Chapter 9

9 Probability

Objectives

- To understand the basic concepts and notation of **set theory**.
- To understand the basic concepts and rules of **probability**.
- To introduce **mutually exclusive events**.
- To apply the **addition rule** to solve problems.
- To use **Venn diagrams**, **tree diagrams** and **probability tables** to calculate probabilities.
- To introduce **conditional probability** and **independence**.
- To use the **multiplication rule** and the **law of total probability** to calculate probabilities.

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

However, whether through our prior observations or through a theoretical understanding of the circumstances, we are often able to assign a numerical value, or **probability**, to each possible outcome of an experiment. This probability, which will take a value between 0 and 1, gives us an indication as to how likely we are to observe the outcome. A probability of 0 means that the outcome is impossible and a probability of 1 means that it is certain. Generally, the probability will be somewhere in between, with a higher value meaning that the outcome is more likely.

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9A Sample spaces and probability

In this section we look at two essential components of probability: a sample space, which is the set of all possible outcomes of an experiment, and a set of probabilities, one for each outcome.

► Sample spaces

Suppose we toss a coin to see whether a head (H) or a tail (T) appears uppermost. The toss of the coin can be termed a single **trial** of a **random experiment**. The word 'random' is used here because, while the outcome observed must be either a head or a tail, on a particular toss we don't know which will be observed. However, we do know that the outcome observed will be one of a known set of possible outcomes, and the set of all possible outcomes is called the **sample space** for the experiment.

Set notation can be used in listing all the elements in the sample space. For example, the sample space for the tossing of a coin would be written as

$$\{H, T\}$$

where H indicates head and T indicates tail. Throughout this chapter, the Greek letter ε (epsilon) will be used to denote the sample space.

For example, the following table lists the sample spaces for each of the random experiments described.

Random experiment	Sample space
The number observed when a die is rolled	$\varepsilon = \{1, 2, 3, 4, 5, 6\}$
The number of brown eggs in a carton of 12 eggs	$\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed	$\varepsilon = \{HH, HT, TH, TT\}$
The number of calls to your phone in the next two hours	$\varepsilon = \{0, 1, 2, 3, 4, \dots\}$
The time, in hours, it takes to complete your homework	$\varepsilon = \{t : t \geq 0\}$

► Events

An **event** is a subset of the sample space. It may consist of a single outcome, or it may consist of several outcomes. For example, when rolling a die, the event of interest may be 'getting a six', which consists of just one outcome and is described by the set {6}. However, the event 'getting an odd number' can be achieved by rolling 1, 3 or 5 and is described by the set $\{1, 3, 5\}$.

It is convenient to use set notation to list the elements of the event. In general we use capital letters, $A, B, C, \dots$, to denote events.

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The following table lists the experiments described earlier and gives the sample space and an example of an event for each one.

Sample space	An event
The number observed when a die is rolled $\varepsilon = \{1, 2, 3, 4, 5, 6\}$	'An even number' $= \{2, 4, 6\}$
The number of brown eggs in a carton of 12 eggs $\varepsilon = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$	'More than half brown' $= \{7, 8, 9, 10, 11, 12\}$
The result when two coins are tossed $\varepsilon = \{HH, HT, TH, TT\}$	'Two heads' $= \{HH\}$
The number of calls to your phone in the next two hours $\varepsilon = \{0, 1, 2, 3, 4, \dots\}$	'Fewer than two phone calls' $= \{0, 1\}$
The time, in hours, it takes to complete your homework $\varepsilon = \{ t : t \geq 0 \}$	'More than two hours' $= \{ t : t > 2 \}$

Note: Both a sample space and an event can be discrete or continuous, finite or infinite.

Example 1

A bag contains seven marbles numbered from 1 to 7 and a marble is withdrawn.

a Give the sample space for this experiment.

b List the outcomes (elements) of the event 'a marble with an odd number is withdrawn'.

Solution	*Explanation*
a $\{1, 2, 3, 4, 5, 6, 7\}$	Any number from 1 to 7 could be observed.
b $\{1, 3, 5, 7\}$	This set contains the odd numbers in the sample space.

Determining probabilities for equally likely outcomes

There are many situations for which we can develop a simple model that can be used to assign a probability to an event. The most obvious of these is when it is reasonable to assume that all of the outcomes are equally likely, such as when a die is rolled.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$. Since six outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6}$$

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When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $\Pr(A) = \Pr(2) + \Pr(4) + \Pr(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(S)$ is used to represent the number of elements in set S.

Of course, there are many situations where the outcomes are not equally likely. For example, it has been established worldwide that the probability of a male birth is in fact 0.51, not 0.5. However, in many situations the assumption of equally likely is justified, and allows us to assign probabilities reasonably.

Example 2

Suppose a number is drawn at random from the numbers 7, 8, 9, 10, 11, 12, 13, 14. What is the probability of choosing a prime number?

Solution

Let A be the event the chosen number is prime. Then

$$\Pr(A) = \frac{3}{8}$$

Explanation

Since the number is drawn at random, we can assume each number is equally likely to be drawn.

$A = \{7, 11, 13\}$, $n(A) = 3$, $n(\varepsilon) = 8$

Example 3

Suppose that a card is drawn from a pack of 52 playing cards, and that each card has equal likelihood of being drawn. Find:

a the probability that the card is black

b the probability that the card is a king

c the probability that the card is a black king.

Solution

a $\Pr(\text{black card}) = \frac{26}{52} = \frac{1}{2}$

Explanation

There are 52 cards in a pack and 26 are black.

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b $\Pr(\text{king}) = \dfrac{4}{52} = \dfrac{1}{13}$

There are 52 cards in a pack and 4 are kings.

c $\Pr(\text{black king}) = \dfrac{2}{52} = \dfrac{1}{26}$

There are 52 cards in a pack and 2 are black kings.

The following rules of probability hold for finite sample spaces:

- $\Pr(A) \geq 0$, for any event A.
- The sum of the probabilities of all the outcomes of a random experiment must equal 1.

The second of these two rules can be used to determine probabilities as follows.

Example 4

A random experiment may result in 1, 2, 3 or 4. If $\Pr(1) = \dfrac{1}{13}$, $\Pr(2) = \dfrac{2}{13}$ and $\Pr(3) = \dfrac{3}{13}$, find the probability of obtaining a 4.

Solution

$$\Pr(4) = 1 - \left(\frac{1}{13} + \frac{2}{13} + \frac{3}{13}\right)$$
$$= 1 - \frac{6}{13} = \frac{7}{13}$$

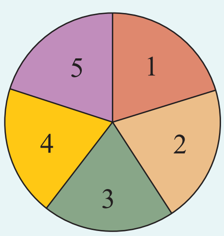
Explanation

The sum of the probabilities is 1.

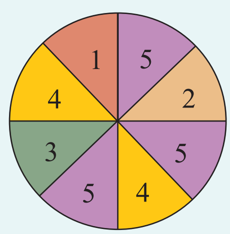
Example 5

Find the probability that each of the possible outcomes is observed for the following spinners:

a



b



Solution

a $\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \dfrac{1}{5}$

b $\Pr(1) = \Pr(2) = \Pr(3) = \dfrac{1}{8} = 0.125$

$\Pr(4) = \dfrac{2}{8} = \dfrac{1}{4} = 0.25$

$\Pr(5) = \dfrac{3}{8} = 0.375$

Note that in both these cases

$\Pr(1) + \Pr(2) + \Pr(3) + \Pr(4) + \Pr(5) = 1$

Explanation

On spinner a, there are five equally likely outcomes.

Since there are 8 equal segments, we assume each has a probability of $\frac{1}{8}$.

The results 1, 2 and 3 appear once.
The result 4 appears twice.
The result 5 appears three times.

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Complementary events

When two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of event A is the event A', which consists of all the outcomes in ε that are not in A. Since the sum of the probabilities is 1, we can write

$$\Pr(A') = 1 - \Pr(A)$$

Example 6

A card is drawn at random from a pack of 52 cards. What is the probability that the card is:

a not a heart

b not an ace?

Solution

a Let H be the event a heart is drawn.

Then
$$\begin{aligned}\Pr(H') &= 1 - \Pr(H)\\ &= 1 - \frac{13}{52}\\ &= 1 - \frac{1}{4}\\ &= \frac{3}{4}\end{aligned}$$

b Let A be the event an ace is drawn.

Then
$$\begin{aligned}\Pr(A') &= 1 - \Pr(A)\\ &= 1 - \frac{4}{52}\\ &= 1 - \frac{1}{13}\\ &= \frac{12}{13}\end{aligned}$$

Combining our knowledge of the rules of probability enables us to solve more complex problems.

Example 7

A random experiment may result in outcomes A, B, C, D or E, where A, B, C, D are equally likely and E is twice as likely as A. Find:

a $\Pr(E)$

b $\Pr(B')$

Solution

a Let $\Pr(A) = \Pr(B) = \Pr(C) = \Pr(D) = x$.
Then $\Pr(E) = 2x$.

Explanation: Summarise the information in the question in terms of one unknown.

$$\begin{aligned}x + x + x + x + 2x &= 1\\ 6x &= 1\\ x &= \frac{1}{6}\end{aligned}$$

Explanation: The sum of the probabilities is 1.

Thus $\Pr(E) = 2x = \frac{1}{3}$

b $\Pr(B') = 1 - \Pr(B) = 1 - \frac{1}{6} = \frac{5}{6}$

Explanation: Since B' is the complement of B, the probabilities will add to 1.

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Section summary

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\Pr(A)$.
- Rules of probability for finite sample spaces:
  - $\Pr(A) \geq 0$, for each event A.
  - The sum of the probabilities of all the outcomes of a random experiment must be equal to 1.
- Equally likely outcomes If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- If two events have no elements in common and together they make up the entire sample space, they are said to be **complementary events**. The complement of any event A is denoted A' and we can write

$$\Pr(A') = 1 - \Pr(A)$$

Exercise 9A

Example 1

1 List the sample space for the toss of a coin.

2 List the sample space for the outcomes when a die is rolled.

3 Answer the following for a normal deck of playing cards:

a How many cards are there?
b How many suits are there?
c What are the suits called?
d Which suits are red and which suits are black?
e How many cards are there in each suit?
f Which cards are known as the ‘picture cards’?
g How many aces are there in the deck?
h How many ‘picture cards’ are there in the deck?

4 List the sample spaces for the following experiments:

a the number of picture cards in a hand of five cards
b the number of female children in a family with six children
c the number of female students on a committee of three students chosen from a class of 10 male and 10 female students

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5 List the sample spaces for the following experiments:

a the number of cars which pass through a particular intersection in a day

b the number of people on board a bus licensed to carry 40 passengers

c the number of times a die is rolled before a six is observed

6 List the outcomes associated with the following events:

a 'an even number' when a die is rolled

b 'more than two female students' when three students are chosen for a committee from a class of 10 male and 10 female students

c 'more than four aces' when five cards are dealt from a standard pack of 52 cards

Example 2

7 A number is drawn at random from the set $\{1, 2, 3, \dots, 20\}$. What is the probability that the number is:

a divisible by 2

b divisible by 3

c divisible by both 2 and 3?

8 A bag has 15 marbles numbered $1, 2, 3, \dots, 15$. If one marble is drawn at random from the bag, what is the probability that the number on the marble is:

a less than 5

b greater than or equal to 6

c a number from 5 to 8 inclusive?

Example 3

9 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a a club

b red

c a picture card (king, queen, jack)

d a red picture card.

10 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:

a less than 10

b less than or equal to 10

c an even number

d an ace.

11 Suppose that in a certain city the same number of people were born on each of the 365 days of the year, and that nobody was born on 29 February. Find the probability that the birthday of a person selected at random:

a is 29 November

b is in November

c falls between 15 January and 15 February, not including either day

d is in the first three months of the year.

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12 One letter is drawn at random from the letters in the word AUSTRALIA. Find the probability that the letter is:

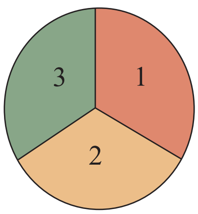
a a T **b** an A **c** a vowel **d** a consonant.

