TOPIC 16

Real numbers

16.1 Overview

Why learn this?

A knowledge of number is crucial if we are to understand the world around us. Over time, you have been building your knowledge of the concept of number, starting with the counting numbers, also known as natural numbers. Moving on, you needed to include zero. You then had to learn about integers and fractions, which are also called rational numbers. But even the rational numbers do not include all of the numbers on the number line, as they do not include numbers that cannot be written as fractions. That brings us to the concept of real numbers, the set of numbers that includes both rational and irrational numbers.

What do you know?

assess on

1 **THINK** List what you know about real numbers. Use a thinking tool such as a concept map to show your list.
2 **PAIR** Share what you know with a partner and then with a small group.
3 **SHARE** As a class, create a thinking tool such as a large concept map that shows your class's knowledge of real numbers.

Learning sequence

16.1 Overview
16.2 Number classification review
16.3 Surds
16.4 Operations with surds
16.5 Fractional indices
16.6 Negative indices
16.7 Logarithms
16.8 Logarithm laws
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The story of mathematics:
Real numbers

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16.2 Number classification review

eBook*plus*

Interactivity
Classifying numbers
int-2792

- The number systems used today evolved from a basic and practical need of primitive people to count and measure magnitudes and quantities such as livestock, people, possessions, time and so on.
- As societies grew and architecture and engineering developed, number systems became more sophisticated. Number use developed from solely whole numbers to fractions, decimals and irrational numbers.

- The real number system contains the set of rational and irrational numbers. It is denoted by the symbol R. The set of real numbers contains a number of subsets which can be classified as shown in the chart below.

Real numbers R

- **Irrational numbers I** (surds, non-terminating and non-recurring decimals, π, e)
- **Rational numbers Q**
  - **Integers Z**
    - Negative Z^-
    - Zero (neither positive nor negative)
    - Positive Z^+ (Natural numbers N)
  - **Non-integer rationals** (terminating and recurring decimals)

Rational numbers (Q)

- A **rational number** (*ratio*-nal) is a number that can be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
  – Rational numbers are given the symbol Q. Examples are:

$$\frac{1}{5}, \frac{2}{7}, \frac{3}{10}, \frac{9}{4}, 7, -6, 0.35, 1.\dot{4}$$

Integers (Z)

- Rational numbers may be expressed as **integers**. Examples are:

$$\tfrac{5}{1} = 5, -\tfrac{4}{1} = -4, \tfrac{27}{1} = 27, -\tfrac{15}{1} = -15$$

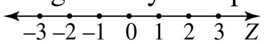
- The set of integers consists of positive and negative whole numbers and 0 (which is neither positive nor negative). They are denoted by the letter Z and can be further divided into subsets. That is:

$$\begin{aligned} Z &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\} \\ Z^{+} &= \{1, 2, 3, 4, 5, 6, \ldots\} \\ Z^{-} &= \{\ldots, -5, -4, -3, -2, -1\} \end{aligned}$$

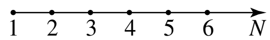
- Positive integers are also known as **natural numbers** (or counting numbers) and are denoted by the letter N. That is:

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$

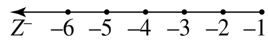
- Integers may be represented on the number line as illustrated below.



The set of integers



The set of positive integers or natural numbers



The set of negative integers

Note: Integers on the number line are marked with a solid dot to indicate that they are the only points in which we are interested.

Non-integer rational numbers

- Rational numbers may be expressed as **terminating decimals**. Examples are:

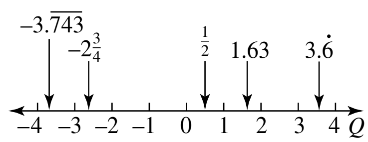
$$\tfrac{7}{10} = 0.7, \tfrac{1}{4} = 0.25, \tfrac{5}{8} = 0.625, \tfrac{9}{5} = 1.8$$

These decimal numbers terminate after a specific number of digits.

- Rational numbers may be expressed as **recurring decimals** (non-terminating or periodic decimals). For example:

$$\begin{aligned} \tfrac{1}{3} &= 0.333\,333\ldots \text{or } 0.\dot{3} \\ \tfrac{9}{11} &= 0.818\,181\ldots \text{or } 0.\dot{8}\dot{1}\ (\text{or } 0.\overline{81}) \\ \tfrac{5}{6} &= 0.833\,333\ldots \text{or } 0.8\dot{3} \\ \tfrac{3}{13} &= 0.230\,769\,230\,769\ldots \text{or } 0.\dot{2}30\,76\dot{9}\ (\text{or } 0.\overline{230\,769}) \end{aligned}$$

- These decimal numbers do not terminate, and the specific digit (or number of digits) is repeated in a pattern. Recurring decimals are represented by placing a dot or line above the repeating digit or pattern.



- Rational numbers are defined in set notation as:
Q = rational numbers

$Q = \left\{\frac{a}{b}, a, b \in Z, b \neq 0\right\}$ where $\in$ means 'an element of'.

Irrational numbers (I)

- An **irrational number** (ir-*ratio*-nal) is a number that cannot be expressed as a ratio of two whole numbers in the form $\frac{a}{b}$, where $b \neq 0$.
- Irrational numbers are given the symbol I. Examples are:

$$\sqrt{7}, \sqrt{13}, 5\sqrt{21}, \frac{\sqrt{7}}{9}, \pi, e$$

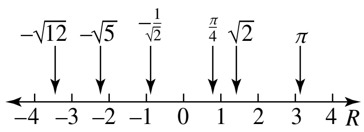
- Irrational numbers may be expressed as decimals. For example:

$$\sqrt{5} = 2.236\,067\,977\,5\ldots \qquad \sqrt{0.03} = 0.173\,205\,080\,757\ldots$$
$$\sqrt{18} = 4.242\,640\,687\,12\ldots \qquad 2\sqrt{7} = 5.291\,502\,622\,13\ldots$$
$$\pi = 3.141\,592\,653\,59\ldots \qquad e = 2.718\,281\,828\,46\ldots$$

- These decimal numbers do not terminate, and the digits do not repeat themselves in any particular pattern or order (that is, they are non-terminating and non-recurring).

Real numbers

- Rational and irrational numbers belong to the set of **real numbers** (denoted by the symbol R). They can be positive, negative or 0. The real numbers may be represented on a number line as shown at right (irrational numbers above the line; rational numbers below it).



- To classify a number as either rational or irrational:
  1. Determine whether it can be expressed as a whole number, a fraction or a terminating or recurring decimal.
  2. If the answer is yes, the number is rational; if the answer is no, the number is irrational.

π (pi)

- The symbol **π (pi)** is used for a particular number; that is, the circumference of a circle whose diameter length is 1 unit.
- It can be approximated as a decimal that is non-terminating and non-recurring. Therefore, π is classified as an irrational number. (It is also called a **transcendental number** and cannot be expressed as a surd.)
- In decimal form, $\pi = 3.141\,592\,653\,589\,793\,23\ldots$ It has been calculated to 29 000 000 (29 million) decimal places with the aid of a computer.

WORKED EXAMPLE 1

Specify whether the following numbers are rational or irrational.

a $\frac{1}{5}$ **b** $\sqrt{25}$ **c** $\sqrt{13}$ **d** 3π **e** 0.54 **f** $\sqrt[3]{64}$ **g** $\sqrt[3]{32}$ **h** $\sqrt[3]{\frac{1}{27}}$

THINK	WRITE
a $\frac{1}{5}$ is already a rational number.	**a** $\frac{1}{5}$ is rational.
b 1 Evaluate $\sqrt{25}$.	**b** $\sqrt{25} = 5$
2 The answer is an integer, so classify $\sqrt{25}$.	$\sqrt{25}$ is rational.
c 1 Evaluate $\sqrt{13}$.	**c** $\sqrt{13} = 3.605\,551\,275\,46\ldots$
2 The answer is a non-terminating and non-recurring decimal; classify $\sqrt{13}$.	$\sqrt{13}$ is irrational.
d 1 Use your calculator to find the value of 3π.	**d** $3\pi = 9.424\,777\,960\,77\ldots$
2 The answer is a non-terminating and non-recurring decimal; classify 3π.	3π is irrational.
e 0.54 is a terminating decimal; classify it accordingly.	**e** 0.54 is rational.

f	1 Evaluate $\sqrt[3]{64}$.	f	$\sqrt[3]{64} = 4$
	2 The answer is a whole number, so classify $\sqrt[3]{64}$.		$\sqrt[3]{64}$ is rational.
g	1 Evaluate $\sqrt[3]{32}$.	g	$\sqrt[3]{32} = 3.174\,802\,103\,94\ldots$
	2 The result is a non-terminating and non-recurring decimal; classify $\sqrt[3]{32}$.		$\sqrt[3]{32}$ is irrational.
h	1 Evaluate $\sqrt[3]{\frac{1}{27}}$.	h	$\sqrt[3]{\frac{1}{27}} = \frac{1}{3}$
	2 The result is a number in a rational form.		$\sqrt[3]{\frac{1}{27}}$ is rational.

Exercise 16.2 Number classification review

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INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–6, 8, 10	Questions: 1–8, 10, 12	Questions: 1–13

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REFLECTION
Why is it important to understand the real number system?

FLUENCY

1 **WE1** Specify whether the following numbers are rational (Q) or irrational (I).

a $\sqrt{4}$ b $\frac{4}{5}$ c $\frac{7}{9}$
d $\sqrt{2}$ e $\sqrt{7}$ f $\sqrt{0.04}$
g $2\frac{1}{2}$ h $\sqrt{5}$ i $\frac{9}{4}$
j 0.15 k -2.4 l $\sqrt{100}$
m $\sqrt{14.4}$ n $\sqrt{1.44}$ o π
p $\sqrt{\frac{25}{9}}$ q 7.32 r $-\sqrt{21}$
s $\sqrt{1000}$ t $7.216\,349\,157\ldots$ u $-\sqrt{81}$
v 3π w $\sqrt[3]{62}$ x $\sqrt{\frac{1}{16}}$
y $\sqrt[3]{0.0001}$

2 Specify whether the following numbers are rational (Q), irrational (I) or neither.

a $\frac{1}{8}$ b $\sqrt{625}$ c $\frac{11}{4}$ d $\frac{0}{8}$ e $-6\frac{1}{7}$
f $\sqrt[3]{81}$ g $-\sqrt{11}$ h $\sqrt{\frac{1.44}{4}}$ i $\sqrt{\pi}$ j $\frac{8}{0}$
k $\sqrt[3]{21}$ l $\frac{\pi}{7}$ m $\sqrt[3]{(-5)^2}$ n $-\frac{3}{11}$ o $\sqrt{\frac{1}{100}}$
p $\frac{64}{16}$ q $\sqrt{\frac{2}{25}}$ r $\frac{\sqrt{6}}{2}$ s $\sqrt[3]{27}$ t $\frac{1}{\sqrt{4}}$
u $\frac{22\pi}{7}$ v $\sqrt[3]{-1.728}$ w $6\sqrt{4}$ x $4\sqrt{6}$ y $(\sqrt{2})^4$

3 **MC** Which of the following best represents a rational number?

A π B $\sqrt{\frac{4}{9}}$ C $\sqrt{\frac{9}{12}}$ D $\sqrt[3]{3}$ E $\sqrt{5}$

4 **MC** Which of the following best represents an irrational number?

A $-\sqrt{81}$ B $\frac{6}{5}$ C $\sqrt[3]{343}$ D $\sqrt{22}$ E $\sqrt{144}$

5 **MC** Which of the following statements regarding the numbers -0.69, $\sqrt{7}$, $\frac{\pi}{3}$, $\sqrt{49}$ is correct?

A $\frac{\pi}{3}$ is the only rational number.
B $\sqrt{7}$ and $\sqrt{49}$ are both irrational numbers.
C -0.69 and $\sqrt{49}$ are the only rational numbers.
D -0.69 is the only rational number.
E $\sqrt{7}$ is the only rational number.

6 **MC** Which of the following statements regarding the numbers $2\frac{1}{2}$, $-\frac{11}{3}$, $\sqrt{624}$, $\sqrt[3]{99}$ is correct?

A $-\frac{11}{3}$ and $\sqrt{624}$ are both irrational numbers.
B $\sqrt{624}$ is an irrational number and $\sqrt[3]{99}$ is a rational number.
C $\sqrt{624}$ and $\sqrt[3]{99}$ are both irrational numbers.
D $2\frac{1}{2}$ is a rational number and $-\frac{11}{3}$ is an irrational number.
E $\sqrt[3]{99}$ is the only rational number.

UNDERSTANDING

7 Simplify $\sqrt{\frac{a^2}{b^2}}$.

8 **MC** If $p < 0$, then $\sqrt{p}$ is:

A positive B negative C rational D none of the above

9 **MC** If $p < 0$, then $\sqrt{p^2}$ must be:

A positive B negative C rational D any of the above

REASONING

10 Simplify $(\sqrt{p} - \sqrt{q}) \times (\sqrt{p} + \sqrt{q})$.

11 Prove that if $c^2 = a^2 + b^2$, it does not follow that $a = b + c$.

PROBLEM SOLVING

12 Find the value of m and n if $\frac{36}{11}$ is written as:

a $3 + \dfrac{1}{\frac{m}{n}}$ b $3 + \dfrac{1}{3 + \frac{m}{n}}$ c $3 + \dfrac{1}{3 + \dfrac{1}{\frac{m}{n}}}$ d $3 + \dfrac{1}{3 + \dfrac{1}{1 + \frac{m}{n}}}$.

13 If x^{-1} means $\frac{1}{x}$, what is the value of $\dfrac{3^{-1} - 4^{-1}}{3^{-1} + 4^{-1}}$?

16.3 Surds

- A **surd** is an irrational number that is represented by a root sign or a radical sign, for example: $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$.
  Examples of surds include: $\sqrt{7}$, $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[4]{15}$.
  Examples that are not surds include:
  $$\sqrt{9},\ \sqrt{16},\ \sqrt[3]{125},\ \sqrt[4]{81}.$$
- Numbers that are not surds can be simplified to rational numbers, that is:
  $$\sqrt{9} = 3, \sqrt{16} = 4, \sqrt[3]{125} = 5, \sqrt[4]{81} = 3.$$

WORKED EXAMPLE 2

Which of the following numbers are surds?

a $\sqrt{16}$ **b** $\sqrt{13}$ **c** $\sqrt{\frac{1}{16}}$ **d** $\sqrt[3]{17}$ **e** $\sqrt[4]{63}$ **f** $\sqrt[3]{1728}$

THINK		WRITE
a	1 Evaluate $\sqrt{16}$.	**a** $\sqrt{16} = 4$
	2 The answer is rational (since it is a whole number), so state your conclusion.	$\sqrt{16}$ is not a surd.
b	1 Evaluate $\sqrt{13}$.	**b** $\sqrt{13} = 3.605\,551\,275\,46\ldots$
	2 The answer is irrational (since it is a non-recurring and non-terminating decimal), so state your conclusion.	$\sqrt{13}$ is a surd.
c	1 Evaluate $\sqrt{\frac{1}{16}}$.	**c** $\sqrt{\frac{1}{16}} = \frac{1}{4}$
	2 The answer is rational (a fraction); state your conclusion.	$\sqrt{\frac{1}{16}}$ is not a surd.
d	1 Evaluate $\sqrt[3]{17}$.	**d** $\sqrt[3]{17} = 2.571\,281\,590\,66\ldots$
	2 The answer is irrational (a non-terminating and non-recurring decimal), so state your conclusion.	$\sqrt[3]{17}$ is a surd.
e	1 Evaluate $\sqrt[4]{63}$.	**e** $\sqrt[4]{63} = 2.817\,313\,247\,26\ldots$
	2 The answer is irrational, so classify $\sqrt[4]{63}$ accordingly.	$\sqrt[4]{63}$ is a surd.
f	1 Evaluate $\sqrt[3]{1728}$.	**f** $\sqrt[3]{1728} = 12$
	2 The answer is rational; state your conclusion.	$\sqrt[3]{1728}$ is not a surd. So **b**, **d** and **e** are surds.

Proof that a number is irrational

- In Mathematics you are required to study a variety of types of proofs. One such method is called proof by contradiction.
- This proof is so named because the logical argument of the proof is based on an assumption that leads to contradiction within the proof. Therefore the original assumption must be false.
- An irrational number is one that cannot be expressed in the form $\frac{a}{b}$ (where a and b are integers). The next worked example sets out to prove that $\sqrt{2}$ is irrational.

WORKED EXAMPLE 3

Prove that $\sqrt{2}$ is irrational.

THINK	WRITE
1 Assume that $\sqrt{2}$ is rational; that is, it can be written as $\frac{a}{b}$ in simplest form. We need to show that a and b have no common factors.	Let $\sqrt{2} = \frac{a}{b}$, where $b \neq 0$.
2 Square both sides of the equation.	$2 = \frac{a^2}{b^2}$
3 Rearrange the equation to make a^2 the subject of the formula.	$a^2 = 2b^2$ [1]
4 $2b^2$ is an even number and $2b^2 = a^2$.	$\therefore$ a^2 is an even number and a must also be even; that is, a has a factor of 2.
5 Since a is even it can be written as $a = 2r$.	$\therefore$ $a = 2r$
6 Square both sides.	$a^2 = 4r^2$ But $a^2 = 2b^2$ from [1] [2]
7 Equate [1] and [2].	$\therefore 2b^2 = 4r^2$ $b^2 = \frac{4r^2}{2}$ $= 2r^2$ $\therefore$ b^2 is an even number and b must also be even; that is, b has a factor of 2.
8 Repeat the steps for b as previously done for a.	Both a and b have a common factor of 2. This contradicts the original assumption that $\sqrt{2} = \frac{a}{b}$, where a and b have no common factor. $\therefore$ $\sqrt{2}$ is not rational. $\therefore$ It must be irrational.

