14

Circular functions

Objectives

- To use **radians** and **degrees** for the measurement of angle.
- To convert radians to degrees, and vice versa.
- To define the circular functions **sine**, **cosine** and **tangent**.
- To explore the **symmetry properties** of circular functions.
- To find standard **exact values** of circular functions.
- To solve equations involving circular functions.
- To understand and sketch the **graphs of circular functions**.

Following on from our study of polynomial, exponential and logarithmic functions, we meet a further three important functions in this chapter. Again we use the notation developed in Chapter 5 for describing functions and their properties.

You have studied trigonometry in earlier years, mainly for angles between 0° and 90°. In this chapter we see how the trigonometry you have studied may be extended to form three new functions: sine, cosine and tangent. We will see that the first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

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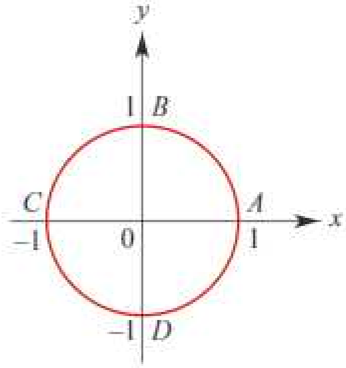
14A Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

Thus, the distance in an anticlockwise direction around the circle from

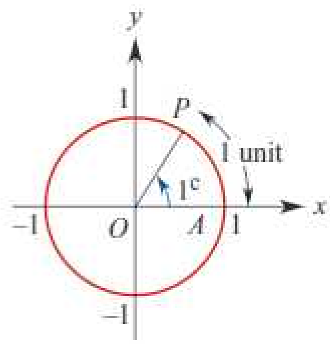
A to $B = \frac{\pi}{2}$ units

A to $C = \pi$ units

A to $D = \frac{3\pi}{2}$ units

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad 1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}$$

Example 1

Convert 30° to radians.

Solution	Explanation
$1^\circ = \frac{\pi^c}{180}$	
$\therefore \quad 30^\circ = \frac{30 \times \pi}{180} = \frac{\pi^c}{6}$	Multiply by $\frac{\pi}{180}$ and simplify by cancelling.

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Example 2

Convert $\frac{\pi^c}{4}$ to degrees.

Solution

$$1^c = \frac{180^\circ}{\pi}$$

$$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ$$

Explanation

Multiply by $\frac{180}{\pi}$ and simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\frac{\pi}{4}$ rather than $\frac{\pi^c}{4}$.

Using the TI-Nspire

To convert 32 degrees to radians, enter 32° ► **Rad** as shown.

- The degree symbol ° can be found using [π▸] or the symbols palette ([ctrl][⌬]).
- The ► **Rad** command can be found in the catalog ([⌬][1][R]).

To convert 2 radians to degrees, enter 2^r ► **DD** as shown.

- The radian symbol ^{r} can be found using [π▸] or the symbols palette ([ctrl][⌬]).
- The ► **DD** command can be found in the catalog ([⌬][1][D]).

Note: If the calculator is in radian mode, you can convert 32° to radians by simply typing 32° then [enter]. If the calculator is in degree mode, type 2^r then [enter].

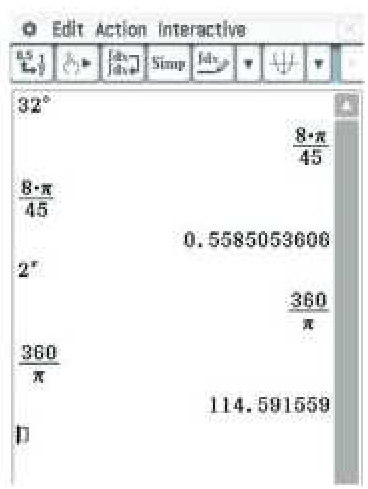
Using the Casio ClassPad

To convert 32 degrees to radians:

- Ensure your calculator is in radian mode (with **Rad** in the status bar at the bottom of the main screen).
- Enter 32° and tap [EXE]. The degree symbol ° is found in the [Math1] keyboard.
- The answer can be displayed exactly, as shown, or highlight the answer and tap [⇄] to convert to decimal.

To convert 2 radians to degrees:

- Ensure your calculator is in degree mode (**Deg**).
- Enter 2^r and tap [EXE]. The radian symbol ^{r} is found in the [Math1] keyboard.



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Summary 14A

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\frac{\pi}{180}$
  - radians to degrees, multiply by $\frac{180}{\pi}$.

Exercise 14A

Example 1

1 Express the following angles in radian measure in terms of π:

a $60°$ **b** $144°$ **c** $240°$ **d** $330°$ **e** $420°$ **f** $480°$

Example 2

2 Express, in degrees, the angles with the following radian measures:

a $\frac{2\pi}{3}$ **b** $\frac{5\pi}{6}$ **c** $\frac{7\pi}{6}$ **d** 0.9π

e $\frac{5\pi}{9}$ **f** $\frac{9\pi}{5}$ **g** $\frac{11\pi}{9}$ **h** 1.8π

3 Use a calculator to convert the following angles from radians to degrees:

a 0.6 **b** 1.89 **c** 2.9 **d** 4.31

e 3.72 **f** 5.18 **g** 4.73 **h** 6.00

4 Use a calculator to express the following in radian measure:

a $38°$ **b** $73°$ **c** $107°$ **d** $161°$

e $84.1°$ **f** $228°$ **g** $136.4°$ **h** $329°$

5 Express, in degrees, the angles with the following radian measures:

a $-\frac{\pi}{3}$ **b** -4π **c** -3π **d** $-\pi$

e $\frac{5\pi}{3}$ **f** $-\frac{11\pi}{6}$ **g** $\frac{23\pi}{6}$ **h** $-\frac{23\pi}{6}$

6 Express each of the following in radian measure in terms of π:

a $-360°$ **b** $-540°$ **c** $-240°$ **d** $-720°$ **e** $-330°$ **f** $-210°$

7 **a** On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{4}$ **ii** $\frac{3\pi}{4}$ **iii** $\frac{5\pi}{4}$ **iv** $\frac{7\pi}{4}$

b On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{3}$ **ii** $-\frac{\pi}{3}$ **iii** $\frac{2\pi}{3}$ **iv** $\frac{4\pi}{3}$

c On a set of axes, draw a unit circle centred at the origin and indicate the position on the unit circle corresponding to each of the following:

i $\frac{\pi}{6}$ **ii** $-\frac{7\pi}{6}$ **iii** $\frac{13\pi}{6}$ **iv** $\frac{17\pi}{6}$

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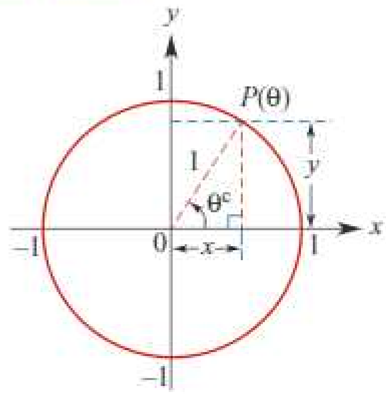
14B Defining circular functions: sine and cosine

Consider the unit circle.

The position of point P on the circle can be described by relating the angle θ to the Cartesian coordinates x and y.

The point P on the circle corresponding to an angle θ is written $P(\theta)$.

The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:



The x-coordinate of $P(\theta)$ is given by $x = \text{cosine}\ \theta$, for $\theta \in \mathbb{R}$.

The y-coordinate of $P(\theta)$ is given by $y = \text{sine}\ \theta$, for $\theta \in \mathbb{R}$.

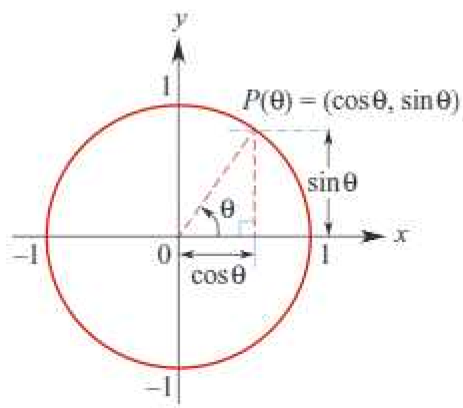
These functions are usually written in an abbreviated form as follows:

$$x = \cos\theta$$

$$y = \sin\theta$$

Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.

Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.



Example 3

Evaluate $\sin\pi$ and $\cos\pi$.

Solution

After moving through an angle of π, the position is $P(\pi) = (-1, 0)$.

$\therefore\ \cos\pi = -1$ and $\sin\pi = 0$

Example 4

Evaluate $\sin\left(-\frac{3\pi}{2}\right)$ and $\cos\left(-\frac{\pi}{2}\right)$.

Solution	Explanation
$\sin\left(-\frac{3\pi}{2}\right) = 1$	The point $P\left(-\frac{3\pi}{2}\right)$ has coordinates $(0, 1)$.
$\cos\left(-\frac{\pi}{2}\right) = 0$	The point $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.

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Example 5

Evaluate $\sin\left(\frac{5\pi}{2}\right)$ and $\sin\left(\frac{7\pi}{2}\right)$.

Solution

$$\sin\left(\frac{5\pi}{2}\right) = \sin\left(2\tfrac{1}{2}\pi\right) = \sin\left(2\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1$$

$$\sin\left(\frac{7\pi}{2}\right) = \sin\left(3\tfrac{1}{2}\pi\right) = \sin\left(2\pi + \frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$$

Example 6

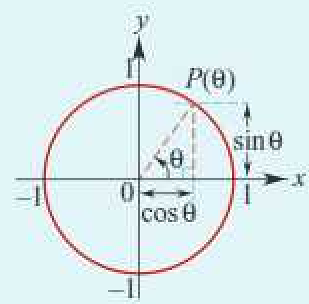
Evaluate $\sin\left(\frac{9\pi}{2}\right)$ and $\cos(27\pi)$.

Solution

$$\sin\left(\frac{9\pi}{2}\right) = \sin\left(4\pi + \frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 \qquad \cos(27\pi) = \cos(26\pi + \pi) = \cos\pi = -1$$

Summary 14B

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$
- If an angle is an odd multiple of $\frac{\pi}{2}$, then its sine is either 1 or −1, and its cosine is zero.
- If an angle is an even multiple of $\frac{\pi}{2}$, then its sine is zero, and its cosine is either 1 or −1.



Exercise 14B

Example 3
Example 4
Example 5
Example 6

1 For each of the following angles, t, determine the values of $\sin t$ and $\cos t$:

a $t = 0$ **b** $t = \frac{3\pi}{2}$ **c** $t = -\frac{3\pi}{2}$ **d** $t = \frac{5\pi}{2}$

e $t = -3\pi$ **f** $t = \frac{9\pi}{2}$ **g** $t = \frac{7\pi}{2}$ **h** $t = 4\pi$

2 Evaluate using your calculator. (Check that your calculator is in radian mode.)

a $\sin 1.9$ **b** $\sin 2.3$ **c** $\sin 4.1$ **d** $\cos 0.3$

e $\cos 2.1$ **f** $\cos(-1.6)$ **g** $\sin(-2.1)$ **h** $\sin(-3.8)$

3 For each of the following angles, θ, determine the values of $\sin\theta$ and $\cos\theta$:

a $\theta = 27\pi$ **b** $\theta = -\frac{5\pi}{2}$ **c** $\theta = \frac{27\pi}{2}$ **d** $\theta = -\frac{9\pi}{2}$

e $\theta = \frac{11\pi}{2}$ **f** $\theta = 57\pi$ **g** $\theta = 211\pi$ **h** $\theta = -53\pi$

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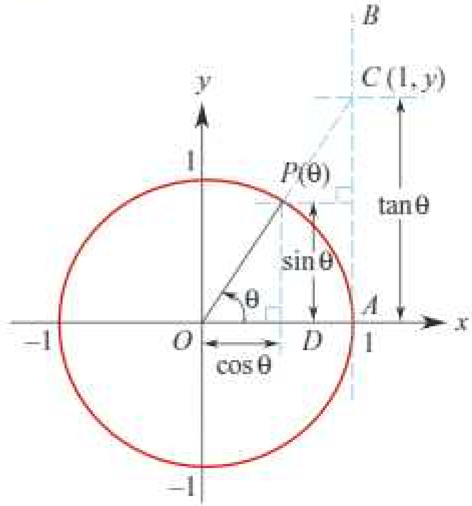
14C Another circular function: tangent

Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the extension of OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).



