5

Functions and relations

Objectives

- ▶ To understand and use the **notation of sets**, including the symbols $\in$, $\subseteq$, $\cap$, $\cup$, $\varnothing$ and $\setminus$.
- ▶ To use the notation for **sets of numbers**.
- ▶ To understand the concepts of **relation** and **function**.
- ▶ To find the **domain** and **range** of a given relation.
- ▶ To decide whether or not a given function is **one-to-one**.
- ▶ To find the **implied (maximal) domain** of a function.
- ▶ To work with **restrictions of a function** and **piecewise-defined functions**.
- ▶ To find the **inverse** of a one-to-one function.
- ▶ To apply a knowledge of functions to solving problems.

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

In Chapters 2 and 3 we looked at linear polynomials and quadratic polynomials, and in Chapter 4 we studied rectangular hyperbolas, truncus graphs, square-root graphs and circles.* These are all examples of relations. You will meet them all again in this chapter, but using a new notation which will be carried through into the following chapters of this book.

* Circles are listed in the 10A Victorian Curriculum, but not explicitly listed in the VCAA study design for Mathematical Methods Units 1 & 2. This chapter includes some simple cases of circles and semicircles to provide important examples of relations.

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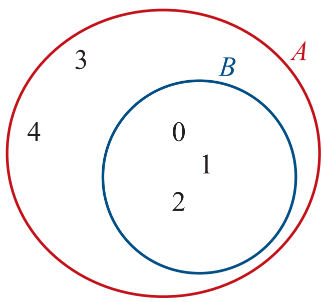
5A Set notation and sets of numbers

Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all the set notation you will need.

- A **set** is a collection of objects.
- The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'. For example: $2 \in$ set of even numbers.
- If x is **not an element** of A, we write $x \notin A$. For example: $2 \notin$ set of odd numbers.
- Set B is called a **subset** of set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A'.

For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then B is a subset of A, as illustrated in the diagram opposite.



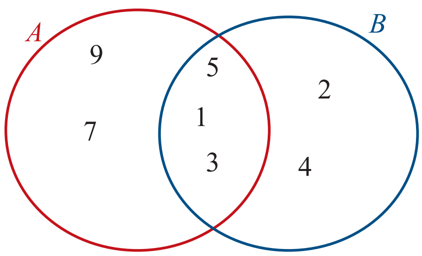
This diagram is called a **Venn diagram**. Venn diagrams are revisited in Chapter 9.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \varnothing$. The set $\varnothing$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.

For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:

$$A \cap B = \{1, 3, 5\}$$

$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$



Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$ **b** $A \cup B$

Solution	Explanation
a $A \cap B = \{3, 7\}$	The elements 3 and 7 are common to sets A and B.
b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$	The set $A \cup B$ contains all elements that belong to A or B (or both).

Note: In Example 1, we have $3 \in A$ and $5 \notin A$ and $\{2, 3\} \subseteq A$.

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Set difference

Finally we introduce the **set difference** of two sets A and B:

$$A \setminus B = \{ x : x \in A,\ x \notin B \}$$

The set $A \setminus B$ contains the elements of A that are not elements of B.

Example 2

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \setminus B$ **b** $B \setminus A$

Solution	Explanation
a $A \setminus B = \{1, 2, 3, 7\} \setminus \{3, 4, 5, 6, 7\}$ $= \{1, 2\}$	The elements 1 and 2 are in A but not in B.
b $B \setminus A = \{3, 4, 5, 6, 7\} \setminus \{1, 2, 3, 7\}$ $= \{4, 5, 6\}$	The elements 4, 5 and 6 are in B but not in A.

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called **natural numbers**, and the elements of $\{\dots, -2, -1, 0, 1, 2, \dots\}$ are called **integers**.

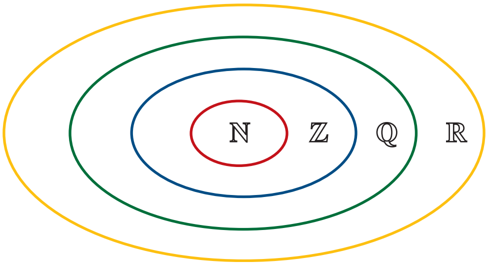
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of natural numbers is denoted by $\mathbb{N}$.
- The set of integers is denoted by $\mathbb{Z}$.
- The set of rational numbers is denoted by $\mathbb{Q}$.
- The set of real numbers is denoted by $\mathbb{R}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{ x : \_\_\_ \}$. Thus, for example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ x : x \neq 0 \}$ is the set of all real numbers other than 0
- $\{ 2n : n \in \mathbb{N} \}$ is the set of all even natural numbers
- $\{ 2n - 1 : n \in \mathbb{N} \}$ is the set of all odd natural numbers.

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Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 3.

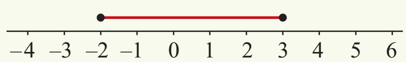
Example 3

Illustrate each of the following intervals of real numbers:

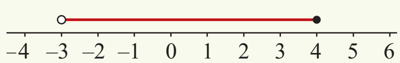
a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-2, 4)$ **d** $(-3, \infty)$ **e** $(-\infty, 5]$

Solution

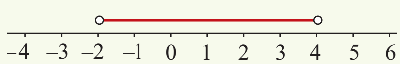
a $[-2, 3]$



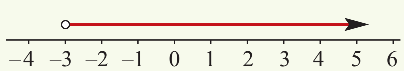
b $(-3, 4]$



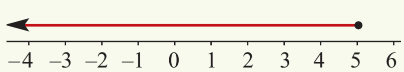
c $(-2, 4)$



d $(-3, \infty)$



e $(-\infty, 5]$



Explanation

a The square brackets indicate that the endpoints are included; this is shown with closed circles.

b The round bracket indicates that the left endpoint is not included; this is shown with an open circle. The right endpoint is included.

c Both brackets are round; the endpoints are not included.

d The symbol ∞ indicates that the interval continues indefinitely (i.e. forever) to the right; it is read as ‘infinity’. The left endpoint is not included.

e The symbol $-\infty$ indicates that the interval continues indefinitely (i.e. forever) to the left; it is read as ‘negative infinity’. The right endpoint is included.

Note: The ‘closed’ circle (●) indicates that the number is included.
The ‘open’ circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

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Summary 5A

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- Intersection The set $A \cap B$ contains the elements in common to A and B.
- Union The set $A \cup B$ contains the elements that are in A or in B (or in both).
- Set difference The set $A \setminus B$ contains the elements of A that are not in B.
- If two sets A and B have no elements in common, we say that A and B are **disjoint** and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- Sets of numbers
  - Natural numbers: $\mathbb{N}$
  - Integers: $\mathbb{Z}$
  - Rational numbers: $\mathbb{Q}$
  - Real numbers: $\mathbb{R}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Skill-sheet

Exercise 5A

Example 1

1 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$ and $C = \{1, 7, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$

d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

Example 2

2 For $A = \{1, 2, 3, 5, 7, 11, 15\}$, $B = \{7, 11, 25, 30, 32\}$ and $C = \{1, 7, 11, 25, 30\}$, find:

a $A \setminus B$ **b** $B \setminus A$ **c** $A \setminus C$ **d** $C \setminus A$

Example 3

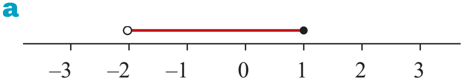
3 Illustrate each of the following intervals on a number line:

a $[-3, 4)$ **b** $(-\infty, 3]$ **c** $[-2, -1]$

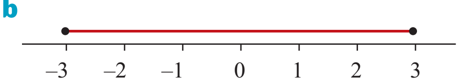
d $(-2, \infty)$ **e** $(-2, 3)$ **f** $(-2, 4]$

4 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.

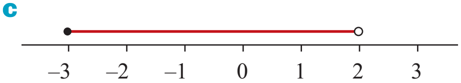
a



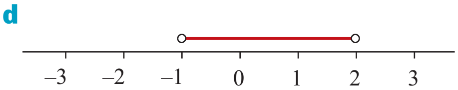
b



c



d



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5 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{x : -1 \le x \le 2\}$ **b** $\{x : -4 < x \le 2\}$ **c** $\{y : 0 < y < \sqrt{2}\}$

d $\left\{y : -\frac{\sqrt{3}}{2} < y \le \frac{1}{\sqrt{2}}\right\}$ **e** $\{x : x > -1\}$ **f** $\{x : x \le -2\}$

g $\mathbb{R}$ **h** $\mathbb{R}^+ \cup \{0\}$ **i** $\mathbb{R}^- \cup \{0\}$

6 For $B = \{7, 11, 25, 30, 32\}$, find:

a $(-2, 10] \cap B$ **b** $(3, \infty) \cap B$ **c** $(2, \infty) \cup B$ **d** $(25, \infty) \cap B$

7 For each of the following, use one number line on which to represent the sets:

a $[-2, 5]$, $[3, 4]$, $[-2, 5] \cap [3, 4]$ **b** $[-2, 5]$, $\mathbb{R} \setminus [-2, 5]$

c $[3, \infty)$, $(-\infty, 7]$, $[3, \infty) \cap (-\infty, 7]$ **d** $[-2, 3]$, $\mathbb{R} \setminus [-2, 3]$

8 Write each of the following sets as a union of two intervals:

a $\mathbb{R} \setminus \{-2\}$ **b** $\mathbb{R} \setminus \{3\}$ **c** $\mathbb{R} \setminus \{4\}$

9 Illustrate each of these sets on a number line:

a $[-3, 2] \cup [4, 8]$ **b** $(-\infty, 2] \cup [4, \infty)$ **c** $(-\infty, -3) \cup (0, \infty)$

d $(-5, -2] \cup (2, 6]$ **e** $(-\infty, 2) \cup (2, \infty)$ **f** $(-\infty, -3) \cup (-3, \infty)$

10 Describe each of the following intersections of intervals as simply as possible:

a $(-\infty, -3) \cap (-6, \infty)$ **b** $(-\infty, 1) \cap (4, \infty)$ **c** $(-\infty, 0] \cap [-6, \infty)$

d $[-3, 2] \cap [-1, 8]$ **e** $[-3, 1] \cap [1, 8]$ **f** $(-\infty, -1] \cap (-10, \infty)$

5B Relations, domain and range

In previous chapters we have looked at how to sketch the graphs of various mathematical relations. We will now look at this aspect of representing relations in a more formal way.

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs. The following are examples of relations:
  - **a** $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$
  - **b** $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$
- Every relation determines two sets:
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

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Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The domain is the set $\{1, 2, 3, 4\}$ and the range is the set $\{2, 3, 4, 5\}$.

Representing relations

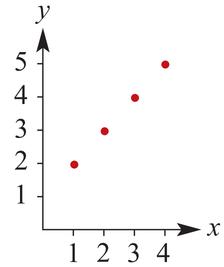
Graphing relations

We can represent a relation as a graph on a set of Cartesian axes.

On the right is the graph of the relation

$$\{ (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \}$$

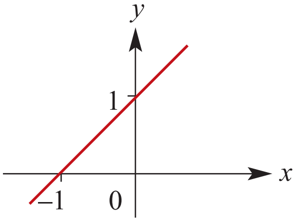
Note that we only graph the individual points of this relation.



If the domain of the relation is the set of real numbers, $\mathbb{R}$, then there are infinitely many points. For example, the graph of

$$\{ (x, y) : y = x + 1,\ x \in \mathbb{R} \}$$

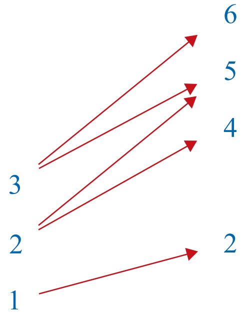
is a continuous straight line.



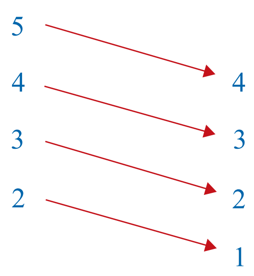
Arrow diagrams

A relation may also be represented by an arrow diagram.

This diagram represents the relation $\{(3, 6), (3, 5), (2, 5), (2, 4), (1, 2)\}$:



This diagram represents the relation $\{(5, 4), (4, 3), (3, 2), (2, 1)\}$:



- A relation may be written as:
  - a listed set of ordered pairs (not always convenient or possible)
  - a rule with a specified or implied domain.
- A relation may be represented by a graph or an arrow diagram.