- *Note:* An irrational number written in surd form gives an exact value of the number; whereas the same number written in decimal form (for example, to 4 decimal places) gives an approximate value.

Exercise 16.3 Surds

assess on

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–8, 10	Questions: 1–10	Questions: 1–11

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REFLECTION
How can you be certain that $\sqrt{a}$ is a surd?

FLUENCY

1 **WE2** Which of the numbers below are surds?

a $\sqrt{81}$ **b** $\sqrt{48}$ **c** $\sqrt{16}$ **d** $\sqrt{1.6}$
e $\sqrt{0.16}$ **f** $\sqrt{11}$ **g** $\sqrt{\frac{3}{4}}$ **h** $\sqrt[3]{\frac{3}{27}}$
i $\sqrt{1000}$ **j** $\sqrt{1.44}$ **k** $4\sqrt{100}$ **l** $2+\sqrt{10}$
m $\sqrt[3]{32}$ **n** $\sqrt{361}$ **o** $\sqrt[3]{100}$ **p** $\sqrt[3]{125}$
q $\sqrt{6}+\sqrt{6}$ **r** 2π **s** $\sqrt[3]{169}$ **t** $\sqrt{\frac{7}{8}}$
u $\sqrt[4]{16}$ **v** $(\sqrt{7})^2$ **w** $\sqrt[3]{33}$ **x** $\sqrt{0.0001}$
y $\sqrt[5]{32}$ **z** $\sqrt{80}$

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2 **MC** The correct statement regarding the set of numbers $\left\{\sqrt{\frac{6}{9}}, \sqrt{20}, \sqrt{54}, \sqrt[3]{27}, \sqrt{9}\right\}$ is:

A $\sqrt[3]{27}$ and $\sqrt{9}$ are the only rational numbers of the set.
B $\sqrt{\frac{6}{9}}$ is the only surd of the set.
C $\sqrt{\frac{6}{9}}$ and $\sqrt{20}$ are the only surds of the set.
D $\sqrt{20}$ and $\sqrt{54}$ are the only surds of the set.
E $\sqrt{9}$ and $\sqrt{20}$ are the only surds of the set.

3 **MC** Which of the numbers of the set $\left\{\sqrt{\frac{1}{4}}, \sqrt[3]{\frac{1}{27}}, \sqrt{\frac{1}{8}}, \sqrt{21}, \sqrt[3]{8}\right\}$ are surds?

A $\sqrt{21}$ only **B** $\sqrt{\frac{1}{8}}$ only **C** $\sqrt{\frac{1}{8}}$ and $\sqrt[3]{8}$
D $\sqrt{\frac{1}{8}}$ and $\sqrt{21}$ only **E** $\sqrt{\frac{1}{4}}$ and $\sqrt{21}$ only

4 **MC** Which statement regarding the set of numbers $\left\{\pi, \sqrt{\frac{1}{49}}, \sqrt{12}, \sqrt{16}, \sqrt{3}+1\right\}$ is *not* true?

A $\sqrt{12}$ is a surd. **B** $\sqrt{12}$ and $\sqrt{16}$ are surds.
C π is irrational but not a surd. **D** $\sqrt{12}$ and $\sqrt{3}+1$ are not rational.
E π is not a surd.

5 **MC** Which statement regarding the set of numbers $\left\{6\sqrt{7}, \sqrt{\frac{144}{16}}, 7\sqrt{6}, 9\sqrt{2}, \sqrt{18}, \sqrt{25}\right\}$ is *not* true?

- **A** $\sqrt{\frac{144}{16}}$ when simplified is an integer.
- **B** $\sqrt{\frac{144}{16}}$ and $\sqrt{25}$ are not surds.
- **C** $7\sqrt{6}$ is smaller than $9\sqrt{2}$.
- **D** $9\sqrt{2}$ is smaller than $6\sqrt{7}$.
- **E** $\sqrt{18}$ is a surd.

UNDERSTANDING

6 Complete the following statement by selecting appropriate words, suggested in brackets:

$\sqrt[6]{a}$ is definitely not a surd, if a is . . . (any multiple of 4; a perfect square and cube).

7 Find the smallest value of m, where m is a positive integer, so that $\sqrt[3]{16m}$ is not a surd.

REASONING

8 **WE3** Prove that the following numbers are irrational, using a proof by contradiction:

a $\sqrt{3}$ **b** $\sqrt{5}$ **c** $\sqrt{7}$.

9 π is an irrational number and so is $\sqrt{3}$. Therefore, determine whether

$$\left(\pi - \sqrt{3}\right)\left(\pi + \sqrt{3}\right)$$

is an irrational number.

PROBLEM SOLVING

10 Many composite numbers have a variety of factor pairs. For example, factor pairs of 24 are 1 and 24, 2 and 12, 3 and 8, 4 and 6.

a Use each pair of possible factors to simplify the following surds.

i $\sqrt{48}$ **ii** $\sqrt{72}$

b Does the factor pair chosen when simplifying a surd affect the way the surd is written in simplified form?

c Does the factor pair chosen when simplifying a surd affect the value of the surd when it is written in simplified form? Explain.

11 Solve $\sqrt{3}x - \sqrt{12} = \sqrt{3}$ and indicate whether the result is rational or irrational and integral or not integral.

16.4 Operations with surds

Simplifying surds

- To simplify a surd means to make a number (or an expression) under the radical sign ($\sqrt{\ }$) as small as possible.
- To simplify a surd (if it is possible), it should be rewritten as a product of two factors, one of which is a perfect square, that is, 4, 9, 16, 25, 36, 49, 64, 81, 100 and so on.
- We must always aim to obtain the largest perfect square when simplifying surds so that there are fewer steps involved in obtaining the answer. For example, $\sqrt{32}$ could be written as $\sqrt{4 \times 8} = 2\sqrt{8}$; however, $\sqrt{8}$ can be further simplified to $2\sqrt{2}$, so $\sqrt{32} = 2 \times 2\sqrt{2}$; that is $\sqrt{32} = 4\sqrt{2}$. If, however, the largest perfect square had been selected and $\sqrt{32}$ had been written as $\sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$, the same answer would be obtained in fewer steps.

WORKED EXAMPLE 4

Simplify the following surds. Assume that *x* and *y* are positive real numbers.

a $\sqrt{384}$ **b** $3\sqrt{405}$ **c** $-\frac{1}{8}\sqrt{175}$ **d** $5\sqrt{180x^3y^5}$

THINK	WRITE
a 1 Express 384 as a product of two factors where one factor is the largest possible perfect square.	**a** $\sqrt{384} = \sqrt{64 \times 6}$
2 Express $\sqrt{64 \times 6}$ as the product of two surds.	$= \sqrt{64} \times \sqrt{6}$
3 Simplify the square root from the perfect square (that is, $\sqrt{64} = 8$).	$= 8\sqrt{6}$
b 1 Express 405 as a product of two factors, one of which is the largest possible perfect square.	**b** $3\sqrt{405} = 3\sqrt{81 \times 5}$
2 Express $\sqrt{81 \times 5}$ as a product of two surds.	$= 3\sqrt{81} \times \sqrt{5}$
3 Simplify $\sqrt{81}$.	$= 3 \times 9\sqrt{5}$
4 Multiply together the whole numbers outside the square root sign (3 and 9).	$= 27\sqrt{5}$
c 1 Express 175 as a product of two factors in which one factor is the largest possible perfect square.	**c** $-\frac{1}{8}\sqrt{175} = -\frac{1}{8}\sqrt{25 \times 7}$
2 Express $\sqrt{25 \times 7}$ as a product of 2 surds.	$= -\frac{1}{8} \times \sqrt{25} \times \sqrt{7}$
3 Simplify $\sqrt{25}$.	$= -\frac{1}{8} \times 5\sqrt{7}$
4 Multiply together the numbers outside the square root sign.	$= -\frac{5}{8}\sqrt{7}$
d 1 Express each of 180, x^3 and y^5 as a product of two factors where one factor is the largest possible perfect square.	**d** $5\sqrt{180x^3y^5} = 5\sqrt{36 \times 5 \times x^2 \times x \times y^4 \times y}$
2 Separate all perfect squares into one surd and all other factors into the other surd.	$= 5 \times \sqrt{36x^2y^4} \times \sqrt{5xy}$
3 Simplify $\sqrt{36x^2y^4}$.	$= 5 \times 6 \times x \times y^2 \times \sqrt{5xy}$
4 Multiply together the numbers and the pronumerals outside the square root sign.	$= 30xy^2\sqrt{5xy}$

Addition and subtraction of surds

- Surds may be added or subtracted only if they are *alike*.
  Examples of *like* surds include $\sqrt{7}$, $3\sqrt{7}$ and $-5\sqrt{7}$. Examples of *unlike* surds include $\sqrt{11}$, $\sqrt{5}$, $2\sqrt{13}$ and $-2\sqrt{3}$.
- In some cases surds will need to be simplified before you decide whether they are like or unlike, and then addition and subtraction can take place. The concept of adding and subtracting surds is similar to adding and subtracting like terms in algebra.

WORKED EXAMPLE 5 TI CASIO

Simplify each of the following expressions containing surds. Assume that *a* and *b* are positive real numbers.

a $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6}$

b $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$

c $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$

THINK	WRITE
a All 3 terms are alike because they contain the same surd ($\sqrt{6}$). Simplify.	**a** $3\sqrt{6} + 17\sqrt{6} - 2\sqrt{6} = (3 + 17 - 2)\sqrt{6}$ $= 18\sqrt{6}$
b 1 Simplify surds where possible.	**b** $5\sqrt{3} + 2\sqrt{12} - 5\sqrt{2} + 3\sqrt{8}$ $= 5\sqrt{3} + 2\sqrt{4 \times 3} - 5\sqrt{2} + 3\sqrt{4 \times 2}$ $= 5\sqrt{3} + 2 \times 2\sqrt{3} - 5\sqrt{2} + 3 \times 2\sqrt{2}$ $= 5\sqrt{3} + 4\sqrt{3} - 5\sqrt{2} + 6\sqrt{2}$
2 Add like terms to obtain the simplified answer.	$= 9\sqrt{3} + \sqrt{2}$
c 1 Simplify surds where possible.	**c** $\frac{1}{2}\sqrt{100a^3b^2} + ab\sqrt{36a} - 5\sqrt{4a^2b}$ $= \frac{1}{2} \times 10\sqrt{a^2 \times a \times b^2} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$ $= \frac{1}{2} \times 10 \times a \times b\sqrt{a} + ab \times 6\sqrt{a} - 5 \times 2 \times a\sqrt{b}$ $= 5ab\sqrt{a} + 6ab\sqrt{a} - 10a\sqrt{b}$
2 Add like terms to obtain the simplified answer.	$= 11ab\sqrt{a} - 10a\sqrt{b}$

Multiplication and division of surds

Multiplying surds

- To multiply surds, multiply together the expressions under the radical signs. For example, $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, where a and b are positive real numbers.
- When multiplying surds it is best to first simplify them (if possible). Once this has been done and a mixed surd has been obtained, the coefficients are multiplied with each other and then the surds are multiplied together. For example,

$$m\sqrt{a} \times n\sqrt{b} = mn\sqrt{ab}.$$

WORKED EXAMPLE 6 TI CASIO

Multiply the following surds, expressing answers in the simplest form. Assume that x and y are positive real numbers.

a $\sqrt{11} \times \sqrt{7}$

b $5\sqrt{3} \times 8\sqrt{5}$

c $6\sqrt{12} \times 2\sqrt{6}$

d $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$

THINK	WRITE
a Multiply the surds together, using $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ (that is, multiply expressions under the square root sign). *Note:* This expression cannot be simplified any further.	**a** $\sqrt{11} \times \sqrt{7} = \sqrt{11 \times 7}$ $= \sqrt{77}$
b Multiply the coefficients together and then multiply the surds together.	**b** $5\sqrt{3} \times 8\sqrt{5} = 5 \times 8 \times \sqrt{3} \times \sqrt{5}$ $= 40 \times \sqrt{3 \times 5}$ $= 40\sqrt{15}$
c 1 Simplify $\sqrt{12}$.	**c** $6\sqrt{12} \times 2\sqrt{6} = 6\sqrt{4 \times 3} \times 2\sqrt{6}$ $= 6 \times 2\sqrt{3} \times 2\sqrt{6}$ $= 12\sqrt{3} \times 2\sqrt{6}$
2 Multiply the coefficients together and multiply the surds together.	$= 24\sqrt{18}$
3 Simplify the surd.	$= 24\sqrt{9 \times 2}$ $= 24 \times 3\sqrt{2}$ $= 72\sqrt{2}$
d 1 Simplify each of the surds.	**d** $\sqrt{15x^5y^2} \times \sqrt{12x^2y}$ $= \sqrt{15 \times x^4 \times x \times y^2} \times \sqrt{4 \times 3 \times x^2 \times y}$ $= x^2 \times y \times \sqrt{15 \times x} \times 2 \times x \times \sqrt{3 \times y}$ $= x^2y\sqrt{15x} \times 2x\sqrt{3y}$
2 Multiply the coefficients together and the surds together.	$= x^2y \times 2x\sqrt{15x \times 3y}$ $= 2x^3y\sqrt{45xy}$
3 Simplify the surd.	$= 2x^3y\sqrt{9 \times 5xy}$ $= 2x^3y \times 3\sqrt{5xy}$ $= 6x^3y\sqrt{5xy}$

- When working with surds, it is sometimes necessary to multiply surds by themselves; that is, square them. Consider the following examples:

$$(\sqrt{2})^2 = \sqrt{2} \times \sqrt{2} = \sqrt{4} = 2$$
$$(\sqrt{5})^2 = \sqrt{5} \times \sqrt{5} = \sqrt{25} = 5$$

- Observe that squaring a surd produces the number under the radical sign. This is not surprising, because squaring and taking the square root are *inverse operations* and, when applied together, leave the original unchanged.
- When a surd is squared, the result is the number (or expression) under the radical sign; that is, $(\sqrt{a})^2 = a$, where a is a positive real number.

WORKED EXAMPLE 7

Simplify each of the following.

a $(\sqrt{6})^2$ **b** $(3\sqrt{5})^2$

THINK	WRITE
a Use $(\sqrt{a})^2 = a$, where $a = 6$.	**a** $(\sqrt{6})^2 = 6$
b 1 Square 3 and apply $(\sqrt{a})^2 = a$ to square $\sqrt{5}$.	**b** $(3\sqrt{5})^2 = 3^2 \times (\sqrt{5})^2$ $= 9 \times 5$
2 Simplify.	$= 45$

Dividing surds

- To divide surds, divide the expressions under the radical signs; that is, $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$, where a and b are whole numbers.
- When dividing surds it is best to simplify them (if possible) first. Once this has been done, the coefficients are divided next and then the surds are divided.

WORKED EXAMPLE 8 TI CASIO

Divide the following surds, expressing answers in the simplest form. Assume that x and y are positive real numbers.

a $\dfrac{\sqrt{55}}{\sqrt{5}}$ **b** $\dfrac{\sqrt{48}}{\sqrt{3}}$ **c** $\dfrac{9\sqrt{88}}{6\sqrt{99}}$ **d** $\dfrac{\sqrt{36xy}}{\sqrt{25x^9y^{11}}}$

THINK	WRITE
a 1 Rewrite the fraction, using $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$.	**a** $\dfrac{\sqrt{55}}{\sqrt{5}} = \sqrt{\dfrac{55}{5}}$
2 Divide the numerator by the denominator (that is, 55 by 5).	$= \sqrt{11}$
3 Check if the surd can be simplified any further.	

b	**1** Rewrite the fraction, using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**b**	$\frac{\sqrt{48}}{\sqrt{3}} = \sqrt{\frac{48}{3}}$
	2 Divide 48 by 3.		$= \sqrt{16}$
	3 Evaluate $\sqrt{16}$.		$= 4$
c	**1** Rewrite surds, using $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$.	**c**	$\frac{9\sqrt{88}}{6\sqrt{99}} = \frac{9}{6}\sqrt{\frac{88}{99}}$
	2 Simplify the fraction under the radical by dividing both numerator and denominator by 11.		$= \frac{9}{6}\sqrt{\frac{8}{9}}$
	3 Simplify surds.		$= \frac{9 \times 2\sqrt{2}}{6 \times 3}$
	4 Multiply the whole numbers in the numerator together and those in the denominator together.		$= \frac{18\sqrt{2}}{18}$
	5 Cancel the common factor of 18.		$= \sqrt{2}$
d	**1** Simplify each surd.	**d**	$\frac{\sqrt{36xy}}{\sqrt{25x^9y^{11}}} = \frac{6\sqrt{xy}}{5\sqrt{x^8 \times x \times y^{10} \times y}}$ $= \frac{6\sqrt{xy}}{5x^4y^5\sqrt{xy}}$
	2 Cancel any common factors — in this case $\sqrt{xy}$.		$= \frac{6}{5x^4y^5}$

Rationalising denominators

- If the **denominator** of a fraction is a surd, it can be changed into a rational number through multiplication. In other words, it can be rationalised.
- As discussed earlier in this chapter, squaring a simple surd (that is, multiplying it by itself) results in a rational number. This fact can be used to rationalise denominators as follows.

$$\frac{\sqrt{a}}{\sqrt{b}} \times \frac{\sqrt{b}}{\sqrt{b}} = \frac{\sqrt{ab}}{b}, \text{ where } \frac{\sqrt{b}}{\sqrt{b}} = 1$$

- If both numerator and denominator of a fraction are multiplied by the surd contained in the denominator, the denominator becomes a rational number. The fraction takes on a different appearance, but its numerical value is unchanged, because multiplying the numerator and denominator by the same number is equivalent to multiplying by 1.

