factorises $x^2 - 2x - 24$ by completing the square.

a Show the student's working to obtain the factorised form of $x^2 - 2x - 24$.

b Now that you have seen the answer from part **a**, what would you suggest is a better way to factorise $x^2 - 2x - 24$?

9 **a** Explain why $x^2 + 9$ cannot be factorised using real numbers.

b Decide whether the following can or cannot be factorised.

i $x^2 - 25$
ii $x^2 - 10$
iii $x^2 + 6$
iv $x^2 + 11$
v $(x + 1)^2 + 4$
vi $(x - 2)^2 - 8$
vii $(x + 3)^2 - 15$
viii $(2x - 1)^2 + 1$

c For what values of m can the following be factorised, using real numbers?

i $x^2 + 4x + m$
ii $x^2 - 6x + m$
iii $x^2 - 10x + m$

ENRICHMENT: Non-monic quadratics and completing the square	–	–	10(½)

10 A non-monic quadratic such as $2x^2 - 5x + 1$ can be factorised in the following way.

$$\begin{aligned} 2x^2 - 5x + 1 &= 2\left(x^2 - \frac{5}{2}x + \frac{1}{2}\right) \\ &= 2\left(x^2 - \frac{5}{2}x + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2 + \frac{1}{2}\right) \qquad \text{Note: } \frac{5}{2} \div 2 = \frac{5}{4} \\ &= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{8}{16}\right) \\ &= 2\left(\left(x - \frac{5}{4}\right)^2 - \frac{17}{16}\right) \\ &= 2\left(x - \frac{5}{4} + \frac{\sqrt{17}}{4}\right)\left(x - \frac{5}{4} - \frac{\sqrt{17}}{4}\right) \end{aligned}$$

Factorise these using a similar technique.

a $2x^2 + 5x - 12$
b $3x^2 + 4x - 3$
c $4x^2 - 7x - 16$
d $3x^2 - 2x + 6$
e $-2x^2 - 3x + 4$
f $-3x^2 - 7x - 3$
g $-4x^2 + 11x - 24$
h $-2x^2 + 3x + 4$
i $2x^2 + 5x - 7$
j $3x^2 + 4x - 5$
k $-2x^2 - 3x + 5$
l $-3x^2 - 7x - 4$

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5A **1** Expand brackets and simplify where possible.

a $-\frac{2x}{3}(12x - 5)$
b $a(3a - 2) - a(5 - a)$
c $(m + 2)(m + 5)$
d $(k - 3)^2$
e $(3m - 2)(3m + 2)$
f $(4h + 7)(2h - 5)$
g $5(x - 4)(x - 3)$
h $(p + 5)(p + 4) - (p - 2)(p - 8)$

5B **2** Factorise the following.

a $4a - 20$
b $-12m^2 + 18m$
c $4(x + 5) - x(x + 5)$
d $a^2 - 81$
e $16a^2 - 121b^2$
f $5m^2 - 125$
g $(k + 2)^2 - 49$
h $(x - 1)^2 - 4$
i $x^2 - 20$ (use surds)
j $(h + 3)^2 - 7$ (use surds)
k $x^2 + 5x + ax + 5a$
l $4x^2 - 8mx - 5x + 10m$

5C **3** Factorise.

a $x^2 + x - 20$
b $a^2 - 10a + 21$
c $3k^2 - 21k - 54$
d $m^2 - 12m + 36$

5C **4** Use factorisation to simplify these algebraic fractions.

a $\dfrac{x^2 + 2x - 15}{x + 5}$

b $\dfrac{x^2 - 25}{x^2 - 9x + 20} \times \dfrac{x^2 + 3x - 28}{2x + 14}$

5E **5** Complete the square and factorise, if possible.

a $x^2 + 8x + 3$
b $x^2 - 12x + 26$
c $x^2 + 14x + 50$
d $x^2 + 5x - \frac{1}{2}$

5D 10A **6** Factorise.

a $6a^2 + 19a + 10$
b $8m^2 - 6m - 9$
c $15x^2 - 22x + 8$
d $6k^2 - 11k - 35$

5D 10A **7** Simplify $\dfrac{9x^2 - 49}{3x^2 - 4x - 7} \times \dfrac{2x^2 + 7x + 5}{6x^2 + 5x - 21}$.

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5F Solving quadratic equations using factorisation

Learning intentions

- To be able to recognise a quadratic equation
- To understand that for the product of two or more numbers to be zero, then one or both of the numbers must be zero
- To know how to rearrange a quadratic equation equal to zero
- To be able to apply the steps required for solving a quadratic equation using the Null Factor Law
- To understand that a quadratic equation can have 0, 1 or 2 solutions

The result of multiplying a number by zero is zero. Consequently, if an expression equals zero then at least one of its factors must be zero. This is called the Null Factor Law and it provides us with an important method that can be utilised to solve a range of mathematical problems involving quadratic equations.

Galileo (17th century) discovered that the path of a thrown or launched object under the influence of gravity follows a precise mathematical rule, the quadratic equation. The flight time, maximum height and range of projectiles could now be calculated.

LESSON STARTER Does factorisation beat trial and error?

Set up two teams.

Team A: Trial and error

Team B: Factorisation

Instructions:

- Team A must try to find the two solutions of $3x^2 - x - 2 = 0$ by guessing and checking values for x that make the equation true.
- Team B must solve the same equation $3x^2 - x - 2 = 0$ by first factorising the left-hand side.

Which team was the first to find the two solutions for x? Discuss the methods used.

KEY IDEAS

- The **Null Factor Law** states that if the product of two numbers is zero, then either or both of the two numbers is zero.
  - If $a \times b = 0$, then either $a = 0$ or $b = 0$.
    For example, if $x(x - 3) = 0$, then either $x = 0$ or $x - 3 = 0$ (i.e. $x = 0$ or $x = 3$).
- To solve a quadratic equation, write it in standard form (i.e. $ax^2 + bx + c = 0$) and factorise. Then use the Null Factor Law.
  - If the coefficients of all the terms have a common factor, then first divide by that common factor.

BUILDING UNDERSTANDING

1 State the solutions to these equations, which are already in factorised form.

a $x(x + 1) = 0$ b $2x(x - 4) = 0$ c $(x - 3)(x + 2) = 0$

d $(x + \sqrt{3})(x - \sqrt{3}) = 0$ e $(2x - 1)(3x + 7) = 0$ f $(8x + 3)(4x + 3) = 0$

2 Rearrange and state in standard form $ax^2 + bx + c = 0$ with $a > 0$. Do not solve.

a $x^2 + 2x = 3$ b $x^2 - 5x = -6$ c $4x^2 = 3 - 4x$

d $2x(x - 3) = 5$ e $x^2 = 4(x - 3)$ f $-4 = x(3x + 2)$

3 How many different solutions for x will these equations have?

a $(x - 2)(x - 1) = 0$ b $(x + 1)(x + 1) = 0$ c $(x + \sqrt{2})(x - \sqrt{2}) = 0$

d $(x + 8)(x - \sqrt{5}) = 0$ e $(x + 2)^2 = 0$ f $3(2x + 1)^2 = 0$

Example 15 Solving quadratic equations using the Null Factor Law

Solve the following quadratic equations.

a $x^2 - 2x = 0$ b $x^2 - 15 = 0$ c $2x^2 = 50$

SOLUTION	EXPLANATION
a $x^2 - 2x = 0$ $x(x - 2) = 0$ $\therefore x = 0$ or $x - 2 = 0$ $\therefore x = 0$ or $x = 2$	Factorise by taking out the common factor x. Apply the Null Factor Law: if $a \times b = 0$, then $a = 0$ or $b = 0$. Solve for x. Check your solutions by substituting back into the equation.
b $x^2 - 15 = 0$ $(x + \sqrt{15})(x - \sqrt{15}) = 0$ $\therefore x + \sqrt{15} = 0$ or $x - \sqrt{15} = 0$ $\therefore x = -\sqrt{15}$ or $x = \sqrt{15}$	Factorise $a^2 - b^2 = (a - b)(a + b)$ using surds. Alternatively, add 15 to both sides to give $x^2 = 15$, then take the positive and negative square root. So $x = \pm\sqrt{15}$.
c $2x^2 = 50$ $2x^2 - 50 = 0$ $2(x^2 - 25) = 0$ $2(x + 5)(x - 5) = 0$ $\therefore x + 5 = 0$ or $x - 5 = 0$ $\therefore x = -5$ or $x = 5$	First, write in standard form (i.e. $ax^2 + bx + c = 0$). Take out the common factor of 2 and then factorise using $a^2 - b^2 = (a + b)(a - b)$. Alternatively, divide first by 2 to give $x^2 = 25$ and $x = \pm 5$.