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Example 4

Sketch a graph of each of the following relations and state its domain and range:

a $\{(-2,-1),(-1,-1),(-1,1),(0,1),(1,-1)\}$

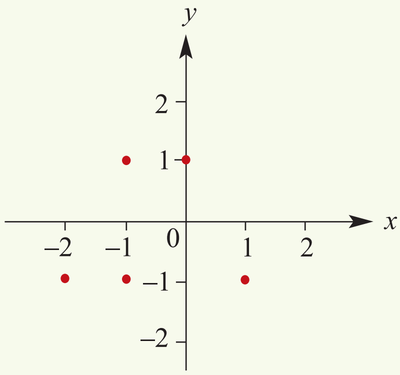
b $\{(x,y) : x^2 + y^2 = 1,\ x \in [-1,1]\}$

c $\{(x,y) : 2x + 3y = 6,\ x \geq 0\}$

d $\{(x,y) : y = 2x - 1,\ x \in [-1,2]\}$

Solution

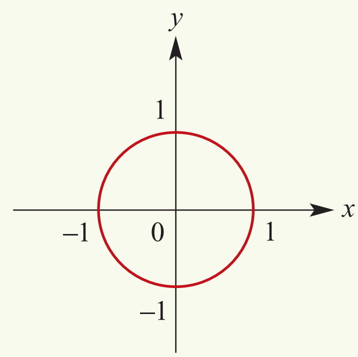
a



Domain = $\{-2,-1,0,1\}$

Range = $\{-1,1\}$

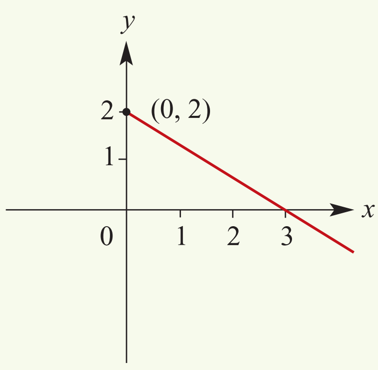
b



Domain = $\{x : -1 \leq x \leq 1\}$

Range = $\{y : -1 \leq y \leq 1\}$

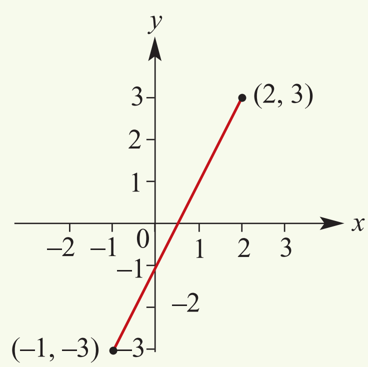
c



Domain = $\mathbb{R}^+ \cup \{0\}$

Range = $(-\infty, 2]$

d



Domain = $[-1,2]$

Range = $[-3,3]$

Often set notation is not used when describing a relation. For example:

- $\{(x,y) : y = x^2\}$ is written as $y = x^2$
- $\{(x,y) : y = x + 1\}$ is written as $y = x + 1$.

This has been the case in your previous considerations of relations.

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Note: In order to determine the range of a relation it is necessary to consider the graph. This strategy is used in the following examples.

Example 5

For each of the following, complete the square, sketch the graph and state the range. The domain is $\mathbb{R}$.

a $y = x^2 - 4x + 5$

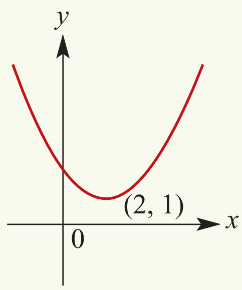
b $y = -x^2 + 4x - 5$

Solution

a $y = x^2 - 4x + 5$

$\quad = (x - 2)^2 + 1$

The range is $[1, \infty)$.



Explanation

Complete the square:

$x^2 - 4x + 5 = x^2 - 4x + 4 - 4 + 5$

$\qquad = (x - 2)^2 + 1$

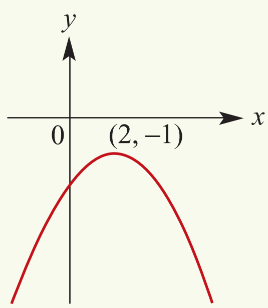
The vertex is at (2, 1).

The minimum y-value is 1.

b $y = -x^2 + 4x - 5$

$\quad = -(x - 2)^2 - 1$

The range is $(-\infty, -1]$.



Complete the square:

$-x^2 + 4x - 5 = -(x^2 - 4x + 5)$

$\qquad = -(x^2 - 4x + 4 + 1)$

$\qquad = -((x - 2)^2 + 1)$

$\qquad = -(x - 2)^2 - 1$

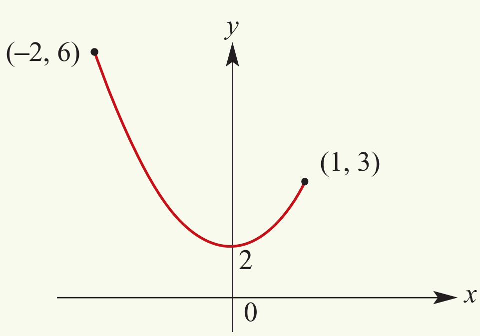
The vertex is at (2, −1).

The maximum y-value is −1.

Example 6

Sketch the graph of the relation $y = x^2 + 2$ for $x \in [-2, 1]$ and state the range.

Solution



The range is $[2, 6]$.

Explanation

Note that the range is not determined by considering the endpoints alone. The minimum y-value is 2, not 3.

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Implied (maximal) domain

When the rule for a relation is written and no domain is specified, then it is understood that the domain taken is the largest for which the rule has meaning. This domain is called the **maximal** or **implied domain**.

For example, the implied domain of $y = x^2$ is $\mathbb{R}$, and the implied domain of $x^2 + y^2 = 1$ is $[-1, 1]$. This concept is considered again in Section 5D.

Example 7

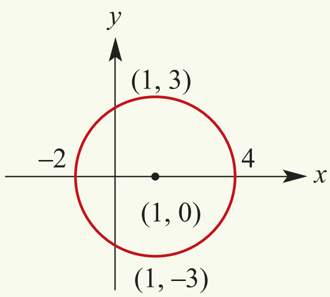
For each of the following relations, state the implied domain and the range:

a $(x - 1)^2 + y^2 = 9$

b $y^2 = x + 2$

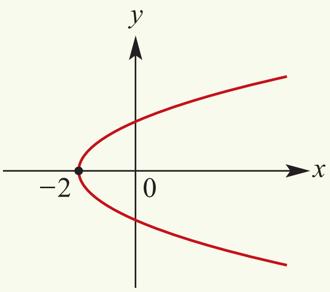
Solution

a This relation is a circle with centre $(1, 0)$ and radius 3.



The implied domain is $[-2, 4]$ and the range is $[-3, 3]$.

b This relation is a parabola with vertex $(-2, 0)$.



The implied domain is $[-2, \infty)$ and the range is $\mathbb{R}$.

Summary 5B

- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example:

$$\{ (x, y) : y = x + 1, \ x \in \mathbb{R}^+ \cup \{0\} \}$$

- For a relation described by a rule with y in terms of x, the domain is the x-values and the range is the y-values.
- The **maximal** or **implied domain** is the largest domain for which the rule of the relation has meaning.

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Skill-sheet

Exercise 5B

Example 4

1 Sketch a graph of each of the following relations and state its domain and range:

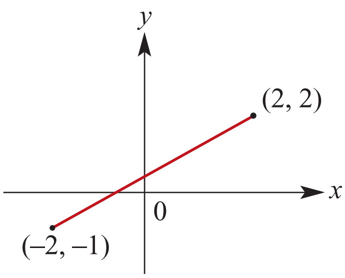
a $\{(-3,-4),(-1,-1),(-6,7),(1,5)\}$ **b** $\{(-4,1),(-4,-1),(-6,7),(-6,8)\}$

c $\{(x,y) : x^2+y^2=4,\ x \in [-2,2]\}$ **d** $\{(x,y) : 3x+2y=12,\ x \geq 0\}$

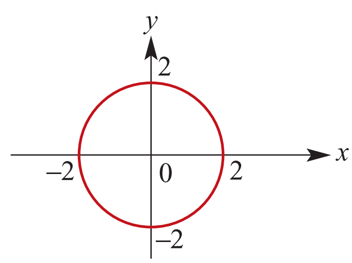
e $\{(x,y) : x-y=4,\ x \in [-1,2]\}$ **f** $\{(x,y) : y=2x+3,\ x \in [-4,1]\}$

2 State the domain and range for the relations represented by each of the following graphs:

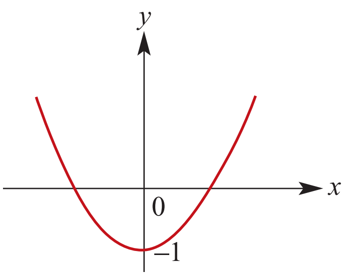
a



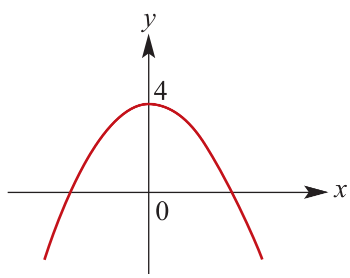
b



c



d



Example 5

3 For each of the following, complete the square, sketch the graph and state the range:

a $y=x^2+6x+10$ **b** $y=-x^2-4x-6$ **c** $y=2x^2-4x+6$

4 Sketch the graphs of each of the following and state the range of each:

a $y=x+1,\ x \in [2,\infty)$ **b** $y=-x+1,\ x \in [2,\infty)$

c $y=2x+1,\ x \in [-4,\infty)$ **d** $y=3x+2,\ x \in (-\infty,3)$

e $y=x+1,\ x \in (-\infty,3]$ **f** $y=-3x-1,\ x \in [-2,6]$

g $y=-3x-1,\ x \in [-5,-1]$ **h** $y=5x-1,\ x \in (-2,4)$

Example 6

5 Sketch the graphs of each of the following and state the range of each:

a $y=x^2+3,\ x \in [-1,1]$ **b** $y=x^2+4,\ x \in [-2,1]$

c $y=x^2-4,\ x \in [-1,2]$ **d** $y=2x^2+1,\ x \in [-2,3]$

6 Sketch the graphs of each of the following relations, stating the range of each:

a $\{(x,y) : y=x^2+1\}$ **b** $\{(x,y) : y=x^2+2x+1\}$

c $\{(x,y) : y=4-x^2,\ x \in [-2,2]\}$ **d** $\{(x,y) : y=x^2+2x+3\}$

e $\{(x,y) : y=-x^2+2x+3\}$ **f** $\{(x,y) : y=x^2-2,\ x \in [-1,2]\}$

g $\{(x,y) : y=2x^2-3x+6\}$ **h** $\{(x,y) : y=6-3x+x^2\}$

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Example 7

7 Sketch the graphs of each of the following relations, stating the implied domain and range of each:

a $x^2 + y^2 = 9$ **b** $y^2 = x$ **c** $y^2 = x - 1$ **d** $y^2 = x + 1$

8 Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $y = \dfrac{2}{2x-5} + 3$ **b** $y = \sqrt{2x-5}$ **c** $y = \sqrt{5-2x}$

d $y = \dfrac{2}{(2x-5)^2}$ **e** $y = (4-2x)^{\frac{1}{2}}$ **f** $(y-3)^2 = x - 1$

***9** *This question involves more challenging examples of circles and semicircles.*
Sketch the graphs of each of the following relations, stating the maximal domain and range of each:

a $(x-2)^2 + (y-3)^2 = 16$ **b** $(2x-1)^2 + (2y-4)^2 = 1$

c $y = \sqrt{25 - x^2}$ **d** $y = -\sqrt{25 - x^2}$

e $y = \sqrt{4 - (x-5)^2}$ **f** $y = -\sqrt{25 - (x-2)^2}$

5C Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 8

Which of the following sets of ordered pairs defines a function?

a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$

b $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a $\{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$ is a function, because for each x-value there is only one y-value.

b $\{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$ is *not* a function, because there is an x-value with two different y-values. The relation contains two ordered pairs, $(-4, 1)$ and $(-4, -1)$, with the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

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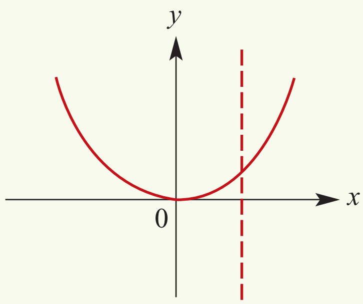
Example 9

a Is $y = x^2$ a function? State the maximal domain and range of $y = x^2$.

b Is $x^2 + y^2 = 4$ a function? State the maximal domain and range of $x^2 + y^2 = 4$.