WORKED EXAMPLE 9 TI CASIO

Express the following in their simplest form with a rational denominator.

a $\dfrac{\sqrt{6}}{\sqrt{13}}$ **b** $\dfrac{2\sqrt{12}}{3\sqrt{54}}$ **c** $\dfrac{\sqrt{17}-3\sqrt{14}}{\sqrt{7}}$

THINK		WRITE
a	**1** Write the fraction.	**a** $\dfrac{\sqrt{6}}{\sqrt{13}}$
	2 Multiply both the numerator and denominator by the surd contained in the denominator (in this case $\sqrt{13}$). This has the same effect as multiplying the fraction by 1, because $\dfrac{\sqrt{13}}{\sqrt{13}} = 1$.	$= \dfrac{\sqrt{6}}{\sqrt{13}} \times \dfrac{\sqrt{13}}{\sqrt{13}}$ $= \dfrac{\sqrt{78}}{13}$
b	**1** Write the fraction.	**b** $\dfrac{2\sqrt{12}}{3\sqrt{54}}$
	2 Simplify the surds. (This avoids dealing with large numbers.)	$\dfrac{2\sqrt{12}}{3\sqrt{54}} = \dfrac{2\sqrt{4 \times 3}}{3\sqrt{9 \times 6}}$ $= \dfrac{2 \times 2\sqrt{3}}{3 \times 3\sqrt{6}}$ $= \dfrac{4\sqrt{3}}{9\sqrt{6}}$
	3 Multiply both the numerator and denominator by $\sqrt{6}$. (This has the same effect as multiplying the fraction by 1, because $\dfrac{\sqrt{6}}{\sqrt{6}} = 1$.) *Note:* We need to multiply only by the surd part of the denominator (that is, by $\sqrt{6}$ rather than by $9\sqrt{6}$).	$= \dfrac{4\sqrt{3}}{9\sqrt{6}} \times \dfrac{\sqrt{6}}{\sqrt{6}}$ $= \dfrac{4\sqrt{18}}{9 \times 6}$
	4 Simplify $\sqrt{18}$.	$= \dfrac{4\sqrt{9 \times 2}}{9 \times 6}$ $= \dfrac{4 \times 3\sqrt{2}}{54}$ $= \dfrac{12\sqrt{2}}{54}$
	5 Divide both the numerator and denominator by 6 (cancel down).	$= \dfrac{2\sqrt{2}}{9}$
c	**1** Write the fraction.	**c** $\dfrac{\sqrt{17}-3\sqrt{14}}{\sqrt{7}}$
	2 Multiply both the numerator and denominator by $\sqrt{7}$. Use grouping symbols (brackets) to make it clear that the whole numerator must be multiplied by $\sqrt{7}$.	$= \dfrac{(\sqrt{17}-3\sqrt{14})}{\sqrt{7}} \times \dfrac{\sqrt{7}}{\sqrt{7}}$

3 Apply the Distributive Law in the numerator. $a(b + c) = ab + ac$

$$= \frac{\sqrt{17} \times \sqrt{7} - 3\sqrt{14} \times \sqrt{7}}{\sqrt{7} \times \sqrt{7}}$$

$$= \frac{\sqrt{119} - 3\sqrt{98}}{7}$$

4 Simplify $\sqrt{98}$.

$$= \frac{\sqrt{119} - 3\sqrt{49 \times 2}}{7}$$

$$= \frac{\sqrt{119} - 3 \times 7\sqrt{2}}{7}$$

$$= \frac{\sqrt{119} - 21\sqrt{2}}{7}$$

Rationalising denominators using conjugate surds

- The product of pairs of **conjugate surds** results in a rational number. (Examples of pairs of conjugate surds include $\sqrt{6} + 11$ and $\sqrt{6} - 11$, $\sqrt{a} + b$ and $\sqrt{a} - b$, $2\sqrt{5} - \sqrt{7}$ and $2\sqrt{5} + \sqrt{7}$.)

  This fact is used to rationalise denominators containing a sum or a difference of surds.
- To rationalise the denominator that contains a sum or a difference of surds, multiply both numerator and denominator by the conjugate of the denominator.

  Two examples are given below:

1. To rationalise the denominator of the fraction $\frac{1}{\sqrt{a} + \sqrt{b}}$, multiply it by $\frac{\sqrt{a} - \sqrt{b}}{\sqrt{a} - \sqrt{b}}$.
2. To rationalise the denominator of the fraction $\frac{1}{\sqrt{a} - \sqrt{b}}$, multiply it by $\frac{\sqrt{a} + \sqrt{b}}{\sqrt{a} + \sqrt{b}}$.

A quick way to simplify the denominator is to use the difference of two squares identity:

$$\begin{aligned}(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) &= (\sqrt{a})^2 - (\sqrt{b})^2 \\ &= a - b\end{aligned}$$

WORKED EXAMPLE 10 TI CASIO

Rationalise the denominator and simplify the following.

a $\mathbf{\frac{1}{4 - \sqrt{3}}}$ **b** $\mathbf{\frac{\sqrt{6} + 3\sqrt{2}}{3 + \sqrt{3}}}$

THINK | **WRITE**

a 1 Write the fraction.

a $\frac{1}{4 - \sqrt{3}}$

2 Multiply the numerator and denominator by the conjugate of the denominator.

(Note that $\frac{(4 + \sqrt{3})}{(4 + \sqrt{3})} = 1$.)

$$= \frac{1}{(4 - \sqrt{3})} \times \frac{(4 + \sqrt{3})}{(4 + \sqrt{3})}$$

3 Apply the Distributive Law in the numerator and the difference of two squares identity in the denominator.

$$= \frac{4 + \sqrt{3}}{(4)^2 - (\sqrt{3})^2}$$

4 Simplify.

$$= \frac{4+\sqrt{3}}{16-3}$$
$$= \frac{4+\sqrt{3}}{13}$$

b 1 Write the fraction.

b $$\frac{\sqrt{6}+3\sqrt{2}}{3+\sqrt{3}}$$

2 Multiply the numerator and denominator by the conjugate of the denominator.

(Note that $\frac{(3-\sqrt{3})}{(3-\sqrt{3})} = 1$.)

$$= \frac{(\sqrt{6}+3\sqrt{2})}{(3+\sqrt{3})} \times \frac{(3-\sqrt{3})}{(3-\sqrt{3})}$$

3 Multiply the expressions in grouping symbols in the numerator, and apply the difference of two squares identity in the denominator.

$$= \frac{\sqrt{6}\times 3 + \sqrt{6}\times -\sqrt{3} + 3\sqrt{2}\times 3 + 3\sqrt{2}\times -\sqrt{3}}{(3)^2 - (\sqrt{3})^2}$$

4 Simplify.

$$= \frac{3\sqrt{6} - \sqrt{18} + 9\sqrt{2} - 3\sqrt{6}}{9-3}$$
$$= \frac{-\sqrt{18}+9\sqrt{2}}{6}$$
$$= \frac{-\sqrt{9\times 2}+9\sqrt{2}}{6}$$
$$= \frac{-3\sqrt{2}+9\sqrt{2}}{6}$$
$$= \frac{6\sqrt{2}}{6}$$
$$= \sqrt{2}$$

assess on

Exercise 16.4 Operations with surds

REFLECTION
Under what circumstance might you need to rationalise the denominator of a fraction?

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–h, 2a–h, 3a–h, 4a–d, 5a–h, 6a–h, 7a–h, 8a–d, 9a–d, 10a–h, 11a–f, 12a–c, 13, 15	Questions: 1e–j, 2e–j, 3e–k, 4c–f, 5c–i, 6e–j, 7g–l, 8d–f, 9g–k, 10f–j, 11e–h, 12d–f, 13–15	Questions: 1g–l, 2g–l, 3g–l, 4e–h, 5g–l, 6g–l, 7j–r, 8e–h, 9i–n, 10k–o, 11i–l, 12g–i, 13–16

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FLUENCY

1 WE4a Simplify the following surds.

a $\sqrt{12}$ b $\sqrt{24}$ c $\sqrt{27}$ d $\sqrt{125}$
e $\sqrt{54}$ f $\sqrt{112}$ g $\sqrt{68}$ h $\sqrt{180}$
i $\sqrt{88}$ j $\sqrt{162}$ k $\sqrt{245}$ l $\sqrt{448}$

2 **WE4b, c** Simplify the following surds.

a $2\sqrt{8}$ **b** $8\sqrt{90}$ **c** $9\sqrt{80}$ **d** $7\sqrt{54}$

e $-6\sqrt{75}$ **f** $-7\sqrt{80}$ **g** $16\sqrt{48}$ **h** $\frac{1}{7}\sqrt{392}$

i $\frac{1}{9}\sqrt{162}$ **j** $\frac{1}{4}\sqrt{192}$ **k** $\frac{1}{9}\sqrt{135}$ **l** $\frac{3}{10}\sqrt{175}$

3 **WE4d** Simplify the following surds. Assume that a, b, c, d, e, f, x and y are positive real numbers.

a $\sqrt{16a^2}$ **b** $\sqrt{72a^2}$ **c** $\sqrt{90a^2b}$ **d** $\sqrt{338a^4}$

e $\sqrt{338a^3b^3}$ **f** $\sqrt{68a^3b^5}$ **g** $\sqrt{125x^6y^4}$ **h** $5\sqrt{80x^3y^2}$

i $6\sqrt{162c^7d^5}$ **j** $2\sqrt{405c^7d^9}$ **k** $\frac{1}{2}\sqrt{88ef}$ **l** $\frac{1}{2}\sqrt{392e^{11}f^{11}}$

4 **WE5a** Simplify the following expressions containing surds. Assume that x and y are positive real numbers.

a $3\sqrt{5} + 4\sqrt{5}$ **b** $2\sqrt{3} + 5\sqrt{3} + \sqrt{3}$

c $8\sqrt{5} + 3\sqrt{3} + 7\sqrt{5} + 2\sqrt{3}$ **d** $6\sqrt{11} - 2\sqrt{11}$

e $7\sqrt{2} + 9\sqrt{2} - 3\sqrt{2}$ **f** $9\sqrt{6} + 12\sqrt{6} - 17\sqrt{6} - 7\sqrt{6}$

g $12\sqrt{3} - 8\sqrt{7} + 5\sqrt{3} - 10\sqrt{7}$ **h** $2\sqrt{x} + 5\sqrt{y} + 6\sqrt{x} - 2\sqrt{y}$

5 **WE5b** Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a $\sqrt{200} - \sqrt{300}$ **b** $\sqrt{125} - \sqrt{150} + \sqrt{600}$

c $\sqrt{27} - \sqrt{3} + \sqrt{75}$ **d** $2\sqrt{20} - 3\sqrt{5} + \sqrt{45}$

e $6\sqrt{12} + 3\sqrt{27} - 7\sqrt{3} + \sqrt{18}$ **f** $\sqrt{150} + \sqrt{24} - \sqrt{96} + \sqrt{108}$

g $3\sqrt{90} - 5\sqrt{60} + 3\sqrt{40} + \sqrt{100}$ **h** $5\sqrt{11} + 7\sqrt{44} - 9\sqrt{99} + 2\sqrt{121}$

i $2\sqrt{30} + 5\sqrt{120} + \sqrt{60} - 6\sqrt{135}$ **j** $6\sqrt{ab} - \sqrt{12ab} + 2\sqrt{9ab} + 3\sqrt{27ab}$

k $\frac{1}{2}\sqrt{98} + \frac{1}{3}\sqrt{48} + \frac{1}{3}\sqrt{12}$ **l** $\frac{1}{8}\sqrt{32} - \frac{7}{6}\sqrt{18} + 3\sqrt{72}$

6 **WE5c** Simplify the following expressions containing surds. Assume that a and b are positive real numbers.

a $7\sqrt{a} - \sqrt{8a} + 8\sqrt{9a} - \sqrt{32a}$ **b** $10\sqrt{a} - 15\sqrt{27a} + 8\sqrt{12a} + 14\sqrt{9a}$

c $\sqrt{150ab} + \sqrt{96ab} - \sqrt{54ab}$ **d** $16\sqrt{4a^2} - \sqrt{24a} + 4\sqrt{8a^2} + \sqrt{96a}$

e $\sqrt{8a^3} + \sqrt{72a^3} - \sqrt{98a^3}$ **f** $\frac{1}{2}\sqrt{36a} + \frac{1}{4}\sqrt{128a} - \frac{1}{6}\sqrt{144a}$

g $\sqrt{9a^3} + \sqrt{3a^5}$ **h** $6\sqrt{a^5b} + \sqrt{a^3b} - 5\sqrt{a^5b}$

i $ab\sqrt{ab} + 3ab\sqrt{a^2b} + \sqrt{9a^3b^3}$ **j** $\sqrt{a^3b} + 5\sqrt{ab} - 2\sqrt{ab} + 5\sqrt{a^3b}$

k $\sqrt{32a^3b^2} - 5ab\sqrt{8a} + \sqrt{48a^5b^6}$ **l** $\sqrt{4a^2b} + 5\sqrt{a^2b} - 3\sqrt{9a^2b}$

7 **WE6** Multiply the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

a $\sqrt{2} \times \sqrt{7}$ **b** $\sqrt{6} \times \sqrt{7}$ **c** $\sqrt{8} \times \sqrt{6}$

d $\sqrt{10} \times \sqrt{10}$ **e** $\sqrt{21} \times \sqrt{3}$ **f** $\sqrt{27} \times 3\sqrt{3}$

g $5\sqrt{3} \times 2\sqrt{11}$ **h** $10\sqrt{15} \times 6\sqrt{3}$ **i** $4\sqrt{20} \times 3\sqrt{5}$

j $10\sqrt{6} \times 3\sqrt{8}$ **k** $\frac{1}{4}\sqrt{48} \times 2\sqrt{2}$ **l** $\frac{1}{9}\sqrt{48} \times 2\sqrt{3}$

m $\frac{1}{10}\sqrt{60} \times \frac{1}{5}\sqrt{40}$ **n** $\sqrt{xy} \times \sqrt{x^3y^2}$ **o** $\sqrt{3a^4b^2} \times \sqrt{6a^5b^3}$

p $\sqrt{12a^7b} \times \sqrt{6a^3b^4}$ **q** $\sqrt{15x^3y^2} \times \sqrt{6x^2y^3}$ **r** $\frac{1}{2}\sqrt{15a^3b^3} \times 3\sqrt{3a^2b^6}$

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Digital docs
SkillSHEET Simplifying surds doc-5355
SkillSHEET Adding and subtracting surds doc-5356
SkillSHEET Multiplying and dividing surds doc-5357
SkillSHEET Rationalising denominators doc-5360
SkillSHEET Conjugate pairs doc-5361
SkillSHEET Applying the difference of two squares rule to surds doc-5362

8 **WE7** Simplify each of the following.

a $(\sqrt{2})^2$ **b** $(\sqrt{5})^2$ **c** $(\sqrt{12})^2$ **d** $(\sqrt{15})^2$

e $(3\sqrt{2})^2$ **f** $(4\sqrt{5})^2$ **g** $(2\sqrt{7})^2$ **h** $(5\sqrt{8})^2$

9 **WE8** Simplify the following surds, expressing answers in the simplest form. Assume that a, b, x and y are positive real numbers.

a $\dfrac{\sqrt{15}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{8}}{\sqrt{2}}$ **c** $\dfrac{\sqrt{60}}{\sqrt{10}}$

d $\dfrac{\sqrt{128}}{\sqrt{8}}$ **e** $\dfrac{\sqrt{18}}{4\sqrt{6}}$ **f** $\dfrac{\sqrt{65}}{2\sqrt{13}}$

g $\dfrac{\sqrt{96}}{\sqrt{8}}$ **h** $\dfrac{7\sqrt{44}}{14\sqrt{11}}$ **i** $\dfrac{9\sqrt{63}}{15\sqrt{7}}$

j $\dfrac{\sqrt{2040}}{\sqrt{30}}$ **k** $\dfrac{\sqrt{x^4y^3}}{\sqrt{x^2y^5}}$ **l** $\dfrac{\sqrt{16xy}}{\sqrt{8x^7y^9}}$

m $\dfrac{\sqrt{xy}}{\sqrt{x^5y^7}} \times \dfrac{\sqrt{12x^8y^{12}}}{\sqrt{x^2y^3}}$ **n** $\dfrac{2\sqrt{2a^2b^4}}{\sqrt{5a^3b^6}} \times \dfrac{\sqrt{10a^9b^3}}{3\sqrt{a^7b}}$

10 **WE9a, b** Express the following in their simplest form with a rational denominator.

a $\dfrac{5}{\sqrt{2}}$ **b** $\dfrac{7}{\sqrt{3}}$ **c** $\dfrac{4}{\sqrt{11}}$ **d** $\dfrac{8}{\sqrt{6}}$ **e** $\dfrac{\sqrt{12}}{\sqrt{7}}$

f $\dfrac{\sqrt{15}}{\sqrt{6}}$ **g** $\dfrac{2\sqrt{3}}{\sqrt{5}}$ **h** $\dfrac{3\sqrt{7}}{\sqrt{5}}$ **i** $\dfrac{5\sqrt{2}}{2\sqrt{3}}$ **j** $\dfrac{4\sqrt{3}}{3\sqrt{5}}$

k $\dfrac{5\sqrt{14}}{7\sqrt{8}}$ **l** $\dfrac{16\sqrt{3}}{6\sqrt{5}}$ **m** $\dfrac{8\sqrt{3}}{7\sqrt{7}}$ **n** $\dfrac{8\sqrt{60}}{\sqrt{28}}$ **o** $\dfrac{2\sqrt{35}}{3\sqrt{14}}$

UNDERSTANDING

11 **WE9c** Express the following in their simplest form with a rational denominator.

a $\dfrac{\sqrt{6}+\sqrt{12}}{\sqrt{3}}$ **b** $\dfrac{\sqrt{15}-\sqrt{22}}{\sqrt{6}}$ **c** $\dfrac{6\sqrt{2}-\sqrt{15}}{\sqrt{10}}$ **d** $\dfrac{2\sqrt{18}+3\sqrt{2}}{\sqrt{5}}$

e $\dfrac{3\sqrt{5}+6\sqrt{7}}{\sqrt{8}}$ **f** $\dfrac{4\sqrt{2}+3\sqrt{8}}{2\sqrt{3}}$ **g** $\dfrac{3\sqrt{11}-4\sqrt{5}}{\sqrt{18}}$ **h** $\dfrac{2\sqrt{7}-2\sqrt{5}}{\sqrt{12}}$

i $\dfrac{7\sqrt{12}-5\sqrt{6}}{6\sqrt{3}}$ **j** $\dfrac{6\sqrt{2}-\sqrt{5}}{4\sqrt{8}}$ **k** $\dfrac{6\sqrt{3}-5\sqrt{5}}{7\sqrt{20}}$ **l** $\dfrac{3\sqrt{5}+7\sqrt{3}}{5\sqrt{24}}$

12 **WE10** Rationalise the denominator and simplify.

a $\dfrac{1}{\sqrt{5}+2}$ **b** $\dfrac{1}{\sqrt{8}-\sqrt{5}}$ **c** $\dfrac{4}{2\sqrt{11}-\sqrt{13}}$ **d** $\dfrac{5\sqrt{3}}{3\sqrt{5}+4\sqrt{2}}$

e $\dfrac{\sqrt{8}-3}{\sqrt{8}+3}$ **f** $\dfrac{\sqrt{12}-\sqrt{7}}{\sqrt{12}+\sqrt{7}}$ **g** $\dfrac{\sqrt{3}-1}{\sqrt{5}+1}$ **h** $\dfrac{3\sqrt{6}-\sqrt{15}}{\sqrt{6}+2\sqrt{3}}$

i $\dfrac{\sqrt{5}-\sqrt{3}}{4\sqrt{2}-\sqrt{3}}$

REASONING

13 Express the average of $\dfrac{1}{2\sqrt{x}}$ and $\dfrac{1}{3-2\sqrt{x}}$, writing your answer with a rational denominator.