Now you try

Solve the following quadratic equations.

a $x^2 - 3x = 0$ b $x^2 - 11 = 0$ c $3x^2 = 27$

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Example 16 Solving $ax^2 + bx + c = 0$

Solve the following quadratic equations.

a $x^2 - 5x + 6 = 0$ **b** $x^2 + 2x + 1 = 0$ 10A **c** $10x^2 - 13x - 3 = 0$

SOLUTION

a
$$x^2 - 5x + 6 = 0$$
$$(x - 3)(x - 2) = 0$$
$$\therefore x - 3 = 0 \text{ or } x - 2 = 0$$
$$\therefore x = 3 \text{ or } x = 2$$

EXPLANATION

Factorise by finding two numbers that multiply to 6 and add to -5: $-3 \times (-2) = 6$ and $-3 + (-2) = -5$. Apply the Null Factor Law and solve for x.

b
$$x^2 + 2x + 1 = 0$$
$$(x + 1)(x + 1) = 0$$
$$(x + 1)^2 = 0$$
$$\therefore x + 1 = 0$$
$$\therefore x = -1$$

$1 \times 1 = 1$ and $1 + 1 = 2$

$(x + 1)(x + 1) = (x + 1)^2$ is a perfect square.

This gives one solution for x.

c
$$10x^2 - 13x - 3 = 0$$
$$10x^2 - 15x + 2x - 3 = 0$$
$$5x(2x - 3) + (2x - 3) = 0$$
$$(2x - 3)(5x + 1) = 0$$

First, factorise using grouping or another method.

$10 \times (-3) = -30$, $-15 \times 2 = -30$ and $-15 + 2 = -13$.

$$\therefore 2x - 3 = 0 \text{ or } 5x + 1 = 0$$

Solve using the Null Factor Law.

$$\therefore 2x = 3 \text{ or } 5x = -1$$
$$\therefore x = \frac{3}{2} \text{ or } x = -\frac{1}{5}$$

Check your solutions by substitution.

Now you try

Solve the following quadratic equations.

a $x^2 - x - 12 = 0$ **b** $x^2 + 6x + 9 = 0$ **c** $6x^2 + x - 2 = 0$

Example 17 Solving disguised quadratics

Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 4(x + 15)$ **b** $\frac{x + 6}{x} = x$

SOLUTION

a
$$x^2 = 4(x + 15)$$
$$x^2 = 4x + 60$$
$$x^2 - 4x - 60 = 0$$
$$(x - 10)(x + 6) = 0$$
$$\therefore x - 10 = 0 \text{ or } x + 6 = 0$$
$$\therefore x = 10 \text{ or } x = -6$$

EXPLANATION

First expand and then write in standard form by subtracting $4x$ and 60 from both sides.

Factorise and apply the Null Factor Law: $-10 \times 6 = -60$ and $-10 + 6 = -4$.

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b $\dfrac{x+6}{x} = x$ — First multiply both sides by x and then write in standard form.

$x + 6 = x^2$

$0 = x^2 - x - 6$

$0 = (x - 3)(x + 2)$ — Factorise and solve using the Null Factor Law.

$\therefore x - 3 = 0$ or $x + 2 = 0$

$\therefore x = 3$ or $x = -2$ — Check your solutions.

Now you try

Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 2(x + 24)$ **b** $\dfrac{x+20}{x} = x$

Exercise 5F

FLUENCY	1, 2–3(1/2)	2–4(1/2)	2–4(1/3)

1 Solve the following quadratic equations.

Example 15a **a i** $x^2 - 5x = 0$ **ii** $x^2 - 12x = 0$

Example 15b **b i** $x^2 - 13 = 0$ **ii** $x^2 - 19 = 0$

Example 15c **c i** $2x^2 = 18$ **ii** $4x^2 = 64$

Example 15 **2** Solve the following quadratic equations.

a $x^2 - 4x = 0$ **b** $x^2 - 3x = 0$ **c** $x^2 + 2x = 0$
d $3x^2 - 12x = 0$ **e** $2x^2 - 10x = 0$ **f** $4x^2 + 8x = 0$
g $x^2 - 7 = 0$ **h** $x^2 - 11 = 0$ **i** $3x^2 - 15 = 0$
j $x^2 = 2x$ **k** $x^2 = -5x$ **l** $7x^2 = -x$
m $5x^2 = 20$ **n** $3x^2 = 27$ **o** $2x^2 = 72$

Example 16a,b **3** Solve the following quadratic equations.

a $x^2 + 3x + 2 = 0$ **b** $x^2 + 5x + 6 = 0$ **c** $x^2 - 6x + 8 = 0$
d $x^2 - 7x + 10 = 0$ **e** $x^2 + 4x - 12 = 0$ **f** $x^2 + 2x - 15 = 0$
g $x^2 - x - 20 = 0$ **h** $x^2 - 5x - 24 = 0$ **i** $x^2 - 12x + 32 = 0$
j $x^2 + 4x + 4 = 0$ **k** $x^2 + 10x + 25 = 0$ **l** $x^2 - 8x + 16 = 0$
m $x^2 - 14x + 49 = 0$ **n** $x^2 - 24x + 144 = 0$ **o** $x^2 + 18x + 81 = 0$

Example 16c 10A **4** Solve the following quadratic equations.

a $2x^2 + 11x + 12 = 0$ **b** $4x^2 + 16x + 7 = 0$ **c** $2x^2 - 17x + 35 = 0$
d $2x^2 - 23x + 11 = 0$ **e** $3x^2 - 4x - 15 = 0$ **f** $5x^2 - 7x - 6 = 0$
g $6x^2 + 7x - 20 = 0$ **h** $7x^2 + 25x - 12 = 0$ **i** $20x^2 - 33x + 10 = 0$

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PROBLEM-SOLVING	6(1/2)	5–6(1/2)	5–7(1/3)

5 Solve by first taking out a common factor.

a $2x^2 + 16x + 24 = 0$ **b** $2x^2 - 20x - 22 = 0$ **c** $3x^2 - 18x + 27 = 0$

d $5x^2 - 20x + 20 = 0$ (10A) **e** $-8x^2 - 4x + 24 = 0$ (10A) **f** $18x^2 - 57x + 30 = 0$

Example 17a

6 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $x^2 = 2(x + 12)$ **b** $x^2 = 4(x + 8)$ **c** $x^2 = 3(2x - 3)$

d $x^2 + 7x = -10$ **e** $x^2 - 8x = -15$ **f** $x(x + 4) = 4x + 9$

g $2x - 16 = x(2 - x)$ **h** $x^2 + 12x + 10 = 2x + 1$ **i** $x^2 + x - 9 = 5x - 4$

j $x^2 - 5x = -15x - 25$ **k** $x^2 - 14x = 2x - 64$ **l** $x(x + 4) = 4(x + 16)$

m $2x(x - 2) = 6$ (10A) **n** $3x(x + 6) = 4(x - 2)$ (10A) **o** $4x(x + 5) = 6x - 4x^2 - 3$

Example 17b

7 Solve the following by first writing in the form $ax^2 + bx + c = 0$.

a $\frac{5x + 84}{x} = x$ **b** $\frac{9x + 70}{x} = x$ **c** $\frac{18 - 7x}{x} = x$

(10A) **d** $\frac{20 - 3x}{x} = 2x$ (10A) **e** $\frac{6x + 8}{5x} = x$ (10A) **f** $\frac{7x + 10}{2x} = 3x$

g $\frac{3}{x} = x + 2$ (10A) **h** $\frac{1}{x} = 3 - 2x$ **i** $\frac{4}{x - 2} = x + 1$

REASONING	8	8, 9	9, 10

8 **a** Write down the solutions to the following equations.

i $2(x - 1)(x + 2) = 0$ **ii** $(x - 1)(x + 2) = 0$

b What difference has the common factor of 2 made to the solutions in the first equation?

c Explain why $x^2 - 5x - 6 = 0$ and $3x^2 - 15x - 18 = 0$ have the same solutions.

9 Explain why $x^2 + 16x + 64 = 0$ has only one solution.

10 When solving $x^2 - 2x - 8 = 7$ a student writes the following.

$x^2 - 2x - 8 = 7$

$(x - 4)(x + 2) = 7$

$x - 4 = 7$ or $x + 2 = 7$

$x = 11$ or $x = 5$

Discuss the problem with this solution and then write a correct solution.