Example 4 **13** A random experiment results in 1, 2, 3, 4, 5 or 6. If $\Pr(1) = \frac{1}{12}$, $\Pr(2) = \frac{1}{6}$, $\Pr(3) = \frac{1}{8}$, $\Pr(5) = \frac{1}{6}$ and $\Pr(6) = \frac{1}{8}$, find the probability of obtaining a 4.

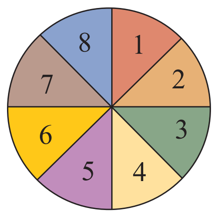
14 A random experiment results in 1, 2, 3 or 4. If $\Pr(1) = 0.2$, $\Pr(3) = 0.1$ and $\Pr(4) = 0.3$, find $\Pr(2)$.

Example 5 **15** Consider the following spinners. In each case, what is the chance of the pointer stopping in region 1?

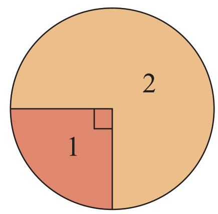
a



b



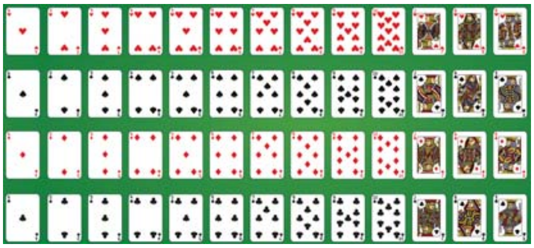
c



Example 6 **16** Assume that the probability of a baby being born on a certain day is the same for each day of the week. Find the probability that a randomly chosen person was born:

a on a Wednesday **b** not on the weekend.

17 A card is drawn at random from a well-shuffled pack of 52 cards. Find the probability that the card is:



a not a club

b not red

c not a picture card

d not a red picture card.

Example 7 **18** A random experiment results in 1, 2, 3 or 4. If 1, 2 and 3 are equally likely to occur, and 4 is twice as likely to occur as 3, find the probability of each of the possible outcomes.

19 For a particular biased six-sided die it is known that the numbers 2, 3, 4 and 5 are equally likely to occur, that the number 6 occurs twice as often as the number 2, and that the number 1 occurs half as often as the number 2.

a Find the probability of each of the possible outcomes.

b Find the probability that the number observed is not a 6.

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9B Estimating probabilities

Skillsheet

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of estimating probability are required.

▶ Subjective probabilities

Sometimes, the probability is assigned a value just on the basis of experience. For example, a sports journalist may suggest that Australia has a 60% chance of winning the next Ashes series relying on his or her own judgement. Another journalist might well assign this probability an entirely different value. Such probabilities are called subjective probabilities, and whether or not they are accurate estimates of the true probability would be open to dispute.

▶ Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment leading to the event of interest many times and recording the results. This information can then be used to estimate the chances of it happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

Suppose, for example, that we are interested in determining the probability that a drawing pin will land 'point up' when it is tossed. Since a drawing pin is not symmetrical, the assumption of equally likely outcomes cannot be used to determine probabilities.

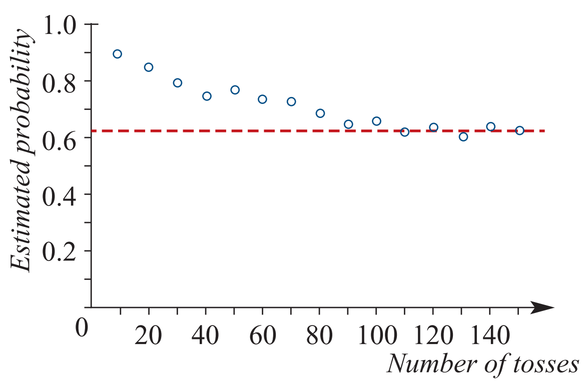
Our strategy to estimate this probability is to toss the drawing pin many times and count the number of times it lands point up. We can then calculate the relative frequency:

$$\text{Relative frequency of 'point up'} = \frac{\text{number of times drawing pin landed 'point up'}}{\text{number of trials}}$$

This proportion, or relative frequency, is an estimate of the probability of a drawing pin landing with the point up.

The graph opposite shows the results of one experiment where a drawing pin is tossed 150 times, with the probability of the drawing pin landing point up estimated every 10 throws.

From the graph it may be seen that, as the number of trials (repetitions of the experiment) increases, the estimated probability converges to a value and then stays fairly stable.



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In general, if the same experiment is repeated many, many times, the relative frequency of any particular event will stabilise to a constant value. This limiting value of the relative frequency is then considered to be the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\Pr(A)$. That is,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and will be the topic of your later studies in mathematics. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 8

In order to investigate the probability that a drawing pin lands point up, Katia decides to toss it 50 times and to count the number of favourable outcomes, which turns out to be 33. Mikki repeats the experiment, but she tosses the same drawing pin 100 times and counts 62 favourable outcomes.

a What is Katia's estimate of the probability of the drawing pin landing point up?

b What is Mikki's estimate?

c Which of these is the preferred estimate of the probability from these experiments?

d Based on the information available, what would be the preferred estimate of the probability?

Solution

a From Katia's information: $\Pr(\text{point up}) \approx \dfrac{33}{50} = 0.66$

b From Mikki's information: $\Pr(\text{point up}) \approx \dfrac{62}{100} = 0.62$

c Since Mikki has estimated the probability from a larger number of trials, her estimate would be preferred to Katia's.

d Based on the information available, the preferred estimate of the probability would be found by combining the data from both experiments, and so maximising the number of trials. In total, 95 favourable outcomes were observed in 150 tosses, and this gives a 'best' estimate of the probability of $\dfrac{95}{150} = 0.63$.

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Thus, probability can be considered as the proportion of times that an event will occur in the long run. This interpretation also defines the minimum and maximum values of probability as 0 (the event never occurs) and 1 (the event always occurs), and confirms that the sum of the probabilities for all possible outcomes will equal 1.

Simulation

The word simulate means to pretend or to imitate. In statistics, simulation is a way to model a random experiment, such that simulated outcomes closely match real-world outcomes. Simulation does not involve repeating the actual experiment. Instead, more complex probabilities can be estimated via multiple trials of an experiment which approximates the actual experiment, but can be carried out quickly and easily. A more detailed discussion of simulation is found in Section 9H.

Probabilities from area

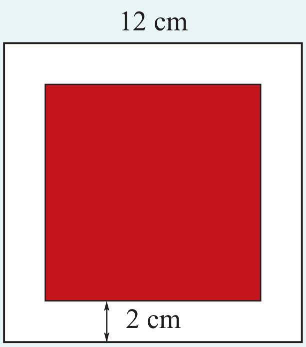
In the previous section we used the model of equally likely outcomes to determine probabilities. We counted both the outcomes in the event and the outcomes in the sample space, and used the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 9

Suppose that a square dartboard consists of a red square drawn inside a larger white square of side length 12 cm, as shown.

If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the red area? (Ignore the possibility that it might land on the line or miss the board altogether!)



Solution

$$\Pr(\text{landing in the red area}) = \frac{\text{area of red square}}{\text{area of dartboard}} = \frac{64}{144} = \frac{4}{9}$$

Explanation

There are really only two outcomes for this experiment: landing in the red area or landing in the white area.

Assume that the probability of landing in an area is proportional to the size of the area.

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Section summary

- When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of estimating probability is used, the rules of probability hold:
  - $\Pr(A) \geq 0$, for each event A
  - $\Pr(\varepsilon) = 1$
  - The sum of the probabilities of all the outcomes of a random experiment equals 1.

Exercise 9B

Example 8

1 Estimate the probability of the event specified occurring, using the data given:

a Pr(head) if a coin is tossed 100 times and 34 heads observed

b Pr(ten) if a spinner is spun 200 times and lands on the 'ten' 20 times

c Pr(two heads) if two coins are tossed 150 times and two heads are observed on 40 occasions

d Pr(three sixes) if three dice are rolled 200 times and three sixes observed only once

2 A student decides to toss two coins and notes the results.

a Do you think relative frequencies obtained from 20 trials would make for a good estimate of the probabilities?

b Perform the experiment 20 times and estimate Pr(two heads), Pr(one head) and Pr(no heads).

c Combine your results with those of your friends, so that you have results from at least 100 trials. Use these results to again estimate the probabilities.

d Do you think the data from 100 trials give better estimates of the probabilities?

e How many trials would you need to find the probabilities exactly?

3 Two misshapen six-sided dice were used for the following experiment. The first die was thrown 500 times and 78 sixes were observed. The second die was thrown 700 times and 102 sixes were observed. If you wished to throw a six, which die would you choose to throw, and why?

4 A bowl contains 340 red and 60 black balls.

a State the proportion of red balls in the bowl.

b A random sample of 60 balls is taken from the bowl and is found to have 48 red balls. Find the proportion of red balls in the sample.

c Another random sample of 60 balls is taken from the bowl and is found to have 54 red balls. Find the proportion of red balls in the sample.

d What is the expected number of red balls in a sample of 60?

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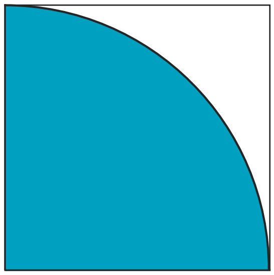
5 In a survey of 2000 people, 890 indicated that they regularly use social media to keep in touch with friends. What is an estimate for the probability that the next person surveyed also uses social media?

Example 9

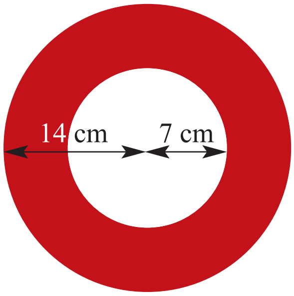
6 A square of side length 1 metre contains a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.



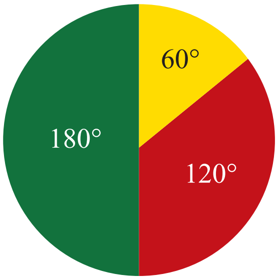
a What proportion of the square is blue?

b If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.

7 A dart is thrown at random onto a board that has the shape of a circle as shown. Calculate the probability that the dart will hit the shaded region.



8 A spinner is as shown in the diagram.
Find the probability that when spun the pointer will land on:

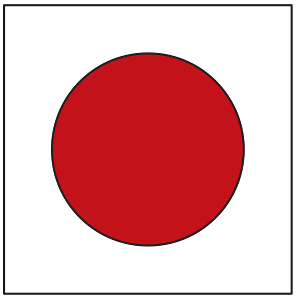


a the red section

b the yellow section

c any section except the yellow section.

9 In a sideshow at a fete a dart is thrown at a square with side length 1 metre. The circle shown has a radius of 0.4 metres. The dart is equally likely to hit any point on the square. Find the probability that the dart will hit:



a the shaded part of the square

b the unshaded part of the square.

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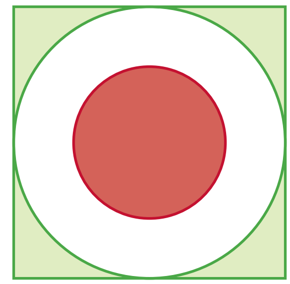
10 A dart is thrown at random onto the board shown. The board is a square of side length x, the larger circle is of radius $\frac{x}{2}$ and the smaller circle is of radius $\frac{x}{4}$.



a Find, in terms of x:

- **i** the area of the square
- **ii** the area of the larger circle
- **iii** the area of the smaller circle.

b Hence find the probability that the dart will land:

- **i** inside the smaller circle
- **ii** in the white region
- **iii** in the outer shaded region.

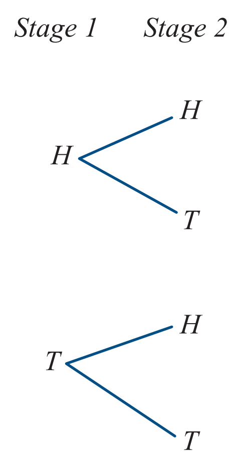
9C Multi-stage experiments

Often we are interested in probabilities which arise from more complex **multi-stage experiments**. That is, they are concerned with experiments which could be considered to take place in more than one stage.