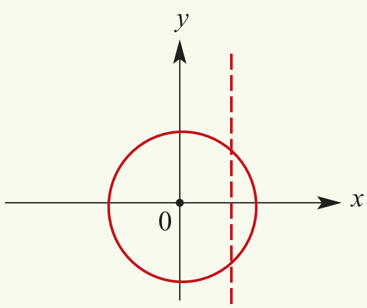
Solution

a



The vertical-line test shows that $y = x^2$ is a function. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+ \cup \{0\}$.

b



The vertical-line test shows that $x^2 + y^2 = 4$ is *not* a function. The maximal domain is $[-2, 2]$ and the range is $[-2, 2]$.

Explanation

For each x-value there is only one y-value.

The ordered pairs of the relation are all of the form (a, a^2).

Note that $(\sqrt{2}, \sqrt{2})$ and $(\sqrt{2}, -\sqrt{2})$ are ordered pairs of the relation.

There is an x-value with more than one y-value.

Function notation

Functions are usually denoted with lowercase letters such as f, g, h.

If f is a function, then for each x in the domain of f there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

Since the y-value obtained is a *function* of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.

For example, instead of $y = 2x + 1$ we can write $f(x) = 2x + 1$. Then $f(2)$ means the y-value obtained when $x = 2$.

e.g.
$$f(2) = 2(2) + 1 = 5$$
$$f(-4) = 2(-4) + 1 = -7$$
$$f(a) = 2a + 1$$

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By incorporating this notation, we have an alternative way of writing functions:

- For the function $\{(x, y) : y = x^2\}$ with domain $\mathbb{R}$, we write $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.
- For the function $\{(x, y) : y = 2x - 1,\ x \in [0, 4]\}$, we write $f : [0, 4] \to \mathbb{R}$, $f(x) = 2x - 1$.
- For the function $\left\{(x, y) : y = \frac{1}{x}\right\}$ with domain $\mathbb{R} \setminus \{0\}$, we write $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$.

If the domain is $\mathbb{R}$, we often just write the rule. For example: $f(x) = x^2$.

Note that in using the notation $f : X \to Y$, the set X is the domain but Y is not necessarily the range. It is a set that contains the range and is called the **co-domain**. With this notation for functions, we write the domain of f as **dom *f*** and the range of f as **ran *f***.

A function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a$ is called a **constant function**. For such a function f, we have dom $f = \mathbb{R}$ and ran $f = \{a\}$. For example, let $f(x) = 7$. Then dom $f = \mathbb{R}$ and ran $f = \{7\}$.

A function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + c$ is called a **linear function**. For example, let $f(x) = 3x + 1$. Then dom $f = \mathbb{R}$ and ran $f = \mathbb{R}$. Note that if the domain of a linear function is $\mathbb{R}$ and $m \neq 0$, then the range is $\mathbb{R}$.

Example 10

Rewrite each of the following using the $f : X \to Y$ notation:

a $\{(x, y) : y = -3x + 2\}$

b $\{(x, y) : y = -2x + 5,\ x \geq 0\}$

c $y = 5x^2 + 6,\ -1 \leq x \leq 2$

d $y = \dfrac{1}{(x-2)^2},\ x \neq 2$

Solution

a $f : \mathbb{R} \to \mathbb{R}$, $f(x) = -3x + 2$

b $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f(x) = -2x + 5$
or $f : [0, \infty) \to \mathbb{R}$, $f(x) = -2x + 5$

c $f : [-1, 2] \to \mathbb{R}$, $f(x) = 5x^2 + 6$

d $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = \dfrac{1}{(x-2)^2}$

Example 11

If $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x - 1)$ **d** $f\left(\frac{1}{a}\right),\ a \neq 0$

Solution

a $f(3) = 2(3)^2 + 3 = 21$

b $f(-2) = 2(-2)^2 - 2 = 6$

c
$$\begin{aligned} f(x-1) &= 2(x-1)^2 + x - 1 \\ &= 2(x^2 - 2x + 1) + x - 1 \\ &= 2x^2 - 3x + 1 \end{aligned}$$

d
$$\begin{aligned} f\left(\frac{1}{a}\right) &= 2\left(\frac{1}{a}\right)^2 + \frac{1}{a} \\ &= \frac{2}{a^2} + \frac{1}{a} \\ &= \frac{2 + a}{a^2} \end{aligned}$$

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Example 12

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$ and $f(t)$.

b Find the value of x for which $f(x) = 6$.

c Find the value of x for which $f(x) = 0$.

d For what values of t is $f(t) = t$?

e For what values of x is $f(x) \ge x$?

f For what values of x is $f(x) \le 3x$?

Solution

a $f(2) = 2(2) - 4 = 0$

$f(t) = 2t - 4$

b
$$\begin{aligned} f(x) &= 6 \\ 2x - 4 &= 6 \\ 2x &= 10 \\ \therefore \quad x &= 5 \end{aligned}$$

c
$$\begin{aligned} f(x) &= 0 \\ 2x - 4 &= 0 \\ 2x &= 4 \\ \therefore \quad x &= 2 \end{aligned}$$

d
$$\begin{aligned} f(t) &= t \\ 2t - 4 &= t \\ t - 4 &= 0 \\ \therefore \quad t &= 4 \end{aligned}$$

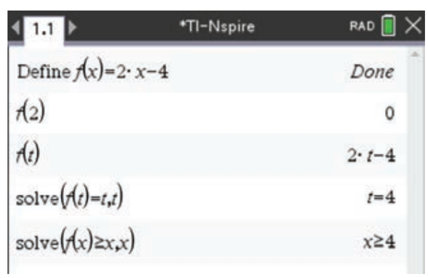
e
$$\begin{aligned} f(x) &\ge x \\ 2x - 4 &\ge x \\ x - 4 &\ge 0 \\ \therefore \quad x &\ge 4 \end{aligned}$$

f
$$\begin{aligned} f(x) &\le 3x \\ 2x - 4 &\le 3x \\ -4 &\le x \\ \therefore \quad x &\ge -4 \end{aligned}$$

Using the TI-Nspire

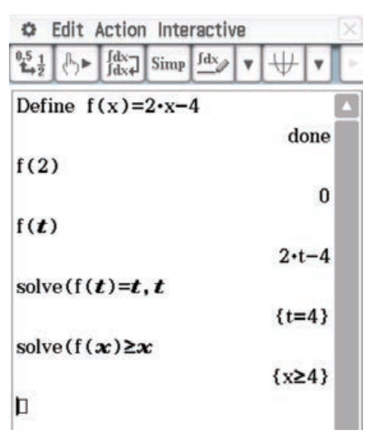
- Use menu > **Actions** > **Define** to define the function $f(x) = 2x - 4$. Find $f(2)$ and $f(t)$.
- Use menu > **Algebra** > **Solve** to solve the equation $f(t) = t$ and the inequality $f(x) \ge x$.

Note: The inequality signs can be accessed using ctrl =.



Using the Casio ClassPad

- In the Main $\sqrt{\alpha}$ screen, select the Math3 keyboard.
- Tap Define and then f. Complete the rule for f as shown. Tap EXE.
- Enter $f(2)$ using f in the Math3 keyboard; tap EXE.
- Enter $f(t)$ using t in the Var keyboard; tap EXE.
- To solve $f(t) = t$ for t, first select solve(from the Math3 keyboard. Enter the equation $f(t) = t$ followed by ‘, t’. Tap EXE.
- Solve $f(x) \ge x$ for x similarly. (Recall that the default setting is to solve for the variable x.)



Note: Alternatively, you can define the function f by first entering the rule $2x - 4$. Highlight the rule and go to **Interactive** > **Define**. Tap OK.

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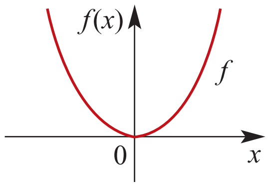
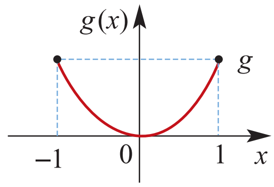
Restriction of a function

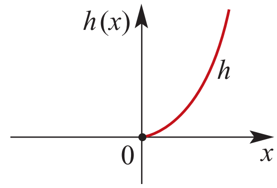
Consider the following functions:



$f(x) = x^2, \ x \in \mathbb{R}$



$g(x) = x^2, \ -1 \le x \le 1$



$h(x) = x^2, \ x \in \mathbb{R}^+ \cup \{0\}$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they all have the same rule. They are different because they are defined for different domains. We say that g and h are **restrictions** of f, since their domains are subsets of the domain of f.

Example 13

Sketch the graph of each of the following functions and state its range:

a $f: [-1, 2] \to \mathbb{R}, f(x) = x$

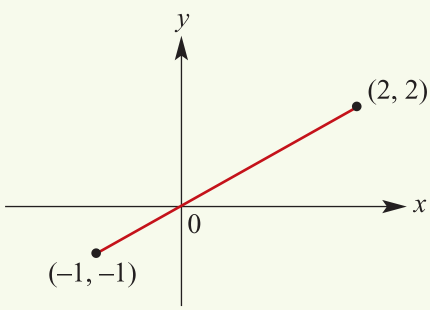
b $f: [-1, 1] \to \mathbb{R}, f(x) = x^2 + x$

c $f: (0, 2] \to \mathbb{R}, f(x) = \dfrac{1}{x}$

d $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2x + 8$

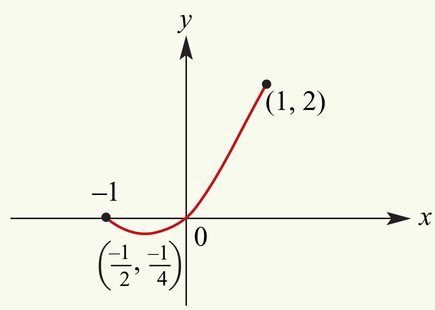
Solution

a



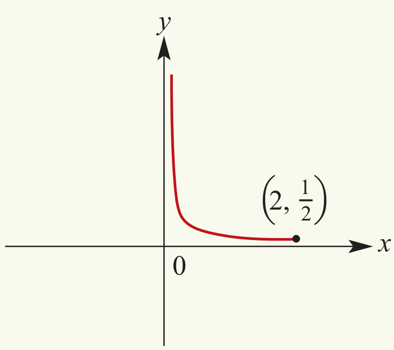
Range is $[-1, 2]$

b



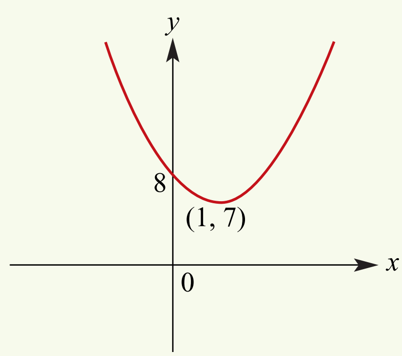
Range is $[-\frac{1}{4}, 2]$

c



Range is $[\frac{1}{2}, \infty)$

d



$f(x) = x^2 - 2x + 8 = (x - 1)^2 + 7$

Range is $[7, \infty)$

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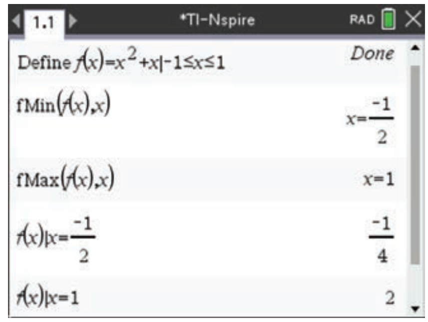
Using the TI-Nspire

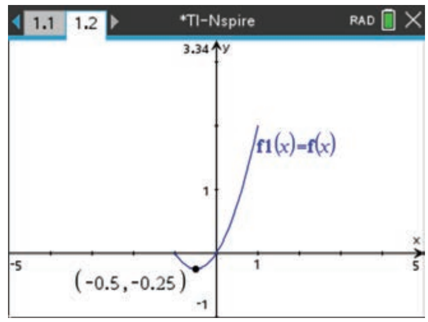
- In a **Calculator** application, use (menu) > **Actions** > **Define** to define the function $f\colon [-1, 1] \to \mathbb{R}, f(x) = x^2 + x$.