ENRICHMENT: More algebraic fractions with quadratics	–	–	11(1/2)

11 Solve these equations by first multiplying by an appropriate expression.

a $x + 3 = -\frac{2}{x}$ **b** $-\frac{1}{x} = x - 2$ **c** $-\frac{5}{x} = 2x - 11$

d $\frac{x^2 - 48}{x} = 2$ **e** $\frac{x^2 + 12}{x} = -8$ **f** $\frac{2x^2 - 12}{x} = -5$

g $\frac{x - 5}{4} = \frac{6}{x}$ **h** $\frac{x - 2}{3} = \frac{5}{x}$ **i** $\frac{x - 4}{2} = -\frac{2}{x}$

j $\frac{x + 4}{2} - \frac{3}{x - 3} = 1$ **k** $\frac{x}{x - 2} - \frac{x + 1}{x + 4} = 1$ **l** $\frac{1}{x - 1} - \frac{1}{x + 3} = \frac{1}{3}$

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Using calculators to solve quadratic equations

1 Solve:

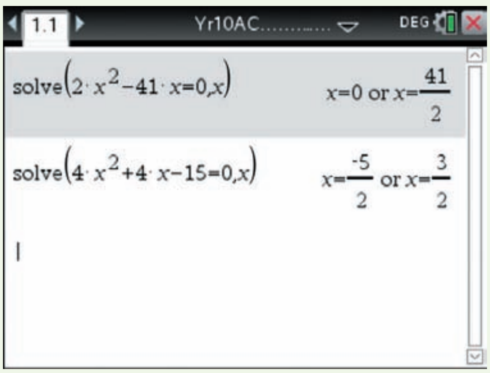
a $2x^2 - 41x = 0$

b $4x^2 + 4x - 15 = 0$

2 Solve $ax^2 + bx + c = 0$.

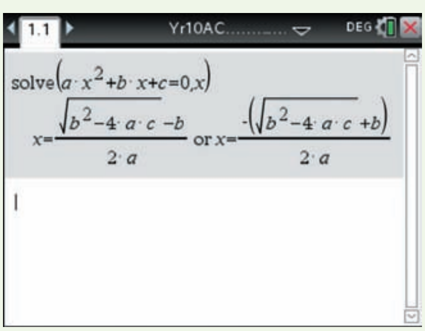
Using the TI-Nspire:

1 In a **Calculator** page use menu **>Algebra>Solve** and type as shown ending with:, x.



Note: if your answers are decimal then change the **Calculation Mode** to **Auto** in **Settings** on the Home screen.

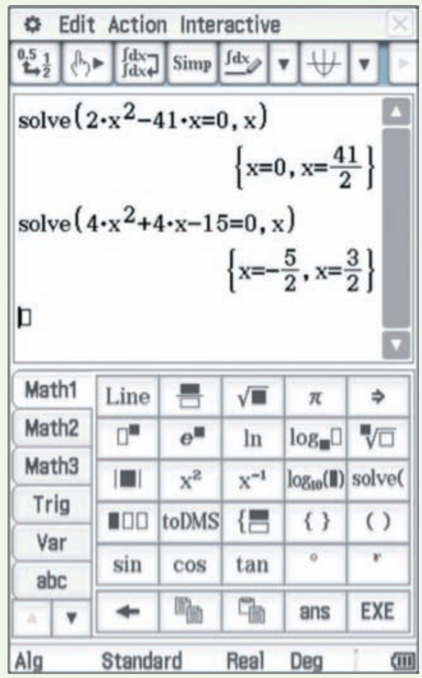
2 Use menu >**Algebra>Solve** and type as shown.



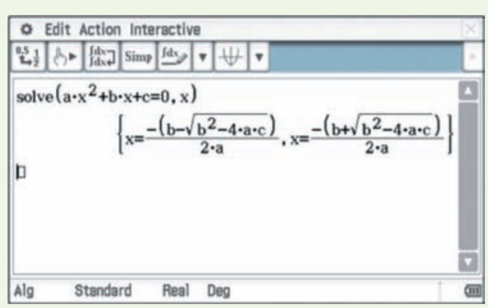
Note: use a multiplication sign between a and x^2 in ax^2 and b and x in bx.
This gives the general quadratic formula studied in **Section 5I**.

Using the ClassPad:

1 In the **Main** application, type and highlight the equation then tap **Interactive**, **Advanced**, **Solve**, **OK**.



2 Use the **VAR** keyboard to type the equation as shown. Highlight the equation and tap **Interactive**, **Advanced**, **Solve**, **OK**. This gives the general quadratic formula studied in **Section 5I**.



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5G Applications of quadratics

Learning intentions

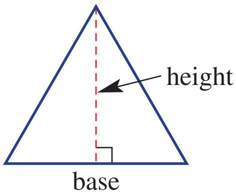
- To be able to set up a quadratic equation from a word problem
- To know how to apply the steps for solving a quadratic equation
- To understand and check the validity of solutions in the context of the given problem

Defining variables, setting up equations, solving equations and interpreting solutions are all important elements of applying quadratic equations to problem solving. The area of a rectangular paddock, for example, that is fenced off using a fixed length of fencing can be found by setting up a quadratic equation, solving it and then interpreting the solutions.

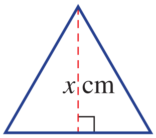
Aerospace engineers model the trajectory of a rocket under the influence of gravity using a quadratic equation of height, h, versus time. The solutions to $h = 0$ are the times when the rocket is at ground level and give its flight time.

LESSON STARTER The 10 cm^2 triangle

There are many base and height measurements for a triangle that give an area of 10 cm^2.



- Draw three different triangles that have a 10 cm^2 area. Include the measurements for the base and the height.
- Do any of your triangles have a base length that is 1 cm more than the height? Find the special triangle with area 10 cm^2 that has a base 1 cm more than its height by following these steps.
  - Let x cm be the height of the triangle.
  - Write an expression for the base length.
  - Write an equation if the area is 10 cm^2.
  - Solve the equation to find two solutions for x.
  - Which solution is to be used to describe the special triangle? Why?



KEY IDEAS

- When applying quadratic equations, follow these steps.
  - Define a variable; i.e. 'Let x be …'.
  - Write an equation.
  - Solve the equation.
  - Choose the solution(s) that solves the equation and answers the question in the context in which it was given. Check that the solutions seem reasonable.

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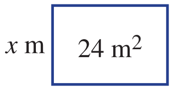
BUILDING UNDERSTANDING

1 A rectangle has an area of 24 m^2. Its length is 5 m longer than its width.



- **a** Complete this sentence: 'Let x m be the ________________.
- **b** State an expression for the rectangle's length.
- **c** State an equation using the rectangle's area.
- **d** Rearrange your equation from part **c** in standard form (i.e. $ax^2 + bx + c = 0$) and solve for x.
- **e** Find the dimensions of the rectangle.

2 Repeat all the steps in Question **1** to find the dimensions of a rectangle with the following properties.

- **a** Its area is 60 m^2 and its length is 4 m more than its width.
- **b** Its area is 63 m^2 and its length is 2 m less than its width.

Example 18 Finding dimensions using quadratics

The area of a rectangle is fixed at 28 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle.

SOLUTION

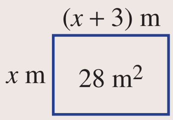
Let x m be the width of the rectangle.

$$\text{Length} = (x + 3)\text{ m}$$
$$x(x + 3) = 28$$
$$x^2 + 3x - 28 = 0$$
$$(x + 7)(x - 4) = 0$$
$$x + 7 = 0 \text{ or } x - 4 = 0$$
$$\therefore x = -7 \text{ or } x = 4$$
$$x > 0 \text{ so, choose } x = 4.$$

Rectangle has width 4 m and length 7 m.

EXPLANATION

Draw a diagram to help.



Write an equation using the given information.

Then write in standard form and solve for x.

Disregard $x = -7$ because $x > 0$.

Answer the question in full. Note: Length is $4 + 3 = 7$.

Now you try

The area of a rectangle is fixed at 48 m^2 and its length is 2 metres more than its width. Find the dimensions of the rectangle.

Exercise 5G

FLUENCY	1–5	2–5	3–6

Example 18 **1** The area of a rectangle is fixed at 12 m^2 and its length is 1 metre more than its width. Find the dimensions of the rectangle.

Example 18 **2** The area of a rectangle is fixed at 54 m^2 and its length is 3 metres more than its width. Find the dimensions of the rectangle.

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3 Find the height and base lengths of a triangle that has an area of 24 cm^2 and height 2 cm more than its base.

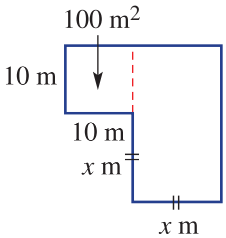
4 Find the height and base lengths of a triangle that has an area of 7 m^2 and height 5 m less than its base.

5 The product of two consecutive numbers is 72. Use a quadratic equation to find the two sets of numbers.

6 The product of two consecutive, even positive numbers is 168. Find the two numbers.

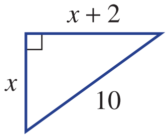
PROBLEM-SOLVING	7, 8	7–10	8–11

7 A 100 m^2 hay shed is to be extended to give 475 m^2 of floor space in total, as shown. All angles are right angles. Find the value of x.

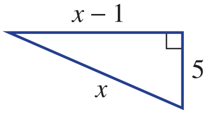


8 Solve for x in these right-angled triangles, using Pythagoras' theorem.

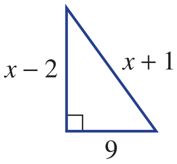
a



b



c



9 A square hut of side length 5 m is to be surrounded by a veranda of width x metres. Find the width of the veranda if its area is to be 24 m^2.