For example, when considering the outcomes from tossing two coins (or tossing one coin twice) we should consider the possible outcomes in two stages:

- the outcome from coin 1
- followed by the outcome from coin 2.

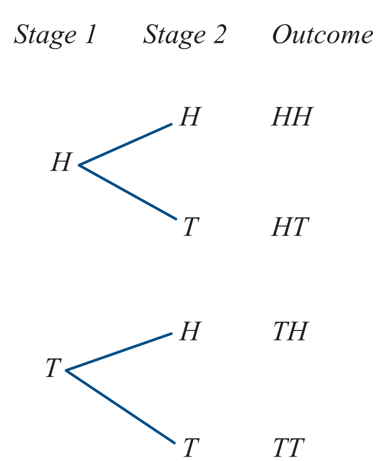
In such cases, it is helpful to list the elements of the sample space systematically by means of a **tree diagram** as shown.



Each path along the branches of the tree gives an outcome, which we determine by reading along the branches, making sure we maintain the order of the outcome at each stage carefully.

Collecting together all the outcomes, we see that the sample space is

$$\varepsilon = \{HH, HT, TH, TT\}$$



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When the outcomes of the multi-stage experiment are equally likely, we can still determine the probability of an event occurring by dividing the number of outcomes in the event by the number of outcomes in the sample space.

Example 10

Find the probability that when a fair coin is tossed twice:

a one head is observed

b at least one head is observed

c both heads or both tails are observed.

Solution

a Pr(one head) $= \frac{2}{4} = \frac{1}{2}$

Explanation

List the outcomes of the event:
'one head' $= \{HT, TH\}$.
There are 2 outcomes in the event and 4 in the sample space (see tree diagram).

b Pr(at least one head) $= \frac{3}{4}$

List the outcomes of the event:
'at least one head' $= \{HH, HT, TH\}$.
There are 3 outcomes in the event and 4 in the sample space.

c Pr(both heads or both tails) $= \frac{2}{4} = \frac{1}{2}$

List the outcomes of the event:
'both heads or both tails' $= \{HH, TT\}$.
There are 2 outcomes in the event and 4 in the sample space.

When listing the outcomes for a two-stage experiment, it can also be convenient to display the sample space in a table. For example, when rolling two dice (or a single die twice) there is the possibility of $\{1, 2, 3, 4, 5, 6\}$ on die 1 (or the first roll), and $\{1, 2, 3, 4, 5, 6\}$ on die 2 (or the second roll). So the sample space for this experiment can be written as:

	Die 2					
Die 1	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

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Example 11

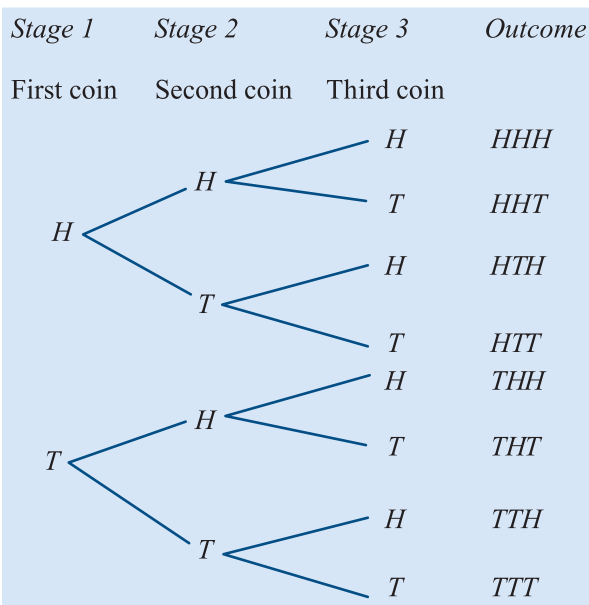
Find the probability that when two fair dice are rolled:

a the same number shows on both dice (a double)

b the sum of the two numbers shown is greater than 10.

Solution	*Explanation*
a $\Pr(\text{double}) = \dfrac{6}{36} = \dfrac{1}{6}$	'double' = $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$. There are 6 outcomes in the event and 36 in the sample space.
b $\Pr(\text{sum} > 10) = \dfrac{3}{36} = \dfrac{1}{12}$	'sum is greater than 10' = $\{(5, 6), (6, 5), (6, 6)\}$. There are 3 outcomes in the event and 36 in the sample space.

When the experiment involves more than two stages, it is best to use a **tree diagram** to determine all of the possible outcomes. Suppose, for example, that three coins are tossed and the outcomes noted. The three-stage tree diagram for listing the sample space for this experiment is as follows:



Thus the required sample space is

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

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Example 12

Find the probability that when a coin is tossed three times:

a one head is observed

b at least one head is observed

c the second toss results in a head

d all heads or all tails are observed.

Solution	Explanation
a Pr(one head) $= \frac{3}{8}$	'one head' $= \{HTT, THT, TTH\}$. There are 3 outcomes in the event and 8 in the sample space.
b Pr(at least one head) $= \frac{7}{8}$	'at least one head' $= \{HHH, HHT, HTH, THH, HTT, THT, TTH\}$. There are 7 outcomes in the event and 8 in the sample space.
c Pr(second toss is a head) $= \frac{4}{8} = \frac{1}{2}$	'second toss is a head' $= \{HHH, HHT, THH, THT\}$. There are 4 outcomes in the event and 8 in the sample space.
d Pr(all heads or all tails) $= \frac{2}{8} = \frac{1}{4}$	'all heads or all tails' $= \{HHH, TTT\}$. There are 2 outcomes in the event and 8 in the sample space.

Section summary

The sample space for a two-stage experiment can be displayed using a tree diagram or a table. If an experiment involves more than two stages, then a tree diagram should be used.

Exercise 9C

Example 10

1 Two fair coins are tossed. Use the sample space for this experiment to find the probability of observing:

a no heads **b** more than one tail.

2 A fair coin is tossed twice. Find the probability that:

a the first toss is a head **b** the second toss is a head **c** both tosses are heads.

Example 11

3 Two regular dice are rolled. Use the sample space for this experiment to find the probability that the sum of the numbers showing is:

a even **b** 3 **c** less than 6.

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4 Two regular dice are rolled. Use the sample space for this experiment to find the probability that the sum of the numbers showing is:

a equal to 10 **b** odd **c** less than or equal to 7.

Example 12

5 A fair coin is tossed three times. Use the sample space for this experiment to find the probability that:

a exactly one tail is observed **b** exactly two tails are observed
c exactly three tails are observed **d** no tails are observed.

6 A fair coin is tossed three times. Use the sample space for this experiment to find the probability that:

a the third toss is a head
b the second and third tosses are heads
c at least one head and one tail are observed.

7 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment. Find the probability of obtaining a head and an even number.

8 Two coins are tossed and a die is rolled.

a Draw a tree diagram to show all the possible outcomes.
b Find the probability of observing:
  i two heads and a 6
  ii one head, one tail and an even number
  iii two tails and an odd number
  iv an odd number on the die.

9 Madison has a choice of two entrees (soup or salad), three main courses (fish, chicken or steak) and three desserts (ice-cream, lemon tart or cheese).

a Draw a tree diagram to show all her possible dinner combinations.
b If Madison chooses all three courses, and is equally likely to choose any of the options at each course, find the probability that:
  i she chooses soup, fish and lemon tart
  ii she chooses fish
  iii she chooses salad and chicken
  iv she doesn't have the lemon tart.
c Suppose Madison has the choice to omit the entree and/or the dessert course altogether. Find the probability that:
  i she chooses soup, fish and lemon tart
  ii she chooses all three courses
  iii she chooses only two courses
  iv she has only the main course.

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10 A bag contains five balls, numbered 1 to 5. A ball is chosen at random, the number noted and the ball replaced. A second ball is then chosen at random and its number noted.

a Draw up a table of ordered pairs to show the sample space for the experiment.

b Find the probability that:

i the sum of the two numbers is 5

ii the two numbers are different

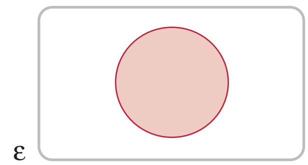
iii the second number is two more than the first.

9D Combining events

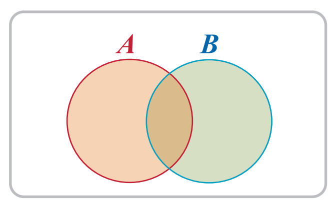
Before proceeding with the discussion of probability, a review of sets and set notation is necessary.

The **empty set**, denoted by $\varnothing$, is the set consisting of no elements. This is different from $\{0\}$, which is a set containing one element, 0.

Sets, and the relationships between sets, can be illustrated clearly by using **Venn diagrams**. The universal set ε is usually shown as a rectangle, and a subset of ε as a circle.



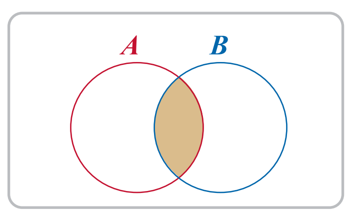
If A and B are any two sets, then the **union** of A and B, denoted $A \cup B$, is the set of all elements in A or B (or both). This is shown on a Venn diagram by shading both sets A and B.



Union

For example, if A is the set of students in a school who play hockey, and B the set of students who play tennis, then the union of A and B is the set of students who play either hockey or tennis or both.

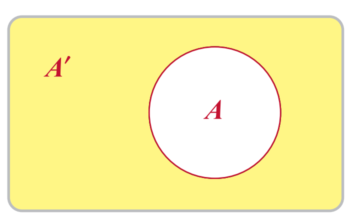
The **intersection** of A and B, denoted $A \cap B$, is the set of elements that are in both A and B. This is shown on a Venn diagram by shading only the area contained in both A and B.



Intersection

For example, the intersection of the two sets previously described is the set of students who play both hockey and tennis.

As previously, note that the **complement** of A, denoted A', is the set of all elements of ε that are not in A. This is shown on a Venn diagram by shading only the area outside A.



Complement

The complement of the set of students who play hockey in a school is the set of students who do not play hockey.

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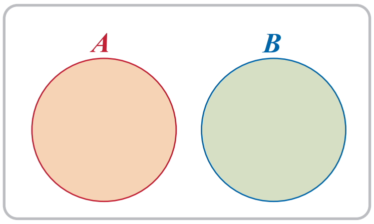
Two sets A and B are said to be **disjoint** or **mutually exclusive** if they have no elements in common, that is, if $A \cap B = \varnothing$. The Venn diagram opposite shows two sets that are mutually exclusive.



Disjoint sets

If A is the set of girls who play hockey in a school and B is the set of boys who play hockey, then A and B are mutually exclusive, as no student can belong to both sets.

Finally, the number of elements in a set A is usually denoted $n(A)$. For example, if $A = \{2, 4, 6\}$, then $n(A) = 3$.

Venn diagrams can be used to help us solve practical problems involving sets.

Example 13

Fifty students were asked what they did on the weekends. A total of 35 said they went to football matches, the movies or both. Of the 22 who went to football matches, 12 said they also went to the movies. Show this information on a Venn diagram.

a How many students went to the movies but not to football matches?

b How many went neither to football matches nor to the movies?

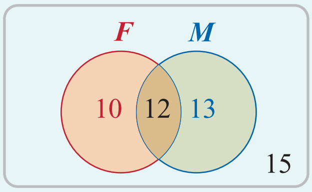
Solution

Let F denote the set of students who attend football matches and M denote the set of students who attend movies.

Hence, from the information given, $n(F \cup M) = 35$, $n(F) = 22$ and $n(F \cap M) = 12$.

a Students who go to the movies but not to football matches are found in the region $F' \cap M$, and from the diagram $n(F' \cap M) = 13$.

b Those who attend neither are found in the region $F' \cap M'$, and from the diagram $n(F' \cap M') = 15$.



$n = 50$

Example 14

Consider Example 13. What is the probability that a student chosen at random from this group of 50:

a went to the movies but not to football matches

b went neither to football matches nor to the movies?

Solution

a $\Pr(F' \cap M) = \dfrac{n(F' \cap M)}{n(\varepsilon)} = \dfrac{13}{50}$

b $\Pr(F' \cap M') = \dfrac{n(F' \cap M')}{n(\varepsilon)} = \dfrac{15}{50} = \dfrac{3}{10}$

Explanation

To determine the probability of these events, divide by the size of the sample space in each case.