Note: The 'with' symbol | and the inequality signs can be accessed using (ctrl) (=).

- Use (menu) > **Calculus** > **Function Minimum** and (menu) > **Calculus** > **Function Maximum** to help determine the range of this restricted function. The range is $[-\frac{1}{4}, 2]$.
- The graph of $y = f(x)$ is plotted by entering $f1(x) = f(x)$ in a **Graphs** application.
- Use (menu) > **Analyze Graph** > **Minimum or Maximum** to show the key points.

Note: You can also enter the restricted function directly in the function entry line in the **Graphs** application if preferred.

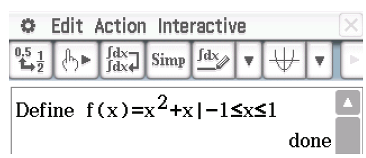


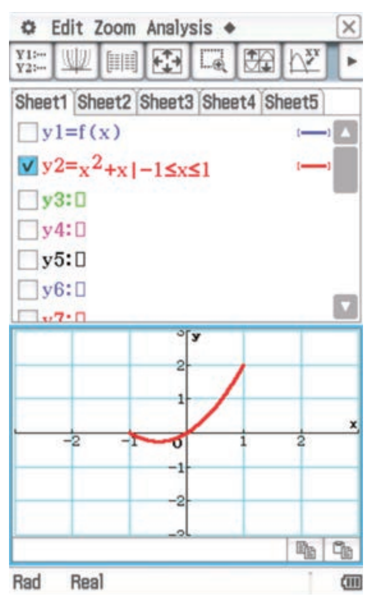


Using the Casio ClassPad

- In the Main $\sqrt{\alpha}$ screen, type $x^2 + x \mid -1 \le x \le 1$. (The symbol | is found in (Math3).)
- Highlight the rule together with the restricted domain and go to **Interactive** > **Define**. Tap OK.
- Graphing in the main screen: Select the graph icon [graph icon]. Highlight the definition of $f(x)$ and drag into the graph window.
- Alternatively, go to the menu [Menu icon] and select **Graph & Table** [Graph & Table icon]. Either enter $f(x)$ in $y1$ or enter $x^2 + x \mid -1 \le x \le 1$ directly.

Note: The window can be adjusted using the [window icon] icon. Alternatively, use **Zoom Box** to manually select an appropriate window.





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Summary 5C

- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- Vertical-line test If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- Functions are usually denoted with lowercase letters such as f, g, h.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y.
- Since the y-value obtained is a function of the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f\colon [0,4] \to \mathbb{R}$, $f(x) = 2x - 1$ to define a function f with domain $[0,4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

Skill-sheet

Exercise 5C

Example 8

1 Which of the following relations are functions? State the domain and range for each:

a $\{(0,1), (0,2), (1,2), (2,3), (3,4)\}$

b $\{(-2,-1), (-1,-2), (0,2), (1,4), (2,-5)\}$

c $\{(0,1), (0,2), (-1,2), (3,4), (5,6)\}$

d $\{(1,3), (2,3), (4,3), (5,3), (6,3)\}$

Example 9

2 Sketch the graph of each of the following relations, then state the range of each and specify whether the relation is a function or not:

a $y = x^2,\ x \in [0,4]$

b $\{(x,y) : x^2 + y^2 = 4,\ x \in [0,2]\}$

c $\{(x,y) : 2x + 8y = 16,\ x \in [0,\infty)\}$

d $y = \sqrt{x},\ x \in \mathbb{R}^+$

e $\left\{(x,y) : y = \dfrac{1}{x^2},\ x \in \mathbb{R} \setminus \{0\}\right\}$

f $\left\{(x,y) : y = \dfrac{1}{x},\ x \in \mathbb{R}^+\right\}$

g $y = x^2,\ x \in [-1,4]$

h $\{(x,y) : x = y^2,\ x \in \mathbb{R}^+ \cup \{0\}\}$

Example 10

3 Rewrite each of the following using the $f\colon X \to Y$ notation:

a $\{(x,y) : y = 3x + 2\}$

b $\{(x,y) : 2y + 3x = 12\}$

c $\{(x,y) : y = 2x + 3,\ x \geq 0\}$

d $y = 5x + 6,\ -1 \leq x \leq 2$

e $y + x^2 = 25,\ -5 \leq x \leq 5$

f $y = 5x - 7,\ 0 \leq x \leq 1$

4 Which of the following relations are functions? State the domain and range for each:

a $\{(x,-2) : x \in \mathbb{R}\}$

b $\{(3,y) : y \in \mathbb{Z}\}$

c $y = -x + 3$

d $y = x^2 + 5$

e $\{(x,y) : x^2 + y^2 = 9\}$

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Example 11

5 **a** Given that $f(x) = 2x - 3$, find:

i $f(0)$ **ii** $f(4)$ **iii** $f(-1)$ **iv** $f(6)$ **v** $f(x-1)$ **vi** $f\left(\frac{1}{a}\right)$

b Given that $g(x) = \frac{4}{x}$, find:

i $g(1)$ **ii** $g(-1)$ **iii** $g(3)$ **iv** $g(2)$

c Given that $g(x) = (x-2)^2$, find:

i $g(4)$ **ii** $g(-4)$ **iii** $g(8)$ **iv** $g(a)$

d Given that $f(x) = 1 - \frac{1}{x}$, find:

i $f(1)$ **ii** $f(1+a)$ **iii** $f(1-a)$ **iv** $f\left(\frac{1}{a}\right)$

Example 12

6 Consider the function defined by $f(x) = 2x + 1$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$ and $f(t)$.
b Find the value of x for which $f(x) = 6$.
c Find the value of x for which $f(x) = 0$.
d For what values of t is $f(t) = t$?
e For what values of x is $f(x) \geq x$?
f For what values of x is $f(x) \leq 3x$?

CAS

7 Find the value(s) of x for which the function has the given value:

a $f(x) = 5x - 2$, $f(x) = 3$
b $f(x) = \frac{1}{x}$, $f(x) = 6$
c $f(x) = x^2$, $f(x) = 9$
d $f(x) = (x+1)(x-4)$, $f(x) = 0$
e $f(x) = x^2 - 2x$, $f(x) = 3$
f $f(x) = x^2 - x - 6$, $f(x) = 0$

8 Let $g(x) = x^2 + 2x$ and $h(x) = 2x^3 - x^2 + 6$.

a Evaluate $g(-1)$, $g(2)$ and $g(-2)$.
b Evaluate $h(-1)$, $h(2)$ and $h(-2)$.
c Express the following in terms of x:

i $g(-3x)$ **ii** $g(x-5)$ **iii** $h(-2x)$ **iv** $g(x+2)$ **v** $h(x^2)$

9 Consider the function $f(x) = 2x^2 - 3$. Find:

a $f(2)$, $f(-4)$ **b** the range of f

10 Consider the function $f(x) = 3x + 1$. Find:

a the image of 2 **b** the pre-image of 7 **c** $\{x : f(x) = 2x\}$

11 Consider the function $f(x) = 3x^2 + 2$. Find:

a the image of 0 **b** the pre-image(s) of 5 **c** $\{x : f(x) = 11\}$

12 Consider the functions $f(x) = 7x + 6$ and $g(x) = 2x + 1$. Find:

a $\{x : f(x) = g(x)\}$ **b** $\{x : f(x) > g(x)\}$ **c** $\{x : f(x) = 0\}$

Example 13

13 Sketch the graphs of each of the following functions and state the range of each:

a $f\colon [-1, 2] \to \mathbb{R},\ f(x) = x^2$
b $f\colon [-2, 2] \to \mathbb{R},\ f(x) = x^2 + 2x$
c $f\colon (0, 3] \to \mathbb{R},\ f(x) = \frac{1}{x}$
d $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 2x + 3$
e $f\colon (1, 6) \to \mathbb{R},\ f(x) = x^2 - 4x + 6$
f $f\colon [-3, 6] \to \mathbb{R},\ f(x) = x^2 - 2x + 1$

CAS

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5D One-to-one functions and implied domains

One-to-one functions

We have seen that a function is a special type of relation such that each x-value maps to only one y-value.

A function is said to be **one-to-one** if different x-values map to different y-values. That is, a function f is one-to-one if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom } f$. Another way to say this is that a one-to-one function cannot contain two different ordered pairs with the same second coordinate.

Example 14

Which of the following functions is one-to-one?

a $\{(1, 4), (2, 2), (3, 4), (4, 6)\}$ **b** $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$

Solution

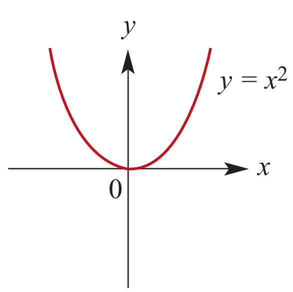
a $\{(1, 4), (2, 2), (3, 4), (4, 6)\}$ is *not* one-to-one, as both 1 and 3 map to 4.

b $\{(1, 4), (2, 3), (3, 2), (4, 1)\}$ is one-to-one, as different x-values map to different y-values.

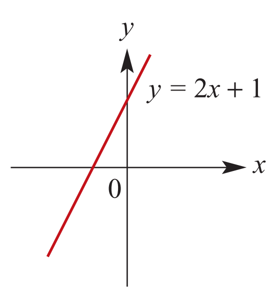
The vertical-line test can be used to determine whether a relation is a function or not. Similarly, there is a geometric test that determines whether a function is one-to-one or not.

Horizontal-line test

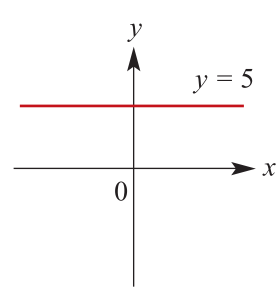
If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is **one-to-one**.



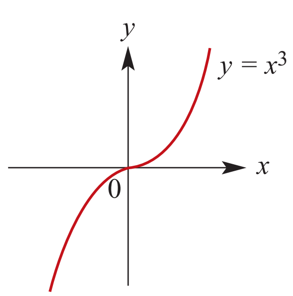
not one-to-one



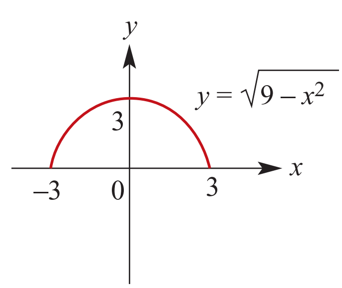
one-to-one



not one-to-one



one-to-one



not one-to-one

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Implied (maximal) domain

We considered implied domains for relations in Section 5B. We recall our definition but this time we do so for functions in particular. The **implied (maximal) domain** of a function is the set of all real numbers for which the rule of the function has meaning.

For example: $f(x) = 3x^2 - 2x$ has implied domain $\mathbb{R}$

$g(x) = \sqrt{x}$ has implied domain $[0, \infty)$

Example 15

State the implied domain, sketch the graph and find the corresponding range of each of the following:

a $f(x) = \sqrt{2x - 5}$ **b** $g(x) = \dfrac{1}{2x - 5}$

Solution

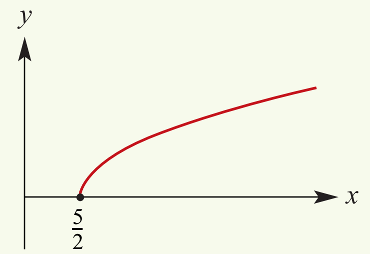
a For $f(x)$ to be defined, we need

$$2x - 5 \geq 0$$

$$\therefore \quad x \geq \frac{5}{2}$$

Hence the implied domain is $[\frac{5}{2}, \infty)$.

The range is $\mathbb{R}^+ \cup \{0\} = [0, \infty)$.