By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note that $\tan\theta$ is undefined when $\cos\theta = 0$.

Hence $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \dots$

The domain of tan is $\mathbb{R} \setminus \{\, \theta : \cos\theta = 0 \,\}$.

Example 7

Evaluate using a calculator. (Give answers to two decimal places.)

a $\tan 1.3$ **b** $\tan 1.9$ **c** $\tan(-2.8)$ **d** $\tan 59^\circ$ **e** $\tan 138^\circ$

Solution	Explanation
a $\tan 1.3 = 3.60$	Don't forget that your calculator must be in radian mode.
b $\tan 1.9 = -2.93$	$\cos 1.9$ is negative.
c $\tan(-2.8) = 0.36$	Both $\cos(-2.8)$ and $\sin(-2.8)$ are negative, so tan is positive.
d $\tan 59^\circ = 1.66$	Calculate in degree mode.
e $\tan 138^\circ = -0.90$	

Exercise 14C

1 Evaluate:

a $\tan\pi$ **b** $\tan(-\pi)$ **c** $\tan\left(\frac{7\pi}{2}\right)$ **d** $\tan(-2\pi)$ **e** $\tan\left(\frac{5\pi}{2}\right)$ **f** $\tan\left(-\frac{\pi}{2}\right)$

Example 7

2 Use a calculator to find correct to two decimal places:

a $\tan 1.6$ **b** $\tan(-1.2)$ **c** $\tan 136^\circ$ **d** $\tan(-54^\circ)$
e $\tan 3.9$ **f** $\tan(-2.5)$ **g** $\tan 239^\circ$

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3 For each of the following values of θ, find $\tan\theta$:

a $\theta = 180^\circ$ **b** $\theta = 360^\circ$ **c** $\theta = 0$
d $\theta = -180^\circ$ **e** $\theta = -540^\circ$ **f** $\theta = 720^\circ$

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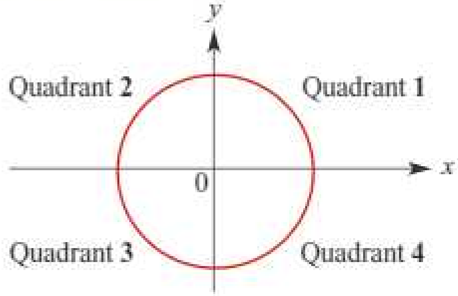
14D Symmetry properties of circular functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.



Using symmetry, we can determine relationships between the circular functions for angles in different quadrants.

Quadrant 2

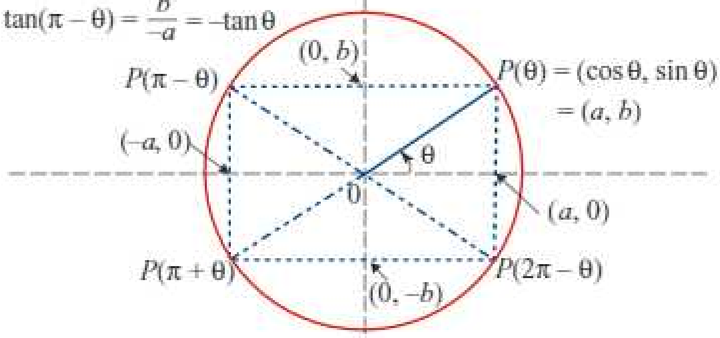
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

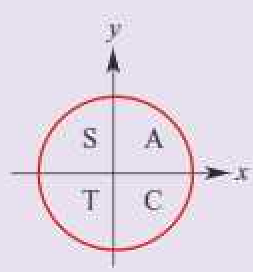
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of circular functions

Using these symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



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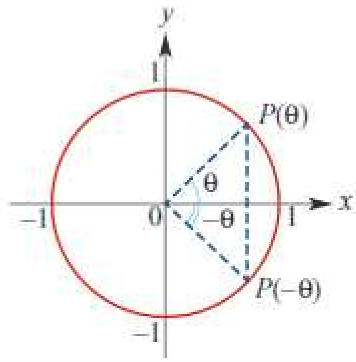
Negative of angles

By symmetry:

$$\cos(-\theta) = \cos\theta$$

$$\sin(-\theta) = -\sin\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Example 8

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a $\sin(\pi - x) = \sin x = 0.6$

b $\sin(\pi + x) = -\sin x = -0.6$

c $\sin(2\pi - x) = -\sin x = -0.6$

d $\sin(-x) = -\sin x = -0.6$

Example 9

If $\cos x^\circ = 0.8$, find the value of:

a $\cos(180 - x)^\circ$ **b** $\cos(180 + x)^\circ$ **c** $\cos(360 - x)^\circ$ **d** $\cos(-x)^\circ$

Solution

a $\cos(180 - x)^\circ = -\cos x^\circ = -0.8$

b $\cos(180 + x)^\circ = -\cos x^\circ = -0.8$

c $\cos(360 - x)^\circ = \cos x^\circ = 0.8$

d $\cos(-x)^\circ = \cos x^\circ = 0.8$

Exercise 14D

Example 8

1 If $\sin\theta = 0.42$, $\cos x = 0.7$ and $\tan\alpha = 0.38$, write down the values of:

a $\sin(\pi + \theta)$ **b** $\cos(\pi - x)$ **c** $\sin(2\pi - \theta)$

d $\tan(\pi - \alpha)$ **e** $\sin(\pi - \theta)$ **f** $\tan(2\pi - \alpha)$

g $\cos(\pi + x)$ **h** $\cos(2\pi - x)$

Example 9

2 If $\sin x^\circ = 0.7$, $\cos\theta^\circ = 0.6$ and $\tan\alpha^\circ = 0.4$, write down the values of:

a $\sin(180 + x)^\circ$ **b** $\cos(180 + \theta)^\circ$ **c** $\tan(360 - \alpha)^\circ$

d $\cos(180 - \theta)^\circ$ **e** $\sin(360 - x)^\circ$ **f** $\sin(-x)^\circ$

g $\tan(360 + \alpha)^\circ$ **h** $\cos(-\theta)^\circ$

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3 **a** If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\frac{\pi}{2} < x < \pi$, find the value of x.

b If $\cos x = -\cos\left(\frac{\pi}{6}\right)$ and $\pi < x < \frac{3\pi}{2}$, find the value of x.

c If $\cos x = \cos\left(\frac{\pi}{6}\right)$ and $\frac{3\pi}{2} < x < 2\pi$, find the value of x.

4 For the diagram shown, write down the values of:

a $a = \cos(\pi - \theta)$

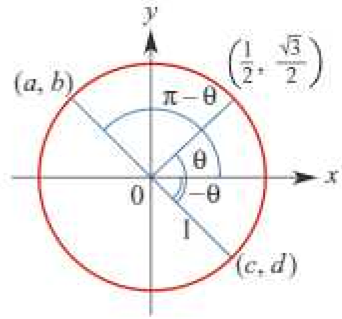
b $b = \sin(\pi - \theta)$

c $c = \cos(-\theta)$

d $d = \sin(-\theta)$

e $\tan(\pi - \theta)$

f $\tan(-\theta)$



5 For the diagram shown, write down the values of:

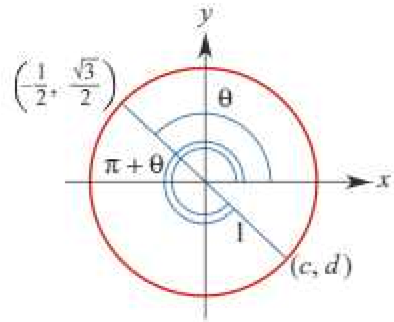
a $d = \sin(\pi + \theta)$

b $c = \cos(\pi + \theta)$

c $\tan(\pi + \theta)$

d $\sin(2\pi - \theta)$

e $\cos(2\pi - \theta)$

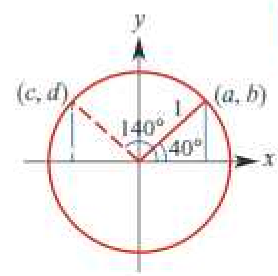


6 **a** For the diagram shown, use your calculator to find a and b correct to four decimal places.

b Hence find the values of c and d.

c **i** Use your calculator to find $\cos 140^\circ$ and $\sin 140^\circ$.

ii Write $\cos 140^\circ$ in terms of $\cos 40^\circ$.



7 **a** If $\sin x^\circ = \sin 60^\circ$ and $90^\circ < x^\circ < 180^\circ$, find the value of x.

b If $\sin x^\circ = -\sin 60^\circ$ and $180^\circ < x^\circ < 270^\circ$, find the value of x.

c If $\sin x^\circ = -\sin 60^\circ$ and $-90^\circ < x^\circ < 0^\circ$, find the value of x.

d If $\cos x^\circ = -\cos 60^\circ$ and $90^\circ < x^\circ < 180^\circ$, find the value of x.

e If $\cos x^\circ = -\cos 60^\circ$ and $180^\circ < x^\circ < 270^\circ$, find the value of x.

f If $\cos x^\circ = \cos 60^\circ$ and $270^\circ < x^\circ < 360^\circ$, find the value of x.

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14E Values of circular functions

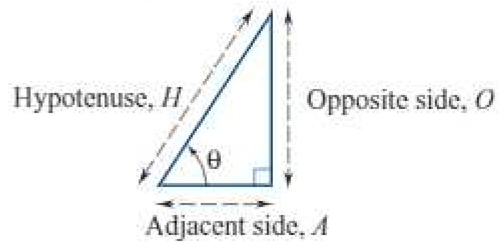
Circular functions and trigonometric ratios

For right-angled triangles:

$$\sin\theta = \frac{O}{H}$$

$$\cos\theta = \frac{A}{H}$$

$$\tan\theta = \frac{O}{A}$$

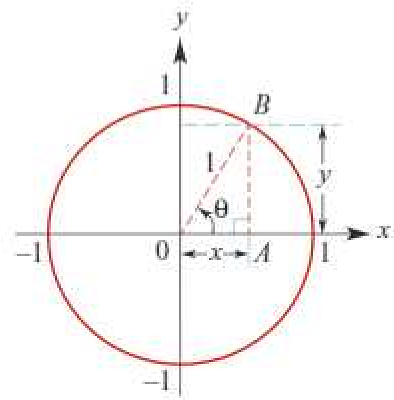


Applying these trigonometric ratios to the right-angled triangle, OAB, in the unit circle:

$$\sin\theta = \frac{O}{H} = \frac{y}{1} = y$$

$$\cos\theta = \frac{A}{H} = \frac{x}{1} = x$$

$$\tan\theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$$



For $0 < \theta < \frac{\pi}{2}$, the functions sin, cos and tan defined by the trigonometric ratios agree with the circular functions introduced in this chapter.