14 **a** Show that $\left(\sqrt{a}+\sqrt{b}\right)^2 = a + b + 2\sqrt{ab}$.

b Use this result to find:

i $\sqrt{8+2\sqrt{15}}$

ii $\sqrt{8-2\sqrt{15}}$

iii $\sqrt{7+4\sqrt{3}}$.

PROBLEM SOLVING

15 Simplify $\dfrac{\sqrt{5}+\sqrt{3}}{\sqrt{3}+\sqrt{3}+\sqrt{5}} - \dfrac{\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{3}-\sqrt{5}}$.

16 Solve for x.

a $\sqrt{9+x} - \sqrt{x} = \dfrac{5}{\sqrt{9+x}}$

b $\dfrac{9\sqrt{x}-7}{3\sqrt{x}} = \dfrac{3\sqrt{x}+1}{\sqrt{x}+5}$

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16.5 Fractional indices

- Consider the expression $a^{\frac{1}{2}}$. Now consider what happens if we square that expression.

$$(a^{\frac{1}{2}})^2 = a \text{ (using the Fourth Index Law, } (a^m)^n = a^{m \times n})$$

- Now, from our work on surds we know that $(\sqrt{a})^2 = a$.
- From this we can conclude that $(a^{\frac{1}{2}})^2 = (\sqrt{a})^2$ and further conclude that $a^{\frac{1}{2}} = \sqrt{a}$.
- We can similarly show that $a^{\frac{1}{3}} = \sqrt[3]{a}$.
- This pattern can be continued and generalised to produce $a^{\frac{1}{n}} = \sqrt[n]{a}$.

WORKED EXAMPLE 11

Evaluate each of the following without using a calculator.

a $9^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$

THINK		WRITE
a	1 Write $9^{\frac{1}{2}}$ as $\sqrt{9}$.	a $9^{\frac{1}{2}} = \sqrt{9}$
	2 Evaluate.	$= 3$
b	1 Write $64^{\frac{1}{3}}$ as $\sqrt[3]{64}$.	b $64^{\frac{1}{3}} = \sqrt[3]{64}$
	2 Evaluate.	$= 4$

WORKED EXAMPLE 12 TI CASIO

Use a calculator to find the value of the following, correct to 1 decimal place.

a $10^{\frac{1}{4}}$ **b** $200^{\frac{1}{5}}$

THINK	WRITE
a Use a calculator to produce the answer.	**a** $10^{\frac{1}{4}} = 1.778\,279\,41$ ≈ 1.8
b Use a calculator to produce the answer.	**b** $200^{\frac{1}{5}} = 2.885\,399\,812$ ≈ 2.9

- Consider the expression $(a^m)^{\frac{1}{n}}$. From earlier, we know that $(a^m)^{\frac{1}{n}} = \sqrt[n]{a^m}$.

  We also know $(a^m)^{\frac{1}{n}} = a^{\frac{m}{n}}$ using the index laws.

  We can therefore conclude that $a^{\frac{m}{n}} = \sqrt[n]{a^m}$.
- Such expressions can be evaluated on a calculator either by using the index function, which is usually either ^ or x^y and entering the fractional index, or by separating the two functions for power and root.

WORKED EXAMPLE 13

Evaluate $3^{\frac{2}{7}}$, correct to 1 decimal place.

THINK	WRITE
Use a calculator to evaluate $3^{\frac{2}{7}}$.	$3^{\frac{2}{7}} \approx 1.4$

- The index law $a^{\frac{1}{2}} = \sqrt{a}$ can be applied to convert between expressions that involve fractional indices and surds.

WORKED EXAMPLE 14

Write each of the following expressions in simplest surd form.

a $10^{\frac{1}{2}}$ **b** $5^{\frac{3}{2}}$

THINK	WRITE
a Since an index of $\frac{1}{2}$ is equivalent to taking the square root, this term can be written as the square root of 10.	**a** $10^{\frac{1}{2}} = \sqrt{10}$
b 1 A power of $\frac{3}{2}$ means the square root of the number cubed.	**b** $5^{\frac{3}{2}} = \sqrt{5^3}$
2 Evaluate 5^3.	$= \sqrt{125}$
3 Simplify $\sqrt{125}$.	$= 5\sqrt{5}$

WORKED EXAMPLE 15

Simplify each of the following.

a $m^{\frac{1}{5}} \times m^{\frac{2}{5}}$ **b** $(a^2b^3)^{\frac{1}{6}}$ **c** $\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

THINK		WRITE
a	1 Write the expression.	$m^{\frac{1}{5}} \times m^{\frac{2}{5}}$
	2 Multiply numbers with the same base by adding the indices.	$= m^{\frac{3}{5}}$
b	1 Write the expression.	$(a^2b^3)^{\frac{1}{6}}$
	2 Multiply each index inside the grouping symbols (brackets) by the index on the outside.	$= a^{\frac{2}{6}}b^{\frac{3}{6}}$
	3 Simplify the fractions.	$= a^{\frac{1}{3}}b^{\frac{1}{2}}$
c	1 Write the expression.	$\left(\dfrac{x^{\frac{2}{3}}}{y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$
	2 Multiply the index in both the numerator and denominator by the index outside the grouping symbols.	$= \dfrac{x^{\frac{1}{3}}}{y^{\frac{3}{8}}}$

Exercise 16.5 Fractional indices

assess on

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–5, 6a–f, 7a–c, 8a–f, 9a–d, 10a–d, 11a–d, 12–14, 16	Questions: 1–5, 6d–g, 7b–d, 8d–f, 9b–d, 10c–f, 11c–f, 12–16	Questions: 1–5, 6g–i, 7d–f, 8f–i, 9c–f, 10e–i, 11e–i, 12–17

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REFLECTION
How will you remember the rule for fractional indices?

FLUENCY

1 **WE11** Evaluate each of the following without using a calculator.

a $16^{\frac{1}{2}}$ b $25^{\frac{1}{2}}$ c $81^{\frac{1}{2}}$

d $8^{\frac{1}{3}}$ e $27^{\frac{1}{3}}$ f $125^{\frac{1}{3}}$

2 **WE12** Use a calculator to evaluate each of the following, correct to 1 decimal place.

a $81^{\frac{1}{4}}$ b $16^{\frac{1}{4}}$ c $3^{\frac{1}{3}}$

d $5^{\frac{1}{2}}$ e $7^{\frac{1}{5}}$ f $8^{\frac{1}{9}}$

3 **WE13** Use a calculator to find the value of each of the following, correct to 1 decimal place.

a $12^{\frac{3}{8}}$ b $100^{\frac{5}{9}}$ c $50^{\frac{2}{3}}$
d $(0.6)^{\frac{4}{5}}$ e $\left(\frac{3}{4}\right)^{\frac{3}{4}}$ f $\left(\frac{4}{5}\right)^{\frac{2}{3}}$

4 **WE14** Write each of the following expressions in simplest surd form.

a $7^{\frac{1}{2}}$ b $12^{\frac{1}{2}}$ c $72^{\frac{1}{2}}$
d $2^{\frac{5}{2}}$ e $3^{\frac{3}{2}}$ f $10^{\frac{5}{2}}$

5 Write each of the following expressions with a fractional index.

a $\sqrt{5}$ b $\sqrt{10}$ c $\sqrt{x}$
d $\sqrt{m^3}$ e $2\sqrt{t}$ f $\sqrt[3]{6}$

6 **WE15a** Simplify each of the following. Leave your answer in index form.

a $4^{\frac{3}{5}} \times 4^{\frac{1}{5}}$ b $2^{\frac{1}{8}} \times 2^{\frac{3}{8}}$ c $a^{\frac{1}{2}} \times a^{\frac{1}{3}}$
d $x^{\frac{3}{4}} \times x^{\frac{2}{5}}$ e $5m^{\frac{1}{3}} \times 2m^{\frac{1}{5}}$ f $\frac{1}{2}b^{\frac{3}{7}} \times 4b^{\frac{2}{7}}$
g $-4y^2 \times y^{\frac{2}{9}}$ h $\frac{2}{5}a^{\frac{3}{8}} \times 0.05a^{\frac{3}{4}}$ i $5x^3 \times x^{\frac{1}{2}}$

7 Simplify each of the following.

a $a^{\frac{2}{3}}b^{\frac{3}{4}} \times a^{\frac{1}{3}}b^{\frac{3}{4}}$ b $x^{\frac{3}{5}}y^{\frac{2}{9}} \times x^{\frac{1}{5}}y^{\frac{1}{3}}$ c $2ab^{\frac{1}{3}} \times 3a^{\frac{3}{5}}b^{\frac{4}{5}}$
d $6m^{\frac{3}{7}} \times \frac{1}{3}m^{\frac{1}{4}}n^{\frac{2}{5}}$ e $x^{\frac{1}{3}}y^{\frac{1}{2}}z^{\frac{1}{3}} \times x^{\frac{1}{6}}y^{\frac{1}{3}}z^{\frac{1}{2}}$ f $2a^{\frac{2}{5}}b^{\frac{3}{8}}c^{\frac{1}{4}} \times 4b^{\frac{3}{4}}c^{\frac{3}{4}}$

8 Simplify each of the following.

a $3^{\frac{1}{2}} \div 3^{\frac{1}{3}}$ b $5^{\frac{2}{3}} \div 5^{\frac{1}{4}}$ c $12^2 \div 12^{\frac{3}{2}}$
d $a^{\frac{6}{7}} \div a^{\frac{3}{7}}$ e $x^{\frac{3}{2}} \div x^{\frac{1}{4}}$ f $\dfrac{m^{\frac{4}{5}}}{m^{\frac{5}{9}}}$
g $\dfrac{2x^{\frac{3}{4}}}{4x^{\frac{3}{5}}}$ h $\dfrac{7n^2}{21n^{\frac{4}{3}}}$ i $\dfrac{25b^{\frac{3}{5}}}{20b^{\frac{1}{4}}}$

9 Simplify each of the following.

a $x^3y^2 \div x^{\frac{4}{3}}y^{\frac{3}{5}}$ b $a^{\frac{5}{9}}b^{\frac{2}{3}} \div a^{\frac{2}{5}}b^{\frac{2}{5}}$ c $m^{\frac{3}{8}}n^{\frac{4}{7}} \div 3n^{\frac{3}{8}}$
d $10x^{\frac{4}{5}}y \div 5x^{\frac{2}{3}}y^{\frac{1}{4}}$ e $\dfrac{5a^{\frac{3}{4}}b^{\frac{3}{5}}}{20a^{\frac{1}{5}}b^{\frac{1}{4}}}$ f $\dfrac{p^{\frac{7}{8}}q^{\frac{1}{4}}}{7p^{\frac{2}{3}}q^{\frac{1}{6}}}$

10 Simplify each of the following.

a $\left(2^{\frac{3}{4}}\right)^{\frac{3}{5}}$ b $\left(5^{\frac{2}{3}}\right)^{\frac{1}{4}}$ c $\left(7^{\frac{1}{5}}\right)^6$
d $(a^3)^{\frac{1}{10}}$ e $\left(m^{\frac{4}{9}}\right)^{\frac{3}{8}}$ f $\left(2b^{\frac{1}{2}}\right)^{\frac{1}{3}}$
g $4\left(p^{\frac{3}{7}}\right)^{\frac{15}{14}}$ h $\left(x^{\frac{m}{n}}\right)^{\frac{n}{p}}$ i $\left(3m^{\frac{a}{b}}\right)^{\frac{b}{c}}$

UNDERSTANDING

11 WE15b, c Simplify each of the following.

a $\left(a^{\frac{1}{2}}b^{\frac{1}{3}}\right)^{\frac{1}{2}}$ b $(a^4b)^{\frac{3}{4}}$ c $\left(x^{\frac{3}{5}}y^{\frac{7}{8}}\right)^2$

d $\left(3a^{\frac{1}{3}}b^{\frac{3}{5}}c^{\frac{3}{4}}\right)^{\frac{1}{3}}$ e $\left(x^{\frac{1}{2}}y^{\frac{2}{3}}z^{\frac{2}{5}}\right)^{\frac{1}{2}}$ f $\left(\dfrac{a^{\frac{3}{4}}}{b}\right)^{\frac{2}{3}}$

g $\left(\dfrac{m^{\frac{4}{5}}}{n^{\frac{7}{8}}}\right)^2$ h $\left(\dfrac{b^{\frac{3}{5}}}{c^{\frac{4}{9}}}\right)^{\frac{2}{3}}$ i $\left(\dfrac{4x^7}{2y^{\frac{3}{4}}}\right)^{\frac{1}{2}}$

12 MC *Note:* There may be more than one correct answer.

If $\left(a^{\frac{3}{4}}\right)^{\frac{m}{n}}$ is equal to $a^{\frac{1}{4}}$, then m and n could not be:

A 1 and 3 B 2 and 6 C 3 and 8 D 4 and 9

13 Simplify each of the following.

a $\sqrt{a^8}$ b $\sqrt[3]{b^9}$ c $\sqrt[4]{m^{16}}$

d $\sqrt{16x^4}$ e $\sqrt[3]{8y^9}$ f $\sqrt[4]{16x^8y^{12}}$

g $\sqrt[3]{27m^9n^{15}}$ h $\sqrt[5]{32p^5q^{10}}$ i $\sqrt[3]{216a^6b^{18}}$

REASONING

14 Manning's formula is used to calculate the flow of water in a river during a flood situation. Manning's formula is $v = \dfrac{R^{\frac{2}{3}}S^{\frac{1}{2}}}{n}$, where R is the hydraulic radius, S is the slope of the river and n is the roughness coefficient. This formula is used by meteorologists and civil engineers to analyse potential flood situations.

a Find the flow of water in metres per second in the river if $R = 8$, $S = 0.0025$ and $n = 0.625$.

b To find the volume of water flowing through the river, we multiply the flow rate by the average cross-sectional area of the river. If the average cross-sectional area is 52 m^2, find the volume of water (in L) flowing through the river each second. (Remember 1 m^3 = 1000 L.)

c If water continues to flow at this rate, what will be the total amount of water to flow through in one hour? Justify your answer.

d Use the internet to find the meaning of the terms 'hydraulic radius' and 'roughness coefficient'.

15 Find x if $m^x = \dfrac{\sqrt{m^{10}}}{\left(\sqrt{m^4}\right)^2}$.

PROBLEM SOLVING

16 Simplify:

a $\dfrac{x + 2x^{\frac{1}{2}}y^{\frac{1}{2}} + y - z}{\left(x^{\frac{1}{2}} + y^{\frac{1}{2}} + z^{\frac{1}{2}}\right)}$

b $\sqrt[5]{\dfrac{t^2}{\sqrt{t^3}}}$.

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17 Expand $\left(m^{\frac{3}{4}} + m^{\frac{1}{2}}n^{\frac{1}{2}} + m^{\frac{1}{4}}n + n^{\frac{3}{2}}\right)\left(m^{\frac{1}{4}} - n^{\frac{1}{2}}\right)$.

16.6 Negative indices

- Consider the following division $\dfrac{2^3}{2^4} = 2^{-1}$ (using the Second Index Law).

  Alternatively, $\dfrac{2^3}{2^4} = \dfrac{8}{16} = \dfrac{1}{2}$.

  We can conclude that $2^{-1} = \dfrac{1}{2}$.
- In general form:

$$a^{-1} = \frac{1}{a} \quad \text{and} \quad a^{-n} = \frac{1}{a^n}.$$

WORKED EXAMPLE 16 | TI | CASIO

Evaluate each of the following using a calculator.

a 4^{-1} **b** 2^{-4}

THINK	WRITE
a Use a calculator to evaluate 4^{-1}.	**a** $4^{-1} = 0.25$
b Use a calculator to evaluate 2^{-4}.	**b** $2^{-4} = 0.0625$

- Consider the index law $a^{-1} = \dfrac{1}{a}$. Now consider the case in which a is fractional.