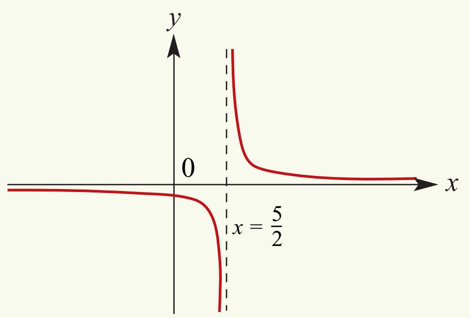
b For $g(x)$ to be defined, we need

$$2x - 5 \neq 0$$

$$\therefore \quad x \neq \frac{5}{2}$$

Hence the implied domain is $\mathbb{R} \setminus \{\frac{5}{2}\}$.

The range is $\mathbb{R} \setminus \{0\}$.



Summary 5D

- A function f is **one-to-one** if different x-values map to different y-values, that is, if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom } f$.
- Horizontal-line test If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is one-to-one.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning. We refer to the **implied (maximal) domain** of a function, because the domain is implied by the rule.

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Exercise 5D

Example 14

1 State which of the following functions are one-to-one:

a $\{(1,3),(2,4),(4,4),(3,6)\}$

b $\{(1,3),(2,4),(3,6),(7,9)\}$

c $\{\,(x,y) : y = x^2\,\}$

d $\{\,(x,y) : y = 3x + 1\,\}$

e $f(x) = x^3 + 1$

f $f(x) = 1 - x^2$

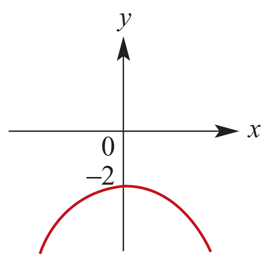
g $y = x^2,\ x \geq 0$

2 Each of the following is the graph of a relation:

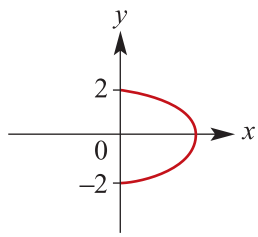
i State which are the graph of a function.

ii State which are the graph of a one-to-one function.

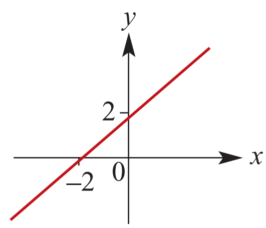
a



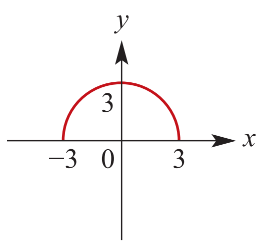
b



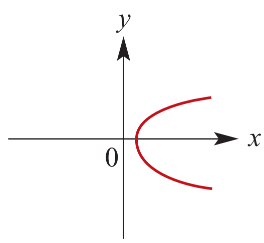
c



d



e



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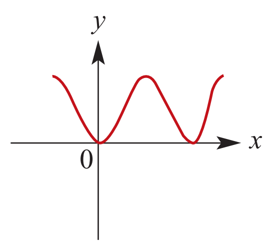
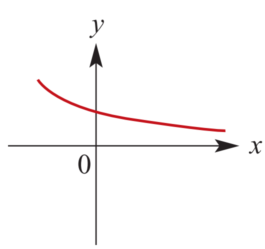


g



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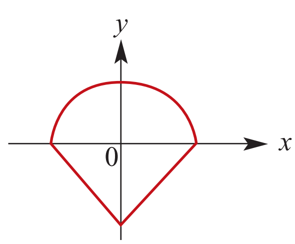

Example 15

3 For each of the following, find the implied domain and the corresponding range for the function defined by the rule:

a $y = 7 - x$

b $y = 2\sqrt{x}$

c $y = x^2 + 1$

d $y = -\sqrt{9 - x^2}$

e $y = \dfrac{1}{\sqrt{x}}$

f $y = 3 - 2x^2$

g $y = \sqrt{x - 2}$

h $y = \sqrt{2x - 1}$

i $y = \sqrt{3 - 2x}$

j $y = \dfrac{1}{2x - 1}$

k $y = \dfrac{1}{(2x - 1)^2} - 3$

l $y = \dfrac{1}{2x - 1} + 2$

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4 For each of the following, state the implied domain and range:

a $f(x) = \sqrt{x - 4}$ **b** $f(x) = \sqrt{4 - x}$

c $f(x) = 2\sqrt{x - 2} + 3$ **d** $f(x) = \dfrac{1}{x - 4}$

e $f(x) = \dfrac{1}{x - 4} + 3$ **f** $f(x) = \dfrac{3}{x + 2} - 3$

5 Each of the following is the rule of a function. In each case write down the maximal domain and the range:

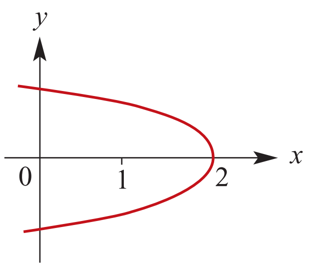
a $f(x) = 3x + 4$ **b** $g(x) = x^2 + 2$

c $y = -\sqrt{16 - x^2}$ **d** $y = \dfrac{1}{x + 2}$

6 The graph shown is of the relation

$$\{ (x, y) : y^2 = -x + 2,\ x \le 2 \}$$

From this relation, form two functions with domain $(-\infty, 2]$ and specify the range of each.



7 **a** Draw the graph of $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2$.

b By restricting the domain of f, form two one-to-one functions that have the same rule as f.

8 **a** Draw the graph of $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2 - 2x + 4$.

b By restricting the domain of f, form two one-to-one functions that have the same rule as f.

9 **a** Draw the graph of $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}, f(x) = \dfrac{1}{(x - 2)^2}$.

b By restricting the domain of f, form two one-to-one functions that have the same rule as f.

10 **a** Draw the graph of $x^2 + y^2 = 4$ and state the domain.

b From this relation form two one-to-one functions that have domain $[0, 2]$.

c From this relation form two one-to-one functions that have domain $[-2, 0]$.

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5E Piecewise-defined functions

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 16

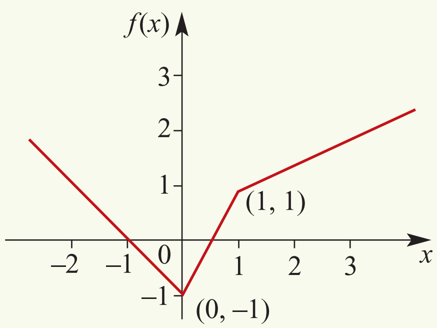
a Sketch the graph of the function:

$$f(x) = \begin{cases} -x - 1 & \text{for } x < 0 \\ 2x - 1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of the function.

Solution

a



b The range is $[-1, \infty)$.

Explanation

- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$. Note that when $x = 1$, $y = 1$ for this rule.

Example 17

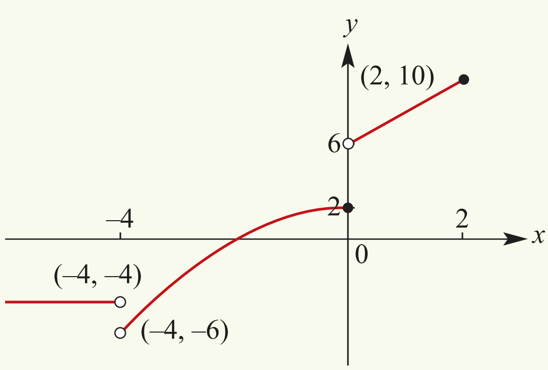
a Sketch the graph of the function:

$$f(x) = \begin{cases} -4 & \text{for } x < -4 \\ -\frac{1}{2}x^2 + 2 & \text{for } -4 < x \le 0 \\ 2x + 6 & \text{for } 0 < x \le 2 \end{cases}$$

b State the range of the function.

Solution

a



b The range is $(-6, 2] \cup (6, 10]$.

Explanation

- The graph of $y = -4$ is sketched for $x < -4$.
- The graph of $y = -\frac{1}{2}x^2 + 2$ is sketched for $-4 < x \le 0$.
- The graph of $y = 2x + 6$ is sketched for $0 < x \le 2$.

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Note: A CAS calculator can be used to plot the graphs of piecewise-defined functions; see the calculator appendices in the Interactive Textbook for instructions.

Exercise 5E

Example 16

1 Sketch the graph of each of the following functions and state its range:

a $h(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ **b** $h(x) = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$ **c** $h(x) = \begin{cases} -x, & x \geq 0 \\ x, & x < 0 \end{cases}$

d $h(x) = \begin{cases} 1+x, & x \geq 0 \\ 1-x, & x < 0 \end{cases}$ **e** $h(x) = \begin{cases} x, & x \geq 1 \\ 2-x, & x < 1 \end{cases}$

Example 17

2 Sketch the graph of each of the following functions:

a $g(x) = \begin{cases} -x-3, & x < 1 \\ x-5, & 1 \leq x \leq 5 \\ 3x-15, & x > 5 \end{cases}$ **b** $g(x) = \begin{cases} x+3, & x < -1 \\ x-2, & 1 \leq x \leq 5 \\ 2x-1, & x > 5 \end{cases}$

CAS

3 Sketch the graph of the following function and state its range:

$$f(x) = \begin{cases} \frac{2}{3}x+3, & x < 0 \\ x+3, & 0 \leq x \leq 1 \\ -2x+6, & x > 1 \end{cases}$$

4 Sketch the graph of the following function and state its range:

$$h(x) = \begin{cases} x^2+1, & x \geq 0 \\ 1-x, & x < 0 \end{cases}$$

5 For each of the following functions, sketch the graph and state the range:

a $f(x) = \begin{cases} x+3, & x < -3 \\ x^2-9, & -3 \leq x \leq 3 \\ x-3, & x > 3 \end{cases}$ **b** $f(x) = \begin{cases} x+2, & x < -3 \\ x^2-2, & -2 \leq x \leq 3 \\ x-2, & x > 4 \end{cases}$

6 For each of the following functions, sketch the graph and state the range:

a $f(x) = \begin{cases} x+3, & x < -3 \\ x, & x > 2 \end{cases}$ **b** $f(x) = \begin{cases} \dfrac{1}{x+3}, & x < -3 \\ x^2, & x > 0 \end{cases}$

c $f(x) = \begin{cases} 2x+3, & x \leq 1 \\ x, & 1 < x < 3 \\ -x, & x \geq 4 \end{cases}$ **d** $f(x) = \begin{cases} \dfrac{1}{x+3}, & x < -3 \\ (x+2)^2, & x > -1 \end{cases}$

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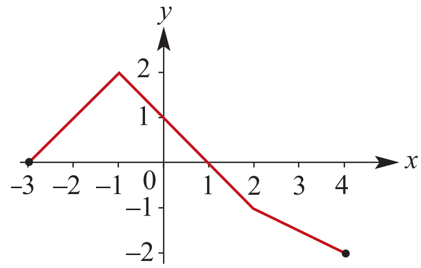
7 Sketch the graph of the following function and state its range:

$$f(x) = \begin{cases} \dfrac{1}{x}, & x > 1 \\ x, & x \le 1 \end{cases}$$

8 Specify the function represented by this graph:



5F Applying function notation

The first four chapters of this book involve functions without using function notation. This section presents further questions which arise from the first four chapters of this book but where function notation can now be used.

Example 18

The volume of a sphere of radius r is determined by the function with rule $V(r) = \frac{4}{3}\pi r^3$. State the practical domain of the function V and find $V(10)$.

Solution

The practical domain is $(0, \infty)$.

$$V(10) = \frac{4}{3} \times \pi \times 10^3 = \frac{4000\pi}{3}$$

(The volume of a sphere of radius 10 is $\frac{4000\pi}{3}$ cubic units.)

Example 19

If $f: \mathbb{R} \to \mathbb{R}$, $f(x) = ax + b$ such that $f(1) = 7$ and $f(5) = 19$, find a and b and sketch the graph of $y = f(x)$.

Solution

Since $f(1) = 7$ and $f(5) = 19$,

$7 = a + b$ (1)

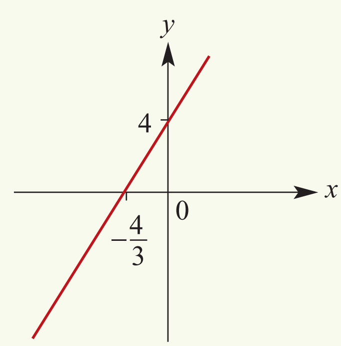
and $19 = 5a + b$ (2)

Subtract (1) from (2):

$12 = 4a$

Thus $a = 3$ and substituting in (1) gives $b = 4$.