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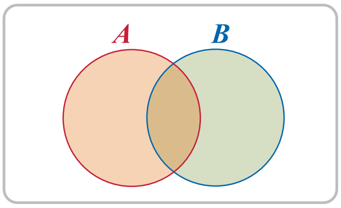
The addition rule

Venn diagrams can be used to illustrate a very important rule that will enable us to calculate probabilities for more complex events. If A and B are two events in a sample space ε and $A \cap B \neq \varnothing$, then the relationship between them can be represented by a Venn diagram, as shown.



From the Venn diagram we can see that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

(As the intersection has been counted twice, in both $n(A)$ and $n(B)$, we must subtract it.)

Dividing through by $n(\varepsilon)$ gives

$$\frac{n(A \cup B)}{n(\varepsilon)} = \frac{n(A)}{n(\varepsilon)} + \frac{n(B)}{n(\varepsilon)} - \frac{n(A \cap B)}{n(\varepsilon)}$$

Now, if each of the outcomes in ε is equally likely to occur, then each term in this expression is equal to the probability of that event occurring. This can be rewritten as:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

So the probability of A or B or both occurring can be calculated using

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

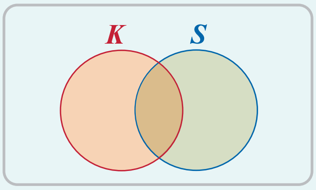
This is called the **addition rule** for combining probabilities. This rule also applies in situations where the outcomes are not equally likely; it is always true.

This rule can be used to help solve more complex problems in probability.

Example 15

If one card is chosen at random from a well-shuffled deck, what is the probability that the card is a king or a spade?

Solution



Let event K be 'a king'. Then K = {king of spades, king of hearts, king of diamonds, king of clubs} and $n(K) = 4$.

Let event S be 'a spade'. Then S = {ace of spades, king of spades, queen of spades, . . . } and $n(S) = 13$.

The event 'a king or a spade' corresponds to the union of sets K and S. We have

$$\Pr(K) = \frac{4}{52}, \qquad \Pr(S) = \frac{13}{52}, \qquad \Pr(K \cap S) = \frac{1}{52}$$

and so, using the addition rule, we find

$$\Pr(K \cup S) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = 0.3077 \quad \text{(correct to 4 decimal places)}$$

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Section summary

- Venn diagrams are often useful for solving problems involving sets.
- For any two events A and B, the **addition rule** can be applied:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

- If the two events A and B are mutually exclusive, then $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Exercise 9D

1 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6\}$.
Show these sets on a Venn diagram and use your diagram to find:

a $A \cup B$ **b** $A \cap B$ **c** A'
d $A \cap B'$ **e** $(A \cap B)'$ **f** $(A \cup B)'$

2 $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $A = \{\text{multiples of four}\}$, $B = \{\text{even numbers}\}$.
Show these sets on a Venn diagram and use your diagram to find:

a A' **b** B' **c** $A \cup B$
d $(A \cup B)'$ **e** $A' \cap B'$

3 $\varepsilon = \{\text{different letters of the word MATHEMATICS}\}$
$A = \{\text{different letters of the word ATTIC}\}$
$B = \{\text{different letters of the word TASTE}\}$
Show ε, A and B on a Venn diagram, entering all the elements. Hence list the sets:

a A' **b** B' **c** $A \cup B$
d $(A \cup B)'$ **e** $A' \cup B'$ **f** $A' \cap B'$

Example 13

4 In a survey of 100 university students, a market research company found that 70 students owned smartphones, 45 owned cars and 35 owned smartphones and cars. Use a Venn diagram to help you answer the following questions:

a How many students owned neither a car nor a smartphone?
b How many students owned either a car or a smartphone, but not both?

Example 14

5 Let $\varepsilon = \{1, 2, 3, 4, 5, 6\}$, where the outcomes are equally likely. If $A = \{2, 4, 6\}$ and $B = \{3\}$, find:

a $\Pr(A \cup B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A')$ **d** $\Pr(B')$

Example 15

6 Let $\varepsilon = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, where the outcomes are equally likely. If A is the event 'an even number' and B is the event 'a multiple of three', find:

a $\Pr(A)$ **b** $\Pr(B)$ **c** $\Pr(A \cap B)$ and hence $\Pr(A \cup B)$.

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7 In Sam's class at school many of the students are on the swimming team. There are 18 students in the class. Seven of the students swim freestyle and four swim backstroke. Two students swim both freestyle and backstroke. Draw a Venn diagram and use it to find the probability that a student randomly chosen from Sam's class:

a swims freestyle **b** swims backstroke

c swims freestyle and backstroke **d** is on the swimming team.

8 Suppose that A is the set of factors of 12, and B is the set of prime numbers less than 10. If a number is chosen at random from the integers from 1 to 20, draw a Venn diagram and use it to find:

a $\Pr(A)$ **b** $\Pr(B)$ **c** $\Pr(A \cap B)$ **d** $\Pr(A \cup B)$

9 Suppose $\Pr(A) = 0.5$, $\Pr(B) = 0.4$ and $\Pr(A \cap B) = 0.2$. Find $\Pr(A \cup B)$.

10 Suppose $\Pr(A) = 0.35$, $\Pr(B) = 0.24$ and $\Pr(A \cap B) = 0.12$. Find $\Pr(A \cup B)$.

11 Suppose $\Pr(A) = 0.28$, $\Pr(B) = 0.45$ and $A \subseteq B$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

12 Suppose $\Pr(A) = 0.58$, $\Pr(B) = 0.45$ and $B \subseteq A$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

13 Suppose $\Pr(A) = 0.3$, $\Pr(B) = 0.4$ and $A \cap B = \varnothing$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

14 Suppose $\Pr(A) = 0.08$, $\Pr(B) = 0.15$, and A and B are disjoint. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$

15 Suppose $\Pr(A) = 0.3$, $\Pr(B) = 0.4$ and $\Pr(A \cup B) = 0.5$. Find $\Pr(A \cap B)$.

16 Suppose $\Pr(A) = 0.24$, $\Pr(B) = 0.44$ and $\Pr(A \cup B) = 0.63$. Find $\Pr(A \cap B)$.

17 Suppose $\Pr(A) = 0.3$, $\Pr(B) = 0.4$ and $\Pr(A \cap B') = 0.2$. Find $\Pr(A \cup B')$.

18 Suppose that in a certain school the probability that a student plays soccer is 0.18, the probability that a student plays tennis is 0.25, and the probability that a student plays both soccer and tennis is 0.11. Find the probability that a student plays either or both of these sports.

19 Suppose that in a certain school the probability that a student studies Chinese is 0.22, the probability that a student studies French is 0.35, and the probability that a student studies both languages is 0.14.

a Find the probability that a student in that school studies at least one of these languages.

b Find the probability that a student in that school studies exactly one of these languages.

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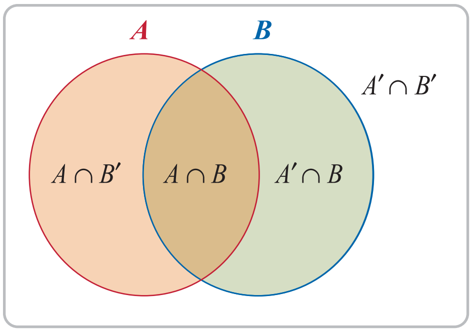
9E Probability tables

Skillsheet

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

In a probability table, the entries give the probabilities of each of these events occurring.

		Column 1	Column 2
		B	B'
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Further, from the Venn diagram we can see that set A is the union of the part of set A that intersects with set B and the part of set A that does not intersect with set B. That is,

$$A = (A \cap B) \cup (A \cap B')$$

The sets $A \cap B$ and $A \cap B'$ are mutually exclusive, so

$$\Pr(A \cap B) + \Pr(A \cap B') = \Pr(A) \qquad \text{(row 1)}$$

and thus summing the probabilities in row 1 gives $\Pr(A)$. Similarly:

$$\Pr(A' \cap B) + \Pr(A' \cap B') = \Pr(A') \qquad \text{(row 2)}$$

$$\Pr(A \cap B) + \Pr(A' \cap B) = \Pr(B) \qquad \text{(column 1)}$$

$$\Pr(A \cap B') + \Pr(A' \cap B') = \Pr(B') \qquad \text{(column 2)}$$

Finally, since $\Pr(A) + \Pr(A') = 1$ and $\Pr(B) + \Pr(B') = 1$, the totals for both column 3 and row 3 are equal to 1. Thus, the completed table becomes:

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
Row 2	A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
Row 3		$\Pr(B)$	$\Pr(B')$	1

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These tables can be useful when solving problems involving probability, as shown in the next two examples.

Example 16

If A and B are events such that $\Pr(A) = 0.7$, $\Pr(A \cap B) = 0.4$ and $\Pr(A' \cap B) = 0.2$, find:

a $\Pr(A \cap B')$ **b** $\Pr(B)$ **c** $\Pr(A' \cap B')$ **d** $\Pr(A \cup B)$

Solution

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B')$	$\Pr(A) = 0.7$
Row 2	A'	$\Pr(A' \cap B) = 0.2$	$\Pr(A' \cap B')$	$\Pr(A')$
Row 3		$\Pr(B)$	$\Pr(B')$	1

The given information has been entered in the table in red.

a From row 1: $\Pr(A \cap B') = \Pr(A) - \Pr(A \cap B) = 0.7 - 0.4 = 0.3$

b From column 1: $\Pr(B) = \Pr(A \cap B) + \Pr(A' \cap B) = 0.4 + 0.2 = 0.6$

c From column 3: $\Pr(A') = 1 - \Pr(A) = 1 - 0.7 = 0.3$
From row 2: $\Pr(A' \cap B') = 0.3 - 0.2 = 0.1$

d Using the addition rule:
$$\begin{aligned}\Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B)\\ &= 0.7 + 0.6 - 0.4\\ &= 0.9\end{aligned}$$

The completed table is shown below.

		Column 1	Column 2	Column 3
		B	B'	
Row 1	A	$\Pr(A \cap B) = 0.4$	$\Pr(A \cap B') = 0.3$	$\Pr(A) = 0.7$
Row 2	A'	$\Pr(A' \cap B) = 0.2$	$\Pr(A' \cap B') = 0.1$	$\Pr(A') = 0.3$
Row 3		$\Pr(B) = 0.6$	$\Pr(B') = 0.4$	1

Example 17

Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students are Australian by birth. They also show that 53% of students are Australian by birth and participate in sport. Use this information to find:

a the probability that a student selected at random is not Australian by birth

b the probability that a student selected at random is not Australian by birth and does not participate in sport.

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Solution

The information in the question may be entered into a table as shown. We use A to represent 'Australian by birth' and S to represent 'participates in sport'.

	S	S'	
A	0.53		0.71
A'			
	0.65		1

All the empty cells in the table may now be filled in by subtraction.

In column 1: $\Pr(A' \cap S) = 0.65 - 0.53 = 0.12$

In column 3: $\Pr(A') = 1 - 0.71 = 0.29$

In row 1: $\Pr(A \cap S') = 0.71 - 0.53 = 0.18$

In row 3: $\Pr(S') = 1 - 0.65 = 0.35$

In row 2: $\Pr(A' \cap S') = 0.29 - 0.12 = 0.17$

	S	S'	
A	0.53	0.18	0.71
A'	0.12	0.17	0.29
	0.65	0.35	1

a The probability that a student selected at random is not Australian by birth is given by $\Pr(A') = 0.29$.

b The probability that a student selected at random is not Australian by birth and does not participate in sport is given by $\Pr(A' \cap S') = 0.17$.