10 A father's age is the square of his son's age (x). In 20 years' time the father will be three times as old as his son. What are the ages of the father and son?

11 A rectangular painting is to have a total area (including the frame) of 1200 cm^2. If the painting is 30 cm long and 20 cm wide, find the width of the frame.

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REASONING	12	12, 13	13, 14

12 The sum of the first n positive integers is given by $\frac{1}{2}n(n + 1)$.

a Find the sum of the first 10 positive integers (i.e. use $n = 10$).

b Find the value of n if the sum of the first n positive integers is:

i 28 **ii** 91 **iii** 276

13 A ball is thrust vertically upwards from a machine on the ground. The height (h metres) after t seconds is given by $h = t(4 - t)$.

a Find the height after 1.5 seconds.

b Find when the ball is at a height of 3 metres.

c Why are there two solutions to part **b**?

d Find when the ball is at ground level. Explain.

e Find when the ball is at a height of 4 metres.

f Why is there only one solution for part **e**?

g Is there a time when the ball is at a height of 5 metres? Explain.

14 The height (h metres) of a golf ball is given by $h = -0.01x(x - 100)$, where x metres is the horizontal distance from where the ball was hit.

a Find the values of x when $h = 0$.

b Interpret your answer from part **a**.

c Find how far the ball has travelled horizontally when the height is 1.96 metres.

ENRICHMENT: Fixed perimeter and area	–	–	15, 16

15 A small rectangular block of land has a perimeter of 100 m and an area of 225 m^2. Find the dimensions of the block of land.

16 A rectangular farm has perimeter 700 m and area 30 000 m^2. Find its dimensions.

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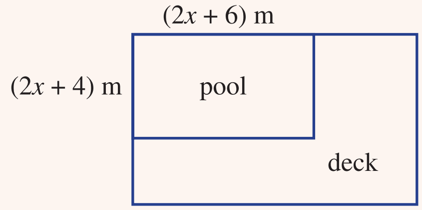
Applications and problem-solving

The following problems will investigate practical situations drawing upon knowledge and skills developed throughout the chapter. In attempting to solve these problems, aim to identify the key information, use diagrams, formulate ideas, apply strategies, make calculations and check and communicate your solutions.

On the pool deck

1 Designs for a rectangular pool are being considered with the dimensions shown such that the length is 2 m more than the width, as shown. The pool will also have a deck built around it as shown. The length and width of the combined rectangular area will be an increase of 50% of the length and width of the pool.



The pool designer wants to explore the areas of possible decks in comparison to the area of the pool.

- a Give the length and width of the combined pool and deck area in terms of x.
- b Find the area of the deck in m^2 in terms of x.
- c If the area of the deck is 100 m^2, determine the dimensions of the pool by first finding the value of x.
- d Use your answer to **b** to determine what fraction the pool area is of the deck area.
- e Repeat parts **a** and **b** to determine what fraction the pool area is of the deck area, if the deck increases the length and width of the rectangular area by 25%.

Round-robin tournament

2 A round-robin tournament with n teams, where every team plays each other once, requires $\frac{n^2 - n}{2}$ games.

Using this rule, the tournament organisers wish to explore the number of games that need to be scheduled and the number of teams required for a given number of games.

- a How many games are played in a round-robin tournament with 6 teams?
- b A round-robin tournament has 28 games, solve an appropriate equation to find the number of teams in the competition.

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c Investigate if doubling the number of teams, doubles the number of matches required. Prove algebraically.

d Give a simplified expression in terms of n for the number of games required for $n + 1$ teams.

e A tournament has n teams. How many more games are required in the tournament if the number of teams increases by

i 1 team? **ii** 2 teams? **iii** x teams?

Kayaking along the river

3 A kayaker is paddling up a river which is flowing at a certain speed. He travels 15 km up the river and then back down the river to where he started, kayaking at the same still-water speed, x km/h. The trip takes 4 hours to go up and down the river.

You wish to investigate the effect of the varying river flow speed on the speed of the kayaker who needs to complete the trip of fixed distance in the given time frame.

a If the river is flowing at a rate of 2 km/h and the man is kayaking at a rate of x km/h, find:

i expressions, in terms of x, for the rate the kayaker is moving upstream and the rate the kayaker is moving downstream

ii the value of x for this 4 hour journey.

Consider the same journey, taking the same time, with the river flowing at y km/h.

b Find a rule for the speed of the kayaker in still-water, x km/h, in terms of y.

c Use your rule from part **b** to confirm your answer to part **a** and to find the kayaker's speed if there was no current.

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5H Solving quadratic equations by completing the square

Learning intentions

- To understand that completing the square can be used to help factorise a quadratic equation when integers cannot be found
- To be able to solve an equation by using the completing the square method to factorise
- To recognise a form of a quadratic equation that gives no solutions

In **Section 5E** we saw that some quadratics cannot be factorised using integers but instead could be factorised by completing the square. Surds were also used to complete the factorisation. We can use this method to solve many quadratic equations.

In the 9th century, the great Persian mathematician Al-Khwarizmi first solved quadratic equations by completing the square. His *Al-jabr* book was the principal maths textbook in European universities for 500 years, introducing algebra, algorithms and surds.

LESSON STARTER Where does $\sqrt{6}$ come in?

Consider the equation $x^2 - 2x - 5 = 0$ and try to solve it by discussing these points.

- Are there any common factors that can be taken out?
- Are there any integers that multiply to give -5 and add to give -2?
- Try completing the square on the left-hand side. Does this help and how?
- Show that the two solutions contain the surd $\sqrt{6}$.

KEY IDEAS

- To solve quadratic equations of the form $ax^2 + bx + c = 0$ for which you cannot factorise using integers:
  - Complete the square for the quadratic expression and factorise if possible.
  - Solve the quadratic equation using the Null Factor Law or an alternate method.
- Expressions such as $x^2 + 5$ and $(x - 1)^2 + 7$ cannot be factorised further and therefore give no solutions as they cannot be expressed as a difference of two squares.

BUILDING UNDERSTANDING

1. What number must be added to the following expressions to form a perfect square?
   a $x^2 + 2x$ b $x^2 + 20x$ c $x^2 - 4x$ d $x^2 + 5x$
2. Factorise using surds.
   a $x^2 - 3 = 0$ b $x^2 - 10 = 0$ c $(x + 1)^2 - 5 = 0$
3. Solve these equations.
   a $(x - \sqrt{2})(x + \sqrt{2}) = 0$ b $(x - \sqrt{7})(x + \sqrt{7}) = 0$
   c $(x - 3 + \sqrt{5})(x - 3 - \sqrt{5}) = 0$ d $(x + 5 + \sqrt{14})(x + 5 - \sqrt{14}) = 0$

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Example 19 Solving quadratic equations by completing the square

Solve these quadratic equations by first completing the square.

a $x^2 + 4x + 2 = 0$ **b** $x^2 + 6x - 11 = 0$ **c** $x^2 - 3x + 1 = 0$

SOLUTION

a

$$x^2 - 4x + 2 = 0$$
$$x^2 - 4x + 4 - 4 + 2 = 0$$
$$(x - 2)^2 - 2 = 0$$
$$(x - 2 + \sqrt{2})(x - 2 - \sqrt{2}) = 0$$
$$\therefore x - 2 + \sqrt{2} = 0 \quad \text{or} \quad x - 2 - \sqrt{2} = 0$$
$$\therefore x = 2 - \sqrt{2} \quad \text{or} \quad x = 2 + \sqrt{2}$$

Alternate method, from

$$(x - 2)^2 - 2 = 0$$
$$(x - 2)^2 = 2$$
$$x - 2 = \pm\sqrt{2}$$
$$x = 2 \pm \sqrt{2}$$

EXPLANATION

Complete the square: $\left(\frac{-4}{2}\right)^2 = 4$.

$x^2 - 4x + 4 = (x - 2)(x - 2) = (x - 2)^2$

Use $a^2 - b^2 = (a + b)(a - b)$.

Apply the Null Factor Law and solve for x.

The solutions can also be written as $2 \pm \sqrt{2}$.

An alternate approach after completing the square is to add 2 to both sides and then take the square root of both sides

$\pm\sqrt{2}$ since $(+\sqrt{2})^2 = 2$ and $(-\sqrt{2})^2 = 2$.

b

$$x^2 + 6x - 11 = 0$$
$$x^2 + 6x + 9 - 9 - 11 = 0$$
$$(x + 3)^2 - 20 = 0$$
$$(x + 3 - \sqrt{20})(x + 3 + \sqrt{20}) = 0$$
$$(x + 3 - 2\sqrt{5})(x + 3 + 2\sqrt{5}) = 0$$
$$\therefore x + 3 - 2\sqrt{5} = 0 \quad \text{or} \quad x + 3 + 2\sqrt{5} = 0$$
$$\therefore x = -3 + 2\sqrt{5} \quad \text{or} \quad x = -3 - 2\sqrt{5}$$

Alternatively, $x = -3 \pm 2\sqrt{5}$.