Hence $f(x) = 3x + 4$.



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Example 20

Find the quadratic function f such that $f(4) = f(-2) = 0$ and $f(0) = 16$.

Solution

Since 4 and −2 are solutions to the quadratic equation $f(x) = 0$, we have

$$f(x) = k(x-4)(x+2)$$

Since $f(0) = 16$, we obtain

$$16 = k(-4)(2)$$

$$\therefore \quad k = -2$$

Hence

$$\begin{aligned} f(x) &= -2(x-4)(x+2) \\ &= -2(x^2 - 2x - 8) \\ &= -2x^2 + 4x + 16 \end{aligned}$$

Skill-sheet

Exercise 5F

Example 18

1 A metal bar is L cm long when its temperature is T°C. The quantities L and T are approximately related by the formula $L = 0.002T + 25$.

a Hence L is a function of T and the rule can be written as $L(T) = 0.002T + 25$. State a possible practical domain for the function L.

b Find:

i $L(30)$ **ii** $L(16)$ **iii** $L(100)$ **iv** $L(500)$

Example 19

2 If $f(x) = a + bx$ with $f(4) = -1$ and $f(8) = 1$:

a find a and b **b** solve the equation $f(x) = 0$.

3 Find a linear function f such that $f(0) = 7$ and its graph is parallel to that of the function with rule $g(x) = 2 - 5x$.

4 If f is a linear function such that $f(-5) = -12$ and $f(7) = 6$:

a find $f(0)$ and $f(1)$ **b** solve the equation $f(x) = 0$.

Example 20

5 Find the quadratic function f such that $f(2) = f(4) = 0$ and 7 is the greatest value of $f(x)$.

6 Write $f(x) = x^2 - 6x + 16$ in the form $f(x) = (x-3)^2 + p$ and hence state the range of f.

7 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = ax^2 + bx + c$. Find a, b and c if $f(0) = 2$, $f(4) = 0$ and $f(5) = 0$.

8 Find two quadratic functions f and g such that $f(1) = 0$, $g(1) = 0$ and $f(0) = 10$, $g(0) = 10$ and both have a maximum value of 18.

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9 **a** Find the set of values of k for which $f(x) = 3x^2 - 5x - k$ is greater than 1 for all real x.

b Show that, for all k, the minimum value of $f(x)$ occurs when $x = \frac{5}{6}$. Find k if this minimum value is zero.

5G Inverse functions

If f is a one-to-one function, then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus if f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \text{ran } f \text{ and } y \in \text{dom } f$$

Example 21

The set of ordered pairs $\{(1, 2), (3, 4), (5, -1), (6, -7), (0, 6)\}$ is a function f.

Describe the inverse function f^{-1} of this function as a set of ordered pairs. Give the domain and range of both f and f^{-1}.

Solution

The inverse function f^{-1} is the set of ordered pairs:
$\{(2, 1), (4, 3), (-1, 5), (-7, 6), (6, 0)\}$

The domain of f is $\{1, 3, 5, 6, 0\}$.
The range of f is $\{2, 4, -1, -7, 6\}$.

The domain of f^{-1} is $\{2, 4, -1, -7, 6\}$.
The range of f^{-1} is $\{1, 3, 5, 6, 0\}$.

Explanation

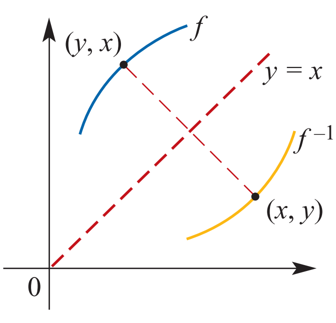
Using function notation we can define the function f as:

$$f(1) = 2, \quad f(3) = 4, \quad f(5) = -1, \quad f(6) = -7, \quad f(0) = 6$$

The inverse function is defined by:

$$f^{-1}(2) = 1, \quad f^{-1}(4) = 3, \quad f^{-1}(-1) = 5, \quad f^{-1}(-7) = 6, \quad f^{-1}(6) = 0$$

It is not difficult to see what the relation between f and f^{-1} means geometrically. The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f. Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.



From this the following is evident:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$

A function has an inverse function if and only if it is one-to-one.

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Example 22

a Find the inverse function f^{-1} of the function $f(x) = 2x - 3$ and sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes.

b Find the inverse function g^{-1} of the function $g\colon [2, 6] \to \mathbb{R}$, $g(x) = 2x - 3$ and state the domain and range of g^{-1}.

Solution

a The graph of f has equation $y = 2x - 3$ and so the graph of f^{-1} has equation $x = 2y - 3$, i.e. x and y are interchanged.

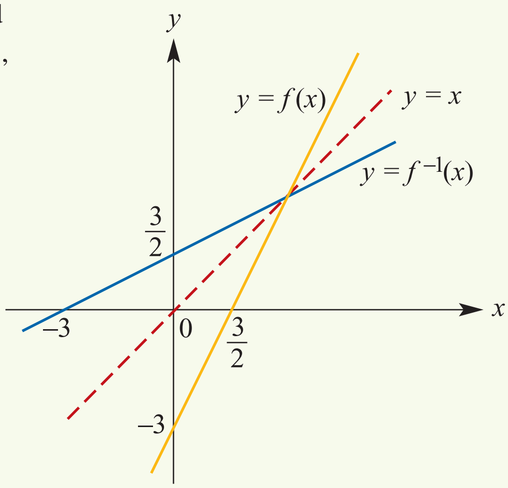
Solve for y:

$$x = 2y - 3$$
$$2y = x + 3$$
$$y = \tfrac{1}{2}(x + 3)$$

Hence $f^{-1}(x) = \tfrac{1}{2}(x + 3)$

with $\operatorname{dom} f^{-1} = \operatorname{ran} f = \mathbb{R}$

and $\operatorname{ran} f^{-1} = \operatorname{dom} f = \mathbb{R}$



b Now consider $g\colon [2, 6] \to \mathbb{R}$, $g(x) = 2x - 3$.

The domain of g is $[2, 6]$. We find $g(2) = 1$ and $g(6) = 9$. The gradient of the graph of $y = g(x)$ is positive. Therefore the maximum value of the function is 9 and the minimum value is 1. The range of g is $[1, 9]$.

From part **a**, the rule for the inverse function is $g^{-1}(x) = \tfrac{1}{2}(x + 3)$.

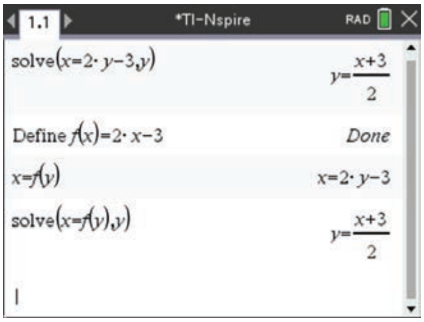
Also $\operatorname{dom} g^{-1} = \operatorname{ran} g = [1, 9]$

and $\operatorname{ran} g^{-1} = \operatorname{dom} g = [2, 6]$

Using the TI-Nspire

To find the inverse of the function with rule $f(x) = 2x - 3$, use menu > **Algebra** > **Solve**.

Two methods are shown.



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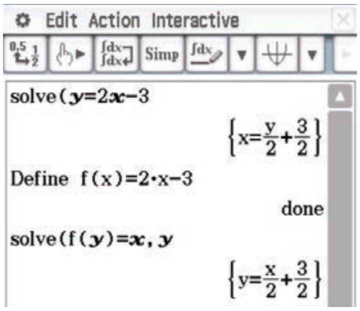
Using the Casio ClassPad

To find the inverse of the function with rule $f(x) = 2x - 3$:

- Select [solve(] from the [Math1] or [Math3] keyboard. Enter the equation $y = 2x - 3$ and tap [EXE].
- Alternatively, define the function $f(x) = 2x - 3$ and then solve the equation $f(y) = x$ for y.



Example 23

Find the inverse of the function $f\colon [1, \infty) \to \mathbb{R}$, $f(x) = (x - 1)^2 + 4$.

Solution

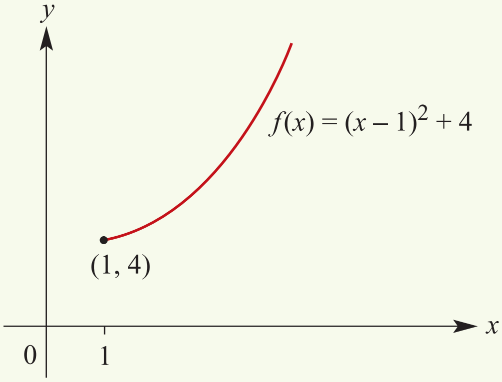
The inverse has rule

$$x = (y-1)^2 + 4$$
$$(y-1)^2 = x - 4$$
$$y - 1 = \pm\sqrt{x-4}$$
$$\therefore\quad y = 1 \pm \sqrt{x-4}$$

But $\operatorname{ran} f^{-1} = \operatorname{dom} f = [1, \infty)$, and so

$$f^{-1}(x) = 1 + \sqrt{x-4}$$

with $\operatorname{dom} f^{-1} = \operatorname{ran} f = [4, \infty)$



Skill-sheet

Exercise 5G

Example 21

1 For each of the following, find the inverse function and state its domain and range:

a $\{(1, 3), (-2, 6), (4, 5), (7, 1)\}$ **b** $\{(2, 3), (-1, 6), (4, -5), (1, 7), (6, -4)\}$

c $\{(3, 3), (-2, -4), (-1, -1), (-8, 1)\}$ **d** $\{(1, 3), (-10, -7), (-7, -6), (2, 8), (11, 4)\}$

Example 22

2 For each of the following, find the inverse function and state its domain and range:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 6 - 2x$ **b** $f\colon [1, 5] \to \mathbb{R}$, $f(x) = 3 - x$

c $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x + 4$ **d** $f\colon (-\infty, 4] \to \mathbb{R}$, $f(x) = x + 4$

e $f\colon [-1, 7] \to \mathbb{R}$, $f(x) = 16 - 2x$

Example 23

3 Find the inverse function of each of the following. State the domain and range of f^{-1}.

a $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ **b** $f\colon [2, \infty) \to \mathbb{R}$, $f(x) = (x - 2)^2 + 3$

c $f\colon (-\infty, 4] \to \mathbb{R}$, $f(x) = (x - 4)^2 + 6$ **d** $f\colon [0, 1] \to \mathbb{R}$, $f(x) = \sqrt{1 - x}$

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CAS

4 Find the inverse function of each of the following:

a $f : [0, 4] \to \mathbb{R}, f(x) = \sqrt{16 - x^2}$

b $f : [0, \infty) \to \mathbb{R}, f(x) = (x + 4)^2 + 6$

5 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f(x) = 2x - 6$.

b Find the coordinates of the point for which $f(x) = f^{-1}(x)$.

6 **a** On the one set of axes sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f : [0, \infty) \to \mathbb{R}, f(x) = x^2$.

b Find the coordinates of the point(s) for which $f(x) = f^{-1}(x)$.

7 Let $f : \mathbb{R} \to \mathbb{R}, f(x) = ax + b$, where a and b are non-zero constants. Given that $f(1) = 2$ and $f^{-1}(1) = 3$, find the values of a and b.

8 Let $f : (-\infty, a] \to \mathbb{R}, f(x) = \sqrt{a - x}$, where a is a constant.

a Find $f^{-1}(x)$.

b If the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect at $x = 1$, find the possible values of a.

5H Functions and modelling exercises

In the following examples we see how function notation can be used when applying mathematics in 'real' situations.

Example 24

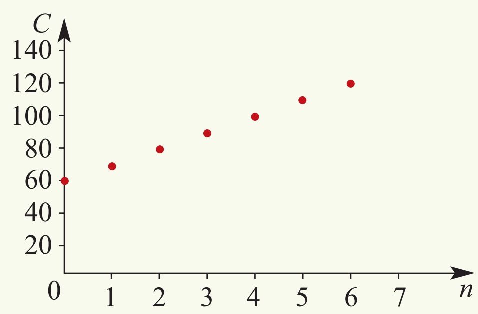
A book club has a membership fee of $60.00 and each book purchased is $10.00. Construct a cost function that can be used to determine the cost of different numbers of books, then sketch its graph.