Exercise 9E

Example 16

1 If A and B are events such that $\Pr(A) = 0.6$, $\Pr(A \cap B) = 0.4$ and $\Pr(A' \cap B) = 0.1$, find:

a $\Pr(A \cap B')$ **b** $\Pr(B)$ **c** $\Pr(A' \cap B')$ **d** $\Pr(A \cup B)$

2 If A and B are events such that $\Pr(A') = 0.25$, $\Pr(A' \cap B) = 0.12$ and $\Pr(B) = 0.52$, find:

a $\Pr(A)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$ **d** $\Pr(B')$

3 If C and D are events such that $\Pr(C \cup D) = 0.85$, $\Pr(C) = 0.45$ and $\Pr(D') = 0.37$, find:

a $\Pr(D)$ **b** $\Pr(C \cap D)$ **c** $\Pr(C \cap D')$ **d** $\Pr(C' \cup D')$

4 If E and F are events such that $\Pr(E \cup F) = 0.7$, $\Pr(E \cap F) = 0.15$ and $\Pr(E') = 0.55$, find:

a $\Pr(E)$ **b** $\Pr(F)$ **c** $\Pr(E' \cap F)$ **d** $\Pr(E' \cup F)$

5 If A and B are events such that $\Pr(A) = 0.8$, $\Pr(B) = 0.7$ and $\Pr(A' \cap B') = 0.1$, find:

a $\Pr(A \cup B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A' \cap B)$ **d** $\Pr(A \cup B')$

Example 17

6 In a recent survey of senior citizens, it was found that 85% favoured giving greater powers of arrest to police, 60% favoured longer sentences for convicted persons, and 50% favoured both propositions.

a What percentage favoured at least one of the two propositions?

b What percentage favoured neither proposition?

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7 Suppose a card is selected at random from an ordinary deck of 52 playing cards.
Let A = event a picture card is selected (i.e. jack, queen, king or ace)
C = event a heart is selected

a List the outcomes corresponding to events A and C.

b Determine the following probabilities and express your results in words:

i $\Pr(A)$ **ii** $\Pr(C)$ **iii** $\Pr(A \cap C)$ **iv** $\Pr(A \cup C)$ **v** $\Pr(A \cup C')$

8 The following information applies to a particular class:

- The probability that a student's name begins with M and the student studies French is $\frac{1}{6}$.
- The probability that a student's name begins with M is $\frac{3}{10}$.
- The probability that a student does not study French is $\frac{7}{15}$.

Find the probability that a student chosen at random from this class:

a studies French

b has a name which does not begin with M

c has a name which does begin with M, but does not study French

d has a name which does not begin with M and does not study French.

9 A frame is chosen at random from a shop where picture frames are sold. It is known that in this shop:

- the probability that the frame is made of wood is 0.72
- the probability that the frame is freestanding is 0.65
- the probability that the frame is not made of wood and is not freestanding is 0.2.

Find the probability that the randomly chosen frame:

a is made of wood or is freestanding

b is made of wood and is freestanding

c is not made of wood

d is not made of wood but is freestanding.

10 A book is chosen at random from a bookshop. It is known that in this bookshop:

- the probability that the book is a hardback but not a novel is 0.05
- the probability that the book is not hardback but is a novel is 0.12
- the probability that the book is not a novel is 0.19.

Find the probability that the randomly chosen book is:

a a novel

b a hardback novel

c a hardback

d a novel or a hardback.

11 At a school camp consisting of 60 students, sailing was offered as an activity one morning, and bushwalking in the afternoon. Every student attended at least one activity. If 32 students went sailing and 40 students went bushwalking, find the probability that a student chosen at random:

a undertook neither of these activities

b has sailed or bushwalked

c has sailed and bushwalked

d has sailed but not bushwalked.

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12 At a barbecue attended by 50 people, hamburgers and sausages were available. It was found that 35 hamburgers and 38 sausages were eaten, and six people were noted to have eaten neither a hamburger nor a sausage. If no person ate more than one hamburger or one sausage, find the probability that a person chosen at random ate:

a a hamburger or a sausage **b** a hamburger and a sausage
c only one serve of food **d** only a hamburger.

9F Conditional probability

Skillsheet

We are often interested in calculating the probability of one event in the light of whether another event has or has not already occurred. For example, consider tossing a coin twice. What is the probability that the second toss shows a head, if we know that the first toss shows a head? Is the probability the same as if the first toss was a tail?

Suppose that we define event A as 'the second toss is a head', and event B as 'the first toss is a head'. Then the probability that the second toss shows a head, given that the first toss shows a head, is written $\Pr(A \mid B)$ and is an example of conditional probability.

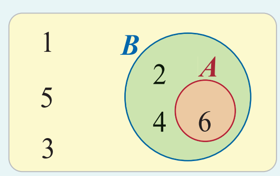
The probability of an event A occurring when it is known that some event B has occurred is called **conditional probability** and is written $\Pr(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Example 18

Suppose we roll a fair die and define event A as 'rolling a six' and event B as 'rolling an even number'. What is the probability of rolling a six given the information that an even number was rolled?

Solution

The events A and B can be shown on a Venn diagram.



We know that event B has already occurred so we know that the outcome was 2, 4 or 6. Thus

Pr(six is rolled given an even number is rolled)

$$= \frac{\text{number of favourable outcomes}}{\text{total number of outcomes}}$$

$$= \frac{n(A)}{n(B)}$$

$$= \frac{1}{3}$$

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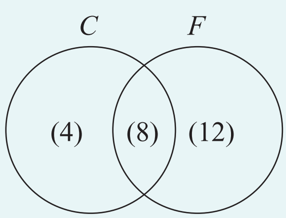
Example 19

In Stephen's class 12 students study Chinese, 20 study French, and 8 study both Chinese and French.

a Given that a student in his class studies Chinese (C), what is the probability that they also study French (F)?

b Given that a student in his class studies French, what is the probability that they also study Chinese?

Solution



a $\Pr(F \mid C) = \dfrac{8}{12} = \dfrac{2}{3}$

b $\Pr(C \mid F) = \dfrac{8}{20} = \dfrac{2}{5}$

Explanation

Display the information in the question in a Venn diagram. The numbers in brackets indicate the number of elements in each region.

If we know that the student studies Chinese, the sample space is restricted to those 12 students. From the Venn diagram we can see that 8 of these students also study French.

If we know that the student studies French, the sample space is restricted to those 20 students. From the Venn diagram we can see that 8 of these students also study Chinese.

This example clearly demonstrates that, in general, $\Pr(A \mid B) \neq \Pr(B \mid A)$. So care needs to be taken when determining conditional probabilities.

Conditional probabilities can also be calculated from a table, as shown in Example 20.

Example 20

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected at random from these 500. Given that the selected person is less than 25 years of age, what is the probability that they regularly use social media?

Solution

$\Pr(\text{Yes} \mid \text{Age} < 25) = \dfrac{200}{240} = \dfrac{5}{6}$

Explanation

If we know the person is less than 25 years old, then the sample space is restricted to those 240 people. Of these, 200 regularly use social media.

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Note that, in Example 20,

$$\Pr(\text{Yes} \cap \text{Age} < 25) = \frac{200}{500} \qquad \text{and} \qquad \Pr(\text{Age} < 25) = \frac{240}{500}$$

Hence we have

$$\frac{\Pr(\text{Yes} \cap \text{Age} < 25)}{\Pr(\text{Age} < 25)} = \frac{\frac{200}{500}}{\frac{240}{500}} = \frac{200}{240} = \frac{5}{6}$$

which is equal to the conditional probability $\Pr(\text{Yes} \mid \text{Age} < 25)$.

This illustrates a general principle which is always true.

The conditional probability of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

Example 21

Given that for two events A and B, $\Pr(A) = 0.7$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.4$, find:

a $\Pr(A \cap B)$

b $\Pr(A \mid B)$

Solution

a $\Pr(A \cap B) = \Pr(B \mid A) \times \Pr(A)$

$= 0.4 \times 0.7 = 0.28$

b $\Pr(A \mid B) = \dfrac{\Pr(A \cap B)}{\Pr(B)} = \dfrac{0.28}{0.3} = \dfrac{14}{15}$

Example 22

In a particular school 55% of the students are male and 45% are female. Of the male students 13% say mathematics is their favourite subject, while of the female students 18% prefer mathematics. Find the probability that:

a a student chosen at random prefers mathematics and is female

b a student chosen at random prefers mathematics and is male.

Solution

Let us use M to represent male, F for female, and P for prefers mathematics. Then

$$\Pr(M) = 0.55, \quad \Pr(F) = 0.45, \quad \Pr(P \mid M) = 0.13, \quad \Pr(P \mid F) = 0.18$$

We can use the multiplication rule to find the required probabilities:

a The event 'prefers mathematics and is female' is represented by $P \cap F$, with

$$\Pr(P \cap F) = \Pr(P \mid F) \times \Pr(F) = 0.18 \times 0.45 = 0.081$$

b The event 'prefers mathematics and is male' is represented by $P \cap M$, with

$$\Pr(P \cap M) = \Pr(P \mid M) \times \Pr(M) = 0.13 \times 0.55 = 0.0715$$

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► The law of total probability

As has already been seen, the tree diagram is an efficient way of listing a multi-stage sample space. If the probabilities associated with each stage are also added to the tree diagram, it becomes a very useful way of calculating the probability for each outcome. The probabilities at each stage are conditional probabilities that the particular path will be followed and the multiplication rule says that the probability of reaching the end of a given branch is the product of the probabilities associated with each segment of that branch.

Example 23

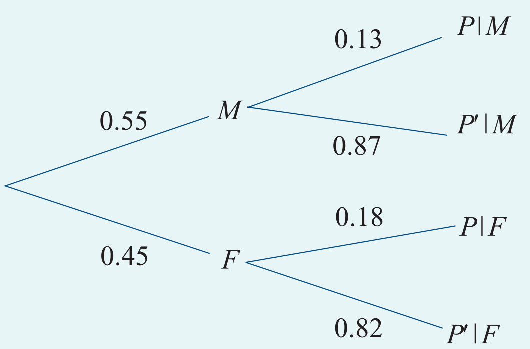
Using the information from Example 22, construct a tree diagram and use it to determine:

a the probability that a student selected is female and does not prefer mathematics

b the overall percentage of students who prefer mathematics.

Solution

The situation described can be represented by a tree diagram as follows:



$\Pr(P \cap M) = 0.55 \times 0.13 = 0.0715$

$\Pr(P' \cap M) = 0.55 \times 0.87 = 0.4785$

$\Pr(P \cap F) = 0.45 \times 0.18 = 0.081$

$\Pr(P' \cap F) = 0.45 \times 0.82 = 0.369$

a To find the probability that a student is female and does not prefer mathematics we multiply along the appropriate branches thus:

$$\Pr(F \cap P') = \Pr(F) \times \Pr(P' \mid F) = 0.45 \times 0.82 = 0.369$$

b Now, to find the overall percentage of students who prefer mathematics we recall that:

$$P = (P \cap F) \cup (P \cap M)$$

Since $P \cap F$ and $P \cap M$ are mutually exclusive,

$$\Pr(P) = \Pr(P \cap F) + \Pr(P \cap M) = 0.081 + 0.0715 = 0.1525$$

Thus 15.25% of all students prefer mathematics.

The solution to part b of Example 23 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

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A further example of the use of the law of total probability is given in the following example.

Example 24

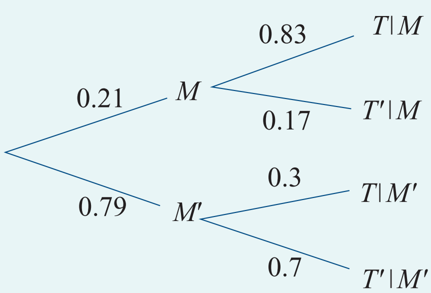
In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

a on both Monday and Tuesday **b** on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$

$\Pr(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$\Pr(T \cap M') = 0.79 \times 0.3 = 0.237$

$\Pr(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$$

b The probability that it rains on Tuesday is given by

$$\Pr(T) = \Pr(T \cap M) + \Pr(T \cap M') = 0.1743 + 0.237 = 0.4113$$

Section summary

- The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $\Pr(A \mid B)$.
- In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{if } \Pr(B) \neq 0$$

  This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

- The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

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Exercise 9F

Example 18 **1** Suppose that a fair die is rolled, and event A is defined as 'rolling a six' and event B as 'rolling a number greater than 2'. Find $\Pr(A \mid B)$.