EXPLANATION

Complete the square: $\left(\frac{6}{2}\right)^2 = 9$.

Use difference of perfect squares with surds.

Recall that $\sqrt{20} = \sqrt{4 \times 5} = 2\sqrt{5}$.

Apply the Null Factor Law and solve for x.

$(x + 3)^2 = 20$ can also be solved by taking the square root of both sides.

Alternatively, write solutions using the $\pm$ symbol.

c

$$x^2 - 3x + 1 = 0$$
$$x^2 - 3x + \frac{9}{4} - \frac{9}{4} + 1 = 0$$
$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$
$$\left(x + \frac{3}{2} + \sqrt{\frac{5}{4}}\right)\left(x - \frac{3}{2} - \sqrt{\frac{5}{4}}\right) = 0$$
$$x - \frac{3}{2} + \sqrt{\frac{5}{4}} = 0 \quad \text{or} \quad x - \frac{3}{2} - \sqrt{\frac{5}{4}} = 0$$
$$\therefore x = \frac{3}{2} - \frac{\sqrt{5}}{2} \quad \text{or} \quad x = \frac{3}{2} + \frac{\sqrt{5}}{2}$$
$$x = \frac{3 - \sqrt{5}}{2} \quad \text{or} \quad x = \frac{3 + \sqrt{5}}{2}$$

So $x = \dfrac{3 \pm \sqrt{5}}{2}$

EXPLANATION

$\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$

$a^2 - b^2 = (a + b)(a - b)$

Use the Null Factor Law.

Recall that $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{\sqrt{4}} = \frac{\sqrt{5}}{2}$

Combine using the $\pm$ symbol.

Continued on next page

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Now you try

Solve these quadratic equations by first completing the square.

a $x^2 - 6x + 2 = 0$ **b** $x^2 + 4x - 14 = 0$ **c** $x^2 - 5x + 2 = 0$

Exercise 5H

FLUENCY	1, 2–4(1/2)	2–4(1/3)	2–4(1/3)

1 Solve these quadratic equations by first completing the square.

Example 19a **a** **i** $x^2 - 8x + 3 = 0$ **ii** $x^2 - 12x + 7 = 0$

Example 19b **b** **i** $x^2 + 4x - 4 = 0$ **ii** $x^2 + 10x - 7 = 0$

Example 19a

2 Solve by first completing the square.

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 4x + 2 = 0$ **c** $x^2 + 10x + 15 = 0$
d $x^2 + 4x - 2 = 0$ **e** $x^2 + 8x - 3 = 0$ **f** $x^2 + 6x - 5 = 0$
g $x^2 - 8x - 1 = 0$ **h** $x^2 - 12x - 3 = 0$ **i** $x^2 - 2x - 16 = 0$
j $x^2 - 10x + 18 = 0$ **k** $x^2 - 6x + 4 = 0$ **l** $x^2 - 8x + 9 = 0$
m $x^2 + 6x - 4 = 0$ **n** $x^2 + 20x + 13 = 0$ **o** $x^2 - 14x - 6 = 0$

Example 19b

3 Solve by first completing the square.

a $x^2 + 8x + 4 = 0$ **b** $x^2 + 6x + 1 = 0$ **c** $x^2 - 10x + 5 = 0$
d $x^2 - 4x - 14 = 0$ **e** $x^2 - 10x - 3 = 0$ **f** $x^2 + 8x - 8 = 0$
g $x^2 - 2x - 31 = 0$ **h** $x^2 + 12x - 18 = 0$ **i** $x^2 + 6x - 41 = 0$

Example 19c

4 Solve by first completing the square.

a $x^2 + 5x + 2 = 0$ **b** $x^2 + 3x + 1 = 0$ **c** $x^2 + 7x + 5 = 0$
d $x^2 - 3x - 2 = 0$ **e** $x^2 - x - 3 = 0$ **f** $x^2 + 5x - 2 = 0$
g $x^2 - 7x + 2 = 0$ **h** $x^2 - 9x + 5 = 0$ **i** $x^2 + x - 4 = 0$
j $x^2 + 9x + 9 = 0$ **k** $x^2 - 3x - \frac{3}{4} = 0$ **l** $x^2 + 5x + \frac{5}{4} = 0$

PROBLEM-SOLVING	5(1/2), 8	5–7(1/3), 8	5–7(1/3), 9

5 Decide how many solutions there are to these equations. Try factorising the equations if you are unsure.

a $x^2 - 2 = 0$ **b** $x^2 - 10 = 0$ **c** $x^2 + 3 = 0$
d $x^2 + 7 = 0$ **e** $(x - 1)^2 + 4 = 0$ **f** $(x + 2)^2 - 7 = 0$
g $(x - 7)^2 - 6 = 0$ **h** $x^2 - 2x + 6 = 0$ **i** $x^2 - 3x + 10 = 0$
j $x^2 + 2x - 4 = 0$ **k** $x^2 + 7x + 1 = 0$ **l** $x^2 - 2x + 17 = 0$

6 Solve the following, if possible, by first factoring out the coefficient of x^2 and then completing the square.

a $2x^2 - 4x + 4 = 0$ **b** $4x^2 + 20x + 8 = 0$ **c** $2x^2 - 10x + 4 = 0$
d $3x^2 + 27x + 9 = 0$ **e** $3x^2 + 15x + 3 = 0$ **f** $2x^2 - 12x + 8 = 0$

7 Solve the following quadratic equations, if possible.

a $x^2 + 3x = 5$ **b** $x^2 + 5x = 9$ **c** $x^2 + 7x = -15$
d $x^2 - 8x = -11$ **e** $x^2 + 12x + 10 = 2x + 5$ **f** $x^2 + x + 9 = 5x - 3$

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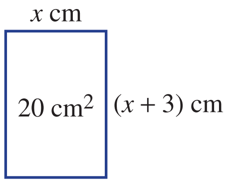
8 A rectangle's length is 3 cm more than its width. Find the dimensions of the rectangle if its area is 20 cm^2.



9 The height, h km, of a ballistic missile launched from a submarine at sea level is given by $h = \frac{x(400 - x)}{20000}$, where x km is the horizontal distance travelled.

a Find the height of a missile that has travelled the following horizontal distances.
  i 100 km ii 300 km

b Find how far the missile has travelled horizontally when the height is:
  i 0 km ii 2 km

c Find the horizontal distance the missile has travelled when its height is 1 km. (*Hint*: Complete the square.)

REASONING	10	10, 11	11, 12

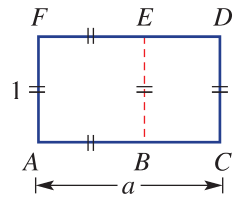
10 Complete the square to show that the following have no (real) solutions.

a $x^2 + 4x + 5 = 0$ b $x^2 - 3x = -3$

11 A friend starts to solve $x^2 + x - 30 = 0$ by completing the square but you notice there is a much quicker way. What method do you describe to your friend?

12 This rectangle is a golden rectangle.

- $ABEF$ is a square.
- Rectangle $BCDE$ is similar to rectangle $ACDF$.



a Show that $\frac{a}{1} = \frac{1}{a - 1}$.

b Find the exact value of a (which will give you the golden ratio) by completing the square.

ENRICHMENT: Completing the square with non-monics	–	–	13(½)

13 In the Enrichment section of **Exercise 5E** we looked at a method to factorise non-monic quadratics by completing the square. It involved taking out the coefficient of x^2. Dividing both sides by that number is possible in these equations and this makes the task easier. Use this technique to solve the following equations.

a $2x^2 + 4x - 1 = 0$ b $3x^2 + 6x - 12 = 0$

c $-2x^2 + 16x - 10 = 0$ d $3x^2 - 9x + 3 = 0$

e $4x^2 + 20x + 8 = 0$ f $5x^2 + 5x - 15 = 0$

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5I Solving quadratic equations using the quadratic formula

Learning intentions

- To know the quadratic formula and when to apply it
- To be able to use the quadratic formula to solve a quadratic equation
- To know what the discriminant is and what it can be used for
- To be able to use the discriminant to determine the number of solutions of a quadratic equation

A general formula for solving quadratic equations can be found by completing the square for the general case.

Surveyors regularly subdivide land into house blocks. When dimensions are linear expressions of the same variable, an area formula forms a quadratic equation. For a given area, surveyors can solve this equation using the quadratic formula.

Consider $ax^2 + bx + c = 0$, where a, b, c are constants and $a \neq 0$. Start by dividing both sides by a.

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2 - 4ac}{4a^2}\right) = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula now gives us a mechanism to solve quadratic equations and to determine how many solutions the equation has.

The expression under the root sign, $b^2 - 4ac$, is called the discriminant ($\triangle$) and helps us to identify the number of solutions. A quadratic equation can have 0, 1 or 2 solutions.

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LESSON STARTER How many solutions?

Complete this table to find the number of solutions for each equation.