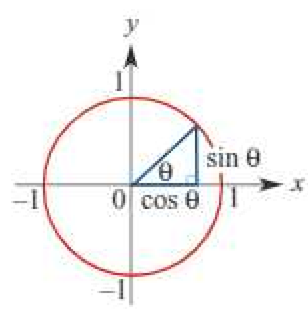
Exact values of circular functions

A calculator can be used to find the values of the circular functions for different values of θ. For many values of θ, the calculator gives an approximation. We now consider some values of θ such that sin, cos and tan can be calculated exactly.

Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

$\sin 0^\circ = 0$	$\sin 90^\circ = 1$
$\cos 0^\circ = 1$	$\cos 90^\circ = 0$
$\tan 0^\circ = 0$	$\tan 90^\circ$ is undefined



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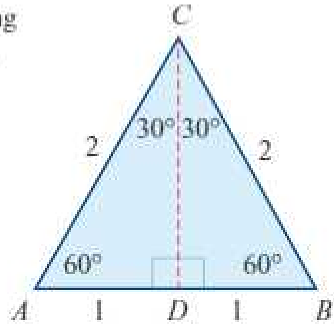
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units. Using Pythagoras' theorem in $\triangle ACD$ gives $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.



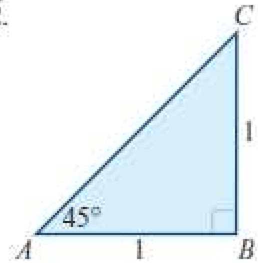
$$\sin 30^\circ = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30^\circ = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60^\circ = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30^\circ = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60^\circ = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.



$$\sin 45^\circ = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45^\circ = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45^\circ = \frac{BC}{AB} = 1$$

Exact values using symmetry

We can now use symmetry to find exact values for some angles outside the first quadrant.

Example 10

a Evaluate $\cos 150^\circ$.

b Evaluate $\sin 690^\circ$.

Solution

a $\cos 150^\circ = \cos(180 - 30)^\circ$

$= -\cos 30^\circ = -\frac{\sqrt{3}}{2}$

b $\sin 690^\circ = \sin(2 \times 360 - 30)^\circ$

$= \sin(-30^\circ) = -\frac{1}{2}$

Example 11

a Evaluate $\cos\left(\frac{5\pi}{4}\right)$.

b Evaluate $\sin\left(\frac{11\pi}{6}\right)$.

Solution

a $\cos\left(\frac{5\pi}{4}\right) = \cos\left(\pi + \frac{\pi}{4}\right)$

$= -\cos\left(\frac{\pi}{4}\right)$ (by symmetry)

$= -\frac{1}{\sqrt{2}}$

b $\sin\left(\frac{11\pi}{6}\right) = \sin\left(2\pi - \frac{\pi}{6}\right)$

$= -\sin\left(\frac{\pi}{6}\right)$ (by symmetry)

$= -\frac{1}{2}$

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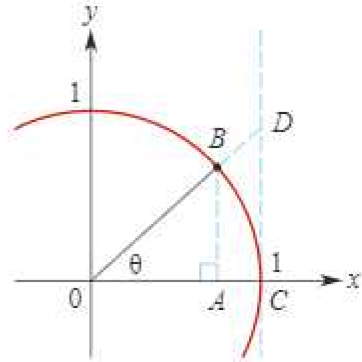
Approximating $\sin\theta$ for small θ

The diagram shows part of the unit circle, where $0 < \theta < \frac{\pi}{2}$.



The arc from C to B has length θ. So we see that

$$AB < \theta < CD$$

$$\therefore \quad \sin\theta < \theta < \tan\theta$$

Dividing through by $\sin\theta$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

Taking reciprocals reverses the inequalities:

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

We know that $\cos\theta \to 1$ as $\theta \to 0$. It follows that $\frac{\sin\theta}{\theta} \to 1$ as $\theta \to 0$.

Hence $\sin\theta \approx \theta$ for small values of θ. (This also holds for small negative values of θ.)

Summary 14E

As an aid to memory, the exact values for circular functions can be tabulated.

θ	0	$\frac{\pi}{6}$ (30°)	$\frac{\pi}{4}$ (45°)	$\frac{\pi}{3}$ (60°)	$\frac{\pi}{2}$ (90°)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

Exercise 14E

Example 10

1 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 120° **b** 135° **c** 210° **d** 240° **e** 315°

f 390° **g** 420° **h** −135° **i** −300° **j** −60°

Example 11

2 Write down the exact values of:

a $\sin\left(\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{3\pi}{4}\right)$ **c** $\tan\left(\frac{5\pi}{6}\right)$ **d** $\sin\left(\frac{7\pi}{6}\right)$ **e** $\cos\left(\frac{5\pi}{4}\right)$

f $\tan\left(\frac{4\pi}{3}\right)$ **g** $\sin\left(\frac{5\pi}{3}\right)$ **h** $\cos\left(\frac{7\pi}{4}\right)$ **i** $\tan\left(\frac{11\pi}{6}\right)$

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3 Write down the exact values of:

a $\sin\left(-\frac{2\pi}{3}\right)$ **b** $\cos\left(\frac{11\pi}{4}\right)$ **c** $\tan\left(\frac{13\pi}{6}\right)$ **d** $\tan\left(\frac{15\pi}{6}\right)$

e $\cos\left(\frac{14\pi}{4}\right)$ **f** $\cos\left(-\frac{3\pi}{4}\right)$ **g** $\sin\left(\frac{11\pi}{4}\right)$ **h** $\cos\left(-\frac{21\pi}{3}\right)$

4 Compare the values of θ and $\sin\theta$ for each of the following:

a $\theta = 0.1$ **b** $\theta = 0.2$ **c** $\theta = -0.1$ **d** $\theta = -0.2$

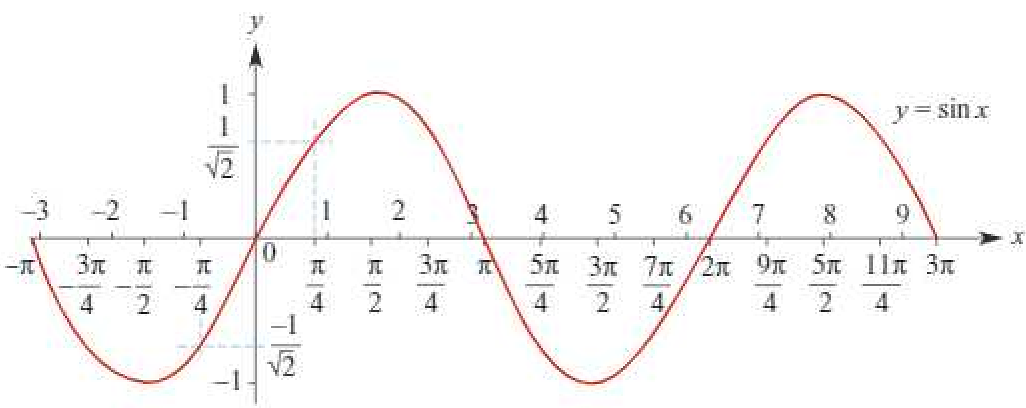
14F Graphs of sine and cosine

Graph of the sine function

A table of exact values for $y = \sin x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0

A calculator can be used to plot the graph of $y = \sin x$ (for $-\pi \le x \le 3\pi$). Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength). Thus $y = \sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the 'mean position' and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

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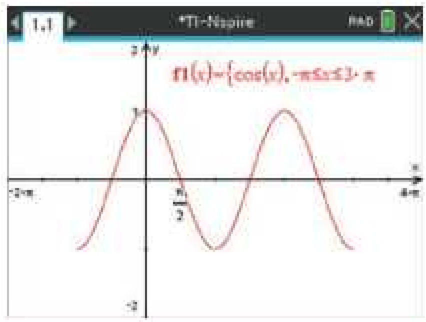
Graph of the cosine function

A table of exact values for $y = \cos x$ is given below.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1	$\frac{-1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$\frac{-1}{\sqrt{2}}$	-1

Using the TI-Nspire

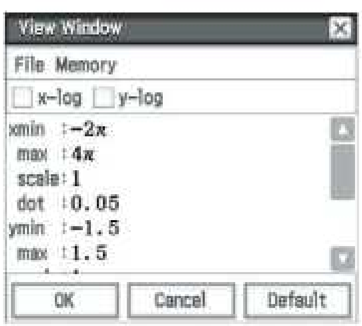
- A graph of $y = \cos x$ for $-\pi \le x \le 3\pi$ can be plotted in a **Graphs** application by entering $f1(x) = \cos(x) \mid -\pi \le x \le 3\pi$.
- Change the window to suit (menu > **Window/Zoom** > **Window Settings**).

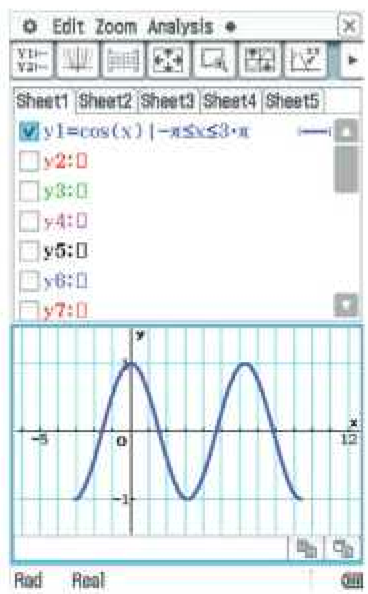


Using the Casio ClassPad

To plot the graph of $y = \cos x$ for $-\pi \le x \le 3\pi$:

- Ensure that your calculator is in radian mode.
- In **Graph & Table**, enter the equation in $y1$ as shown, then tick to select and tap to produce the graph.
- Select the icon to adjust the window settings as shown below.





Observations from the graph of $y = \cos x$

- The period is 2π and the amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

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Sketch graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

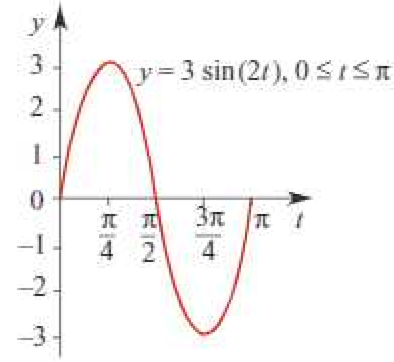
The graphs of functions of the forms $y = a\sin(nt)$ and $y = a\cos(nt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and n are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{2}$ from the y-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	−3	0

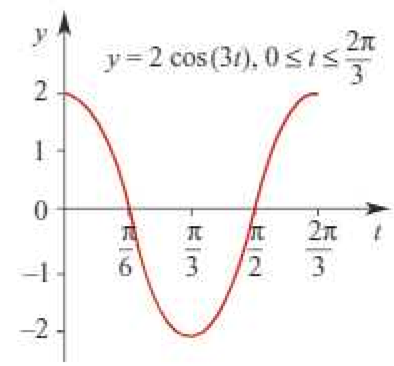


We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 from the t-axis and a dilation of factor $\frac{1}{3}$ from the y-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	−2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	Amplitude	Period
$y = a\sin(nt)$	a	$\frac{2\pi}{n}$
$y = a\cos(nt)$	a	$\frac{2\pi}{n}$

Note: The dilation from the t-axis determines the amplitude.
The dilation from the y-axis determines the period.