Solution

Let $C(n)$ denote the cost (in dollars) when n books are purchased. Then

$$C(n) = 60 + 10n$$

The domain of this function is $\mathbb{N} \cup \{0\}$, the set of non-negative integers, and its graph will be as shown.



Note: The graph of this function consists of discrete points. Sometimes to simplify the situation we represent such functions by a continuous line. Strictly, this is not mathematically correct, but it may aid our understanding of the situation.

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Example 25

The following table shows the Younanistan Post rates for sending letters.

Mass, m (g)	Cost, C (\$)
Up to 50 g	\$0.70
Over 50 g up to 100 g	\$1.15
Over 100 g up to 250 g	\$1.70
Over 250 g up to 500 g	\$3.00

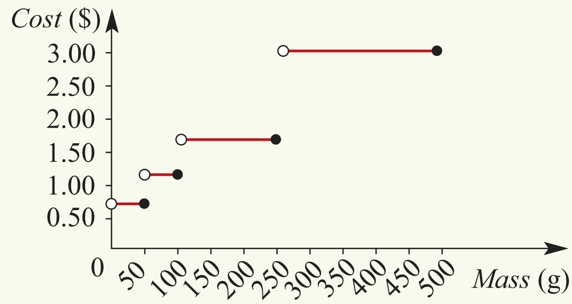
Sketch a graph of the cost function, C, giving its domain and range and the rules that define it.

Solution

$$C(m) = \begin{cases} 0.70 & \text{for } 0 < m \le 50 \\ 1.15 & \text{for } 50 < m \le 100 \\ 1.70 & \text{for } 100 < m \le 250 \\ 3.00 & \text{for } 250 < m \le 500 \end{cases}$$

Domain = $(0, 500]$

Range = $\{0.70, 1.15, 1.70, 3.00\}$



Example 26

A householder has six laying hens and wishes to construct a rectangular enclosure to provide a maximum area for the hens, using a 12 m length of fencing wire. Construct a function that will give the area of the enclosure, A, in terms of its length, ℓ. By sketching a graph, find the maximum area that can be fenced.

Solution

Let ℓ = length of the enclosure

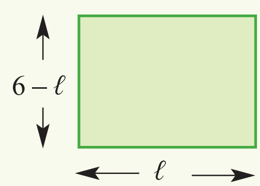
Then width $= \dfrac{12 - 2\ell}{2} = 6 - \ell$

The area is

$$A(\ell) = \ell(6 - \ell)$$
$$= 6\ell - \ell^2$$

The domain of A is the interval $[0, 6]$.

The maximum area is 9 m^2 and occurs when $\ell = 3$ m, i.e. when the enclosure is a square.



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Exercise 5H

Example 24

1 A small printing company produces the printed menus for a take-away restaurant. The charge for a print run is \$80 plus 30 cents for each menu printed. Construct a cost function that describes the charge (in dollars) for such a print run.

Example 25

2 Suppose that Younanistan Post charges the following rates for airmail letters to Africa: \$1.20 up to 20 g; \$2.00 over 20 g and up to 50 g; \$3.00 over 50 g and up to 150 g.

a Write a cost function, C (\$), in terms of the mass, m (g), for letters up to 150 g.

b Sketch the graph of the function, stating the domain and range.

3 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 \$64 per day + 25 cents per kilometre

Method 2 \$89 per day with unlimited travel

a Write a rule for each method if C_1 is the cost, in dollars, using method 1 for x kilometres travelled, and C_2 is the cost using method 2.

b Draw a graph of each rule on the same axes.

c Determine, from the graph, the distance which must be travelled per day if method 2 is cheaper than method 1.

Example 26

4 A piece of wire 100 cm long is bent to form a rectangle. Let x cm be the width of the rectangle.

a Find the length of the rectangle in terms of x.

b Find the rule $A(x)$ for the function A that gives the area of the rectangle in cm^2.

c Find the allowable values for x.

d Find the maximum possible area of the rectangle and the value of x for which this maximum occurs.

5 The table gives the rates charged by MD Couriers for delivering parcels within Bendigo.

Weight, w kg	Cost
$w < 1$	\$3.50
$1 \le w < 2.5$	\$6.50
$2.5 \le w < 5$	\$12.00
$5 \le w < 10$	\$20.00

a Sketch a graph representing the information given in the table.

b Find the cost of sending a parcel that weighs:

i 2 kg **ii** 4 kg **iii** 5 kg

c Alexa has two items to send to the same destination: one item weighs 2.6 kg and the other weighs 2.3 kg. Should she send the two items separately, or package them together? Justify your answer.

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6 Two taxi services use the following different systems for charging for a journey:

Gold Taxi — Initial charge of \$10, plus a charge of 50 cents for each 200 m travelled

Purple Taxi — Flat fee of \$25 for travelling up to 20 km, plus a charge of \$1 for each kilometre travelled beyond 20 km

a Let $G(d)$ be the cost (in dollars) of a journey of d km in a Gold Taxi. Show that

$$G(d) = 2.5d + 10 \quad \text{for } d \geq 0$$

b Let $P(d)$ be the cost (in dollars) of a journey of d km in a Purple Taxi. Show that

$$P(d) = \begin{cases} 25 & \text{for } 0 \leq d \leq 20 \\ d + 5 & \text{for } d > 20 \end{cases}$$

c On the same coordinate axes, sketch graphs to represent $G(d)$ and $P(d)$.

d Find the cost of a journey of:

i 7 km in a Gold Taxi **ii** 12 km in a Purple Taxi

e Art wants to travel a distance of 15 km. Which taxi service will be cheaper?

f Find the distances for which a Purple Taxi is the cheaper option.

7 A gardener charges \$40 per hour for work that takes up to 2 hours and then charges \$30 per hour for work after the first 2 hours, up to a maximum of 6 hours.

a Write the rule for a function that describes the total fee, $\$F$, for work that takes the gardener t hours. (Assume a continuous model.)

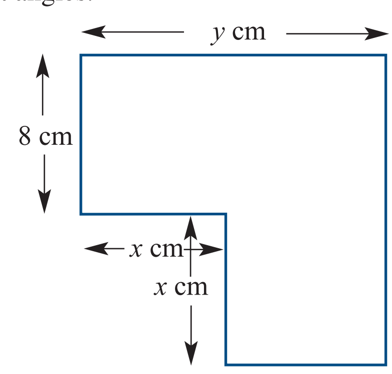
b Sketch a graph of your function.

c Find the fee for work that takes the gardener:

i 1.5 hours **ii** 2.5 hours **iii** 3.5 hours

d What is the effective hourly rate for work that takes 4 hours?

8 Assume that angles that look like right angles are right angles.



a **i** Find an expression for the area A in terms of x and y.

ii Find an expression for the perimeter P in terms of x and y.

b **i** If $P = 64$ cm, find A in terms of x.

ii Find the allowable values for x.

iii Sketch the graph of A against x for these values.

iv What is the maximum area?

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Review

Chapter summary

Assign-ment

Nrich

- **Set notation**

$x \in A$	x is an element of A	
$x \notin A$	x is not an element of A	
$A \subseteq B$	A is a subset of B	
$A \cap B$	A intersect B	the set of elements in common to A and B
$A \cup B$	A union B	the set of elements that are in A or in B (or in both)
$A \setminus B$	A take away B	the set of elements of A that are not in B

- **Sets of numbers**

$\mathbb{N}$	Natural numbers	$\mathbb{Z}$	Integers
$\mathbb{Q}$	Rational numbers	$\mathbb{R}$	Real numbers

- **Interval notation**

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

- **Relations**
  - A **relation** is a set of ordered pairs.
  - The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
  - The **range** is the set of all the second coordinates of the ordered pairs in the relation.

- **Functions**
  - For a function f and an element x of the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f, and the element x is called a **pre-image** of y.
  - A function f is said to be **one-to-one** if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom } f$. In other words, f is one-to-one if every image under f has a unique pre-image.
  - The **implied domain** (or **maximal domain**) of a function is the largest subset of $\mathbb{R}$ for which the rule is defined.
  - For a function f with domain D, a new function g may be defined with domain $A \subseteq D$ and rule given by $g(x) = f(x)$ for all $x \in A$. The function g is called a **restriction** of f.

- **Inverse functions**

If f is a one-to-one function, then for each element y in the range of f there is exactly one element x in the domain of f such that $f(x) = y$.
Thus if f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \text{ran } f \text{ and } y \in \text{dom } f$$

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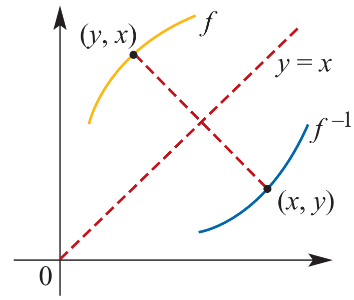
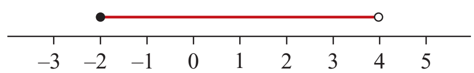
The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f.
Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.
From this the following is evident:

$$\text{dom } f^{-1} = \text{ran } f$$
$$\text{ran } f^{-1} = \text{dom } f$$



Technology-free questions

1 Describe each of the following using interval notation:

a


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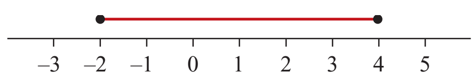
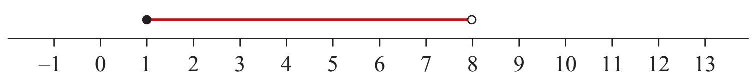


c


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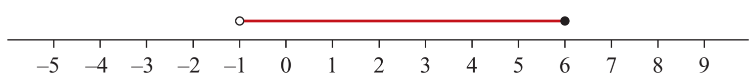
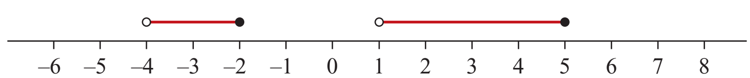
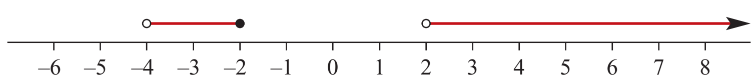
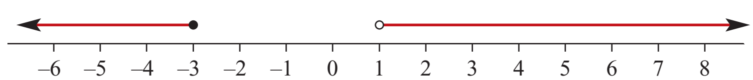


e


f


g


2 If f is the function with rule $f(x) = 2 - 6x$, find:

a $f(3)$ **b** $f(-4)$ **c** the value of x for which f maps x to 6.

3 For $f\colon [-1, 6] \to \mathbb{R}$, $f(x) = 6 - x$:

a sketch the graph of f **b** state the range of f.

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4 Sketch the graphs of each of the following, stating the range of each:

a $\{(x, y) : 3x + y = 6\}$

b $\{(x, y) : y = 3x - 2,\ x \in [-1, 2]\}$

c $\{(x, y) : y = x^2,\ x \in [-2, 2]\}$

d $\{(x, y) : y = 9 - x^2\}$

e $\{(x, y) : y = x^2 + 4x + 6\}$

f $\{(1, 2), (3, 4), (2, -6)\}$

g $f : \mathbb{R} \to \mathbb{R},\ f(x) = (x - 2)^2$

h $f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{x} + 2$

i $(y - 3)^2 = 2x + 1$

j $f : [-1, 3] \to \mathbb{R},\ f(x) = x$

5 The function f has rule $f(x) = \frac{a}{x} + b$ such that $f(1) = \frac{3}{2}$ and $f(2) = 9$.

a Find the values of a and b.

b State the implied domain of f.

6 Given that $f : [0, 2] \to \mathbb{R},\ f(x) = 2x - x^2$:

a sketch the graph

b state the range.