2 Suppose that a fair die is rolled, and event A is defined as 'rolling a three' and event B as 'rolling an odd number'. Draw a Venn diagram and use it to find $\Pr(A \mid B)$.

Example 19 **3** Suppose that a card is drawn from a pack of 52 cards, and event A is defined as 'selecting an ace' and event B as 'selecting a club'. Draw a Venn diagram and use it to find the probability that the card drawn is an ace, given that it is a club.

4 In Sam's class 12 students play violin, 12 play piano and 7 play both. Draw a Venn diagram and use it to find the probability that a randomly chosen student plays the violin given that they play the piano.

5 Two dice are rolled and the outcomes observed. Given that the dice both show the same number, what is the probability that it's a 'double six'?

6 In Annabelle's class 17 students own an iPhone, 7 own an iPad, and 4 own both.

a What is the probability that a student owns an iPad, given that they own an iPhone?

b What is the probability that a student owns an iPhone, given that they own an iPad?

Example 20 **7** 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

One person is selected at random from these 100. Given that the selected person is male, what is the probability that they think private individuals should be allowed to carry guns?

8 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

One student is selected at random from these 500. Find:

a the probability that the student prefers sport

b the probability that the student prefers sport, given that they are male.

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9 The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	Probable cause: Speed	Alcohol	Reckless driving	Other	Total
Fatal	42	61	22	12	137
Non-fatal	88	185	98	60	431
Total	130	246	120	72	568

Use the table to estimate:

a the probability that speed is the cause of the accident
b the probability that the accident is fatal
c the probability that the accident is fatal, given that speed is the cause
d the probability that the accident is fatal, given that alcohol is the cause.

Example 21 **10** Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.1$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \mid B)$

11 For events A and B:

a if $\Pr(A) = 0.7$ and $\Pr(A \cap B) = 0.4$, find $\Pr(B \mid A)$
b if $\Pr(A \mid B) = 0.6$ and $\Pr(B) = 0.5$, find $\Pr(A \cap B)$
c if $\Pr(A \mid B) = 0.44$ and $\Pr(A \cap B) = 0.3$, find $\Pr(B)$.

12 In a random experiment $\Pr(A) = 0.5$, $\Pr(B) = 0.4$ and $\Pr(A \cup B) = 0.7$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \mid B)$ **c** $\Pr(B \mid A)$

13 In a random experiment $\Pr(A) = 0.6$, $\Pr(B) = 0.54$ and $\Pr(A \cap B') = 0.4$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(A \mid B)$ **c** $\Pr(B \mid A)$

14 In a random experiment $\Pr(A) = 0.4$, $\Pr(A \mid B) = 0.6$ and $\Pr(B) = 0.5$. Find:

a $\Pr(A \cap B)$ **b** $\Pr(B \mid A)$

Example 22 **15** The current football fixture has the local team playing at home for 60% of its matches. When it plays at home, the team wins 80% of the time. When it plays away, the team wins only 40% of the time. What percentage of its games does the team play away and win?

16 The probability that a car will need an oil change is 0.15, the probability that it needs a new oil filter is 0.08, and the probability that both the oil and the filter need changing is 0.03. Given that the oil has to be changed, what is the probability that a new oil filter is also needed?

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17 A card is selected from a pack of 52 playing cards. The card is replaced and a second card chosen. Find the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d both cards are picture cards.

18 A card is selected from a pack of 52 playing cards and **not** replaced. Then a second card is chosen. Find the probability that:

a both cards are hearts
b both cards are aces
c the first card is red and the second is black
d both cards are picture cards.

19 A person is chosen at random from the employees of a large company. Let W be the event that the person chosen is a woman, and let A be the event that the person chosen is 25 years or older. Suppose the probability of selecting a woman is $\Pr(W) = 0.652$ and the probability of a woman being 25 years or older is $\Pr(A \mid W) = 0.354$. Find the probability that a randomly chosen employee is a woman aged 25 years or older.

20 In a class of 28 students there are 15 girls. Of the students in the class, six girls and eight boys play basketball. A student is chosen at random from the class. If G represents the event that a girl student is chosen and B represents the event that the student chosen plays basketball, find:

a $\Pr(G)$ **b** $\Pr(B)$ **c** $\Pr(B')$ **d** $\Pr(B \mid G)$
e $\Pr(G \mid B)$ **f** $\Pr(B \mid G')$ **g** $\Pr(B' \cap G')$ **h** $\Pr(B \cap G)$

21 In a recent survey it was found that 85% of the population eats red meat. Of those who eat red meat, 60% preferred lamb. A person is chosen at random from the population. If R represents the event that the person eats red meat and L represents the event that the person prefers lamb, find:

a $\Pr(R)$ **b** $\Pr(L \mid R)$ **c** $\Pr(L \cap R)$ **d** $\Pr(L)$

Example 23 **22** In a senior college, 25% of the Year 11 students and 40% of the Year 12 students would prefer not to wear school uniform. This particular college has 320 Year 11 students and 280 Year 12 students. Find the probability that a randomly chosen student is in Year 11 and is in favour of wearing school uniform. What is the overall percentage of students who are in favour of wearing school uniform?

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23 At a certain school it was found that 35% of the 500 boys and 40% of the 400 girls enjoyed bushwalking. One student from the school is chosen at random. Let G represent the event that the student is a girl, and B represent the event that the student enjoys bushwalking.

a Find, correct to two decimal places:

i $\Pr(G)$ **ii** $\Pr(B \mid G)$ **iii** $\Pr(B \mid G')$

iv $\Pr(B \cap G)$ **v** $\Pr(B \cap G')$

b Find $\Pr(B)$.

c Hence find:

i $\Pr(G \mid B)$ **ii** $\Pr(G \mid B')$

24 In a factory two machines produce a particular circuit board. The older machine produces 480 boards each day, of which an average of 12% are defective. The newer machine produces 620 boards each day, of which an average of 5% are defective. A board is chosen at random and checked. Let N represent the event that the board comes from the newer machine, and D represent the event that the board is defective.

a Find, correct to two decimal places:

i $\Pr(N)$ **ii** $\Pr(D \mid N)$

iii $\Pr(D \mid N')$ **iv** $\Pr(D \cap N)$

v $\Pr(D \cap N')$

b Find $\Pr(D)$.

c Hence find $\Pr(N \mid D)$, correct to two decimal places.

25 Jane has three bags of lollies. In bag 1 there are three mints and three toffees, in bag 2 there are three mints and two toffees, and in bag 3 there are two mints and one toffee. Jane selects a bag at random, and then selects a lolly at random. Find:

a the probability she chooses a mint from bag 1

b the probability she chooses a mint

c the probability that Jane chose bag 1, given that she selects a mint.

26 Assuming a finite sample space, describe the relationship between events A and B if:

a $\Pr(A \mid B) = 1$ **b** $\Pr(A \mid B) = 0$ **c** $\Pr(A \mid B) = \dfrac{\Pr(A)}{\Pr(B)}$

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9G Independent events

Let us again consider the question of the probability that the second toss shows a head, given that the first toss shows a head, when tossing a coin twice. If we define A as the event 'the second toss is a head' and B as the event 'the first toss is a head', then what is $\Pr(A \mid B)$?

Using the definition of conditional probability:

$$\begin{aligned}\Pr(A \mid B) &= \frac{\Pr(A \cap B)}{\Pr(B)} \\ &= \frac{\Pr(\text{both tosses show heads})}{\Pr(\text{first toss shows a head})} \\ &= \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}\end{aligned}$$

That is, the probability of the second toss showing a head does not seem to be affected by the outcome of the first toss. This is an example of independent events.

> Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if
>
> $$\Pr(A \mid B) = \Pr(A)$$

If $\Pr(B) \neq 0$, then the multiplication rule of probability gives

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

Thus, when events A and B are independent, we can equate the two expressions for $\Pr(A \mid B)$ to obtain

$$\Pr(A) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

and therefore

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

In fact, we can also use this final equation as a test for independence:

> Events A and B are independent if and only if
>
> $$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

Notes:

- For events A and B with $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:
  - $\Pr(A \mid B) = \Pr(A)$
  - $\Pr(B \mid A) = \Pr(B)$
  - $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$
- In the special case that $\Pr(A) = 0$ or $\Pr(B) = 0$, the condition $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ holds since both sides are zero, and so we say that A and B are independent.
- Sometimes this definition of independence is referred to as **pairwise independence**.

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Example 25

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Is the regular use of social media independent of the respondent's age?

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

Solution

From the table:

$$\Pr(\text{Age} < 25 \cap \text{Yes}) = \frac{200}{500} = 0.4$$

$$\Pr(\text{Age} < 25) \times \Pr(\text{Yes}) = \frac{240}{500} \times \frac{300}{500} = 0.48 \times 0.6 = 0.288$$

Hence

$$\Pr(\text{Age} < 25 \cap \text{Yes}) \neq \Pr(\text{Age} < 25) \times \Pr(\text{Yes})$$

and therefore these events are not independent.

Example 26

An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{4, 6, 8\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

Solution

Explanation

$\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{2}$, $\Pr(C) = \frac{3}{8}$

a $\Pr(A \cap B) = \frac{1}{4}$ — since $A \cap B = \{1, 3\}$

$\Pr(A) \times \Pr(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

$\therefore \Pr(A) \times \Pr(B) = \Pr(A \cap B)$

Thus A and B are independent — since these two probabilities are equal.

b $\Pr(A \cap C) = \frac{1}{8}$ — since $A \cap C = \{4\}$

$\Pr(A) \times \Pr(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore \Pr(A) \times \Pr(C) \neq \Pr(A \cap C)$

Thus A and C are not independent — since these two probabilities are not equal.

c $\Pr(B \cap C) = 0$ — since $B \cap C = \varnothing$

$\Pr(B) \times \Pr(C) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16}$

$\therefore \Pr(B) \times \Pr(C) \neq \Pr(B \cap C)$

Thus B and C are not independent — since these two probabilities are not equal.

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The concept of mathematical independence is sometimes confused with that of physical independence. If two events are physically independent, then they are also mathematically independent, but the converse is not necessarily true. The following example illustrates this.

Example 27

Suppose we roll a die twice and define the following events:

A = the first roll shows a 4

B = the second roll shows a 4

C = the sum of the numbers showing is at least 10

a Are A and B independent events?

b What about A and C?

Solution

a Since A and B are physically independent, they must also be mathematically independent, but we can also check this directly.

We have

$$\Pr(A) \times \Pr(B) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

If we write the sample space as ordered pairs, in which the first entry is the result of the first throw and the second is the result of the second throw, then

$$\varepsilon = \{(1,1), (1,2), (1,3), \dots, (6,5), (6,6)\}$$

and $n(\varepsilon) = 36$

The event $A \cap B$ corresponds to the outcome $(4, 4)$, and so $n(A \cap B) = 1$.

Thus

$$\Pr(A \cap B) = \frac{1}{36} = \Pr(A) \times \Pr(B)$$

and so A and B are independent.

b We have $C = \{(4,6), (5,5), (5,6), (6,4), (6,5), (6,6)\}$ and so $n(C) = 6$.

Thus

$$\Pr(A) \times \Pr(C) = \frac{1}{6} \times \frac{6}{36} = \frac{1}{36}$$

The event $A \cap C$ corresponds to the outcome $(4, 6)$, and so $n(A \cap C) = 1$.

Thus

$$\Pr(A \cap C) = \frac{1}{36} = \Pr(A) \times \Pr(C)$$

This means that A and C are also independent events.

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Knowing that events are independent means that we can determine the probability of their intersection by multiplying together their individual probabilities. This is illustrated in the following example.

Example 28

Suppose that the probability that a family in a certain town owns a television set (T) is 0.75, and the probability that a family owns a station wagon (S) is 0.25. If these events are independent, find the following probabilities:

a A family chosen at random owns both a television set and a station wagon.

b A family chosen at random owns at least one of these items.