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Example 12

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$ **b** $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$ **c** $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

In general, for a and n positive numbers, the graph of $y = a\sin(nt)$ (or $y = a\cos(nt)$) is obtained from the graph of $y = \sin t$ (or $y = \cos t$) by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

Example 13

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$

b $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 from the x-axis and a dilation of factor 2 from the y-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

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Example 14

Sketch the graph of each of the following functions:

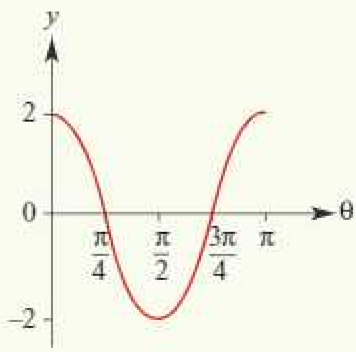
a $y = 2\cos(2\theta)$

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a $y = 2\cos(2\theta)$



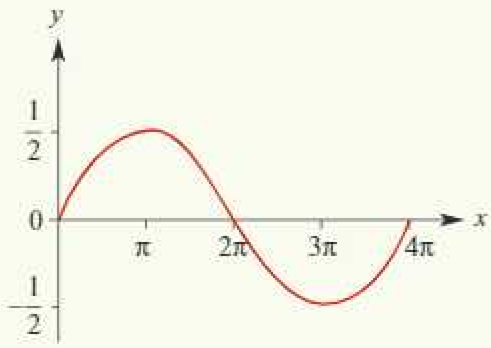
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

b $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$



The amplitude is $\frac{1}{2}$.

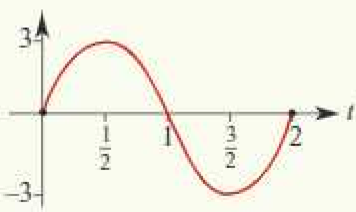
The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin(\frac{x}{2})$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 2 from the y-axis.

Example 15

Sketch the graph of $f\colon [0, 2] \to \mathbb{R}$, $f(t) = 3\sin(\pi t)$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{\pi}$ from the y-axis.

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Transformations: reflection in the horizontal axis

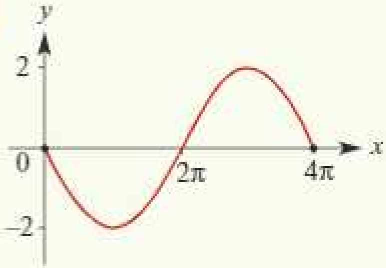
Example 16

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

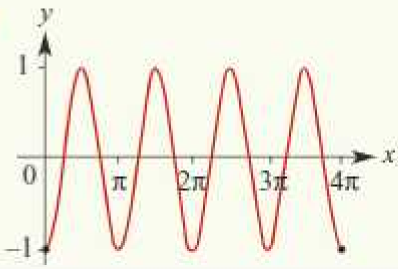


Explanation

The graph of $f(x) = -2\sin(\frac{x}{2})$ is obtained from the graph of $y = 2\sin(\frac{x}{2})$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Transformations: reflection in the vertical axis

Remember that $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$.

Hence, when reflected in the y-axis:

- The graph of $y = \cos x$ transforms onto itself. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = \cos x$ by the observation above.)
- The graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$. (The point (α, β) is mapped to the point $(-\alpha, \beta)$. This second point is on the curve $y = -\sin x$ by the observation above.)

Summary 14F

For positive numbers a and n, the graphs of $y = a\sin(nt)$, $y = -a\sin(nt)$, $y = a\cos(nt)$ and $y = -a\cos(nt)$ all have the following properties:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Skill-sheet

Exercise 14F

Example 12

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $2\sin\theta$ **b** $3\sin(2\theta)$ **c** $\frac{1}{2}\cos(3\theta)$

d $3\sin\left(\frac{1}{2}\theta\right)$ **e** $4\cos(3\theta)$ **f** $-\frac{1}{2}\sin(4\theta)$

g $-2\cos\left(\frac{1}{2}\theta\right)$ **h** $2\cos(\pi t)$ **i** $-3\sin\left(\frac{\pi t}{2}\right)$

Example 13

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin x$ **b** $g(x) = \sin(5x)$

c $g(x) = \sin\left(\frac{x}{3}\right)$ **d** $g(x) = 2\sin(5x)$

3 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = -\sin(5x)$ **b** $g(x) = \sin(-x)$ **c** $g(x) = 2\sin\left(\frac{x}{3}\right)$

d $g(x) = -4\sin\left(\frac{x}{2}\right)$ **e** $g(x) = 2\sin\left(\frac{-x}{3}\right)$

Example 14

4 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 3\sin(2x)$ **b** $y = 2\cos(3\theta)$ **c** $y = 4\sin\left(\frac{\theta}{2}\right)$

d $y = \frac{1}{2}\cos(3x)$ **e** $y = 4\sin(3x)$ **f** $y = 5\cos(2x)$

g $y = -3\cos\left(\frac{\theta}{2}\right)$ **h** $y = 2\cos(4\theta)$ **i** $y = -2\sin\left(\frac{\theta}{3}\right)$

Example 15

5 Sketch the graph of:

a $f\colon [0,2] \to \mathbb{R},\ f(t) = 2\cos(\pi t)$ **b** $f\colon [0,2] \to \mathbb{R},\ f(t) = 3\sin(2\pi t)$

Example 16

6 Sketch the graph of:

a $f(x) = \sin(2x)$ for $x \in [-2\pi, 2\pi]$ **b** $f(x) = 2\sin\left(\frac{x}{3}\right)$ for $x \in [-6\pi, 6\pi]$

c $f(x) = 2\cos(3x)$ for $x \in [0, 2\pi]$ **d** $f(x) = -2\sin(3x)$ for $x \in [0, 2\pi]$

7 Sketch the graph of $f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \frac{5}{2}\cos\left(\frac{2x}{3}\right)$.
Hint: For the endpoints, find $f(0)$ and $f(2\pi)$.

8 **a** On the one set of axes, sketch the graphs of $f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = \sin x$ and $g\colon [0, 2\pi] \to \mathbb{R},\ g(x) = \cos x$.

b By inspection from these graphs, state the values of x for which $\sin x = \cos x$.

9 With the help of your calculator, sketch the graphs of $y = \sin x$ and $y = x$ on the same set of axes for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. CAS

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14G Solution of trigonometric equations

In this section we introduce methods for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$.

Solving equations of the form $\sin t = b$ and $\cos t = b$

First we look at the techniques for solving equations of the form $\sin t = b$ and $\cos t = b$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 17

Find all solutions to the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

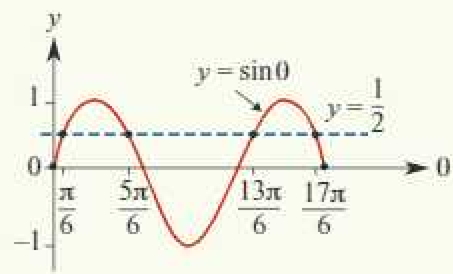
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

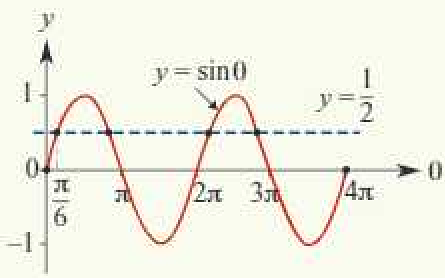
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



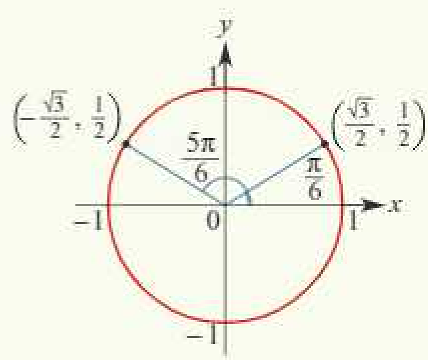
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

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Example 18

Find two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$

b $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

Solution

a First solve the equation $\sin \alpha = 0.3$ for $\alpha \in \left[0, \frac{\pi}{2}\right]$. Use your calculator to find the solution $\alpha = 0.30469\ldots$

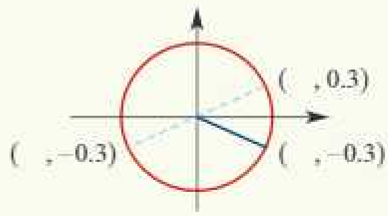
Now the value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants. From the symmetry relationships (or from the graph of $y = \sin x$):

3rd quadrant: $x = \pi + 0.30469\ldots$

$= 3.446$ (to 3 d.p.)

4th quadrant: $x = 2\pi - 0.30469\ldots$

$= 5.978$ (to 3 d.p.)



$\therefore$ If $\sin x = -0.3$, then $x = 3.446$ or $x = 5.978$.

b First solve the equation $\cos \alpha^\circ = 0.7$ for $\alpha^\circ \in [0^\circ, 90^\circ]$. Use your calculator to find the solution $\alpha^\circ = 45.57^\circ$.

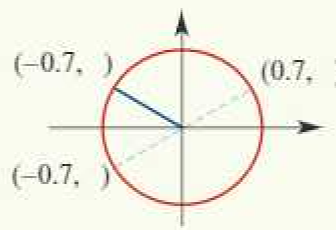
Now the value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

2nd quadrant: $x^\circ = 180^\circ - 45.57^\circ$

$= 134.43^\circ$

3rd quadrant: $x^\circ = 180^\circ + 45.57^\circ$

$= 225.57^\circ$



$\therefore$ If $\cos x^\circ = -0.7$, then $x^\circ = 134.43^\circ$ or $x^\circ = 225.57^\circ$.

Example 19

Find all the values of θ° between 0° and 360° for which:

a $\cos \theta^\circ = \dfrac{\sqrt{3}}{2}$ **b** $2 \sin \theta^\circ = -1$ **c** $\cos \theta^\circ - \dfrac{1}{\sqrt{2}} = 0$

Solution	Explanation
a $\cos \theta^\circ = \dfrac{\sqrt{3}}{2}$	$\cos \theta^\circ$ is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.
$\theta^\circ = 30^\circ$ or $\theta^\circ = 360^\circ - 30^\circ$	$\cos(360^\circ - \theta^\circ) = \cos \theta^\circ$
$\theta^\circ = 30^\circ$ or $\theta^\circ = 330^\circ$	

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b $2\sin\theta^\circ = -1$

$$\sin\theta^\circ = -\frac{1}{2}$$

sin θ° is negative, and so $P(\theta^\circ)$ lies in the 3rd or 4th quadrant.

$$\theta^\circ = 180^\circ + 30^\circ \quad \text{or} \quad \theta^\circ = 360^\circ - 30^\circ$$

$$\theta^\circ = 210^\circ \quad \text{or} \quad \theta^\circ = 330^\circ$$

$\sin(180^\circ + \theta^\circ) = -\sin\theta^\circ$

$\sin(360^\circ - \theta^\circ) = -\sin\theta^\circ$

c $\cos\theta^\circ - \frac{1}{\sqrt{2}} = 0$

$$\cos\theta^\circ = \frac{1}{\sqrt{2}}$$

cos θ° is positive, and so $P(\theta^\circ)$ lies in the 1st or 4th quadrant.