$ax^2 + bx + c = 0$	a	b	c	$b^2 - 4ac$	$\frac{-b + \sqrt{b^2 - 4ac}}{2a}$	$\frac{-b - \sqrt{b^2 - 4ac}}{2a}$
$2x^2 + 7x + 1 = 0$						
$9x^2 - 6x + 1 = 0$						
$x^2 - 3x + 4 = 0$						

Discuss under what circumstances a quadratic equation has:

- 2 solutions
- 1 solution
- 0 solutions.

KEY IDEAS

- If $ax^2 + bx + c = 0$ (where a, b, c are constants and $a \neq 0$), then

  $$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

  - This is called the **quadratic formula**.
  - The quadratic formula is useful when a quadratic cannot be factorised easily.
- The **discriminant** is $\triangle = b^2 - 4ac$.
  - When $\triangle < 0$, the quadratic equation has 0 real solutions (since $\sqrt{\triangle}$ is undefined when $\triangle$ is negative).
  - When $\triangle = 0$, the quadratic equation has 1 real solution $\left(x = -\frac{b}{2a}\right)$.
  - When $\triangle > 0$, the quadratic equation has 2 real solutions $\left(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right)$.

BUILDING UNDERSTANDING

1. For these quadratic equations in the form $ax^2 + bx + c = 0$, state the values of a, b and c.
   - a $3x^2 + 2x + 1 = 0$
   - b $5x^2 + 3x - 2 = 0$
   - c $2x^2 - x - 5 = 0$
   - d $-3x^2 + 4x - 5 = 0$
2. Find the value of the discriminant ($b^2 - 4ac$) for each part in Question 1 above.
3. State the number of solutions of a quadratic that has:
   - a $b^2 - 4ac = 0$
   - b $b^2 - 4ac < 0$
   - c $b^2 - 4ac > 0$

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Example 20 Using the discriminant

Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 5x - 3 = 0$

b $2x^2 - 3x + 4 = 0$

c $x^2 + 6x + 9 = 0$

SOLUTION	EXPLANATION
a $a = 1, b = 5, c = -3$	State the values of a, b and c in $ax^2 + bx + c = 0$.
$\triangle = b^2 - 4ac$ $= (5)^2 - 4(1)(-3)$ $= 25 + 12$ $= 37$	Calculate the value of the discriminant by substituting values.
$\triangle > 0$, so there are 2 solutions.	Interpret the result with regard to the number of solutions.
b $a = 2, b = -3, c = 4$ $\triangle = b^2 - 4ac$ $= (-3)^2 - 4(2)(4)$ $= 9 - 32$ $= -23$	State the values of a, b and c and substitute to evaluate the discriminant. Recall that $(-3)^2 = -3 \times (-3) = 9$.
$\triangle < 0$, so there are no solutions.	Interpret the result.
c $a = 1, b = 6, c = 9$ $\triangle = b^2 - 4ac$ $= (6)^2 - 4(1)(9)$ $= 36 - 36$ $= 0$ $\triangle = 0$, so there is 1 solution.	Substitute the values of a, b and c to evaluate the discriminant and interpret the result. Note: $x^2 + 6x + 9 = (x + 3)^2$ is a perfect square.

Now you try

Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 7x - 1 = 0$

b $3x^2 - x + 2 = 0$

c $x^2 + 8x + 16 = 0$

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Example 21 Solving quadratic equations using the quadratic formula

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 5x + 3 = 0$

b $2x^2 - 2x - 1 = 0$

SOLUTION

a $a = 1, b = 5, c = 3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-5 \pm \sqrt{(5)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{-5 \pm \sqrt{25 - 12}}{2}$$

$$= \frac{-5 \pm \sqrt{13}}{2}$$

b $a = 2, b = -2, c = -1$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{2 \pm \sqrt{4 + 8}}{4}$$

$$= \frac{2 \pm \sqrt{12}}{4}$$

$$= \frac{2 \pm 2\sqrt{3}}{4}$$

$$= \frac{1 \pm \sqrt{3}}{2}$$

EXPLANATION

a Determine the values of a, b and c in $ax^2 + bx + c = 0$. Write out the quadratic formula and substitute the values.

Simplify.

Two solutions: $x = \frac{-5 - \sqrt{13}}{2}, \frac{-5 + \sqrt{13}}{2}$.

b Determine the values of a, b and c.

Simplify: $\sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3}$.

Cancel using the common factor:

$$\frac{2 \pm 2\sqrt{3}}{4} = \frac{2(1 \pm \sqrt{3})}{4} = \frac{1 \pm \sqrt{3}}{2}$$

Now you try

Find the exact solutions to the following using the quadratic formula.

a $x^2 + 3x + 1 = 0$

b $4x^2 - 2x - 3 = 0$

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Exercise 5I

FLUENCY	1, 2–3(1/2)	2–4(1/3)	2–4(1/3)

Example 20

1 Determine the number of solutions to the following quadratic equations using the discriminant.

a $x^2 + 3x - 1 = 0$ **b** $4x^2 - 2x + 5 = 0$ **c** $x^2 + 4x + 4 = 0$

Example 20

2 Using the discriminant, determine the number of solutions for these quadratic equations.

a $x^2 + 5x + 3 = 0$ **b** $x^2 + 3x + 4 = 0$ **c** $x^2 + 6x + 9 = 0$
d $x^2 + 7x - 3 = 0$ **e** $x^2 + 5x - 4 = 0$ **f** $x^2 + 4x - 4 = 0$
g $4x^2 + 5x + 3 = 0$ **h** $4x^2 + 3x + 1 = 0$ **i** $2x^2 + 12x + 9 = 0$
j $-x^2 - 6x - 9 = 0$ **k** $-2x^2 + 3x - 4 = 0$ **l** $-4x^2 - 6x + 3 = 0$

Example 21a

3 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 3x - 2 = 0$ **b** $x^2 + 7x - 4 = 0$ **c** $x^2 - 7x + 5 = 0$
d $x^2 - 8x + 16 = 0$ **e** $-x^2 - 5x - 4 = 0$ **f** $-x^2 - 8x - 7 = 0$
g $4x^2 + 7x - 1 = 0$ **h** $3x^2 + 5x - 1 = 0$ **i** $3x^2 - 4x - 6 = 0$
j $-2x^2 + 5x + 5 = 0$ **k** $-3x^2 - x + 4 = 0$ **l** $5x^2 + 6x - 2 = 0$

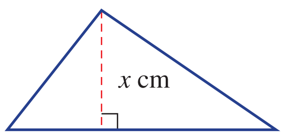
Example 21b

4 Find the exact solutions to the following quadratic equations, using the quadratic formula.

a $x^2 + 4x + 1 = 0$ **b** $x^2 - 6x + 4 = 0$ **c** $x^2 + 6x - 2 = 0$
d $-x^2 - 3x + 9 = 0$ **e** $-x^2 + 4x + 4 = 0$ **f** $-3x^2 + 8x - 2 = 0$
g $2x^2 - 2x - 3 = 0$ **h** $3x^2 - 6x - 1 = 0$ **i** $-5x^2 + 8x + 3 = 0$

PROBLEM-SOLVING	5, 6(1/2)	5, 6(1/2), 8	6(1/2), 7, 9

5 A triangle's base is 5 cm more than its height of x cm. Find its height if the triangle's area is 10 cm^2.

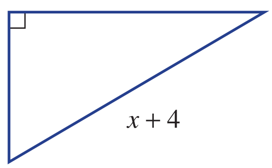


6 Solve the following using the quadratic formula.

a $3x^2 = 1 + 6x$ **b** $2x^2 = 3 - 4x$ **c** $5x = 2 - 4x^2$
d $2x - 5 = -\frac{1}{x}$ **e** $\frac{3}{x} = 3x + 4$ **f** $-\frac{5}{x} = 2 - x$
g $5x = \frac{2x + 2}{x}$ **h** $x = \frac{3x + 4}{2x}$ **i** $3x = \frac{10x - 1}{2x}$

7 Two positive numbers differ by 3 and their product is 11. Find the numbers.

8 Find the exact perimeter of this right-angled triangle.



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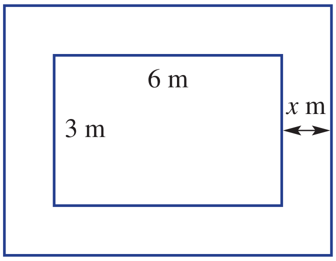
9 A rectangular pool measuring 6 m by 3 m is to have a path surrounding it. If the total area of the pool and path is to be 31 m^2, find the width (x m) of the path, correct to the nearest centimetre.



REASONING	10	10, 11	11, 12

10 Explain why the rule $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ gives only one solution when $b^2 - 4ac = 0$.