  Consider the expression $\left(\dfrac{a}{b}\right)^{-1}$.

$$\begin{aligned}\left(\frac{a}{b}\right)^{-1} &= \frac{1}{\frac{a}{b}} \\ &= 1 \times \frac{b}{a} \\ &= \frac{b}{a}\end{aligned}$$

We can therefore consider an index of -1 to be a reciprocal function.

WORKED EXAMPLE 17

Write down the value of each of the following without the use of a calculator.

a $\left(\frac{2}{3}\right)^{-1}$ b $\left(\frac{1}{5}\right)^{-1}$ c $\left(1\frac{1}{4}\right)^{-1}$

THINK	WRITE
a To evaluate $\left(\frac{2}{3}\right)^{-1}$ take the reciprocal of $\frac{2}{3}$.	a $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$
b 1 To evaluate $\left(\frac{1}{5}\right)^{-1}$ take the reciprocal of $\frac{1}{5}$.	b $\left(\frac{1}{5}\right)^{-1} = \frac{5}{1}$
2 Write $\frac{5}{1}$ as a whole number.	$= 5$
c 1 Write $1\frac{1}{4}$ as an improper fraction.	c $\left(1\frac{1}{4}\right)^{-1} = \left(\frac{5}{4}\right)^{-1}$
2 Take the reciprocal of $\frac{5}{4}$.	$= \frac{4}{5}$

Exercise 16.6 Negative indices

assess on

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2a–e, 3a–e, 4a–e, 5a–e, 6a–d, 7a–d, 8–12	Questions: 1d–f, 2d–f, 3d–f, 4d–f, 5e–h, 6c–f, 7c–f, 8–12	Questions: 1e–h, 2e–h, 3e–h, 4e–h, 5g–l, 6e–h, 7e–h, 8–13

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REFLECTION
How can division be used to explain negative indices?

FLUENCY

1 WE16 Evaluate each of the following using a calculator.

a 5^{-1} b 3^{-1} c 8^{-1} d 10^{-1}
e 2^{-3} f 3^{-2} g 5^{-2} h 10^{-4}

2 Find the value of each of the following, correct to 3 significant figures.

a 6^{-1} b 7^{-1} c 6^{-2} d 9^{-3}
e 6^{-3} f 15^{-2} g 16^{-2} h 5^{-4}

3 Find the value of each of the following, correct to 2 significant figures.

a $(2.5)^{-1}$ b $(0.4)^{-1}$ c $(1.5)^{-2}$ d $(0.5)^{-2}$
e $(2.1)^{-3}$ f $(10.6)^{-4}$ g $(0.45)^{-3}$ h $(0.125)^{-4}$

4 Find the value of each of the following, correct to 2 significant figures.

a $(-3)^{-1}$ b $(-5)^{-1}$ c $(-2)^{-2}$ d $(-4)^{-2}$
e $(-1.5)^{-1}$ f $(-2.2)^{-1}$ g $(-0.6)^{-1}$ h $(-0.85)^{-2}$

5 WE17 Write down the value of each of the following without the use of a calculator.

a $\left(\frac{4}{5}\right)^{-1}$ b $\left(\frac{3}{10}\right)^{-1}$ c $\left(\frac{7}{8}\right)^{-1}$ d $\left(\frac{13}{20}\right)^{-1}$
e $\left(\frac{1}{2}\right)^{-1}$ f $\left(\frac{1}{4}\right)^{-1}$ g $\left(\frac{1}{8}\right)^{-1}$ h $\left(\frac{1}{10}\right)^{-1}$
i $\left(1\frac{1}{2}\right)^{-1}$ j $\left(2\frac{1}{4}\right)^{-1}$ k $\left(1\frac{1}{10}\right)^{-1}$ l $\left(5\frac{1}{2}\right)^{-1}$

6 Find the value of each of the following, leaving your answer in fraction form if necessary.

a $\left(\frac{1}{2}\right)^{-2}$ **b** $\left(\frac{2}{5}\right)^{-2}$ **c** $\left(\frac{2}{3}\right)^{-3}$ **d** $\left(\frac{1}{4}\right)^{-2}$

e $\left(1\frac{1}{2}\right)^{-2}$ **f** $\left(2\frac{1}{4}\right)^{-2}$ **g** $\left(1\frac{1}{3}\right)^{-3}$ **h** $\left(2\frac{1}{5}\right)^{-3}$

7 Find the value of each of the following.

a $\left(-\frac{2}{3}\right)^{-1}$ **b** $\left(-\frac{3}{5}\right)^{-1}$ **c** $\left(-\frac{1}{4}\right)^{-1}$ **d** $\left(-\frac{1}{10}\right)^{-1}$

e $\left(-\frac{2}{3}\right)^{-2}$ **f** $\left(-\frac{1}{5}\right)^{-2}$ **g** $\left(-1\frac{1}{2}\right)^{-1}$ **h** $\left(-2\frac{3}{4}\right)^{-2}$

UNDERSTANDING

8 Without using a calculator, evaluate $\dfrac{\left(\dfrac{2^{-1}}{3}\right)^{-1}}{\left(\dfrac{4}{5^{-1}}\right)}$.

9 Simplify $\left(\sqrt{\dfrac{a^2}{b^2}}\right)^{-1}$.

REASONING

10 Consider the equation $y = \frac{6}{x}$. Clearly $x \neq 0$, as $\frac{6}{x}$ would be undefined.

What happens to the value of y as x gets closer to zero coming from:

a the positive direction

b the negative direction?

11 Consider the expression 2^{-n}. Explain what happens to the value of this expression as n increases.

PROBLEM SOLVING

12 Solve the following pair of simultaneous equations.

$$3^{y+1} = \frac{1}{9} \text{ and } \frac{5^y}{125^x} = 125$$

13 Simplify $\dfrac{x^{n+2} + x^{n-2}}{x^{n-4} + x^n}$.

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16.7 Logarithms

- The index, power or exponent in the statement $y = a^x$ is also known as a **logarithm** (or log for short).

Logarithm or index or power or exponent

$$y = a^x$$

Base

- This statement $y = a^x$ can be written in an alternative form as $\log_a y = x$, which is read as 'the logarithm of y to the base a is equal to x'. These two statements are equivalent.

$a^x = y$	$\Leftrightarrow$	$\log_a y = x$
Index form		Logarithmic form

- For example, $3^2 = 9$ can be written as $\log_3 9 = 2$. The log form would be read as 'the logarithm of 9, to the base of 3, is 2'. In both forms, the base is 3 and the logarithm is 2.

WORKED EXAMPLE 18

Write the following in logarithmic form.

a $10^4 = 10\,000$ **b** $6^x = 216$

THINK		WRITE
a 1	Write the given statement.	**a** $10^4 = 10\,000$
2	Identify the base (10) and the logarithm (4) and write the equivalent statement in logarithmic form. (Use $a^x = y \Leftrightarrow \log_a y = x$, where the base is a and the log is x.)	$\log_{10} 10\,000 = 4$
b 1	Write the given statement.	**b** $6^x = 216$
2	Identify the base (6) and the logarithm (x) and write the equivalent statement in logarithmic form.	$\log_6 216 = x$

WORKED EXAMPLE 19

Write the following in index form.

a $\log_2 8 = 3$ **b** $\log_{25} 5 = \frac{1}{2}$

THINK		WRITE
a 1	Write the statement.	**a** $\log_2 8 = 3$
2	Identify the base (2) and the log (3), and write the equivalent statement in index form. Remember that the log is the same as the index.	$2^3 = 8$
b 1	Write the statement.	**b** $\log_{25} 5 = \frac{1}{2}$
2	Identify the base (25) and the log $\left(\frac{1}{2}\right)$, and write the equivalent statement in index form.	$25^{\frac{1}{2}} = 5$

- In the previous examples, we found that:

$$\log_2 8 = 3 \Leftrightarrow 2^3 = 8 \text{ and } \log_{10} 10\,000 = 4 \Leftrightarrow 10^4 = 10\,000.$$

We could also write $\log_2 8 = 3$ as $\log_2 2^3 = 3$ and $\log_{10} 10\,000 = 4$ as $\log_{10} 10^4 = 4$.

- Can this pattern be used to work out the value of $\log_3 81$? We need to find the power when the base of 3 is raised to that power to give 81.

WORKED EXAMPLE 20 TI CASIO

Evaluate $\log_3 81$.

THINK		WRITE
1	Write the log expression.	$\log_3 81$
2	Express 81 in index form with a base of 3.	$= \log_3 3^4$
3	Write the value of the logarithm.	$= 4$

assess on

Exercise 16.7 Logarithms

REFLECTION
How are indices and logarithms related?

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–e, 2, 3a–e, 4, 5a–e, 6–8, 10	Questions: 1e–k, 2, 3d–i, 4, 5e–h, 6–10	Questions: 1i–p, 2, 3g–l, 4, 5g–l, 6–11

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FLUENCY

1 WE18 Write the following in logarithmic form.

- **a** $4^2 = 16$
- **b** $2^5 = 32$
- **c** $3^4 = 81$
- **d** $6^2 = 36$
- **e** $1000 = 10^3$
- **f** $25 = 5^2$
- **g** $4^3 = x$
- **h** $5^x = 125$
- **i** $7^x = 49$
- **j** $p^4 = 16$
- **k** $9^{\frac{1}{2}} = 3$
- **l** $0.1 = 10^{-1}$
- **m** $2 = 8^{\frac{1}{3}}$
- **n** $2^{-1} = \frac{1}{2}$
- **o** $a^0 = 1$
- **p** $4^{\frac{3}{2}} = 8$

2 MC The statement $w = h^t$ is equivalent to:

- **A** $w = \log_t h$
- **B** $h = \log_t w$
- **C** $t = \log_w h$
- **D** $t = \log_h w$

3 WE19 Write the following in index form.

- **a** $\log_2 16 = 4$
- **b** $\log_3 27 = 3$
- **c** $\log_{10} 1\,000\,000 = 6$
- **d** $\log_5 125 = 3$
- **e** $\log_{16} 4 = \frac{1}{2}$
- **f** $\log_4 64 = x$
- **g** $\frac{1}{2} = \log_{49} 7$
- **h** $\log_3 x = 5$
- **i** $\log_{81} 9 = \frac{1}{2}$
- **j** $\log_{10} 0.01 = -2$
- **k** $\log_8 8 = 1$
- **l** $\log_{64} 4 = \frac{1}{3}$

4 MC The statement $q = \log_r p$ is equivalent to:

- **A** $q = r^p$
- **B** $p = r^q$
- **C** $r = p^q$
- **D** $r = q^p$

5 WE20 Evaluate the following logarithms.

- **a** $\log_2 16$
- **b** $\log_4 16$
- **c** $\log_{11} 121$
- **d** $\log_{10} 100\,000$
- **e** $\log_3 243$
- **f** $\log_2 128$
- **g** $\log_5 1$
- **h** $\log_9 3$
- **i** $\log_3 \left(\frac{1}{3}\right)$
- **j** $\log_6 6$
- **k** $\log_{10} \left(\frac{1}{100}\right)$
- **l** $\log_{125} 5$

6 Write the value of each of the following.

- **a** $\log_{10} 1$
- **b** $\log_{10} 10$
- **c** $\log_{10} 100$
- **d** $\log_{10} 1000$
- **e** $\log_{10} 10\,000$
- **f** $\log_{10} 100\,000$

UNDERSTANDING

7 Use your results to question **6** to answer the following.

- **a** Between which two whole numbers would $\log_{10} 7$ lie?
- **b** Between which two whole numbers would $\log_{10} 4600$ lie?
- **c** Between which two whole numbers would $\log_{10} 85$ lie?
- **d** Between which two whole numbers would $\log_{10} 12\,750$ lie?
- **e** Between which two whole numbers would $\log_{10} 110$ lie?
- **f** Between which two whole numbers would $\log_{10} 81\,000$ lie?

REASONING

8 a If $\log_{10} g = k$, find the value of $\log_{10} g^2$. Justify your answer.
 b If $\log_x y = 2$, find the value of $\log_y x$. Justify your answer.
 c By referring to the equivalent index statement, explain why x must be a positive number given $\log_4 x = y$, for all values of y.

9 Calculate each of the following logarithms.
 a $\log_2 (64)$
 b $\log_3 \left(\frac{1}{81}\right)$
 c $\log_{10} (0.000\,01)$

PROBLEM SOLVING

10 Find the value of x.
 a $\log_x \left(\frac{1}{243}\right) = -5$
 b $\log_x (343) = 3$
 c $\log_{64} (x) = -\frac{1}{2}$

11 Simplify $10^{\log_{10}(x)}$.

16.8 Logarithm laws

- The index laws are:

1. $a^m \times a^n = a^{m+n}$
2. $\frac{a^m}{a^n} = a^{m-n}$
3. $(a^m)^n = a^{mn}$
4. $a^0 = 1$
5. $a^1 = a$
6. $a^{-1} = \frac{1}{a}$

- The index laws can be used to produce equivalent logarithm laws.

Law 1

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

Now	$xy = a^m \times a^n$	
or	$xy = a^{m+n}$	(First Index Law).
So	$\log_a (xy) = m + n$	(equivalent log form)
or	$\log_a (xy) = \log_a x + \log_a y$	(substituting for m and n).

$$\boxed{\log_a x + \log_a y = \log_a (xy)}$$

- This means that the sum of two logarithms with the same base is equal to the logarithm of the product of the numbers.

WORKED EXAMPLE 21 TI CASIO

Evaluate $\log_{10} 20 + \log_{10} 5$.

THINK	WRITE
1 Since the same base of 10 is used in each log term, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.	$\log_{10} 20 + \log_{10} 5 = \log_{10} (20 \times 5)$ $= \log_{10} 100$
2 Evaluate. (Remember that $100 = 10^2$.)	$= 2$

Law 2

- If $x = a^m$ and $y = a^n$, then $\log_a x = m$ and $\log_a y = n$ (equivalent log form).

Now $\quad \dfrac{x}{y} = \dfrac{a^m}{a^n}$

or $\quad \dfrac{x}{y} = a^{m-n}$ (Second Index Law).

So $\quad \log_a\left(\dfrac{x}{y}\right) = m - n$ (equivalent log form)

or $\quad \log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$ (substituting for m and n).

$$\boxed{\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)}$$

- This means that the difference of two logarithms with the same base is equal to the logarithm of the quotient of the numbers.

WORKED EXAMPLE 22

Evaluate $\log_4 20 - \log_4 5$.

THINK	WRITE
1 Since the same base of 4 is used in each log term, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.	$\log_4 20 - \log_4 5 = \log_4\left(\frac{20}{5}\right)$ $= \log_4 4$
2 Evaluate. (Remember that $4 = 4^1$.)	$= 1$

WORKED EXAMPLE 23 TI CASIO

Evaluate $\log_5 35 + \log_5 15 - \log_5 21$.

THINK	WRITE
1 Since the first two log terms are being added, use $\log_a x + \log_a y = \log_a (xy)$ and simplify.	$\log_5 35 + \log_5 15 - \log_5 21$ $= \log_5 (35 \times 15) - \log_5 21$ $= \log_5 525 - \log_5 21$
2 To find the difference between the two remaining log terms, use $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ and simplify.	$= \log_5\left(\frac{525}{21}\right)$ $= \log_5 25$
3 Evaluate. (Remember that $25 = 5^2$.)	$= 2$

- Once you have gained confidence in using the first two laws, you can reduce the number of steps of working by combining the application of the laws. In Worked example 23, we could write:

$$\begin{aligned}\log_5 35 + \log_5 15 - \log_5 21 &= \log_5\left(\frac{35 \times 15}{21}\right)\\ &= \log_5 25\\ &= 2\end{aligned}$$

Law 3

- If $x = a^m$, then $\log_a x = m$ (equivalent log form).

Now	$x^n = (a^m)^n$	
or	$x^n = a^{mn}$	(Third Index Law).
So	$\log_a x^n = mn$	(equivalent log form)
or	$\log_a x^n = (\log_a x) \times n$	(substituting for m)
or	$\log_a x^n = n \log_a x$	

$$\boxed{\log_a x^n = n \log_a x}$$

- This means that the logarithm of a number raised to a power is equal to the product of the power and the logarithm of the number.

WORKED EXAMPLE 24

Evaluate $2 \log_6 3 + \log_6 4$.

THINK	WRITE
1 The first log term is not in the required form to use the log law relating to sums. Use $\log_a x^n = n \log_a x$ to rewrite the first term in preparation for applying the first log law.	$2 \log_6 3 + \log_6 4 = \log_6 3^2 + \log_6 4$ $= \log_6 9 + \log_6 4$
2 Use $\log_a x + \log_a y = \log_a (xy)$ to simplify the two log terms to one.	$= \log_6 (9 \times 4)$ $= \log_6 36$
3 Evaluate. (Remember that $36 = 6^2$.)	$= 2$

Law 4

- As $a^0 = 1$ (Fourth Index Law),
  $\log_a 1 = 0$ (equivalent log form).

$$\boxed{\log_a 1 = 0}$$

- This means that the logarithm of 1 with any base is equal to 0.

Law 5

- As $a^1 = a$ (Fifth Index Law),
  $\log_a a = 1$ (equivalent log form).

$$\boxed{\log_a a = 1}$$

- This means that the logarithm of any number a with base a is equal to 1.

Law 6

- Now $\log_a\left(\frac{1}{x}\right) = \log_a x^{-1}$ (Sixth Index Law)

or $\log_a\left(\frac{1}{x}\right) = -1 \times \log_a x$ (using the fourth log law)

or $\log_a\left(\frac{1}{x}\right) = -\log_a x.$

$$\boxed{\log_a\left(\frac{1}{x}\right) = -\log_a x}$$

Law 7

- Now $\log_a a^x = x\log_a a$ (using the third log law)

or $\log_a a^x = x \times 1$ (using the fifth log law)

or $\log_a a^x = x.$

$$\boxed{\log_a a^x = x}$$

assess on

Exercise 16.8 Logarithm laws

REFLECTION
What technique will you use to remember the log laws?