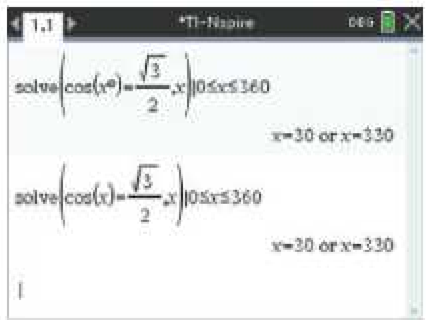
$$\theta^\circ = 45^\circ \quad \text{or} \quad \theta^\circ = 360^\circ - 45^\circ$$

$$\theta^\circ = 45^\circ \quad \text{or} \quad \theta^\circ = 315^\circ$$

Using the TI-Nspire

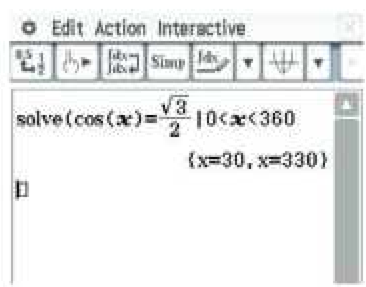
We show two methods for Example 19 **a**:

- For the first method, the calculator is in radian mode, and the degree symbol must be used in the equation as shown.
- For the second method, the calculator is in degree mode, and the degree symbol is not required.



Using the Casio ClassPad

- Ensure that your calculator is in degree mode (with **Deg** in the status bar at the bottom of the main screen).
- Select [solve(] from the [Math1] or [Math3] keyboard.
- Use the [Math1], [Math3] and [Trig] keyboards to enter the equation and domain as shown.
- Tap [EXE].



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Solving equations of the form $a \sin(nt) = b$ and $a \cos(nt) = b$

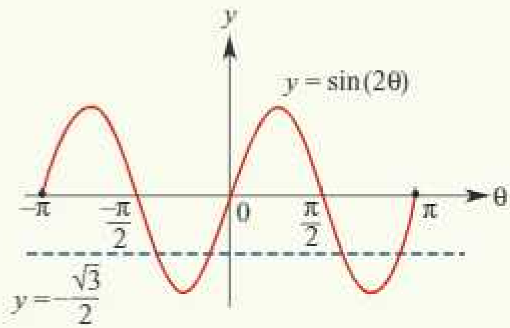
The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 20

Solve the equation $\sin(2\theta) = -\frac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.



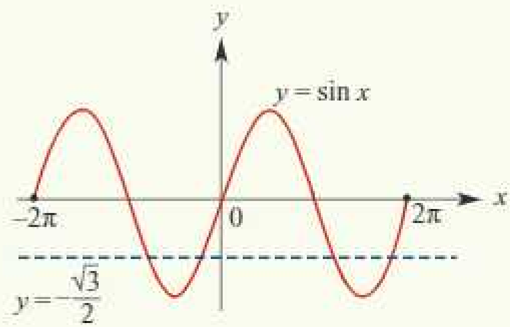
To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have $x = 2\theta \in [-2\pi, 2\pi]$.

Now consider the equation

$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$

The 1st quadrant solution to the equation $\sin \alpha = \frac{\sqrt{3}}{2}$ is $\alpha = \frac{\pi}{3}$.



Using symmetry, the solutions to $\sin x = -\frac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \frac{\pi}{3}$ and $x = 2\pi - \frac{\pi}{3}$

i.e. $x = \frac{4\pi}{3}$ and $x = \frac{5\pi}{3}$

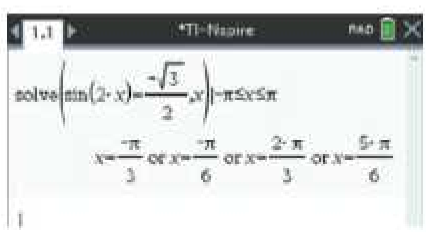
The other two solutions (obtained by subtracting 2π) are $x = \frac{4\pi}{3} - 2\pi$ and $x = \frac{5\pi}{3} - 2\pi$.

$\therefore$ The required solutions for x are $-\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\frac{\pi}{3}, -\frac{\pi}{6}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

Using the TI-Nspire

Ensure that the calculator is in radian mode and complete as shown.



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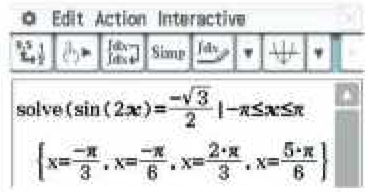
Using the Casio ClassPad

- Ensure that your calculator is in radian mode (with **Rad** in the status bar at bottom of the main screen).
- Use [solve(] and complete as shown.



Summary 14G

- For solving equations of the form $\sin t = b$ and $\cos t = b$:
  - First find the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$:
  - First substitute $x = nt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Skill-sheet

Exercise 14G

Example 17

1 Find all the values of x between 0 and 4π for which:

a $\cos x = \frac{1}{2}$ **b** $\sin x = \frac{1}{\sqrt{2}}$ **c** $\sin x = \frac{\sqrt{3}}{2}$

Example 18

2 Find, correct to two decimal places, all the values of x between 0 and 2π for which:

a $\sin x = 0.8$ **b** $\cos x = -0.4$ **c** $\sin x = -0.35$
d $\sin x = 0.4$ **e** $\cos x = -0.7$ **f** $\cos x = -0.2$

Example 19

3 Without using a calculator, find all the values of θ° between 0° and 360° for each of the following:

a $\cos\theta^\circ = -\frac{\sqrt{3}}{2}$ **b** $\sin\theta^\circ = \frac{1}{2}$ **c** $\cos\theta^\circ = -\frac{1}{2}$
d $2\cos(\theta^\circ) + 1 = 0$ **e** $2\sin\theta^\circ = \sqrt{3}$ **f** $\sqrt{2}\sin(\theta^\circ) - 1 = 0$

4 Without using a calculator, find all the values of x between 0 and 2π for each of the following:

a $2\cos x = \sqrt{3}$ **b** $\sqrt{2}\sin(x) + 1 = 0$ **c** $\sqrt{2}\cos(x) - 1 = 0$

5 Find all the values of x between $-\pi$ and π for which:

a $\cos x = -\frac{1}{\sqrt{2}}$ **b** $\sin x = \frac{\sqrt{3}}{2}$ **c** $\cos x = -\frac{1}{2}$

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6 **a** Sketch the graph of $f: [-2\pi, 2\pi] \to \mathbb{R}$, $f(x) = \cos x$.

b On the graph, mark the points with y-coordinate $\frac{1}{2}$ and give the associated x-values.

c On the graph, mark the points with y-coordinate $-\frac{1}{2}$ and give the associated x-values.

Example 20

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\dfrac{1}{2}$ **b** $\cos(2\theta) = \dfrac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \dfrac{1}{2}$

d $\sin(3\theta) = -\dfrac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\dfrac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\dfrac{1}{\sqrt{2}}$

8 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

14H Sketch graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$

In this section, we consider translations of graphs of functions of the form $f(t) = a \sin(nt)$ and $g(t) = a \cos(nt)$ in the direction of the t-axis.

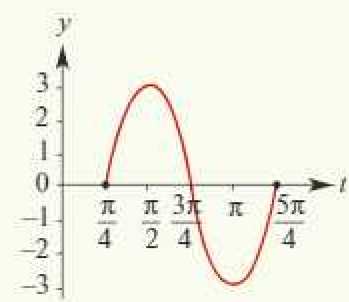
Example 21

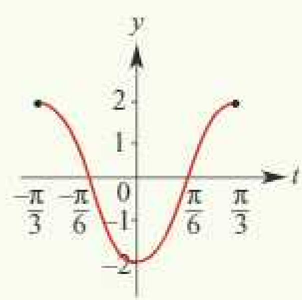
On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

a $y = 3 \sin 2\left(t - \dfrac{\pi}{4}\right)$, $\dfrac{\pi}{4} \le t \le \dfrac{5\pi}{4}$ **b** $y = 2 \cos 3\left(t + \dfrac{\pi}{3}\right)$, $-\dfrac{\pi}{3} \le t \le \dfrac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π. **b** The range is $[-2, 2]$ and the period is $\dfrac{2\pi}{3}$.





Observations from the example

a The graph of $y = 3 \sin 2\left(t - \dfrac{\pi}{4}\right)$ is the same shape as $y = 3\sin(2t)$, but is translated $\dfrac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2 \cos 3\left(t + \dfrac{\pi}{3}\right)$ is the same shape as $y = 2\cos(3t)$, but is translated $\dfrac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm\varepsilon$ is to translate the graph parallel to the t-axis. (Here $\pm\varepsilon$ is called the phase.)

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Determining the transformation

To determine the sequence of transformations, the techniques of Chapter 7 can also be used.

The graph of $y = \sin t$ is transformed to the graph of $y = 3\sin 2\left(t - \dfrac{\pi}{4}\right)$.

Write the second equation as $\dfrac{y'}{3} = \sin 2\left(t' - \dfrac{\pi}{4}\right)$.

From this it can be seen that $y = \dfrac{y'}{3}$ and $t = 2\left(t' - \dfrac{\pi}{4}\right)$. Thus $y' = 3y$ and $t' = \dfrac{t}{2} + \dfrac{\pi}{4}$.

Hence the sequence of transformations is:

- dilation of factor 3 from the t-axis
- dilation of factor $\frac{1}{2}$ from the y-axis
- translation of $\dfrac{\pi}{4}$ units in the positive direction of the t-axis.

Alternatively, we can find this sequence by noting that the graph of $y = f(t)$ is transformed to the graph of $y = 3f\left(2\left(t - \dfrac{\pi}{4}\right)\right)$, where $f(t) = \sin t$.

Summary 14H

The graphs of $y = a\sin n(t \pm \varepsilon)$ and $y = a\cos n(t \pm \varepsilon)$ are translations of the graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase.

Exercise 14H

Example 21

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3\sin\left(0 - \dfrac{\pi}{2}\right)$ **b** $y = \sin 2(0 + \pi)$ **c** $y = 2\sin 3\left(0 + \dfrac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(0 - \dfrac{\pi}{2}\right)$ **e** $y = 3\sin(2x)$ **f** $y = 2\cos 3\left(0 + \dfrac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(0 - \dfrac{\pi}{3}\right)$ **h** $y = -3\sin(2x)$ **i** $y = -3\cos 2\left(0 + \dfrac{\pi}{2}\right)$

2 For the function $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = \cos\left(x - \dfrac{\pi}{3}\right)$:

a find $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

3 For the function $f : [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin 2\left(x - \dfrac{\pi}{3}\right)$:

a find $f(0)$, $f(2\pi)$ **b** sketch the graph of f.

4 For the function $f : [-\pi, \pi] \to \mathbb{R}$, $f(x) = \sin 3\left(x + \dfrac{\pi}{4}\right)$:

a find $f(-\pi)$, $f(\pi)$ **b** sketch the graph of f.

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5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

- **a** dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis
- **b** dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis
- **c** dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis
- **d** dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis
- **e** dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis

14I Sketch graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

We now consider translations parallel to the y-axis.

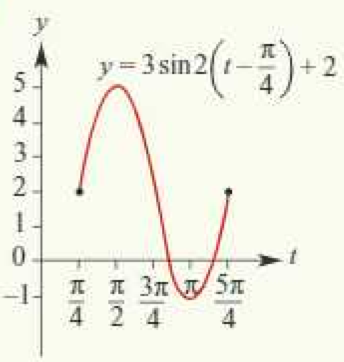
Example 22

Sketch each of the following graphs. Use a calculator to help establish the shape.