11 Make up three quadratic equations that have:
- a no solutions
- b 1 solution
- c 2 solutions.

12 For what two values of k does $x^2 + kx + 9 = 0$ have only one solution?

ENRICHMENT: k determines the number of solutions	–	–	13(½)

13 The discriminant for $x^2 + 2x + k = 0$ is $4 - 4k$, so there:
- are no solutions for $4 - 4k < 0, \therefore k > 1$
- is 1 solution for $4 - 4k = 0, \therefore k = 1$
- are 2 solutions for $4 - 4k > 0, \therefore k < 1$

a For what values of k does $x^2 + 4x + k = 0$ have:
 i no solutions? ii 1 solution? iii 2 solutions?

b For what values of k does $kx^2 + 3x + 2 = 0$ have:
 i no solutions? ii 1 solution? iii 2 solutions?

c For what values of k does $x^2 + kx + 1 = 0$ have:
 i no solutions? ii 1 solution? iii 2 solutions?

d For what values of k does $3x^2 + kx - 1 = 0$ have:
 i no solutions? ii 1 solution? iii 2 solutions?

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Problems and challenges

Up for a challenge? If you get stuck on a question, check out the 'Working with unfamiliar problems' poster at the end of the book to help you.

1 Find the monic quadratic in the form $x^2 + bx + c = 0$ with solutions $x = 2 - \sqrt{3}$ and $x = 2 + \sqrt{3}$.

2 If $x + \frac{1}{x} = 7$, what is $x^2 + \frac{1}{x^2}$?

3 Find all the solutions to each equation. (*Hint*: Consider letting $a = x^2$ in each equation.)

a $x^4 - 5x^2 + 4 = 0$

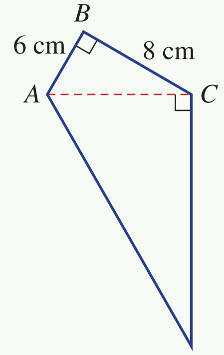
b $x^4 - 7x^2 - 18 = 0$

4 Make a substitution as you did in Question 3 to obtain a quadratic equation to help you solve the following.

a $3^{2x} - 4 \times 3^x + 3 = 0$

b $4 \times 2^{2x} - 9 \times 2^x + 2 = 0$

5 Quadrilateral *ABCD* has a perimeter of 64 cm with measurements as shown. What is the area of the quadrilateral?



6 A cyclist in a charity ride rides 300 km at a constant average speed. If the average speed had been 5 km/h faster, the ride would have taken 2 hours less. What was the average speed of the cyclist?

7 Find the value of x, correct to one decimal place, in this diagram if the area is to be 20 square units.

8 Prove that $x^2 - 2x + 2 > 0$ for all values of x.

9 A square has the same perimeter as a rectangle of length x cm and width y cm. Determine a simplified expression for the difference in their areas and, hence, show that when the perimeters are equal the square has the greatest area.

10 The equation $x^2 + wx + t = 0$ has solutions α and β, where the equation $x^2 + px + q = 0$ has solutions 3α and 3β. Determine the ratios $w : p$ and $t : q$.

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Quadratic equations

Factorising non-monic quadratics $ax^2 + bx + c$ (10A)

Use grouping, split up bx using two numbers that multiply to $a \times c$ and add to give b:

e.g. $6x^2 - 5x - 4$
$a \times c = 6 \times (-4) = -24$
$-8 \times 3 = -24$
$-8 + 3 = -5$
$\therefore$ $6x^2 + 3x - 8x - 4$
$= 3x(2x + 1) - 4(2x + 1)$
$= (2x + 1)(3x - 4)$

Factorising $x^2 + bx + c$

Two numbers $\times$ to give c.
Two numbers $+$ to give b.
e.g. $x^2 - 7x - 18 = (x - 9)(x + 2)$
$-9 \times 2 = -18$
$-9 + 2 = -7$

Factorising and DOPS

Always take out common factors first.
Difference of perfect squares
$a^2 - b^2 = (a - b)(a + b)$
e.g. $4x^2 - 9 = (2x)^2 - (3)^2$
$= (2x - 3)(2x + 3)$
$x^2 - 7 = (x - \sqrt{7})(x + \sqrt{7})$

Factorising by completing the square

e.g. $x^2 + 4x - 3$
$= (x^2 + 4x + (\frac{4}{2})^2) - (\frac{4}{2})^2 - 3$
$= (x + \frac{4}{2})^2 - 4 - 3$
$= (x + 2)^2 - 7$
$= (x + 2 - \sqrt{7})(x + 2 + \sqrt{7})$
Note, for example, $(x + 2)^2 + 5$ cannot be factorised.

Expanding brackets

$a(b + c) = ab + ac$
$(a + b)(c + d) = ac + ad + bc + bd$
$(a + b)(a - b) = a^2 - b^2$
$(a + b)^2 = a^2 + 2ab + b^2$
$(a - b)^2 = a^2 - 2ab + b^2$

Solving quadratic equations

Null Factor Law:
If $ab = 0$ then $a = 0$ or $b = 0$.
Write each quadratic in standard form $ax^2 + bx + c = 0$, factorise then apply the Null Factor Law to solve.

e.g. **1** $x^2 - 4x = 0$
$x(x - 4) = 0$
$x = 0$ or $x - 4 = 0$
$x = 0$ or $x = 4$

2 $x^2 = 3x - 10$
$x^2 - 3x + 10 = 0$
$(x - 5)(x + 2) = 0$
$x - 5 = 0$ or $x + 2 = 0$
$x = 5$ or $x = -2$

Quadratic formula

If $ax^2 + bx + c = 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The discriminant $\Delta = b^2 - 4ac$ tells us how many solutions:

$\Delta > 0$	2 solutions
$\Delta = 0$	1 solution
$\Delta < 0$	0 solutions

Applications

1. Define the variable.
2. Set up the equation.
3. Solve by factorising and using the Null Factor Law or quadratic formula.
4. Determine the suitable answer(s).

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Chapter checklist

Chapter checklist: Success criteria

		✔
5A	**1. I can apply the distributive law to expand and simplify.** e.g. Expand and simplify $2x(3x - 5) - 3(3x - 5)$.	
5A	**2. I can expand a binomial product.** e.g. Expand and simplify $(2x - 3)(x + 4)$.	
5A	**3. I can expand to form a difference of perfect squares.** e.g. Expand $(3x + 2)(3x - 2)$.	
5A	**4. I can expand a perfect square.** e.g. Expand $(x + 5)^2$.	
5B	**5. I can factorise by taking out a common factor.** e.g. Factorise $12x^2 - 18x$.	
5B	**6. I can factorise a difference of perfect squares.** e.g. Factorise $9x^2 - 16$.	
5B	**7. I can factorise a difference of perfect squares involving surds.** e.g. Factorise $x^2 - 7$ using surds.	
5B	**8. I can factorise using grouping.** e.g. Factorise $x^2 - ax + 2x - 2a$ by grouping.	
5C	**9. I can factorise a monic trinomial.** e.g. Factorise $x^2 - 8x - 20$.	
5C	**10. I can factorise a trinomial with a common factor.** e.g. Factorise $3x^2 - 24x + 45$.	
5C	**11. I can multiply and divide algebraic fractions by first factorising.** e.g. Simplify by first factorising $\frac{x^2 - 4}{x + 2} \times \frac{3x + 12}{x^2 + 2x - 8}$.	
5D	**12. I can factorise a non-monic quadratic.** 10A e.g. Factorise $5x^2 + 13x - 6$.	
5E	**13. I can factorise by completing the square.** e.g. Factorise $x^2 + 6x + 2$ by completing the square.	
5E	**14. I can recognise when a quadratic cannot be factorised.** e.g. Factorise $x^2 - 3x + 4$ by completing the square if possible.	
5F	**15. I can solve a quadratic equation by factorising and applying the Null Factor Law.** e.g. Solve $3x^2 - 9x = 0$.	
5F	**16. I can solve a quadratic equation by first rearranging into standard form.** e.g. Solve $x^2 = 2x + 3$.	
5G	**17. I can solve a word problem using a quadratic model.** e.g. The area of a rectangle is 60 m^2 and its length is 4 metres more than its width. Find the dimensions of the rectangle.	
5H	**18. I can solve a quadratic equation using completing the square.** e.g. Solve $x^2 + 4x + 22 = 0$ by first completing the square.	
5I	**19. I can determine the number of solutions of a quadratic equation.** e.g. Use the discriminant to determine the number of solutions of the equation $2x^2 - 3x - 5 = 0$.	
5I	**20. I can use the quadratic formula to solve a quadratic equation.** e.g. Find the exact solutions to $2x^2 + 3x - 4 = 0$ using the quadratic formula.	

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Short-answer questions

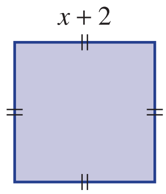
5A **1** Expand the following and simplify where possible.

a $2(x + 3) - 4(x - 5)$ **b** $(x + 5)(3x - 4)$ **c** $(5x - 2)(5x + 2)$

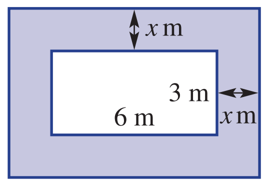
d $(x - 6)^2$ **e** $(x + 4)^2 - (x + 3)(x - 2)$ **f** $(3x - 2)(4x - 5)$

5A **2** Write, in expanded form, an expression for the shaded areas. All angles are right angles.