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1–7, 8a–f, 9a–f, 10, 11a–g, 12, 13, 15	Questions: 1–7, 8d–i, 9e–j, 10, 11e–i, 12–15	Questions: 1–7, 8g–l, 9g–l, 10, 11g–l, 12–16

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FLUENCY

1 Use a calculator to evaluate the following, correct to 5 decimal places.

a $\log_{10} 50$ **b** $\log_{10} 25$ **c** $\log_{10} 5$ **d** $\log_{10} 2$

2 Use your answers to question **1** to show that each of the following statements is true.

a $\log_{10} 25 + \log_{10} 2 = \log_{10} 50$

b $\log_{10} 50 - \log_{10} 2 = \log_{10} 25$

c $\log_{10} 25 = 2\log_{10} 5$

d $\log_{10} 50 - \log_{10} 25 - \log_{10} 2 = \log_{10} 1$

3 **WE21** Evaluate the following.

a $\log_6 3 + \log_6 2$ **b** $\log_4 8 + \log_4 8$

c $\log_{10} 25 + \log_{10} 4$ **d** $\log_8 32 + \log_8 16$

e $\log_6 108 + \log_6 12$ **f** $\log_{14} 2 + \log_{14} 7$

4 **WE22** Evaluate the following.

a $\log_2 20 - \log_2 5$ **b** $\log_3 54 - \log_3 2$

c $\log_4 24 - \log_4 6$ **d** $\log_{10} 30\,000 - \log_{10} 3$

e $\log_6 648 - \log_6 3$ **f** $\log_2 224 - \log_2 7$

5 **WE23** Evaluate the following.

a $\log_3 27 + \log_3 2 - \log_3 6$ **b** $\log_4 24 - \log_4 2 - \log_4 6$

c $\log_6 78 - \log_6 13 + \log_6 1$ **d** $\log_2 120 - \log_2 3 - \log_2 5$

6 Evaluate $2\log_4 8$.

7 **WE24** Evaluate the following.

a $2\log_{10} 5 + \log_{10} 4$

b $\log_3 648 - 3\log_3 2$

c $4\log_5 10 - \log_5 80$

d $\log_2 50 + \frac{1}{2}\log_2 16 - 2\log_2 5$

8 Evaluate the following.

a $\log_8 8$

b $\log_5 1$

c $\log_2\left(\frac{1}{2}\right)$

d $\log_4 4^5$

e $\log_6 6^{-2}$

f $\log_{20} 20$

g $\log_2 1$

h $\log_3\left(\frac{1}{9}\right)$

i $\log_4\left(\frac{1}{2}\right)$

j $\log_5 \sqrt{5}$

k $\log_3\left(\frac{1}{\sqrt{3}}\right)$

l $\log_2 8\sqrt{2}$

UNDERSTANDING

9 Use the logarithm laws to simplify each of the following.

a $\log_a 5 + \log_a 8$

b $\log_a 12 + \log_a 3 - \log_a 2$

c $4\log_x 2 + \log_x 3$

d $\log_x 100 - 2\log_x 5$

e $3\log_a x - \log_a x^2$

f $5\log_a a - \log_a a^4$

g $\log_x 6 - \log_x 6x$

h $\log_a a^7 + \log_a 1$

i $\log_p \sqrt{p}$

j $\log_k k\sqrt{k}$

k $6\log_a\left(\frac{1}{a}\right)$

l $\log_a\left(\frac{1}{\sqrt[3]{a}}\right)$

10 **MC** *Note:* There may be more than one correct answer.

a The equation $y = 10^x$ is equivalent to:

A $x = 10^y$

B $x = \log_{10} y$

C $x = \log_x 10$

D $x = \log_y 10$

b The equation $y = 10^{4x}$ is equivalent to:

A $x = \log_{10}\sqrt{4y}$

B $x = \log_{10}\sqrt[4]{y}$

C $x = 10^{\frac{1}{4}y}$

D $x = \frac{1}{4}\log_{10} y$

c The equation $y = 10^{3x}$ is equivalent to:

A $x = \frac{1}{3}\log_{10} y$

B $x = \log_{10} y^{\frac{1}{3}}$

C $x = \log_{10} y - 3$

D $x = 10^{y-3}$

d The equation $y = ma^{nx}$ is equivalent to:

A $x = \frac{1}{n}a^{my}$

B $x = \log_a\left(\frac{m}{y}\right)^n$

C $x = \frac{1}{n}(\log_a y - \log_a m)$

D $x = \frac{1}{n}\log_a\left(\frac{y}{m}\right)$

11 Simplify, and evaluate where possible, each of the following without a calculator.

a $\log_2 8 + \log_2 10$

b $\log_3 7 + \log_3 15$

c $\log_{10} 20 + \log_{10} 5$

d $\log_6 8 + \log_6 7$

e $\log_2 20 - \log_2 5$

f $\log_3 36 - \log_3 12$

g $\log_5 100 - \log_5 8$

h $\log_2 \frac{1}{3} + \log_2 9$

i $\log_4 25 + \log_4 \frac{1}{5}$

j $\log_{10} 5 - \log_{10} 20$

k $\log_3 \frac{4}{5} - \log_3 \frac{1}{5}$

l $\log_2 9 + \log_2 4 - \log_2 12$

m $\log_3 8 - \log_3 2 + \log_3 5$

n $\log_4 24 - \log_4 2 - \log_4 6$

12 MC a The expression $\log_{10} xy$ is equal to:

A $\log_{10} x \times \log_{10} y$ B $\log_{10} x - \log_{10} y$ C $\log_{10} x + \log_{10} y$ D $y \log_{10} x$

b The expression $\log_{10} x^y$ is equal to:

A $x \log_{10} y$ B $y \log_{10} x$ C $10 \log_x y$ D $\log_{10} x + \log_{10} y$

c The expression $\frac{1}{3} \log_2 64 + \log_2 10$ is equal to:

A $\log_2 40$ B $\log_2 80$ C $\log_2 \frac{64}{10}$ D 1

REASONING

13 For each of the following, write the possible strategy you intend to use.

a Evaluate $(\log_3 81)(\log_3 27)$.

b Evaluate $\dfrac{\log_a 81}{\log_a 3}$.

c Evaluate $5^{\log_5 7}$.

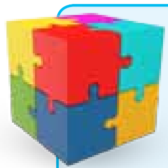
In each case, explain how you obtained your final answer.

14 Simplify $\log_2 \left(\frac{8}{125}\right) - 3 \log_2 \left(\frac{3}{5}\right) - 4 \log_2 \left(\frac{1}{2}\right)$.

PROBLEM SOLVING

15 Simplify $\log_a (a^5 + a^3) - \log_a (a^4 + a^2)$.

16 If $2 \log_a (x) = 1 + \log_a (8x - 15a)$, find x in terms of a where a is a positive constant and x is positive.



CHALLENGE 16.1

Evaluate $\dfrac{\log_2 8 \times \log_2 16}{4^{\log_4 8}}$.

16.9 Solving equations

- The equation $\log_a y = x$ is an example of a general **logarithmic equation**. Laws of logarithms and indices are used to solve these equations.

WORKED EXAMPLE 25

Solve for x in the following equations.

a $\log_2 x = 3$ **b** $\log_6 x = -2$ **c** $\log_3 x^4 = -16$ **d** $\log_5 (x - 1) = 2$

THINK		WRITE
a 1	Write the equation.	a $\log_2 x = 3$
2	Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = x$
3	Rearrange and simplify.	$x = 8$

b 1 Write the equation. — **b** $\log_6 x = -2$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $6^{-2} = x$

3 Rearrange and simplify. — $x = \frac{1}{6^2}$

$= \frac{1}{36}$

c 1 Write the equation. — **c** $\log_3 x^4 = -16$

2 Rewrite using $\log_a x^n = n \log_a x$. — $4 \log_3 x = -16$

3 Divide both sides by 4. — $\log_3 x = -4$

4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $3^{-4} = x$

5 Rearrange and simplify. — $x = \frac{1}{3^4}$

$= \frac{1}{81}$

d 1 Write the equation. — **d** $\log_5 (x - 1) = 2$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $5^2 = x - 1$

3 Solve for x. — $x - 1 = 25$

$x = 26$

WORKED EXAMPLE 26 TI CASIO

Solve for x in $\log_x 25 = 2$, given that $x > 0$.

THINK — **WRITE**

1 Write the equation. — $\log_x 25 = 2$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $x^2 = 25$

3 Solve for x. — $x = 5$ (because $x > 0$)

Note: $x = -5$ is rejected as a solution because $x > 0$.

WORKED EXAMPLE 27

Solve for x in the following.

a $\log_2 16 = x$ **b** $\log_3 \left(\frac{1}{3}\right) = x$ **c** $\log_9 3 = x$

THINK — **WRITE**

a 1 Write the equation. — **a** $\log_2 16 = x$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $2^x = 16$

3 Write 16 with base 2. — $= 2^4$

4 Equate the indices. — $x = 4$

b 1 Write the equation. — **b** $\log_3\left(\frac{1}{3}\right) = x$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.

$$3^x = \frac{1}{3} = \frac{1}{3^1}$$

3 Write $\frac{1}{3}$ with base 3. — $3^x = 3^{-1}$

4 Equate the indices. — $x = -1$

c 1 Write the equation. — **c** $\log_9 3 = x$

2 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$. — $9^x = 3$

3 Write 9 with base 3. — $(3^2)^x = 3$

4 Remove the grouping symbols. — $3^{2x} = 3^1$

5 Equate the indices. — $2x = 1$

6 Solve for x. — $x = \frac{1}{2}$

WORKED EXAMPLE 28 TI CASIO

Solve for x in the equation $\log_2 4 + \log_2 x - \log_2 8 = 3$.

THINK	WRITE
1 Write the equation.	$\log_2 4 + \log_2 x - \log_2 8 = 3$
2 Simplify the left-hand side. Use $\log_a x + \log_a y = \log_a (xy)$ and $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$.	$\log_2\left(\frac{4 \times x}{8}\right) = 3$
3 Simplify.	$\log_2\left(\frac{x}{2}\right) = 3$
4 Rewrite using $a^x = y \Leftrightarrow \log_a y = x$.	$2^3 = \frac{x}{2}$
5 Solve for x.	$x = 2 \times 2^3 = 2 \times 8 = 16$

- When solving an equation like $\log_2 8 = x$, it could be rewritten in index form as $2^x = 8$. This can be written with the same base of 2 to produce $2^x = 2^3$. Equating the indices gives us a solution of $x = 3$.
- Can we do this to solve the equation $2^x = 7$? Consider the method shown in the next worked example. It involves the use of logarithms and the $\log_{10}$ function on a calculator.

WORKED EXAMPLE 29 TI CASIO

Solve for x, correct to 3 decimal places, if:

a $2^x = 7$ **b** $3^{-x} = 0.4$.

THINK	WRITE
a 1 Write the equation.	**a** $2^x = 7$
2 Take $\log_{10}$ of both sides.	$\log_{10} 2^x = \log_{10} 7$
3 Use the logarithm-of-a-power law to bring the power, x, to the front of the logarithmic equation.	$x \log_{10} 2 = \log_{10} 7$
4 Divide both sides by $\log_{10} 2$ to get x by itself.	Therefore, $x = \dfrac{\log_{10} 7}{\log_{10} 2}$
5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$= 2.807$
b 1 Write the equation.	**b** $3^{-x} = 0.4$
2 Take $\log_{10}$ of both sides.	$\log_{10} 3^{-x} = \log_{10} 0.4$
3 Use the logarithm of a power law to bring the power, x, to the front of the logarithmic equation.	$-x \log_{10} 3 = \log_{10} 0.4$
4 Divide both sides by $\log_{10} 3$ to get the $-x$ by itself.	$-x = \dfrac{\log_{10} 0.4}{\log_{10} 3}$
5 Use a calculator to evaluate the logarithms and write the answer correct to 3 decimal places.	$-x = -0.834$
6 Divide both sides by -1 to get x by itself.	$x = 0.834$

- Therefore, we can state the following rule:

$$\textbf{If } a^x = b\textbf{, then } x = \frac{\log_{10} b}{\log_{10} a}.$$

This rule applies to any base, but since your calculator has base 10, this is the most commonly used for this solution technique.

Exercise 16.9 Solving equations

assess on

INDIVIDUAL PATHWAYS

PRACTISE	CONSOLIDATE	MASTER
Questions: 1a–h, 2a–e, 3a–f, 4a–h, 5, 6a–h, 7a–f, 8, 9, 11	Questions: 1d–k, 2d–f, 3c–f, 4e–j, 5, 6e–l, 7d–i, 8–11	Questions: 1g–l, 2d–h, 3e–j, 4i–n, 5, 6i–o, 7g–l, 8–12

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REFLECTION
Tables of logarithms were used in classrooms before calculators were used there. Would using logarithms have any effect on the accuracy of calculations?

FLUENCY

1 WE25 Solve for x in the following.

a $\log_5 x = 2$ **b** $\log_3 x = 4$ **c** $\log_2 x = -3$

d $\log_4 x = -2$ **e** $\log_{10} x^2 = 4$ **f** $\log_2 x^3 = 12$

g $\log_3(x+1) = 3$
h $\log_5(x-2) = 3$
i $\log_4(2x-3) = 0$
j $\log_{10}(2x+1) = 0$
k $\log_2(-x) = -5$
l $\log_3(-x) = -2$
m $\log_5(1-x) = 4$
n $\log_{10}(5-2x) = 1$

2 **WE26** Solve for x in the following, given that $x > 0$.
a $\log_x 9 = 2$
b $\log_x 16 = 4$
c $\log_x 25 = \frac{2}{3}$
d $\log_x 125 = \frac{3}{4}$
e $\log_x\left(\frac{1}{8}\right) = -3$
f $\log_x\left(\frac{1}{64}\right) = -2$
g $\log_x 6^2 = 2$
h $\log_x 4^3 = 3$

3 **WE27** Solve for x in the following.
a $\log_2 8 = x$
b $\log_3 9 = x$
c $\log_5\left(\frac{1}{5}\right) = x$
d $\log_4\left(\frac{1}{16}\right) = x$
e $\log_4 2 = x$
f $\log_8 2 = x$
g $\log_6 1 = x$
h $\log_8 1 = x$
i $\log_{\frac{1}{2}} 2 = x$
j $\log_{\frac{1}{3}} 9 = x$

4 **WE28** Solve for x in the following.
a $\log_2 x + \log_2 4 = \log_2 20$
b $\log_5 3 + \log_5 x = \log_5 18$
c $\log_3 x - \log_3 2 = \log_3 5$
d $\log_{10} x - \log_{10} 4 = \log_{10} 2$
e $\log_4 8 - \log_4 x = \log_4 2$
f $\log_3 10 - \log_3 x = \log_3 5$
g $\log_6 4 + \log_6 x = 2$
h $\log_2 x + \log_2 5 = 1$
i $3 - \log_{10} x = \log_{10} 2$
j $5 - \log_4 8 = \log_4 x$
k $\log_2 x + \log_2 6 - \log_2 3 = \log_2 10$
l $\log_2 x + \log_2 5 - \log_2 10 = \log_2 3$
m $\log_3 5 - \log_3 x + \log_3 2 = \log_3 10$
n $\log_5 4 - \log_5 x + \log_5 3 = \log_5 6$

5 **MC** a The solution to the equation $\log_7 343 = x$ is:
A $x = 2$
B $x = 3$
C $x = 1$
D $x = 0$

b If $\log_8 x = 4$, then x is equal to:
A 4096
B 512
C 64
D 2

c Given that $\log_x 3 = \frac{1}{2}$, x must be equal to:
A 3
B 6
C 81
D 9

d If $\log_a x = 0.7$, then $\log_a x^2$ is equal to:
A 0.49
B 1.4
C 0.35
D 0.837

6 Solve for x in the following equations.
a $2^x = 128$
b $3^x = 9$
c $7^x = \frac{1}{49}$
d $9^x = 1$
e $5^x = 625$
f $64^x = 8$
g $6^x = \sqrt{6}$
h $2^x = 2\sqrt{2}$
i $3^x = \frac{1}{\sqrt{3}}$
j $4^x = 8$
k $9^x = 3\sqrt{3}$
l $2^x = \frac{1}{4\sqrt{2}}$
m $3^{x+1} = 27\sqrt{3}$
n $2^{x-1} = \frac{1}{32\sqrt{2}}$
o $4^{x+1} = \frac{1}{8\sqrt{2}}$

UNDERSTANDING

7 WE29 Solve the following equations, correct to 3 decimal places.

a $2^x = 11$ b $2^x = 0.6$ c $3^x = 20$ d $3^x = 1.7$

e $5^x = 8$ f $0.7^x = 3$ g $0.4^x = 5$ h $3^{x+2} = 12$

i $7^{-x} = 0.2$ j $8^{-x} = 0.3$ k $10^{-2x} = 7$ l $8^{2-x} = 0.75$

8 The decibel (dB) scale for measuring loudness, d, is given by the formula $d = 10\log_{10}(I \times 10^{12})$, where I is the intensity of sound in watts per square metre.

a Find the number of decibels of sound if the intensity is 1.

b Find the number of decibels of sound produced by a jet engine at a distance of 50 metres if the intensity is 10 watts per square metre.

c Find the intensity of sound if the sound level of a pneumatic drill 10 metres away is 90 decibels.

d Find how the value of d changes if the intensity is doubled. Give your answer to the nearest decibel.

e Find how the value of d changes if the intensity is 10 times as great.

f By what factor does the intensity of sound have to be multiplied in order to add 20 decibels to the sound level?