7 Given that $f(x) = ax + b$, $f(5) = 10$ and $f(1) = -2$, find the values of a and b.

8 Given that $f(x) = ax^2 + bx + c$, $f(0) = 0$, $f(4) = 0$ and $f(-2) = -6$, find the values of a, b and c.

9 State the implied (maximal) domain for each of the following:

a $y = \frac{1}{x - 2}$

b $f(x) = \sqrt{x - 2}$

c $y = \sqrt{25 - x^2}$

d $f(x) = \frac{1}{2x - 1}$

e $g(x) = \sqrt{100 - x^2}$

f $h(x) = \sqrt{4 - x}$

10 State which of the following functions are one-to-one:

a $y = x^2 + 2x + 3$

b $f : [2, \infty) \to \mathbb{R},\ f(x) = (x - 2)^2$

c $f(x) = 3x + 2$

d $f(x) = \sqrt{x - 2}$

e $f(x) = \frac{1}{x - 2}$

f $f : [-1, \infty) \to \mathbb{R},\ f(x) = (x + 2)^2$

g $f : [-3, 5] \to \mathbb{R},\ f(x) = 3x - 2$

h $f(x) = 7 - x^2$

i $f(x) = \frac{1}{(x - 2)^2}$

j $h(x) = \frac{1}{x - 2} + 4$

11 Sketch the graphs of each of the following:

a $f(x) = \begin{cases} 3x - 1, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ 9, & x \in (-\infty, -3) \end{cases}$

b $h(x) = \begin{cases} 1 - 2x, & x \in [0, \infty) \\ x^2, & x \in [-3, 0) \\ -x^2, & x \in (-\infty, -3) \end{cases}$

12 Sketch the graph of each of the following and state the range:

a $f : [0, 3] \to \mathbb{R},\ f(x) = (x - 1)^2$

b $f : [-4, 1] \to \mathbb{R},\ f(x) = (x + 2)^2 + 1$

c $f : [1, 5] \to \mathbb{R},\ f(x) = \frac{1}{(2x - 1)^2}$

d $f : [-2, 3] \to \mathbb{R},\ f(x) = -x^2 + 3$

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13 State the maximal domain and range of each of the following:

a $f(x) = \sqrt{x-1}$ **b** $f(x) = \sqrt{1-x}$ **c** $f(x) = 1 - \sqrt{x}$

14 State the maximal domain and range of each of the following:

a $f(x) = \dfrac{2}{x-1}$ **b** $f(x) = \dfrac{2}{x+1}$ **c** $f(x) = \dfrac{2}{x-1} + 3$

15 State the maximal domain and range of each of the following:

a $f(x) = \sqrt{1-x^2}$ **b** $f(x) = \sqrt{9-x^2}$ **c** $f(x) = \sqrt{1-x^2} + 3$

16 For each of the following, find the inverse function, stating its rule and domain:

a $f\colon [-1, 5] \to \mathbb{R},\ f(x) = 3x - 2$ **b** $f\colon [-2, \infty) \to \mathbb{R},\ f(x) = \sqrt{x+2} + 2$

c $f\colon [-1, \infty) \to \mathbb{R},\ f(x) = 3(x+1)^2$ **d** $f\colon (-\infty, 1) \to \mathbb{R},\ f(x) = (x-1)^2$

17 If $f(x) = 2x + 5$, find:

a $f(p)$ **b** $f(p+h)$ **c** $f(p+h) - f(p)$ **d** $f(p+1) - f(p)$

18 If $f(x) = 3 - 2x$, find $f(p+1) - f(p)$.

19 State the range of each of the following:

a $f(x) = -2x^2 + x - 2$ **b** $f(x) = 2x^2 - x + 4$

c $f(x) = -x^2 + 6x + 11$ **d** $g(x) = -2x^2 + 8x - 5$

20 Define $f\colon [-1, 6] \to \mathbb{R},\ f(x) = 5 - 3x$.

a Sketch the graph of f. **b** State the range of f.

21 Define $f\colon [-1, 8] \to \mathbb{R},\ f(x) = (x-2)^2$.

a Sketch the graph of f. **b** State the range of f.

22 The domain of the function f is $\{1, 2, 3, 4\}$. Find the range of f if:

a $f(x) = 2x$ **b** $f(x) = 5 - x$ **c** $f(x) = x^2 - 4$ **d** $f(x) = \sqrt{x}$

Multiple-choice questions

1 For $f(x) = 10x^2 + 2$, $f(2a)$ equals

A $20a^2 + 2$ **B** $40a^2 + 2$ **C** $2a^2 + 2a$ **D** $100a^2 + 2$ **E** $10a^2 + 2a$

2 The maximal domain of the function f with rule $f(x) = \sqrt{3x+5}$ is

A $(0, \infty)$ **B** $\left(-\dfrac{5}{3}, \infty\right)$ **C** $(5, \infty)$ **D** $[-5, \infty)$ **E** $\left[-\dfrac{5}{3}, \infty\right)$

3 The maximal domain of the function f with rule $f(x) = \sqrt{6-2x}$ is

A $(0, \infty)$ **B** $[3, \infty)$ **C** $(-\infty, 2)$ **D** $(-\infty, 3]$ **E** $[6, \infty)$

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4 The range of the function $f: (-1, 2] \to \mathbb{R}$ with rule $f(x) = x^2 + 1$ is

A $(2, \infty)$ **B** $(2, 5]$ **C** $(1, 5]$ **D** $[0, 5]$ **E** $[1, 5]$

5 If $f(x) = 7x - 6$, then $f^{-1}(x)$ equals

A $7x + 4$ **B** $\frac{1}{7}x + 6$ **C** $\frac{1}{7}x + \frac{6}{7}$ **D** $\frac{1}{7x - 6}$ **E** $\frac{1}{7}x - 6$

6 For $f: (a, b] \to \mathbb{R}$, $f(x) = 3 - x$, the range is

A $(3 - a, 3 - b)$ **B** $(3 - a, 3 - b]$ **C** $(3 - b, 3 - a)$
D $(3 - b, 3 - a]$ **E** $[3 - b, 3 - a)$

7 Which of the following functions is not one-to-one?

A $f(x) = 9 - x^2,\ x \ge 0$ **B** $f(x) = (x - 3)^2$ **C** $f(x) = 1 - 9x$
D $f(x) = \sqrt{x}$ **E** $f(x) = \frac{9}{x}$

8 The graph of $y = \frac{2}{x} + 3$ is reflected in the x-axis and then in the y-axis. The equation of the final image is

A $y = -\frac{2}{x} + 3$ **B** $y = -\frac{2}{x} - 3$ **C** $y = \frac{2}{x} + 3$ **D** $y = \frac{2}{x} - 3$ **E** $y = 2x - 3$

9 For $f: [-1, 5) \to \mathbb{R}$, $f(x) = x^2$, the range is

A $\mathbb{R}$ **B** $[0, \infty)$ **C** $[0, 25)$ **D** $[1, 25]$ **E** $[0, 5]$

10 Which of the following rules does *not* describe a function?

A $y = x^2 - x$ **B** $y = \sqrt{4 - x}$ **C** $y = 3,\ x > 0$
D $x = 3$ **E** $y = 3x$

Extended-response questions

1 After taking a medication, the concentration of the drug in a patient's bloodstream first increases and then gradually decreases. When a second medication is taken, the concentration starts to increase again. The concentration of the drug, C mg/L, in the patient's bloodstream over a 12-hour period is modelled by the function

$$C(t) = \begin{cases} -5(t-3)^2 + 45 & \text{for } 0 \le t < 3 \\ -(t-3)^2 + 45 & \text{for } 3 \le t < 9 \\ -5(t-12)^2 + 54 & \text{for } 9 \le t \le 12 \end{cases}$$

a Find the concentration of the drug 2 hours after the first medication was taken.

b What was the maximum concentration of the drug during the first 3 hours?

c Find the concentration of the drug 6 hours after the first medication was taken.

d When did the patient take the second medication, and what was the concentration of the drug at that time?

e What was the maximum concentration of the drug during the 12-hour period?

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2 Sections of the rail of a ride at an amusement park are modelled by the function

$$h(x) = \begin{cases} -0.2(x-5)^2 + 8 & \text{for } 0 \le x < 7 \\ -0.8x + 12.8 & \text{for } 7 \le x < 11 \\ 0.2(x-13)^2 + 3.2 & \text{for } 11 \le x \le 16 \end{cases}$$

where $h(x)$ m is the height of the rail above the ground at a horizontal distance of x m from the start of the ride.

a Find the height at the start of the ride.

b Find the height at each of the following horizontal distances from the start of the ride:

i 5 m **ii** 10 m **iii** 15 m

c Find the horizontal distance(s) from the start of the ride when the height is 6 m.

3 An Easyride coach leaves town X and maintains a constant speed of 80 km/h for 4 hours, stops at town Y for $\frac{3}{4}$ hours before travelling for a further $2\frac{1}{2}$ hours at 80 km/h to its destination at town Z. A second coach leaves town Z at the same time and runs express to town X, completing its journey in $5\frac{1}{2}$ hours.

a Construct functions that describe the distance, d km, from X of each coach at time t, stating the domain, range and rule for each.

b Calculate the distance, from X, at which the two coaches pass each other.

4 The Exhibition Centre hires a graphics company to produce a poster for an exhibit. The graphics company charges \$1000 and an additional \$5 for each poster produced.

a **i** Write down the rule for a function, $C(n)$, which describes the cost to the Exhibition Centre of obtaining n posters.

ii Sketch the graph of C against n. (Use a continuous model.)

b The Exhibition Centre is going to sell the posters for \$15 each.

i Write down the rule for a function, $P(n)$, which describes the profit when the Exhibition Centre sells n posters.

ii Sketch the graph of P against n. (Use a continuous model.)

5 The organisers of a sporting event know that, on average, 50 000 people will visit the venue each day. They are presently charging \$15.00 for an admission ticket. Each time in the past when they have raised the admission price, an average of 2500 fewer people have come to the venue for each \$1.00 increase in ticket price. Let x represent the number of \$1.00 increases.

a Write the rule for a function which gives the revenue, R, in terms of x.

b Sketch the graph of R against x.

c Find the price which will maximise the revenue.

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Review

6 A thin wire of length a cm is bent to form the perimeter of a pentagon $ABCDE$ in which $BCDE$ is a rectangle and ABE is an equilateral triangle. Let x cm be the length of CD and let $A(x)$ be the area of the pentagon.

a Find $A(x)$ in terms of x.

b State the allowable values for x.

c Show that the maximum area is $\dfrac{a^2}{4(6-\sqrt{3})}$ cm^2.

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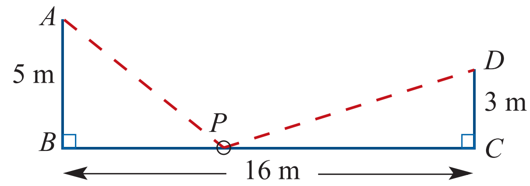
7 Let P be a point between B and C on the line BC.
Let $d(x)$ be the distance $(PA + PD)$ m, where x is the distance of P from B.



a **i** Find an expression for $d(x)$.

ii Find the allowable values of x.

b **i** Use a calculator to plot the graph of $y = d(x)$ for a suitable window setting.

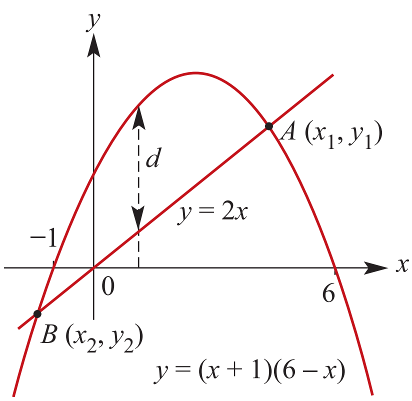
ii Find the value of x if $d(x) = 20$ (correct to two decimal places).

iii Find the values of x for which $d(x) = 19$ (correct to two decimal places).

c **i** Find the minimum value of $d(x)$ and the value of x for which this occurs.

ii State the range of the function.

8 **a** Find the coordinates of the points $A(x_1, y_1)$ and $B(x_2, y_2)$.



b Let $d(x)$ be the 'vertical' distance between the graphs for $x \in [x_2, x_1]$.

i Find $d(x)$ in terms of x.

ii Use a calculator to plot the graph of $d(x)$ against x for $x \in [x_2, x_1]$, and on the same screen plot the graphs of $y = 2x$ and $y = (x + 1)(6 - x)$.

c **i** State the maximum value of the function defined by $d(x)$ for $x \in [x_2, x_1]$.

ii State the range of this function.

d Repeat with the graphs $y = 5x$ and $y = (x + 1)(6 - x)$.

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