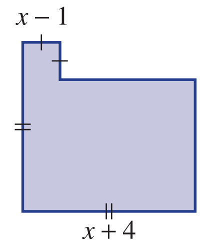
a



b



c



5B **3** Factorise the following difference of perfect squares. Remember to look for a common factor first.

a $x^2 - 49$ **b** $9x^2 - 16$ **c** $4x^2 - 1$

d $3x^2 - 75$ **e** $2x^2 - 18$ **f** $x^2 - 11$

g $-2x^2 + 40$ **h** $(x + 1)^2 - 16$ **i** $(x - 3)^2 - 10$

5C **4** Factorise these quadratic trinomials.

a $x^2 - 8x + 12$ **b** $x^2 + 10x - 24$ **c** $-3x^2 + 21x - 18$

5D 10A **5** Factorise these non-monic quadratic trinomials.

a $3x^2 + 17x + 10$ **b** $4x^2 + 4x - 15$ **c** $12x^2 - 16x - 3$ **d** $12x^2 - 23x + 10$

5C/D **6** Simplify.

a $\dfrac{12x}{x^2 + 2x - 3} \times \dfrac{x^2 - 1}{6x + 6}$ 10A **b** $\dfrac{4x^2 - 9}{2x^2 + x - 6} \div \dfrac{8x + 12}{x^2 - 2x - 8}$

5E **7** Factorise the following by completing the square.

a $x^2 + 8x + 10$ **b** $x^2 + 10x - 4$ **c** $x^2 - 6x - 3$

d $x^2 + 3x - 2$ **e** $x^2 + 5x + 3$ **f** $x^2 + 7x + \dfrac{9}{2}$

5F **8** Solve these quadratic equations by factorising and applying the Null Factor Law.

a $x^2 + 4x = 0$ **b** $3x^2 - 9x = 0$ **c** $x^2 - 25 = 0$

d $x^2 - 10x + 21 = 0$ **e** $x^2 - 8x + 16 = 0$ **f** $x^2 + 5x - 36 = 0$

10A **g** $2x^2 + 3x - 2 = 0$ 10A **h** $6x^2 + 11x - 10 = 0$ 10A **i** $18x^2 + 25x - 3 = 0$

5F **9** Solve the following quadratic equations by first writing them in standard form.

a $3x^2 = 27$ **b** $x^2 = 4x + 5$

c $2x^2 - 28 = x(x - 3)$ **d** $\dfrac{3x + 18}{x} = x$

5G **10** A rectangular sandpit is 2 m longer than it is wide. If it occupies an area of 48 m^2, determine the dimensions of the sandpit by solving a suitable equation.

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5H **11** Solve these quadratic equations by first completing the square.

a $x^2 + 4x - 3 = 0$ **b** $x^2 - 6x + 1 = 0$

c $x^2 - 3x - 2 = 0$ **d** $x^2 + 5x - 5 = 0$

5I **12** For each quadratic equation, determine the number of solutions by finding the value of the discriminant.

a $x^2 + 2x + 1 = 0$ **b** $x^2 - 3x - 3 = 0$

c $2x^2 - 4x + 3 = 0$ **d** $-3x^2 + x + 5 = 0$

5I **13** Use the quadratic formula to give exact solutions to these quadratic equations.

a $x^2 + 3x - 6 = 0$ **b** $x^2 - 2x - 4 = 0$

c $2x^2 - 4x - 5 = 0$ **d** $-3x^2 + x + 3 = 0$

Multiple-choice questions

5A **1** $(x + 5)^2$ is equivalent to:

A $x^2 + 25$ **B** $x^2 + 5x$ **C** $x^2 + 5x + 25$

D $x^2 + 10x + 25$ **E** $x^2 + 50$

5A **2** $2(2x - 1)(x + 4)$ is equivalent to:

A $4x^2 + 15x - 4$ **B** $4x^2 + 14x - 8$ **C** $8x^2 + 28x - 16$

D $8x^2 + 18x - 4$ **E** $4x^2 + 10x + 8$

5B **3** $4x^2 - 25$ in factorised form is:

A $4(x - 5)(x + 5)$ **B** $(2x - 5)^2$ **C** $(2x - 5)(2x + 5)$

D $(4x + 5)(x - 5)$ **E** $2(2x + 1)(x - 25)$

5C **4** The fully factorised form of $2x^2 - 10x - 28$ is:

A $2(x + 2)(x - 7)$ **B** $(2x + 7)(x + 4)$ **C** $2(x - 4)(x - 1)$

D $(2x - 2)(x + 14)$ **E** $(x - 2)(x + 7)$

5C **5** $\frac{x^2 + x - 20}{8x} \times \frac{2x + 8}{x^2 - 16}$ simplifies to:

A $\frac{x - 20}{8}$ **B** $\frac{x + 5}{4x}$ **C** $\frac{x + 5}{x - 4}$ **D** $x - 5$ **E** $\frac{x^2 - 20}{16}$

5E **6** The term that needs to be added to make $x^2 - 6x$ a perfect square is:

A 18 **B** −9 **C** −3 **D** 9 **E** 3

5F **7** The solution(s) to $2x^2 - 8x = 0$ are:

A $x = 0, x = -4$ **B** $x = 2$ **C** $x = 0, x = 4$

D $x = 4$ **E** $x = 0, x = 2$

5F 10A **8** For $8x^2 - 14x + 3 = 0$, the solutions for x are:

A $\frac{1}{8}, -\frac{1}{3}$ **B** $\frac{3}{4}, -\frac{1}{2}$ **C** $\frac{1}{4}, \frac{3}{2}$

D $\frac{3}{4}, -\frac{1}{2}$ **E** $-\frac{1}{2}, -\frac{3}{8}$

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5F **9** When written in the standard form $ax^2 + bx + c = 0$, $\frac{x-3}{x} = 2x$ is:

A $x^2 + 2x + 3 = 0$ **B** $x^2 + 3 = 0$ **C** $2x^2 + x - 3 = 0$
D $2x^2 - x - 3 = 0$ **E** $2x^2 - x + 3 = 0$

5G **10** The product of two consecutive numbers is 72. If x is the smaller number, an equation to represent this would be:

A $x^2 + x + 72 = 0$ **B** $2x - 71 = 0$ **C** $x^2 + x - 72 = 0$
D $x^2 + 1 = 72$ **E** $x^2 = x + 72$

5H **11** For $(x - 7)^2 - 3 = 0$, the solutions for x are:

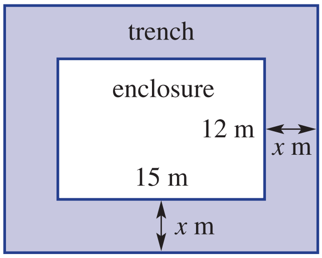
A $7 - \sqrt{3}, 7 + \sqrt{3}$ **B** $-7 - \sqrt{3}, -7 + \sqrt{3}$ **C** $7, -3$
D $-7 - \sqrt{3}, 7 + \sqrt{3}$ **E** $4, 10$

5I **12** If $ax^2 + bx + c = 0$ has exactly two solutions, then:

A $b^2 - 4ac = 0$ **B** $b^2 - 4ac > 0$ **C** $b^2 - 4ac \leqslant 0$
D $b^2 - 4ac \neq 0$ **E** $b^2 - 4ac < 0$

Extended-response questions

1 A zoo enclosure for a rare tiger is rectangular in shape and has a trench of width x m all the way around it to ensure the tiger doesn't get far if it tries to escape. The dimensions are as shown.



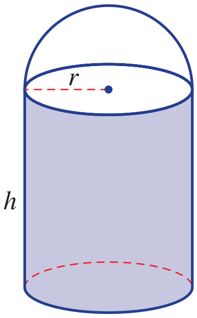
a Write an expression in terms of x for:
  i the length of the enclosure and trench combined
  ii the width of the enclosure and trench combined.

b Use your answers from part **a** to find the area of the overall enclosure and trench, in expanded form.

c Hence, find an expression for the area of the trench alone.

d Zoo restrictions state that the trench must have an area of at least 58 m^2. By solving a suitable equation, find the minimum width of the trench.

2 The surface area S of a cylindrical tank with a hemispherical top is given by the equation $S = 3\pi r^2 + 2\pi rh$, where r is the radius and h is the height of the cylinder.



a If the radius of a tank with height 6 m is 3 m, determine its exact surface area.

b If the surface area of a tank with radius 5 m is 250 m^2, determine its height, to two decimal places.

c The surface area of a tank of height 6 m is found to be 420 m^2.
  i Substitute the values and rewrite the equation in terms of r only.
  ii Rearrange the equation and write it in the form $ar^2 + br + c = 0$.
  iii Solve for r using the quadratic formula and round your answer to two decimal places.

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