REASONING

9 The Richter scale is used to describe the energy of earthquakes. A formula for the Richter scale is $R = \frac{2}{3}\log_{10}K - 0.9$, where R is the Richter scale value for an earthquake that releases K kilojoules (kJ) of energy.

a Find the Richter scale value for an earthquake that releases the following amounts of energy:

i 1000 kJ ii 2000 kJ iii 3000 kJ

iv 10 000 kJ v 100 000 kJ vi 1 000 000 kJ

b Does doubling the energy released double the Richter scale value? Justify your answer.

c Find the energy released by an earthquake of:

i magnitude 4 on the Richter scale

ii magnitude 5 on the Richter scale

iii magnitude 6 on the Richter scale.

d What is the effect (on the amount of energy released) of increasing the Richter scale value by 1?

e Why is an earthquake measuring 8 on the Richter scale so much more devastating than one that measures 5?

10 Solve for x.

a $3^{x+1} = 7$

b $3^{x+1} = 7^x$

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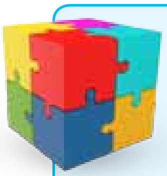
PROBLEM SOLVING

11 Solve for x.

$(27 \times 3^x)^3 = 81^x \times 3^2$

12 Solve $\{x : (3^x)^2 = 30 \times 3^x - 81\}$.



CHALLENGE 16.2

This challenge explores an interesting association between logarithms and quadratics.

Consider solving the logarithmic equation $\log_{10}(x + 1) + \log_{10} x - \log_{10} 6 = 0$.

This first step in the solution could be $\log_{10}\left[\dfrac{x(x+1)}{6}\right] = 0$.

Continue the solution by converting the logarithmic equation into a quadratic equation, then solving for x.

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16.10 Review

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The Maths Quest Review is available in a customisable format for students to demonstrate their knowledge of this topic.

The Review contains:

- **Fluency** questions — allowing students to demonstrate the skills they have developed to efficiently answer questions using the most appropriate methods
- **Problem Solving** questions — allowing students to demonstrate their ability to make smart choices, to model and investigate problems, and to communicate solutions effectively.

A summary of the key points covered and a concept map summary of this topic are available as digital documents.

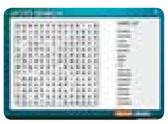
Review questions

Download the Review questions document from the links found in your eBookPLUS.

eBook*plus*

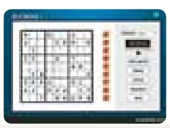
Interactivities

Word search
int-2871



Crossword
int-2872

Sudoku
int-3891



Language

It is important to learn and be able to use correct mathematical language in order to communicate effectively. Create a summary of the topic using the key terms below. You can present your summary in writing or using a concept map, a poster or technology.

base	integer	pi
conjugate	irrational	power
contradiction	laws of logarithms	rational
exponent	logarithm	rational denominator
fractional power	logarithmic equation	real
index	negative index	surd
indices	number base	transcendental number

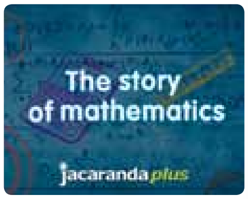
Link to assessON for questions to test your readiness **FOR** learning, your progress **AS** you learn and your levels **OF** achievement.

assess on

assessON provides sets of questions for every topic in your course, as well as giving instant feedback and worked solutions to help improve your mathematical skills.

www.assesson.com.au

The story of mathematics is an exclusive Jacaranda video series that explores the history of mathematics and how it helped shape the world we live in today.



Real numbers (eles-2019) shows how number systems have evolved over time. A concept unknown to Western mathematicians for centuries, the existence of zero, enabled many of the greatest mathematics discoveries.

RICH TASK

Other number systems

Throughout history, different systems have been used to aid with counting. Ancient tribes are known to have used stones, bones and knots in rope to help keep count. The counting system that is used around the world today is called the Hindu-Arabic system. This system had its origin in India around 300–200 BC. The Arabs brought this method of counting to Europe in the Middle Ages.

The Hindu–Arabic method is known as the decimal or base 10 system, as it is based on counting in lots of ten. This system uses the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Notice that the largest digit is one less than the base number, that is, the largest digit in base 10 is 9. To make larger numbers, digits are grouped together. The position of the digit tells us about its value. We call this *place value*. For example, in the number 325, the 3 has a value of 'three lots of a hundred', the 2 has a value of 'two lots of ten' and the 5 has a value of 'five lots of units'. Another way to write this is:

$$3 \times 100 + 2 \times 10 + 5 \times 1 \textit{ or } 3 \times 10^2 + 2 \times 10^1 + 5 \times 10^0$$

In a decimal system, every place value is based on the number 10 raised to a power. The smallest place value (units) is described by 10^0, the tens place value by 10^1, the hundreds place value by 10^2, the thousands by 10^3 and so on.

Computers do not use a decimal system. The system for computer languages is based on the number 2 and is known as the binary system. The only digits needed in the binary system are the digits 0 and 1. Can you see why?

Decimal number	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Binary number	0	1	10	11	100	101	110	111	1000	1001	1010	1011	1100	1101

Consider the decimal number 7. From the table above, you can see that its binary equivalent is 111. How can you be sure this is correct?

$$111 = 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 = 4 + 2 + 1 = 7$$

Notice that this time each place value is based on the number 2 raised to a power. You can use this technique to change any binary number into a decimal number. (The same pattern applies to other bases, for example, in base 6 the place values are based on the number 6 raised to a power.)

Binary operations

When adding in the decimal system, each time the addition is greater than 9, we need to 'carry over' into the next place value. In the example below, the units column adds to more than 9, so we need to carry over into the next place value.

$$\begin{array}{r} {}^{1}17 \\ +\ 13 \\ \hline 30 \end{array}$$

The same is true when adding in binary, except we need to 'carry over' every time the addition is greater than 1.

$$\begin{array}{r} {}^{1}01 \\ +\ 01 \\ \hline 10 \end{array}$$

1 Perform the following binary additions.

a $\begin{array}{r} 11_2 \\ +\ 01_2 \\ \hline \\ \hline \end{array}$ **b** $\begin{array}{r} 111_2 \\ +\ 110_2 \\ \hline \\ \hline \end{array}$ **c** $\begin{array}{r} 1011_2 \\ +\ 101_2 \\ \hline \\ \hline \end{array}$

2 Perform the following binary subtractions. Remember that if you need to borrow a number from a column on the left-hand side, you will actually be borrowing a 2 (not a 10).

a $\begin{array}{r} 11_2 \\ -\ 01_2 \\ \hline \\ \hline \end{array}$ **b** $\begin{array}{r} 111_2 \\ -\ 110_2 \\ \hline \\ \hline \end{array}$ **c** $\begin{array}{r} 1011_2 \\ -\ 101_2 \\ \hline \\ \hline \end{array}$

3 Try some multiplication. Remember to carry over lots of 2.

a $\begin{array}{r} 11_2 \\ \times\ 01_2 \\ \hline \\ \hline \end{array}$ **b** $\begin{array}{r} 111_2 \\ \times\ 110_2 \\ \hline \\ \hline \end{array}$ **c** $\begin{array}{r} 1011_2 \\ \times\ 101_2 \\ \hline \\ \hline \end{array}$

4 What if our number system had an 8 as its basis (that is, we counted in lots of 8)? The only digits available for use would be 0, 1, 2, 3, 4, 5, 6 and 7. (Remember the maximum digit is 1 less than the base value.) Give examples to show how numbers would be added, subtracted and multiplied using this base system. Remember that you would 'carry over' or 'borrow' lots of 8.

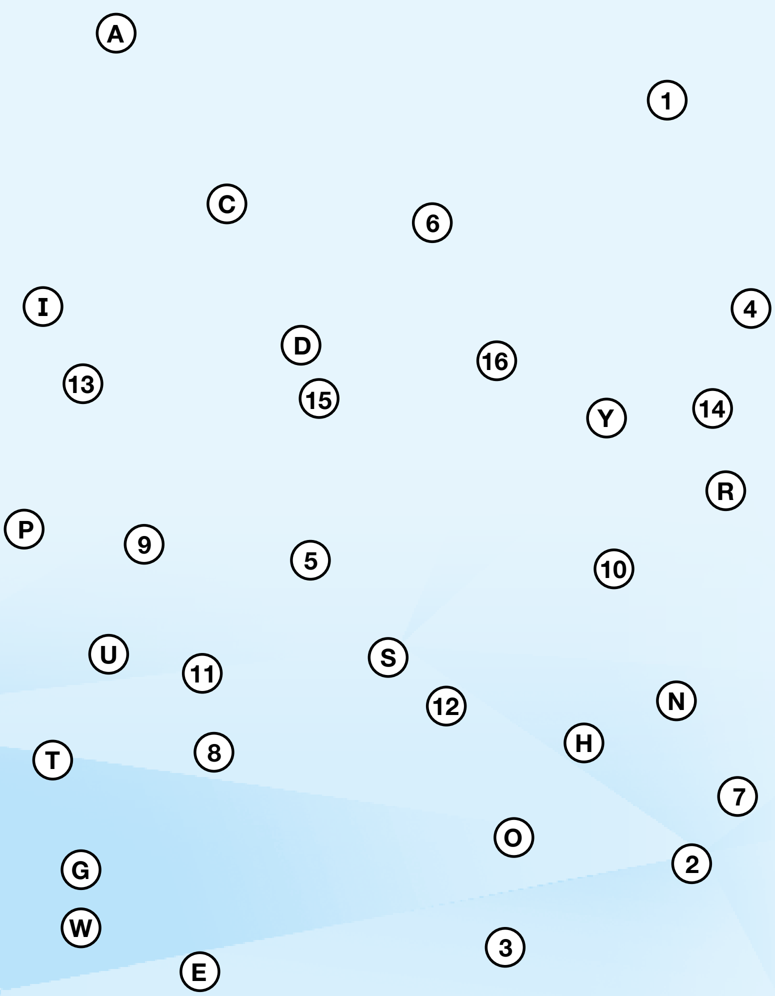
5 The hexadecimal system has 16 as its basis. Investigate this system. Explain how it would be possible to have 15, for example, in a single place position. Give examples to show how the system would add, subtract and multiply.

CODE PUZZLE

Who is Jørn Utzon?

Match the equation on the left-hand side with the answer on the right-hand side by ruling a line between the dots. The line will pass through a letter and a number to help answer the puzzle.

$\log_3 x = 2$ •	• $x = 18$
$\log_2 x = -3$ •	• $x = 5$
$\log_2 x + \log_2 4 = 4$ •	• $x = 8$
$\log_3 (x - 1) = 2$ •	• $x = 12$
$\log_4 2 = x$ •	• $x = 10$
$\log_7 x = 1$ •	• $x = 9$
$\log_5 (\frac{1}{25}) = x$ •	• $x = \frac{1}{3}$
$\log_{27} 3 = x$ •	• $x = 13$
$\log_4 (5x + 4) = 3$ •	• $x = 2$
$\log_5 (2x - 1) = 2$ •	• $x = -1$
$\log_{\frac{1}{3}} 3 = x$ •	• $x = 4$
$\log_2 x = 3$ •	• $x = 7$
$\log_3 x - \log_3 2 = 2$ •	• $x = \frac{1}{2}$
$2 - \log_4 2x = \log_4 4$ •	• $x = 3$
$\log_3 27 = x$ •	• $x = \frac{1}{8}$
$\log_5 x + \log_5 5 = 2$ •	• $x = -2$



1	2	3	4	5	6	2	7	1	3	6	1	8	2	9	10	3	11	7	12	13	3	10

1	2	3	11	14	10	13	3	14	9	15	3	5	4	2	9	16	11	3

eBookplus Activities

16.1 Overview

Video
- The story of mathematics (eles-2019)

16.2 Number classification review

Interactivities
- Classifying numbers (int-2792)
- IP interactivity 16.2 (int-4639): Number classification review

16.3 Surds

Interactivity
- IP interactivity 16.3 (int-4640): Surds

Digital doc
- SkillSHEET (doc-5354): Identifying surds

16.4 Operations with surds

Interactivity
- IP interactivity 16.4 (int-4641): Operations with surds

Digital docs
- SkillSHEET (doc-5355): Simplifying surds
- SkillSHEET (doc-5356): Adding and subtracting surds
- SkillSHEET (doc-5357): Multiplying and dividing surds
- SkillSHEET (doc-5360): Rationalising denominators
- SkillSHEET (doc-5361): Conjugate pairs
- SkillSHEET (doc-5362): Applying the difference of two squares rule to surds
- WorkSHEET 16.1 (doc-14612): Real numbers I

16.5 Fractional indices

Interactivity
- IP interactivity 16.5 (int-4642): Fractional indices

Digital doc
- WorkSHEET 16.2 (doc-14613): Real numbers II

16.6 Negative indices

Interactivity
- IP interactivity 16.6 (int-4643): Negative indices

Digital doc
- WorkSHEET 16.3 (doc-14614): Real numbers III

16.7 Logarithms

Interactivity
- IP interactivity 16.7 (int-4644): Logarithms

16.8 Logarithm laws

Interactivity
- IP interactivity 16.8 (int-4645): Logarithm laws

16.9 Solving equations

Interactivity
- IP interactivity 16.9 (int-4646): Solving equations

Digital doc
- WorkSHEET 16.4 (doc-14615): Real numbers IV

16.10 Review

Interactivities
- Word search (int-2871)
- Crossword (int-2872)
- Sudoku (int-3891)

Digital docs
- Chapter summary (doc-14616)
- Concept map (doc-14617)

To access eBookPLUS activities, log on to www.jacplus.com.au

Answers

TOPIC 16 Real numbers

Exercise 16.2 — Number classification review

1 **a** Q **b** Q **c** Q **d** I **e** I
f Q **g** Q **h** I **i** Q **j** Q
k Q **l** Q **m** I **n** Q **o** I
p Q **q** Q **r** I **s** I **t** I
u Q **v** I **w** I **x** Q **y** I

2 **a** Q **b** Q **c** Q **d** Q **e** Q
f I **g** I **h** Q **i** I **j** Undefined
k I **l** I **m** I **n** Q **o** Q
p Q **q** I **r** I **s** Q **t** Q
u I **v** Q **w** Q **x** I **y** Q

3 B
4 D
5 C
6 C
7 $\frac{a}{b}$
8 D
9 A
10 $p - q$
11 Check with your teacher.
12 **a** $m = 11, n = 3$ **b** $m = 2, n = 3$
c $m = 3, n = 2$ **d** $m = 1, n = 2$
13 $\frac{1}{7}$ or 7^{-1}

Exercise 16.3 — Surds

1 **b d f g h i l m o q s t w z**
2 A
3 D
4 B
5 C
6 Any perfect square
7 $m = 4$
8 Check with your teacher.
9 Irrational
10 **a** **i** $4\sqrt{3}$ **ii** $6\sqrt{2}$
b Yes. If you don't choose the largest perfect square, then you will need to simplify again.
c No
11 Integral and rational

Exercise 16.4 — Operations with surds

1 **a** $2\sqrt{3}$ **b** $2\sqrt{6}$ **c** $3\sqrt{3}$ **d** $5\sqrt{5}$
e $3\sqrt{6}$ **f** $4\sqrt{7}$ **g** $2\sqrt{17}$ **h** $6\sqrt{5}$
i $2\sqrt{22}$ **j** $9\sqrt{2}$ **k** $7\sqrt{5}$ **l** $8\sqrt{7}$

2 **a** $4\sqrt{2}$ **b** $24\sqrt{10}$ **c** $36\sqrt{5}$ **d** $21\sqrt{6}$
e $-30\sqrt{3}$ **f** $-28\sqrt{5}$ **g** $64\sqrt{3}$ **h** $2\sqrt{2}$
i $\sqrt{2}$ **j** $2\sqrt{3}$ **k** $\frac{1}{3}\sqrt{15}$ **l** $\frac{3}{2}\sqrt{7}$

3 **a** $4a$ **b** $6a\sqrt{2}$ **c** $3a\sqrt{10b}$
d $13a^2\sqrt{2}$ **e** $13ab\sqrt{2ab}$ **f** $2ab^2\sqrt{17ab}$
g $5x^3y^2\sqrt{5}$ **h** $20xy\sqrt{5x}$ **i** $54c^3d^2\sqrt{2cd}$
j $18c^3d^4\sqrt{5cd}$ **k** $\sqrt{22ef}$ **l** $7e^5f^5\sqrt{2ef}$

4 **a** $7\sqrt{5}$ **b** $8\sqrt{3}$ **c** $15\sqrt{5} + 5\sqrt{3}$
d $4\sqrt{11}$ **e** $13\sqrt{2}$ **f** $-3\sqrt{6}$
g $17\sqrt{3} - 18\sqrt{7}$ **h** $8\sqrt{x} + 3\sqrt{y}$

5 **a** $10(\sqrt{2} - \sqrt{3})$ **b** $5(\sqrt{5} + \sqrt{6})$
c $7\sqrt{3}$ **d** $4\sqrt{5}$
e $14\sqrt{3} + 3\sqrt{2}$ **f** $3\sqrt{6} + 6\sqrt{3}$
g $15\sqrt{10} - 10\sqrt{15} + 10$ **h** $-8\sqrt{11} + 22$
i $12\sqrt{30} - 16\sqrt{15}$ **j** $12\sqrt{ab} + 7\sqrt{3ab}$
k $\frac{7}{2}\sqrt{2} + 2\sqrt{3}$ **l** $15\sqrt{2}$

6 **a** $31\sqrt{a} - 6\sqrt{2a}$ **b** $52\sqrt{a} - 29\sqrt{3a}$
c $6\sqrt{6ab}$ **d** $32a + 2\sqrt{6a} + 8a\sqrt{2}$
e $a\sqrt{2a}$ **f** $\sqrt{a} + 2\sqrt{2a}$
g $3a\sqrt{a} + a^2\sqrt{3a}$ **h** $(a^2 + a)\sqrt{ab}$
i $4ab\sqrt{ab} + 3a^2b\sqrt{b}$ **j** $3\sqrt{ab}(2a + 1)$
k $-6ab\sqrt{2a} + 4a^2b^3\sqrt{3a}$ **l** $-2a\sqrt{b}$