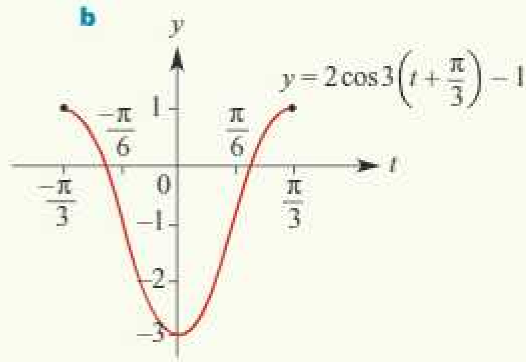
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2, \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2$ is the same shape as the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$, but is translated 2 units in the positive direction of the y-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$ is the same shape as the graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$, but is translated 1 unit in the negative direction of the y-axis.

In general, the effect of $\pm b$ is to translate the graph $\pm b$ units parallel to the y-axis.

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Finding axis intercepts

Example 23

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, the equation $\sqrt{2}\sin(x) + 1 = 0$ must be solved.

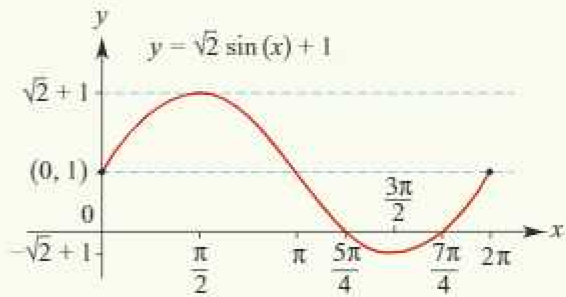
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



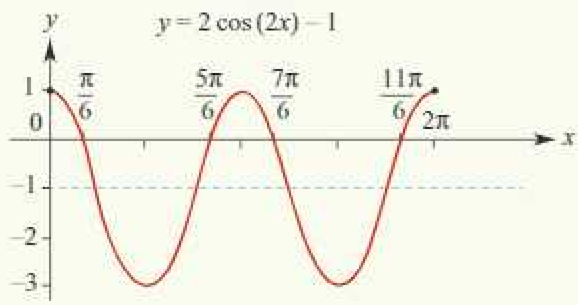
b $$2\cos(2x) - 1 = 0$$

$$\therefore \quad \cos(2x) = \frac{1}{2}$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{11\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

The x-axis intercepts are $\frac{\pi}{6}, \frac{5\pi}{6}$, $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$.



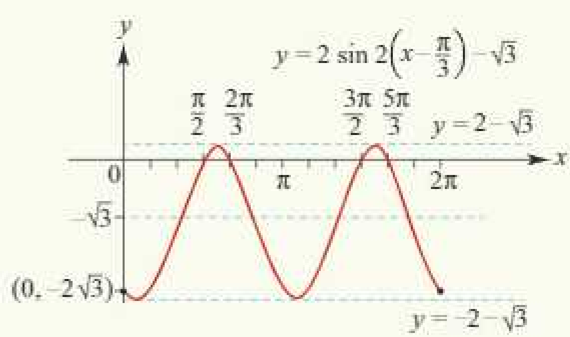
c $$\sin 2\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$\therefore 2\left(x - \frac{\pi}{3}\right) = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3} \text{ or } \frac{8\pi}{3}$$

$$\therefore \quad x - \frac{\pi}{3} = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6} \text{ or } \frac{4\pi}{3}$$

$$\therefore \quad x = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{2} \text{ or } \frac{5\pi}{3}$$

The x-axis intercepts are $\frac{\pi}{2}, \frac{2\pi}{3}$, $\frac{3\pi}{2}$ and $\frac{5\pi}{3}$.



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Skill-sheet

Exercise 14I

Example 23

1 Sketch the graph of each of the following for $x \in [0, 2\pi]$. Label the axis intercepts.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right) + 1$

2 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$. Label the axis intercepts.

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

3 Sketch the graph of each of the following for $x \in [-\pi, \pi]$. Label the axis intercepts.

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$ **c** $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

14J Further symmetry properties and the Pythagorean identity

Complementary relationships

From the diagram to the right,

$$\sin\left(\frac{\pi}{2} - \theta\right) = a$$

and, since $a = \cos\theta$,

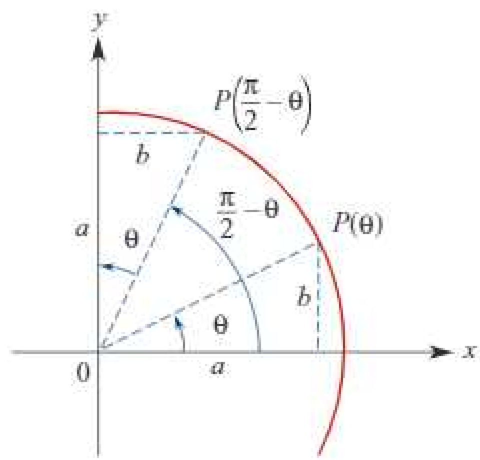
$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$$

From the same diagram,

$$\cos\left(\frac{\pi}{2} - \theta\right) = b$$

and, since $b = \sin\theta$,

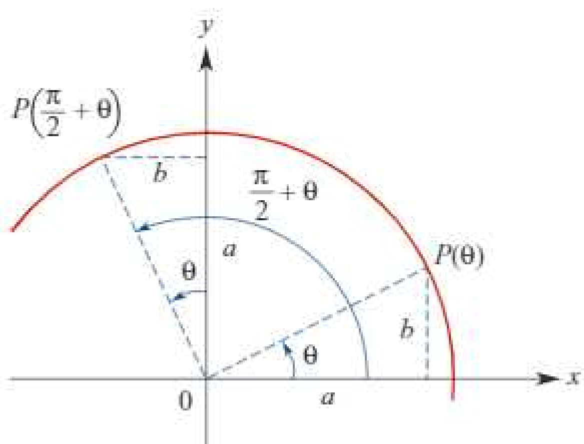
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$

$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



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Example 24

If $\sin\theta = 0.3$ and $\cos\alpha = 0.8$, find the values of:

a $\sin\left(\frac{\pi}{2} - \alpha\right)$ **b** $\cos\left(\frac{\pi}{2} + \theta\right)$ **c** $\sin(-\theta)$

Solution

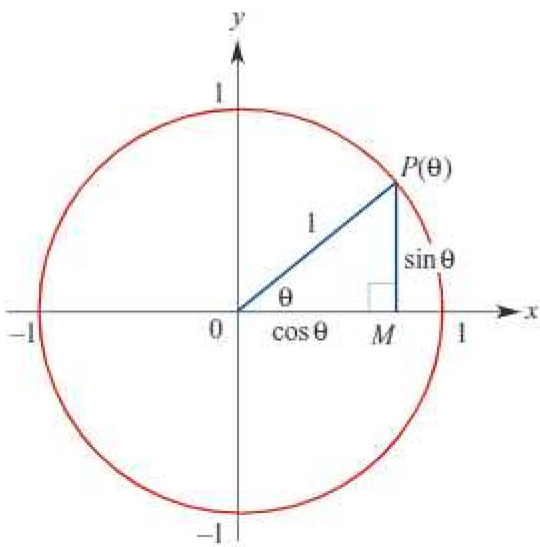
a $\sin\left(\frac{\pi}{2} - \alpha\right) = \cos\alpha$
$= 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$
$= -0.3$

c $\sin(-\theta) = -\sin\theta$
$= -0.3$

The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.



By Pythagoras' theorem,

$$OP^2 = OM^2 + MP^2$$

$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$.

$$\therefore \quad 1 = \cos^2\theta + \sin^2\theta$$

As this is true for all values of θ, it is called an identity. In particular, this is called the **Pythagorean identity**:

$$\cos^2\theta + \sin^2\theta = 1$$

Example 25

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$ **b** $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$

$$\cos^2 x + \frac{9}{25} = 1$$

$$\cos^2 x = 1 - \frac{9}{25}$$

$$= \frac{16}{25}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part **a**, we have

$$\tan x = \frac{\sin x}{\cos x}$$

$$= \frac{3}{5} \div \left(-\frac{4}{5}\right)$$

$$= \frac{3}{5} \times \left(-\frac{5}{4}\right)$$

$$= -\frac{3}{4}$$

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Summary 14J

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta$$

$$\sin\left(\frac{\pi}{2}+\theta\right)=\cos\theta \qquad \cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta+\sin^2\theta=1$$

Exercise 14J

Example 24

1 If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, find the values of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2}+\alpha\right)$ **c** $\tan(-\theta)$ **d** $\cos\left(\frac{\pi}{2}-x\right)$

e $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2}-\theta\right)$ **g** $\cos\left(\frac{\pi}{2}+x\right)$ **h** $\sin\left(\frac{\pi}{2}-\alpha\right)$

i $\sin\left(\frac{3\pi}{2}+\alpha\right)$ **j** $\cos\left(\frac{3\pi}{2}-x\right)$

2 **a** Given that $0<\theta<\frac{\pi}{2}$ and $\cos\theta=\sin\left(\frac{\pi}{6}\right)$, find the value of θ.

b Given that $0<\theta<\frac{\pi}{2}$ and $\sin\theta=\cos\left(\frac{\pi}{6}\right)$, find the value of θ.

c Given that $0<\theta<\frac{\pi}{2}$ and $\cos\theta=\sin\left(\frac{\pi}{12}\right)$, find the value of θ.

d Given that $0<\theta<\frac{\pi}{2}$ and $\sin\theta=\cos\left(\frac{3\pi}{7}\right)$, find the value of θ.

Example 25

3 Given that $\cos x=\frac{3}{5}$ and $\frac{3\pi}{2}<x<2\pi$, find $\sin x$ and $\tan x$.

4 Given that $\sin x=\frac{5}{13}$ and $\frac{\pi}{2}<x<\pi$, find $\cos x$ and $\tan x$.

5 Given that $\cos x=\frac{1}{5}$ and $\frac{3\pi}{2}<x<2\pi$, find $\sin x$ and $\tan x$.

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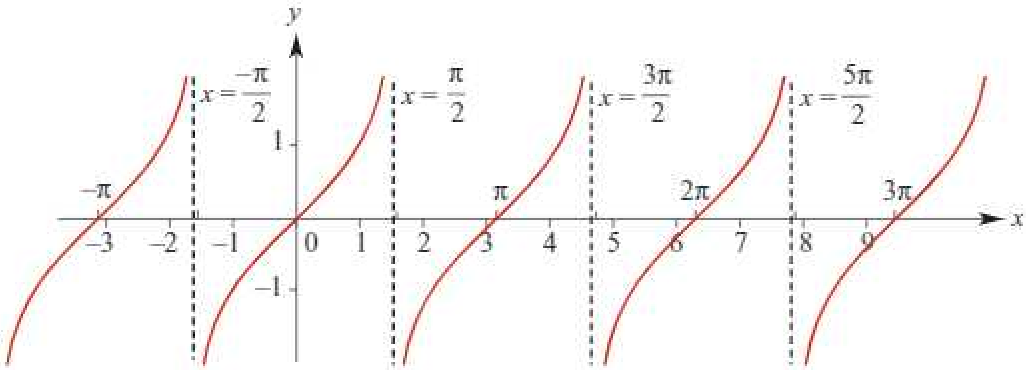
14K The tangent function

A table of values for $y = \tan x$ is given below. Use a calculator to check these values and plot the graph of $y = \tan x$.

x	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0	1	ud	−1	0



Note: The lines $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, $x = \frac{3\pi}{2}$ and $x = \frac{5\pi}{2}$ are asymptotes.