Solution

a The event 'owns both a television set and a station wagon' is represented by $T \cap S$, with

$$\begin{aligned} \Pr(T \cap S) &= \Pr(T) \times \Pr(S) && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.75 \times 0.25 = 0.1875 \end{aligned}$$

b The event 'owns at least one of these items' is represented by $T \cup S$, with

$$\begin{aligned} \Pr(T \cup S) &= \Pr(T) + \Pr(S) - \Pr(T \cap S) && \text{(from the addition rule)} \\ &= 0.75 + 0.25 - 0.75 \times 0.25 && \text{(as } T \text{ and } S \text{ are independent)} \\ &= 0.8125 \end{aligned}$$

Confusion often arises between independent and mutually exclusive events. That two events A and B are mutually exclusive means that $A \cap B = \varnothing$ and hence that $\Pr(A \cap B) = 0$. Thus, if two events are independent, they cannot also be mutually exclusive, unless the probability of at least one of the events is zero.

Section summary

- The probability of an event A occurring when it is known that some event B has already occurred is called **conditional probability** and is written $\Pr(A \mid B)$, where

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad \text{if } \Pr(B) \neq 0$$

- Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

$$\Pr(A \mid B) = \Pr(A)$$

- Events A and B are independent if and only if

$$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

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Exercise 9G

Example 25 **1** 100 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	35	30	65
No	25	10	35
Total	60	40	100

Is support for private individuals carrying guns independent of sex?

2 A group of 500 students were asked whether they would rather spend their recreational time playing sport or listening to music. The results, as well as the sex of the student, are given in the following table.

	Male	Female	Total
Sport	225	150	375
Music	75	50	125
Total	300	200	500

Is preference for playing sport or listening to music independent of sex?

3 An analysis of traffic accidents in a certain city classified the accident as serious or minor, as well as whether the driver was speeding or not.

Type of accident	Speeding		
	Yes	No	Total
Serious	42	61	103
Minor	88	185	273
Total	130	246	376

Is the seriousness of the accident independent of whether the driver was speeding or not?

Example 26 **4** An experiment consists of drawing a number at random from $\{1, 2, 3, \dots, 12\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9, 11\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent?

b Are A and C independent?

c Are B and C independent?

Example 27 **5** A die is thrown and the number uppermost is recorded. Events A and B are defined as 'an even number' and 'a square number' respectively. Show that A and B are independent.

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6 Two events A and B are such that $\Pr(A) = 0.3$, $\Pr(B) = 0.1$ and $\Pr(A \cap B) = 0.1$. Are A and B independent?

7 If A and B are independent events with $\Pr(A) = 0.6$ and $\Pr(B) = 0.7$, find:

a $\Pr(A \mid B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$

8 If A and B are independent events with $\Pr(A) = 0.5$ and $\Pr(B) = 0.2$, find $\Pr(A \cup B)$.

Example 28

9 A man and a woman decide to marry. Assume that the probability that each will have a specific blood group is as follows:

Blood group	O	A	B	AB
Probability	0.5	0.35	0.1	0.05

If the blood group of the husband is independent of that of his wife, find the probability that:

a the husband is group A

b the husband is group A and his wife is group B

c both are group A

d the wife is group AB and her husband is group O.

10 The 165 subjects volunteering for a medical study are classified by sex and blood pressure (high (H), normal (N) and low (L)).

	H	N	L
M	88	22	10
F	11	22	12
Total	99	44	22

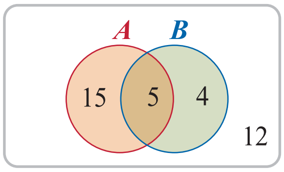
If a subject is selected at random, find:

a $\Pr(N)$ **b** $\Pr(F \cap H)$ **c** $\Pr(F \cup H)$

d $\Pr(F \mid L)$ **e** $\Pr(L \mid F)$

Are F and L independent? Explain.

11 Events A and B are as shown in the Venn diagram. Show that A and B are independent.



12 The probability that a married woman watches a certain television show is 0.4, and the probability that her husband watches the show is 0.5. The television viewing habits of a husband and wife are clearly not independent. In fact, the probability that a married woman watches the show, given that her husband does, is 0.7. Find the probability that:

a both the husband and wife watch the show

b the husband watches the show given that his wife watches it.

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13 The 65 middle managers in a company are classified by age and income as follows:

	Age		
Income	30–39 (T)	40–49 (F)	50–69 (S)
Low (L)	13	4	1
Moderate (M)	8	10	3
High (H)	2	16	8
Total	23	30	12

A middle manager is selected at random from the company. Find:

a $\Pr(L)$ **b** $\Pr(S)$ **c** $\Pr(T)$ **d** $\Pr(M)$

e $\Pr(L \cap F)$ **f** $\Pr(T \cap M)$ **g** $\Pr(L \mid F)$ **h** $\Pr(T \mid M)$

Is income independent of age? Explain your answer.

14 A consumer research organisation has studied the services provided by the 150 TV repair persons in a certain city and their findings are summarised in the following table.

	Good service (G)	Poor service (G')
Factory trained (F)	48	16
Not factory trained (F')	24	62

a One of the TV repairers is randomly selected. Calculate the following probabilities:

i $\Pr(G \mid F)$, the probability that a factory-trained repairer is one who gives good service

ii $\Pr(G \cap F)$, the probability that the repairer is giving good service and is factory trained

iii $\Pr(G \cup F)$, the probability that the repairer is giving good service or is factory trained or both

b Are events G and F independent?

c Are the events G and F mutually exclusive?

9H Solving probability problems using simulation

Simulation is a very powerful and widely used procedure which enables us to find approximate answers to difficult probability questions. It is a technique which imitates the operation of the real-world system being investigated. Some problems are not able to be solved directly and simulation allows a solution to be obtained where otherwise none would be possible. In this section some specific probability problems are looked at which may be solved by using simulation, a valuable and legitimate tool for the statistician.

Suppose, for example, we would like to know the probability that a family of five children will include at least four girls. There are ways of determining this probability exactly, which will be discussed in Chapter 11, but we don't know how to do this yet. What we can do, however, is estimate the probability by simulation.

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Firstly, we need to make some assumptions so we can decide on a suitable model for the simulation. We will assume:

- There is a probability of 0.5 of each child being female.
- The sex of each child is independent of the sex of the other children. That is, the probability of a female child is always 0.5.

Since the probability of a female child is 0.5, a suitable simulation model would be tossing a fair coin. Let a head represent a female child and a tail a male child. A trial consists of tossing the coin five times to represent one complete family of five children, and the result of the trial is the number of heads obtained in the trial.

To estimate the required probability, several trials need to be conducted. How many trials are needed to estimate the probability? As we have already noted in Section 9B, the more repetitions of an experiment the better the estimate of the probability. Initially about 50 trials could be considered.

An example of the results that might be obtained from 10 trials is:

Trial number	Simulation results	Number of heads
1	$T\,H\,H\,T\,T$	2
2	$H\,H\,H\,T\,H$	4
3	$H\,H\,H\,T\,H$	4
4	$H\,T\,T\,T\,H$	2
5	$H\,T\,H\,H\,H$	4
6	$H\,T\,T\,T\,H$	2
7	$T\,T\,H\,H\,H$	3
8	$H\,T\,H\,H\,T$	3
9	$T\,T\,T\,H\,H$	2
10	$H\,H\,T\,T\,T$	2

Continuing in this way, the following results were obtained for 50 trials:

Number of heads	0	1	2	3	4	5
Number of times obtained	1	8	17	13	10	1

The results in the table can be used to estimate the required probability. Since at least four heads were obtained in 11 trials, estimate the probability of at least four female children as $\frac{11}{50}$ or 0.22. Of course, since this probability has been estimated experimentally, repeating the simulations would give a slightly different result, but we would expect to obtain approximately this value most of the time.

Simulation is also widely used to estimate the values of other quantities which are of interest in a probability problem. We may wish to know the average result, the largest result, the number of trials required to achieve a certain result, and so on. An example of this type of problem is given in Example 29.

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Example 29

A pizza shop is giving away football cards with each pizza bought. There are six different cards available, and a fan decides to continue buying the pizzas until all six are obtained. How many pizzas will need to be bought, on average, to obtain the complete set of cards?

Solution

As there are more than two outcomes of interest, a coin is not a suitable simulation model, but a fair six-sided die could be used. Each of the six different cards is represented by one of the six sides of the die. Rolling the die and observing the outcome is equivalent to buying a pizza and noting which card was obtained. This simulation model is based on the following assumptions:

- The six cards all occur with equal frequency.
- The card obtained with one pizza is independent of the cards obtained with the other pizzas.

A trial would consist of rolling the die until all of the six numbers 1, 2, 3, 4, 5 and 6 have been observed, and the result of the trial is the number of rolls necessary to do this. The results of one trial are shown:

5 2 5 2 2 2 3 3 1 2 6 3 5 4

In this instance, 14 pizzas were bought before the whole set was obtained. Of course, we would not expect to buy 14 pizzas every time – this is just the result from one trial. To obtain an appropriate estimate, we would need to conduct several trials.

The following is an example of the results that might be obtained from 50 trials. In each case the number listed represents the number of pizzas that were bought to obtain a complete set of football cards:

14	8	12	11	16	8	8	11	15	26	14	20	11	13	35
23	19	14	10	10	20	9	10	14	29	13	7	15	15	22
9	10	14	16	14	17	12	10	24	13	19	27	31	11	9
16	21	22	8	9										

To estimate the number of pizzas that need to be bought, the average of the numbers obtained in these simulations is calculated. Thus we estimate that, in order to collect the complete set of cards, it would be necessary to buy approximately

$$\frac{14 + 8 + 12 + 11 + 16 + \cdots + 16 + 21 + 22 + 8 + 9}{50} \approx 15 \text{ pizzas}$$

In practice there are situations where coins and dice may not be useful. Other methods of simulation need to be adopted to deal with a wide range of situations. Suppose we wished to determine how many pizzas would need to be bought, on average, to obtain a complete set of eight souvenirs. This time we need to generate random numbers from 1 to 8 and a six-sided die would no longer be appropriate, but there are other methods that could be used.

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We could construct a spinner with eight equal sections marked from 1 to 8, or we could mark eight balls from 1 to 8 and draw them (with replacement) from a bowl, or one of a number of other methods. Generally, when we wish to simulate we use random number generators on a calculator or computer.

Section summary

- Simulation is a simple and legitimate method for finding solutions to problems when an exact solution is difficult, or impossible, to find.
- In order to use simulation to solve a problem, a clear statement of the problem and the underlying assumptions must be made.
- A model must be selected to generate outcomes for a simulation. Possible choices for physical simulation models are coins, dice and spinners. Random number tables, calculators and computers may also be used.
- Each trial should be defined and repeated several times (at least 50).
- The results from all the trials should be recorded and summarised appropriately to provide an answer to a problem.

Exercise 9H

1 Use simulation to estimate the probability that a family with three children have all boys.

2 A teacher gives her class a test consisting of five ‘true or false’ questions. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least three correct.

3 A teacher gives the class a test consisting of 10 multiple-choice questions, each with five alternatives. Use simulation to estimate the probability that a student who guesses the answer to every question gets at least five correct.

4 Use simulation to estimate the number of pizzas we would need to buy if the number of football cards described in Example 29 was extended to 10.

5 Eight players are entered into a tennis tournament.
In round one, every player plays (four matches).
In round two, the four winners from round one play (two matches).
In round three, the two winners from round two play (one match).

a Suppose Shaun has a 50% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.

b Suppose he has a 70% chance of winning each match he plays. Use simulation to determine how many matches he will play, on average, in the tournament.

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Review

AS

Nrich

Chapter summary

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le \Pr(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varepsilon) = 1$
  - $\Pr(\varnothing) = 0$
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$.
- If events A and B are mutually exclusive, then $\Pr(A \cap B) = 0$ and $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

giving $\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$ (the **multiplication rule**)

- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Two events A and B are **independent** if

$$\Pr(A \mid B) = \Pr(A)$$

so whether or not B has occurred has no effect on the probability of A occurring.
- Events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Technology-free questions

1 Two six-sided dice are tossed. Find the probability that:

a the sum of the values of the uppermost faces is 7 **b** the sum is not 7.

2 The probability that a computer chip is operational is 0.993. What is the probability that it is not operational?

3 A whole number between 1 and 300 (inclusive) is chosen at random. Find the probability that the number is:

a divisible by 3 **b** divisible by 4 **c** divisible by 3 or by 4.