7 **a** $\sqrt{14}$ **b** $\sqrt{42}$ **c** $4\sqrt{3}$ **d** 10
e $3\sqrt{7}$ **f** 27 **g** $10\sqrt{33}$ **h** $180\sqrt{5}$
i 120 **j** $120\sqrt{3}$ **k** $2\sqrt{6}$ **l** $2\frac{2}{3}$
m $\frac{2}{5}\sqrt{6}$ **n** $x^2y\sqrt{y}$ **o** $3a^4b^2\sqrt{2ab}$ **p** $6a^5b^2\sqrt{2b}$
q $3x^2y^2\sqrt{10xy}$ **r** $\frac{9}{2}a^2b^4\sqrt{5ab}$

8 **a** 2 **b** 5 **c** 12 **d** 15
e 18 **f** 80 **g** 28 **h** 200

9 **a** $\sqrt{5}$ **b** 2 **c** $\sqrt{6}$ **d** 4
e $\frac{\sqrt{3}}{4}$ **f** $\frac{\sqrt{5}}{2}$ **g** $2\sqrt{3}$ **h** 1
i $1\frac{4}{5}$ **j** $2\sqrt{17}$ **k** $\frac{x}{y}$ **l** $\frac{\sqrt{2}}{x^3y^4}$
m $2xy\sqrt{3y}$ **n** $\frac{4\sqrt{a}}{3}$

10 **a** $\frac{5\sqrt{2}}{2}$ **b** $\frac{7\sqrt{3}}{3}$ **c** $\frac{4\sqrt{11}}{11}$ **d** $\frac{4\sqrt{6}}{3}$
e $\frac{2\sqrt{21}}{7}$ **f** $\frac{\sqrt{10}}{2}$ **g** $\frac{2\sqrt{15}}{5}$ **h** $\frac{3\sqrt{35}}{5}$
i $\frac{5\sqrt{6}}{6}$ **j** $\frac{4\sqrt{15}}{15}$ **k** $\frac{5\sqrt{7}}{14}$ **l** $\frac{8\sqrt{15}}{15}$
m $\frac{8\sqrt{21}}{49}$ **n** $\frac{8\sqrt{105}}{7}$ **o** $\frac{\sqrt{10}}{3}$

11 **a** $\sqrt{2} + 2$ **b** $\frac{3\sqrt{10} - 2\sqrt{33}}{6}$
c $\frac{12\sqrt{5} - 5\sqrt{6}}{10}$ **d** $\frac{9\sqrt{10}}{5}$
e $\frac{3\sqrt{10} + 6\sqrt{14}}{4}$ **f** $\frac{5\sqrt{6}}{3}$
g $\frac{3\sqrt{22} - 4\sqrt{10}}{6}$ **h** $\frac{\sqrt{21} - \sqrt{15}}{3}$
i $\frac{14 - 5\sqrt{2}}{6}$ **j** $\frac{12 - \sqrt{10}}{16}$
k $\frac{6\sqrt{15} - 25}{70}$ **l** $\frac{\sqrt{30} + 7\sqrt{2}}{20}$

12 **a** $\sqrt{5} - 2$ **b** $\frac{2\sqrt{2} + \sqrt{5}}{3}$
c $\frac{8\sqrt{11} + 4\sqrt{13}}{31}$ **d** $\frac{15\sqrt{15} - 20\sqrt{6}}{13}$
e $12\sqrt{2} - 17$ **f** $\frac{19 - 4\sqrt{21}}{5}$

g $\dfrac{\sqrt{15} - \sqrt{3} - \sqrt{5} + 1}{4}$ **h** $\dfrac{-6 + 6\sqrt{2} + \sqrt{10} - 2\sqrt{5}}{2}$

i $\dfrac{4\sqrt{10} + \sqrt{15} - 4\sqrt{6} - 3}{29}$

13 $\dfrac{9\sqrt{x} + 6x}{36x - 16x^2}$

14 a Check with your teacher.

b i $\sqrt{5} + \sqrt{3}$ **ii** $\sqrt{5} - \sqrt{3}$ **iii** $\sqrt{3} + 2$

15 $\dfrac{2}{7}$

16 a $x = 16$ **b** $x = 1$

Exercise 16.5 — Fractional indices

1 a 4 **b** 5 **c** 9
d 2 **e** 3 **f** 5

2 a 3 **b** 2 **c** 1.4
d 2.2 **e** 1.5 **f** 1.3

3 a 2.5 **b** 12.9 **c** 13.6
d 0.7 **e** 0.8 **f** 0.9

4 a $\sqrt{7}$ **b** $2\sqrt{3}$ **c** $6\sqrt{2}$
d $4\sqrt{2}$ **e** $3\sqrt{3}$ **f** $100\sqrt{10}$

5 a $5^{\frac{1}{2}}$ **b** $10^{\frac{1}{2}}$ **c** $x^{\frac{1}{2}}$
d $m^{\frac{3}{2}}$ **e** $2t^{\frac{1}{2}}$ **f** $6^{\frac{1}{3}}$

6 a $4^{\frac{4}{5}}$ **b** $2^{\frac{1}{2}}$ **c** $a^{\frac{5}{6}}$
d $x^{\frac{23}{20}}$ **e** $10m^{\frac{8}{15}}$ **f** $2b^{\frac{5}{7}}$
g $-4y^{\frac{20}{9}}$ **h** $0.02a^{\frac{9}{8}}$ **i** $5x^{\frac{7}{2}}$

7 a $ab^{\frac{3}{2}}$ **b** $x^{\frac{4}{5}}y^{\frac{5}{9}}$ **c** $6a^{\frac{8}{5}}b^{\frac{17}{15}}$
d $2m^{\frac{19}{28}}n^{\frac{2}{5}}$ **e** $x^{\frac{19}{6}}y^{\frac{5}{6}}z^{\frac{5}{6}}$ **f** $8a^{\frac{2}{5}}b^{\frac{9}{8}}c$

8 a $3^{\frac{1}{6}}$ **b** $5^{\frac{5}{12}}$ **c** $12^{\frac{1}{2}}$
d $a^{\frac{3}{7}}$ **e** $x^{\frac{5}{4}}$ **f** $m^{\frac{11}{45}}$
g $\frac{1}{2}x^{\frac{3}{20}}$ **h** $\frac{1}{3}n^{\frac{2}{3}}$ **i** $\frac{5}{4}b^{\frac{7}{20}}$

9 a $x^{\frac{5}{3}}y^{\frac{7}{5}}$ **b** $a^{\frac{7}{45}}b^{\frac{4}{15}}$ **c** $\frac{1}{3}m^{\frac{3}{8}}n^{\frac{11}{56}}$
d $2x^{\frac{2}{15}}y^{\frac{3}{4}}$ **e** $\frac{1}{4}a^{\frac{11}{20}}b^{\frac{7}{20}}$ **f** $\frac{1}{7}p^{\frac{5}{24}}q^{\frac{1}{12}}$

10 a $2^{\frac{9}{20}}$ **b** $5^{\frac{1}{6}}$ **c** $7^{\frac{6}{5}}$
d $a^{\frac{3}{10}}$ **e** $m^{\frac{1}{6}}$ **f** $2^{\frac{1}{3}}b^{\frac{1}{6}}$
g $4p^{\frac{2}{5}}$ **h** $x^{\frac{m}{p}}$ **i** $3^{\frac{b}{c}}m^{\frac{a}{c}}$

11 a $a^{\frac{1}{4}}b^{\frac{1}{6}}$ **b** $a^3b^{\frac{3}{4}}$ **c** $x^{\frac{6}{5}}y^{\frac{7}{4}}$
d $3^{\frac{1}{3}}a^{\frac{1}{9}}b^{\frac{1}{5}}c^{\frac{1}{4}}$ **e** $x^{\frac{1}{4}}y^{\frac{1}{3}}z^{\frac{1}{5}}$ **f** $\dfrac{a^{\frac{1}{2}}}{b^{\frac{2}{3}}}$
g $\dfrac{m^{\frac{8}{5}}}{n^{\frac{7}{4}}}$ **h** $\dfrac{b^{\frac{2}{5}}}{c^{\frac{8}{27}}}$ **i** $\dfrac{2^{\frac{1}{2}}x^{\frac{7}{2}}}{y^{\frac{3}{8}}}$

12 C, D

13 a a^4 **b** b^3 **c** m^4 **d** $4x^2$
e $2y^3$ **f** $2x^2y^3$ **g** $3m^3n^5$ **h** $2pq^2$
i $6a^2b^6$

14 a 0.32 m/s
b 16 640 L/s
c 59 904 000 L/hr
That is 16 640 × 60 × 60.
d The hydraulic radius is the measure of a channel flow efficiency. The roughness coefficient is the resistance of the bed of a channel to the flow of water in it.

15 $x = 1$

16 a $x^{\frac{1}{2}} + y^{\frac{1}{2}} - z^{\frac{1}{2}}$ **b** $t^{\frac{1}{10}}$

17 $m - n^2$

Exercise 16.6 — Negative indices

1 a $\frac{1}{5} = 0.2$ **b** $\frac{1}{3} = 0.\dot{3}$
c $\frac{1}{8} = 0.125$ **d** $\frac{1}{10} = 0.1$
e $\frac{1}{8} = 0.125$ **f** $\frac{1}{9} = 0.\dot{1}$
g $\frac{1}{25} = 0.04$ **h** $\frac{1}{10\,000} = 0.0001$

2 a 0.167 **b** 0.143 **c** 0.0278
d 0.001 37 **e** 0.004 63 **f** 0.004 44
g 0.003 91 **h** 0.001 60

3 a 0.40 **b** 2.5 **c** 0.44
d 4.0 **e** 0.11 **f** 0.000 079
g 11 **h** 4100

4 a −0.33 **b** −0.20 **c** 0.25 **d** 0.063
e −0.67 **f** −0.45 **g** −1.7 **h** 1.4

5 a $\frac{5}{4}$ or $1\frac{1}{4}$ **b** $\frac{10}{3}$ or $3\frac{1}{3}$ **c** $\frac{8}{7}$ or $1\frac{1}{7}$ **d** $\frac{20}{13}$ or $1\frac{7}{13}$
e 2 **f** 4 **g** 8 **h** 10
i $\frac{2}{3}$ **j** $\frac{4}{9}$ **k** $\frac{10}{11}$ **l** $\frac{2}{11}$

6 a 4 **b** $6\frac{1}{4}$ **c** $3\frac{3}{8}$ **d** 16
e $\frac{4}{9}$ **f** $\frac{16}{81}$ **g** $\frac{27}{64}$ **h** $\frac{125}{1331}$

7 a $-\frac{3}{2}$ **b** $-\frac{5}{3}$ **c** −4 **d** −10
e $\frac{9}{4}$ **f** 25 **g** $-\frac{2}{3}$ **h** $\frac{16}{121}$

8 $\frac{3}{10}$

9 $\dfrac{b}{a}$

10 a $y \to \infty$ **b** $y \to -\infty$

11 As the value of n increases, the value of 2^{-n} gets closer to 0.

12 $x = -2$, $y = -3$

13 x^2

Exercise 16.7 — Logarithms

1 a $\log_4 16 = 2$ **b** $\log_2 32 = 5$
c $\log_3 81 = 4$ **d** $\log_6 36 = 2$
e $\log_{10} 1000 = 3$ **f** $\log_5 25 = 2$
g $\log_4 x = 3$ **h** $\log_5 125 = x$
i $\log_7 49 = x$ **j** $\log_p 16 = 4$
k $\log_9 3 = \frac{1}{2}$ **l** $\log_{10} 0.1 = -1$
m $\log_8 2 = \frac{1}{3}$ **n** $\log_2 \frac{1}{2} = -1$
o $\log_a 1 = 0$ **p** $\log_4 8 = \frac{3}{2}$

2 D

3 a $2^4 = 16$ **b** $3^3 = 27$ **c** $10^6 = 1\,000\,000$
d $5^3 = 125$ **e** $16^{\frac{1}{2}} = 4$ **f** $4^x = 64$
g $49^{\frac{1}{2}} = 7$ **h** $3^5 = x$ **i** $81^{\frac{1}{2}} = 9$
j $10^{-2} = 0.01$ **k** $8^1 = 8$ **l** $64^{\frac{1}{3}} = 4$

4 B

5 a 4 **b** 2 **c** 2 **d** 5
e 5 **f** 7 **g** 0 **h** $\frac{1}{2}$
i −1 **j** 1 **k** −2 **l** $\frac{1}{3}$

6 a 0 **b** 1 **c** 2
d 3 **e** 4 **f** 5

7 a 0 and 1 **b** 3 and 4 **c** 1 and 2
d 4 and 5 **e** 2 and 3 **f** 4 and 5

8 a $\log_{10} g = k$ implies that $g = 10^k$ so $g^2 = (10^k)^2$. That is, $g^2 = 10^{2k}$; therefore, $\log_{10} g^2 = 2k$.
b $\log_x y = 2$ implies that $y = x^2$, so $x = y^{\frac{1}{2}}$ and therefore $\log_y x = \frac{1}{2}$.
c The equivalent exponential statement is $x = 4^y$, and we know that 4^y is greater than zero for all values of y. Therefore, x is a positive number.

9 a 6 **b** −4 **c** −5

10 a 3 **b** 7 **c** $\frac{1}{8}$

11 x

Exercise 16.8 — Logarithm laws

1 a 1.698 97 **b** 1.397 94 **c** 0.698 97 **d** 0.301 03

2 Teacher to check.

3 a 1 **b** 3 **c** 2
d 3 **e** 4 **f** 1

4 a 2 **b** 3 **c** 1
d 4 **e** 3 **f** 5

5 a 2 **b** $\frac{1}{2}$ **c** 1 **d** 3

6 3

7 a 2 **b** 4 **c** 3 **d** 3

8 a 1 **b** 0 **c** −1 **d** 5
e −2 **f** 1 **g** 0 **h** −2
i $-\frac{1}{2}$ **j** $\frac{1}{2}$ **k** $-\frac{1}{2}$ **l** $\frac{7}{2}$

9 a $\log_a 40$ **b** $\log_a 18$ **c** $\log_x 48$ **d** $\log_x 4$
e $\log_a x$ **f** 1 **g** −1 **h** 7
i $\frac{1}{2}$ **j** $\frac{3}{2}$ **k** −6 **l** $-\frac{1}{3}$

10 a B **b** B, D **c** A, B **d** C, D

11 a $\log_2 80$ **b** $\log_3 105$ **c** $\log_{10} 100 = 2$ **d** $\log_6 56$
e $\log_2 4 = 2$ **f** $\log_3 3 = 1$ **g** $\log_5 12.5$ **h** $\log_2 3$
i $\log_4 5$ **j** $\log_{10} \frac{1}{4}$ **k** $\log_3 4$ **l** $\log_2 3$
m $\log_3 20$ **n** $\log_4 2 = \frac{1}{2}$

12 a C **b** B **c** A

13 a 12 (Evaluate each logarithm separately and then find the product.)
b 4 (First simplify the numerator by expressing 81 as a power of 3.)
c 7 (Let $y = 5^{\log_5 7}$ and write an equivalent statement in logarithmic form.)

14 $7 - 3\log_2(3)$

15 1

16 $x = 3a, 5a$

Challenge 16.1

$\frac{3}{2}$

Exercise 16.9 — Solving equations

1 a 25 **b** 81 **c** $\frac{1}{8}$
d $\frac{1}{16}$ **e** 100, −100 **f** 16
g 26 **h** 127 **i** 2
j 0 **k** $-\frac{1}{32}$ **l** $-\frac{1}{9}$
m −624 **n** −2.5

2 a 3 **b** 2 **c** 125 **d** 625
e 2 **f** 8 **g** 6 **h** 4

3 a 3 **b** 2 **c** −1 **d** −2
e $\frac{1}{2}$ **f** $\frac{2}{5}$ **g** 0 **h** 0
i −1 **j** −2

4 a 5 **b** 6 **c** 10 **d** 8 **e** 4
f 2 **g** 9 **h** $\frac{2}{5}$ **i** 500 **j** 128
k 5 **l** 6 **m** 1 **n** 2

5 a B **b** A **c** D **d** B

6 a 7 **b** 2 **c** −2 **d** 0 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{2}$ **h** $\frac{3}{2}$ **i** $-\frac{1}{2}$ **j** $\frac{3}{2}$
k $\frac{3}{4}$ **l** $-\frac{5}{2}$ **m** $\frac{5}{2}$ **n** $-\frac{9}{2}$ **o** $-\frac{11}{4}$

7 a 3.459 **b** −0.737 **c** 2.727 **d** 0.483
e 1.292 **f** −3.080 **g** −1.756 **h** 0.262
i 0.827 **j** 0.579 **k** −0.423 **l** 2.138

8 a 120 **b** 130 **c** 0.001
d 3 dB are added.
e 10 dB are added.
f 100

9 a **i** 1.1
ii 1.3
iii 1.418
iv 1.77
v 2.43
vi 3.1
b No; see answers to **9a i** and **ii** above.
c **i** 22 387 211 kJ
ii 707 945 784 kJ
iii 22 387 211 385 kJ.
d The energy is increased by a factor of 31.62.
e It releases 31.62^3 times more energy.

10 a $x = 0.7712$
b $x = 1.2966$

11 $x = 7$

12 $x = 1, 3$

Challenge 16.2

The remaining steps of the solution are $\frac{x(x+1)}{6}$, $x^2 + x - 6 = 0$, $x = -3$ or 2.

Investigation — Rich task

1 a 100_2 **b** 1101_2 **c** 10000_2

2 a 10_2 **b** 101_2 **c** 110_2

3 a 11_2 **b** 1001_2 **c** 10101_2

4 Answers will vary; teacher to check.

5 Answers will vary; teacher to check. The numbers 10, 11, 12, 13, 14 and 15 are allocated the letters A, B, C, D, E and F respectively.

Code puzzle

The architect who designed the Sydney Opera House