The x-axis intercepts occur when $\sin x = 0$, which is for $x = 0, \pi, 2\pi$, etc. In general, $x = k\pi$, where k is an integer.

Observations from the graph of $y = \tan x$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The equations of the asymptotes are of the form $x = \frac{(2k+1)\pi}{2}$, where k is an integer.
- The x-axis intercepts occur for $x = k\pi$, where k is an integer.

Transformations of $y = \tan x$

Consider a dilation of factor $\frac{1}{2}$ from the y-axis and a dilation of factor 3 from the x-axis:

$$(x, y) \to (\tfrac{1}{2}x, 3y)$$

If the image of (x, y) under the transformation is (x', y'), then $x' = \frac{1}{2}x$ and $y' = 3y$. Hence $x = 2x'$ and $y = \frac{y'}{3}$.

Thus the graph of $y = \tan x$ is transformed to the graph of $\frac{y'}{3} = \tan(2x')$. That is, it is transformed to the graph of $y = 3\tan(2x)$. The period of the graph will be $\frac{\pi}{2}$.

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Graph of $y = a\tan(nt)$

In general, for a and n positive numbers, the graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The following are important properties of the function $f(t) = a\tan(nt)$:

- The period is $\frac{\pi}{n}$.
- The range is $\mathbb{R}$.
- The asymptotes have equations $t = \frac{(2k+1)\pi}{2n}$, where k is an integer.
- The t-axis intercepts are $t = \frac{k\pi}{n}$, where k is an integer.

Example 26

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(2x)$

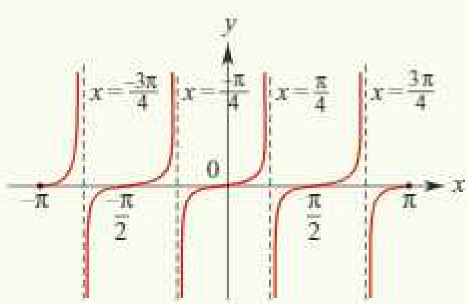
b $y = -2\tan(3x)$

Solution

a Period $= \frac{\pi}{n} = \frac{\pi}{2}$

Asymptotes: $x = \frac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$

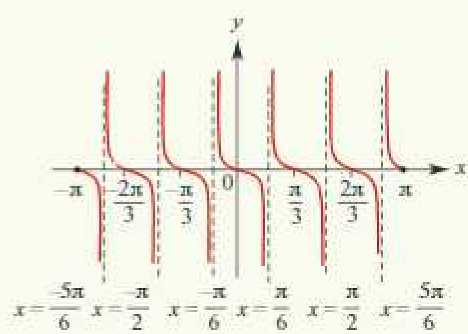
Axis intercepts: $x = \frac{k\pi}{2}$, $k \in \mathbb{Z}$



b Period $= \frac{\pi}{n} = \frac{\pi}{3}$

Asymptotes: $x = \frac{(2k+1)\pi}{6}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \frac{k\pi}{3}$, $k \in \mathbb{Z}$



Solution of equations

The techniques for solving equations of the form $a\tan(nt) = b$ are similar to those for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$ discussed in Section 14G. An important difference is that the period of tan is π. The method for obtaining further solutions is to add and subtract multiples of π.

Example 27

Solve each of the following equations for $x \in [-\pi, \pi]$:

a $\tan x = -1$

b $\tan(2x) = \sqrt{3}$

c $2\tan(3x) = 0$

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Solution

a $\tan x = -1$

$\therefore \quad x = \frac{3\pi}{4}$ or $\frac{-\pi}{4}$

Explanation

Since $\tan x$ is negative, the point $P(x)$ lies in the 2nd or 4th quadrant. Solutions are required for $x \in [-\pi, \pi]$.

Use $\tan(\pi - x) = -\tan x$ and $\tan(-x) = -\tan x$.

b $\tan(2x) = \sqrt{3}$

Let $a = 2x$. The equation becomes

$\tan a = \sqrt{3}$

$\therefore \quad a = \frac{\pi}{3}$ or $\frac{4\pi}{3}$ or $\frac{-2\pi}{3}$ or $\frac{-5\pi}{3}$

$\therefore \quad x = \frac{\pi}{6}$ or $\frac{4\pi}{6}$ or $\frac{-2\pi}{6}$ or $\frac{-5\pi}{6}$

$= \frac{\pi}{6}$ or $\frac{2\pi}{3}$ or $\frac{-\pi}{3}$ or $\frac{-5\pi}{6}$

Consider solutions for $a \in [-2\pi, 2\pi]$.

Since $\tan a$ is positive, the point $P(a)$ lies in the 1st or 3rd quadrant.

Use $\tan(\pi + x) = \tan x$.

Subtract π from each of the first two solutions to obtain the second two.

c $2\tan(3x) = 0$

$3x = -3\pi, -2\pi, -\pi, 0, \pi, 2\pi$ or 3π

$\therefore x = -\pi, \frac{-2\pi}{3}, \frac{-\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}$ or π

The solutions for $3x$ are to be in the interval $[-3\pi, 3\pi]$.

Example 28

Sketch the graph of $y = \tan(2x) + 1$ for $x \in [-\pi, \pi]$.

Solution

The graph of $y = \tan(2x) + 1$ is obtained from the graph of $y = \tan(2x)$ by a translation of 1 unit in the positive direction of the y-axis.

For the y-axis intercept, let $x = 0$. Then $y = \tan 0 + 1 = 1$.

For the x-axis intercepts, consider $\tan(2x) + 1 = 0$.

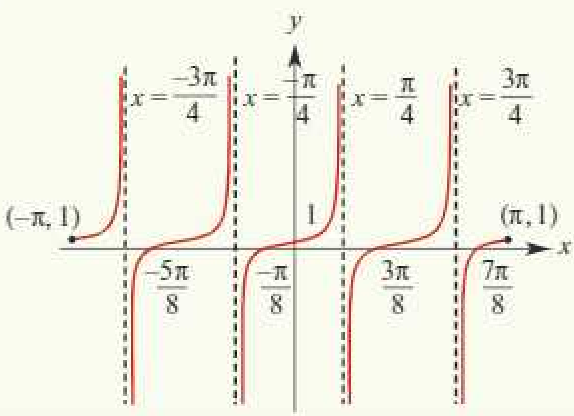
This implies $\tan(2x) = -1$.

Hence $\quad 2x = \frac{3\pi}{4}, \frac{-\pi}{4}, \frac{7\pi}{4}$ or $\frac{-5\pi}{4}$

and so $\quad x = \frac{3\pi}{8}, \frac{-\pi}{8}, \frac{7\pi}{8}$ or $\frac{-5\pi}{8}$

The asymptotes are the same as those for $y = \tan(2x)$,

that is, $x = \frac{(2k+1)\pi}{4}, k \in \mathbb{Z}$



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Exercise 14K

1 For each of the following, state the period:

a $y = \tan(4x)$ **b** $y = \tan\left(\frac{2x}{3}\right)$ **c** $y = -3\tan(2x)$

Example 26

2 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -\tan x$

Example 27

3 Solve each of the following equations for $x \in [-\pi, \pi]$:

a $2\tan(2x) = 2$ **b** $3\tan(3x) = \sqrt{3}$

c $2\tan(2x) = 2\sqrt{3}$ **d** $3\tan(3x) = -\sqrt{3}$

Example 28

4 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 3\tan(x) + \sqrt{3}$ **b** $y = \tan(x) + 2$ **c** $y = 3\tan(x) - 3$

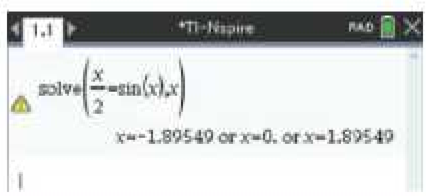
14L Numerical methods with a CAS calculator

Example 29

Solve the equation $\frac{x}{2} = \sin x$, giving your answer correct to two decimal places.

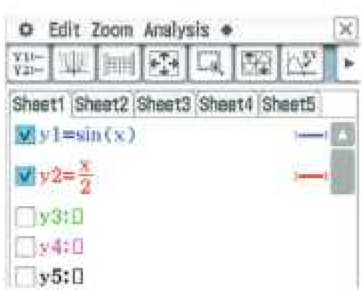
Using the TI-Nspire

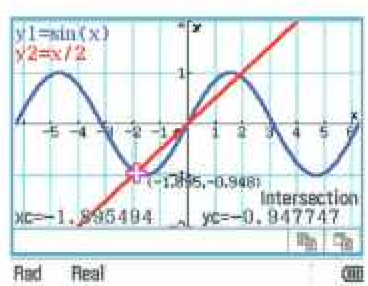
- Use (menu) > **Algebra** > **Solve** to solve as shown.
- Press (ctrl)(enter), if necessary, to obtain the answer as a decimal approximation.



Using the Casio ClassPad

The equation $\frac{x}{2} = \sin(x)$ can be solved numerically by drawing a graph for each side of the equation and finding the intersection points.





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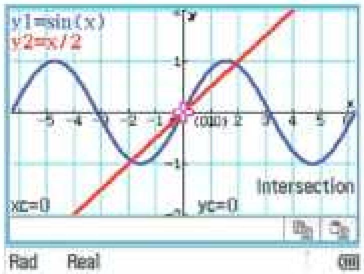
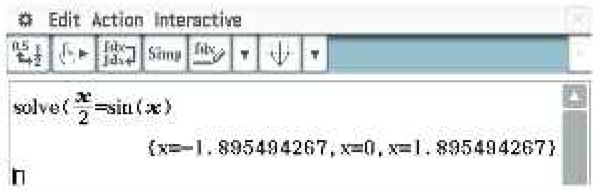
- To see all the intersection points, set the window to $-2\pi \le x \le 2\pi$ and $-2 \le y \le 2$.
- Select **Analysis** > **G-Solve** > **Intersection** to obtain the first point. To find the other points, navigate across using the cursor arrows on the hard keyboard.



Alternatively, the solutions can be found in the main screen.



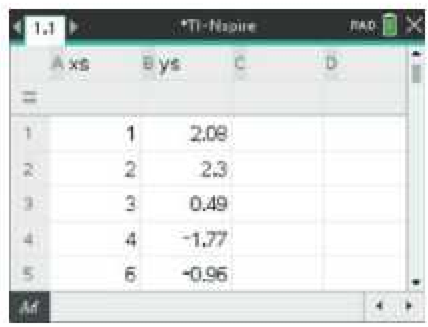
Fitting data

Example 30

Fit a sine curve to the points (1, 2.08), (2, 2.3), (3, 0.49), (4, −1.77) and (6, −0.96).

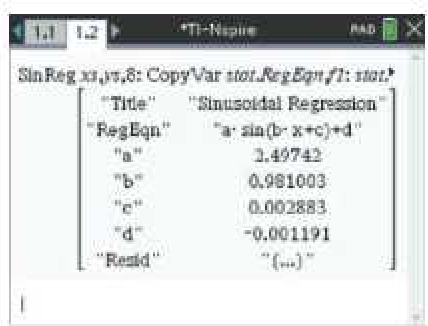
Using the TI-Nspire

- Enter the data either in a **Calculator** application as lists or in a **Lists & Spreadsheet** application as shown.



- In a **Calculator** application, choose menu > **Statistics** > **Stat Calculations** > **Sinusoidal Regression**.
- This now gives the values of a, b, c and d, and the function has been stored as $f1$.

Note: This regression analysis can also be done in the **Lists & Spreadsheet** application.