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Review

4 A drawer contains 30 red socks and 20 blue socks.

a If a sock is chosen at random, its colour noted, the sock replaced and a second sock withdrawn, what is the probability that both socks are red?

b If replacement doesn't take place, what is the probability that both socks are red?

5 Box A contains five pieces of paper numbered 1, 3, 5, 7, 9.
Box B contains three pieces of paper numbered 1, 4, 9.
One piece of paper is removed at random from each box. Find the probability that the two numbers obtained have a sum that is divisible by 3.

6 A three-digit number is formed by arranging the digits 1, 5 and 6 in a random order.

a List the sample space.

b Find the probability of getting a number larger than 400.

c What is the probability that an even number is obtained?

7 A letter is chosen at random from the word STATISTICIAN.

a What is the probability that it is a vowel?

b What is the probability that it is a T?

8 Ivan and Joe are chess players. In any game the probabilities of Ivan beating Joe, Joe beating Ivan or the game resulting in a draw are 0.6, 0.1 or 0.3 respectively. They play a series of three games. Calculate the probability that:

a they win alternate games, with Ivan winning the first game

b the three games are drawn

c exactly two of the games are drawn

d Joe does not win a game.

9 A die with two red faces and four blue faces is thrown three times. Each face is equally likely to face upward. Find the probability of obtaining the following:

a three red faces

b a blue on the first, a red on the second and a blue on the third

c exactly one red face

d at least two blue faces

10 If $\Pr(A) = 0.6$ and $\Pr(B) = 0.5$, can A and B be mutually exclusive? Why or why not?

11 Events A and B are such that $\Pr(A) = 0.6$, $\Pr(B) = 0.5$ and $\Pr(A' \cap B) = 0.4$. Construct a probability table and use it to find:

a $\Pr(A \cap B')$

b $\Pr(A' \cap B')$

c $\Pr(A \cup B)$

12 In Minh's class 18 students study mathematics, 14 study music, and 7 study both mathematics and music.

a Given that a student in his class studies mathematics, what is the probability that they also study music?

b Given that a student in his class studies music, what is the probability that they also study mathematics?

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13 Given $\Pr(B) = \frac{1}{3}$, $\Pr(A \mid B) = \frac{2}{3}$ and $\Pr(A \mid B') = \frac{3}{7}$, determine:

a $\Pr(A \cap B')$ **b** $\Pr(A)$ **c** $\Pr(B' \mid A)$

14 A group of executives is classified according to body weight and incidence of hypertension. The proportion of the various categories is as shown.

	Overweight	Normal weight	Underweight
Hypertensive	0.10	0.08	0.02
Not hypertensive	0.15	0.45	0.20

a What is the probability that a person selected at random from this group will have hypertension?

b A person, selected at random from this group, is found to be overweight. What is the probability that this person is also hypertensive?

15 Given an experiment such that $\Pr(A) = 0.3$, $\Pr(B) = 0.6$ and $\Pr(A \cap B) = 0.2$, find:

a $\Pr(A \cup B)$ **b** $\Pr(A' \cap B')$ **c** $\Pr(A \mid B)$ **d** $\Pr(B \mid A)$

16 For a finite sample space, explain the implication of each of the following in terms of the relationship between events A and B:

a $\Pr(A \mid B) = 1$ **b** $\Pr(A \mid B) = 0$ **c** $\Pr(A \mid B) = \Pr(A)$

Multiple-choice questions

1 If the probability of Chris scoring 50 or more marks in the exam is 0.7, then the probability he scores less than 50 marks is

A 0 **B** 0.3 **C** 0.4 **D** 0.7 **E** 0.8

2 A spinner is coloured red, yellow, blue and green. When spun the probability that it lands on red is 0.1, yellow is 0.2 and blue is 0.4. What is the probability that it lands on green?

A 0.1 **B** 0.2 **C** 0.3 **D** 0.4 **E** 0.5

3 Phillip is making a sign, and has cut the letters of the word THEATRETTE out of wood and placed them in his toolbox. If a letter is selected at random from the toolbox, then the probability that it is a T is

A $\frac{2}{5}$ **B** $\frac{3}{10}$ **C** $\frac{1}{5}$ **D** $\frac{1}{6}$ **E** $\frac{3}{5}$

4 Of a group of 25 people in a restaurant, three chose a vegetarian meal, five chose fish, ten chose beef and the rest chose chicken for their main course. What is the probability that a randomly chosen diner chose chicken?

A $\frac{3}{25}$ **B** $\frac{6}{25}$ **C** $\frac{7}{25}$ **D** $\frac{2}{5}$ **E** $\frac{7}{18}$

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5 Suppose that a card is chosen at random from a well-shuffled deck of 52 playing cards. What is the probability that the card is a spade or a jack?

A $\frac{1}{4}$ **B** $\frac{1}{13}$ **C** $\frac{17}{52}$ **D** $\frac{4}{13}$ **E** $\frac{9}{26}$

6 A square has side length of 4 metres. Inside the square is a circle of radius 1.5 metres. If a dart thrown at the square is equally likely to land at any point inside the square, then the probability that it will land outside the circle is closest to

A 0.442 **B** 0.295 **C** 0.558 **D** 0.250 **E** 0.375

7 An experiment consists of tossing a coin and then rolling a fair six-sided die. What is the probability of observing a head and a 'six'?

A $\frac{1}{2}$ **B** $\frac{1}{4}$ **C** $\frac{1}{35}$ **D** $\frac{1}{12}$ **E** $\frac{7}{12}$

8 If A and B are events such that $\Pr(A) = 0.35$, $\Pr(A \cap B) = 0.18$ and $\Pr(B) = 0.38$, then $\Pr(A \cup B)$ is equal to

A 0.73 **B** 0.133 **C** 0.15 **D** 0.21 **E** 0.55

9 If A and B are events such that $\Pr(A) = 0.47$, $\Pr(B) = 0.28$ and $B \subseteq A$, then $\Pr(A \cup B)$ is equal to

A 0.47 **B** 0.75 **C** 0.62 **D** 0.13 **E** 0

10 Suppose that 57% of the swimmers in a club are female (F), that 32% of the swimmers in the club swim butterfly (B), and that 11% of the swimmers in the club are female and swim butterfly. Which of the following probability tables correctly summarises this information?

A

	B	B'	
F	0.11	0.21	0.32
F'	0.46	0.22	0.68
	0.57	0.43	1

B

	B	B'	
F	0.11	0.46	0.57
F'	0.21	0.22	0.43
	0.32	0.68	1

C

	B	B'	
F	0.04	0.53	0.57
F'	0.28	0.15	0.43
	0.32	0.68	1

D

	B	B'	
F	0.18	0.39	0.57
F'	0.14	0.29	0.43
	0.32	0.68	1

E

	B	B'	
F	0.11	0.32	0.43
F'	0.21	0.36	0.57
	0.32	0.68	1

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11 The following information applies to a particular class:

- The probability that a student studies mathematics is $\frac{2}{3}$.
- The probability that a student studies German is $\frac{3}{10}$.
- The probability that a student studies mathematics and does not study German is $\frac{7}{15}$.

The probability that a randomly chosen student does not study either mathematics or German is

A $\dfrac{4}{5}$ **B** $\dfrac{7}{30}$ **C** $\dfrac{7}{15}$ **D** $\dfrac{7}{10}$ **E** $\dfrac{1}{3}$

12 In Imogen's class 15 students play tennis, 14 play basketball and 7 play both. The probability that a randomly chosen student plays basketball, given that they play tennis, is

A $\dfrac{14}{15}$ **B** $\dfrac{7}{15}$ **C** $\dfrac{7}{29}$ **D** $\dfrac{15}{29}$ **E** $\dfrac{1}{2}$

13 The following data was derived from accident records on a highway noted for its above-average accident rate.

Type of accident	Probable cause: Speed	Alcohol	Reckless driving	Other	Total
Fatal	42	61	22	12	137
Non-fatal	88	185	98	60	431
Total	130	246	120	72	568

The probability that the accident is not fatal, given that reckless driving is the cause, is closest to

A 0.82 **B** 0.17 **C** 0.21 **D** 0.23 **E** 0.29

14 If for two events A and B, $\Pr(A) = \dfrac{3}{8}$, $\Pr(B) = \dfrac{4}{7}$ and $\Pr(A \cap B) = \dfrac{8}{21}$, then $\Pr(A \mid B)$ is equal to

A $\dfrac{3}{8}$ **B** $\dfrac{3}{14}$ **C** $\dfrac{63}{64}$ **D** $\dfrac{21}{32}$ **E** $\dfrac{2}{3}$

The following information relates to Questions 15 and 16.

The probability that Miller goes to the gym on Monday is 0.6. If he goes to the gym on Monday, then the probability that he will go again on Tuesday is 0.7. If he doesn't go to the gym on Monday, then the probability that Miller will go on Tuesday is only 0.4.

15 The probability that Miller goes to the gym on both Monday and Tuesday is

A 0.36 **B** 0.24 **C** 0.42 **D** 0.16 **E** 0.28

16 The probability that Miller goes to the gym on Tuesday is

A 0.58 **B** 0.42 **C** 0.16 **D** 0.84 **E** 0.32

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17 If A and B are independent events such that $\Pr(A) = 0.35$ and $\Pr(B) = 0.46$, then $\Pr(A \cup B)$ is equal to

A 0.810 **B** 0.649 **C** 0.161
D 0.110 **E** cannot be determined

18 The primary cooling unit in a nuclear power plant has a reliability of 0.95. There is also a back-up cooling unit to substitute for the primary unit when it fails. The reliability of the back-up unit is 0.85. The cooling system of the plant is considered reliable if either one of the systems is working. Assuming that the two systems are independent, the reliability of the cooling system of the power plant is:

A 0.95 **B** 0.85 **C** 0.8075 **D** 0.9925 **E** 1.0

Extended-response questions

1 To have a stage production ready for opening night there are three tasks which must be done and, as the same people are involved in each task, these must be done in sequence. The following probabilities are estimated for the duration of the activities:

Task	6 days	7 days	8 days
Build scenery	0.3	0.3	0.4
Paint scenery	0.6	0.3	0.1
Print programs	0.4	0.4	0.2

a What is the probability that the building and painting of the scenery will together take exactly 15 days?

b What is the probability that all three tasks will together take exactly 22 days?

2 Two bowls each contain eight pieces of fruit. In bowl A there are five oranges and three apples; in bowl B there is one orange and seven apples.

a For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, if the first piece of fruit is not replaced before the second piece of fruit is chosen.

b For each bowl, find the probability that two pieces of fruit chosen at random will both be apples, when the first piece of fruit is replaced before the second is chosen.

c One bowl is chosen at random and from it two pieces of fruit are chosen at random without replacement. If both pieces of fruit are apples, find the probability that bowl A was chosen.

d One bowl is chosen at random and from it two pieces of fruit are chosen at random, the first piece of fruit being replaced before the second is chosen. If both pieces of fruit are apples, find the probability that bowl A was chosen.

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3 Rachel is a keen runner. She is supposed to attend running training five days per week. Rachel finds that if she runs one day, the probability that she will run again the next day is $\frac{4}{5}$, and if she does not run one day, the probability that she will not run the next day is $\frac{3}{4}$. Suppose that Rachel runs one day:

a What is the probability that she runs the next day?

b What is the probability that she runs the day after that?

c What is the probability that she runs exactly twice in the next three days?

4 Sixteen players are entered in a tennis tournament.

- In round one, every player plays (eight matches).
- In round two, the eight winners from round one play (four matches).
- In round three, the four winners from round two play (two matches).
- In round four, the two winners from round three play (one match).

Use simulation to estimate how many matches a player will play, on average:

a if the player has a 50% chance of winning each match

b if the player has a 70% chance of winning each match.

5 Consider a finals series of games in which the top four teams play off as follows:

Game 1 Team A vs Team B

Game 2 Team C vs Team D

Game 3 Winner of game 2 plays loser of game 1

Game 4 Winner of game 3 plays winner of game 1

The winner of game 4 is then the winner of the series.

a Assuming all four teams are equally likely to win any game, use simulation to model the series.

b Use the results of the simulation to estimate the probability that each of the four teams wins the series.

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