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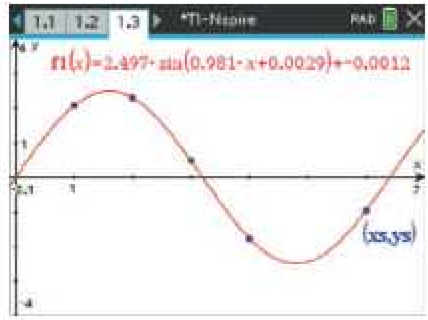
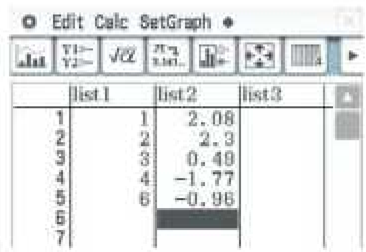
- The curve can be shown in a **Graphs** application together with the scatter plot ((menu) > **Graph Type** > **Scatter Plot**) using an appropriate window ((menu) > **Window/Zoom**).



Note: The scatter plot and regression curve can also be obtained using the **Data & Statistics** application.

Using the Casio ClassPad

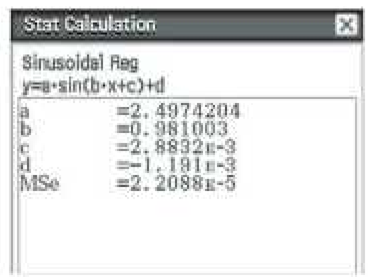
- In **Statistics**, enter the data in list1 and list2 as shown.

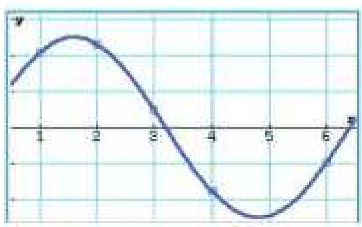


- Select **Calc** > **Regression** > **Sinusoidal Reg** and check the settings are correct. Tap OK.

Note: Set **Copy Formula** to $y1$, as this will store the formula for the graph for later use.

- Take note of the formula. Then tap OK again to produce the graph.





Exercise 14L

CAS

Example 29 **1** Solve each of the following equations for x, correct to two decimal places:

a $\cos x = x$ **b** $\sin x = 1 - x$ **c** $\cos x = x^2$ **d** $\sin x = x^2$

Example 30 **2** For each of the following sets of data, find a suitable trigonometric rule (model):

a

0	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	1	2.4	−1	2.4	1

b

0	0	0.2	0.4	0.6	0.8
y	0	1.77	2.85	2.85	1.77

c

0	0	0.2	0.4	0.6	0.8
y	5	2.18	0.34	0.13	1.62

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14M General solution of trigonometric equations

In Section 14G, we considered the solution of equations involving circular functions over a restricted domain. In this section, we consider the general solutions of such equations over the maximal domain for each function.

If an equation involving a circular function has one or more solutions in one 'cycle', then it will have corresponding solutions in each cycle of its domain, i.e. there will be an infinite number of solutions.

For example, if $\cos x = \frac{1}{4}$, then the solution in the interval $[0, \pi]$ is given by:

$$x = \cos^{-1}(\tfrac{1}{4})$$

By the symmetry properties of the cosine function, other solutions are given by:

$$-\cos^{-1}(\tfrac{1}{4}),\ \pm 2\pi + \cos^{-1}(\tfrac{1}{4}),\ \pm 2\pi - \cos^{-1}(\tfrac{1}{4}),\ \pm 4\pi + \cos^{-1}(\tfrac{1}{4}),\ \pm 4\pi - \cos^{-1}(\tfrac{1}{4}),\ \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

Example 31

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$\begin{aligned} x &= 2n\pi \pm \cos^{-1}(0.5) \\ &= 2n\pi \pm \frac{\pi}{3} \\ &= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z} \end{aligned}$$

b $\tan(3x) = \dfrac{1}{\sqrt{3}}$

$$\begin{aligned} 3x &= n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= n\pi + \frac{\pi}{6} \\ &= \frac{(6n+1)\pi}{6} \\ x &= \frac{(6n+1)\pi}{18}, \quad n \in \mathbb{Z} \end{aligned}$$

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c $\sin x = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

$$x = 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

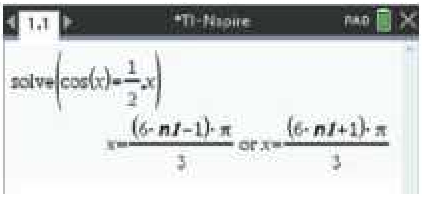
$$= 2n\pi + \frac{\pi}{4} \qquad\qquad = (2n+1)\pi - \frac{\pi}{4}$$

$$= \frac{(8n+1)\pi}{4}, \quad n \in \mathbb{Z} \qquad\qquad = \frac{(8n+3)\pi}{4}, \quad n \in \mathbb{Z}$$

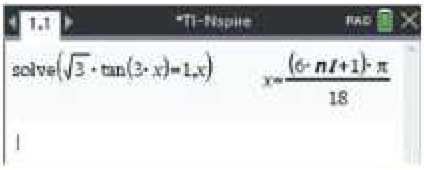
Using the TI-Nspire

Check that the calculator is in radian mode.

a Use (menu) > **Algebra** > **Solve** and complete as shown. Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



b

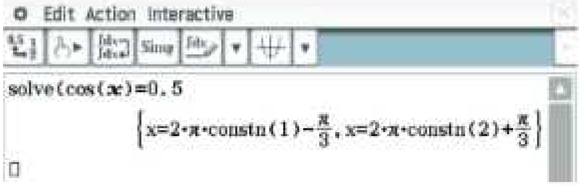


c

Using the Casio ClassPad

Check that the calculator is in radian mode.

a
- In Main $\sqrt{\alpha}$, select [solve( ] from the [Math1] or [Math3] keyboard.
- Enter the equation $\cos(x) = 0.5$ and tap [EXE].
- To view the entire solution, rotate the screen by selecting Rotate.



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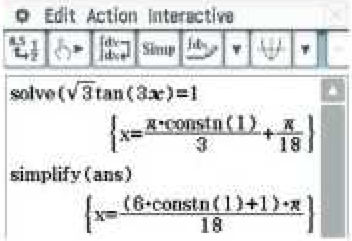

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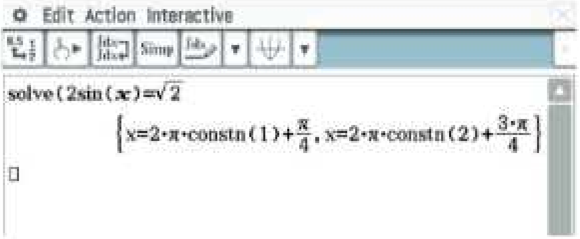

Note: The constn(1) and constn(2) are replaced with n in the written answer.

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Example 32

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a The general solution (from Example 31 **a**) is given by $x = \frac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\frac{\pi}{3}$, and when $n = 1$, $x = \frac{5\pi}{3}$ or $x = \frac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$.

b The general solution (from Example 31 **b**) is given by $x = \frac{(6n+1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{18}$, and when $n = 1$, $x = \frac{7\pi}{18}$, and when $n = 2$, $x = \frac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \frac{\pi}{18}, \frac{7\pi}{18}, \frac{13\pi}{18}$.

c The general solution (from Example 31 **c**) is $x = \frac{(8n+1)\pi}{4}$ or $x = \frac{(8n+3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \frac{\pi}{4}$ or $x = \frac{3\pi}{4}$, and when $n = 1$, $x = \frac{9\pi}{4}$ or $x = \frac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}$.

Summary 14M

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

  $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

  $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Exercise 14M

Example 31

1 Find the general solution of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = -3$

Example 32

2 Find the first two positive solutions of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = -3$

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3 Find the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$, and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

4 Find the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$, and hence find all the solutions for x in the interval $[-\pi, 0]$.

5 Find the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$, and hence find all the solutions for x in the interval $[-1, 1]$.

14N Applications of circular functions

Example 33

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule

$$h(t) = 4\sin\left(\frac{\pi}{6}t\right)$$

where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

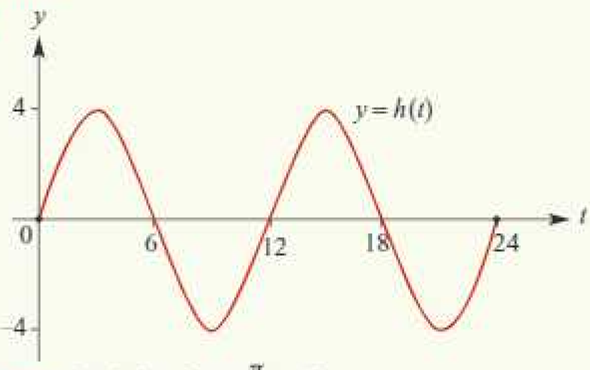
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$:

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\sin\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$

The water is $2\sqrt{3}$ metres below the mean height at 8 a.m.

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e We first consider $4\sin\left(\frac{\pi}{6}t\right) = 1$.

Thus $\sin\left(\frac{\pi}{6}t\right) = \frac{1}{4}$

$\therefore \quad \frac{\pi}{6}t = 0.2526,\ 2.889,\ 6.5358,\ 9.172$

$\therefore \quad t = 0.4824,\ 5.5176,\ 12.4824,\ 17.5173$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.

Thus the boat can pass across the harbour bar between 00:29 and 05:31, and between 12:29 and 17:31.

Skill-sheet

Exercise 14N

Example 33

1 It is suggested that the height, $h(t)$ metres, of the tide above mean sea level during a particular day at Seabreak is given approximately by the rule

$$h(t) = 5\sin\left(\frac{\pi}{6}t\right)$$

where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

c What was the height of the high tide?

d What was the height of the tide at 2 a.m.?

e What was the height of the tide at 2 p.m.?

f A boat can only cross the harbour bar when the tide is at least 2.5 metres above mean sea level. When could the boat cross the harbour bar on this day?

2 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 3 + 2\sin(3t)$.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 5 metres from O for $0 \le t \le 5$.

d Find the times at which it is 3 metres from O for $0 \le t \le 3$.

e Describe the motion of the particle.

3 A particle moves on a straight line, OX, and its distance x metres from O at time t seconds is given by $x = 5 + 2\sin(2\pi t)$.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 7 metres from O for $0 \le t \le 5$.

d Find the times at which it is 6 metres from O for $0 \le t \le 3$.

e Describe the motion of the particle.

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4 A particle moves in a vertical circle of radius 10 m. The height, $h(t)$ m, of the particle above the ground at time t seconds is given by the function

$$h(t) = 10\sin\left(\frac{\pi t}{3}\right) + 10 \quad \text{where } t \geq 0$$

a Find the height of the particle above the ground for:

i $t = 0$ **ii** $t = 1$ **iii** $t = 2$ **iv** $t = 4$ **v** $t = 5$

b Find the period of the motion of the particle.

c Find the greatest height of the particle above the ground.

d Find the first four times that the particle is at a height 15 m above the ground.

e Find the first four times that the particle is at a height 5 m above the ground.

5 The temperature, T°C, in a town over a day is modelled by the function with rule

$$T = 17 - 8\cos\left(\frac{\pi t}{12}\right)$$

where t is the time in hours after midnight, $0 \leq t \leq 24$.

a What is the temperature at midnight?

b What are the maximum and minimum temperatures reached?

c At what times of the day, to the nearest minute, are temperatures warmer than 20°C?

d Sketch the graph for the temperatures over a day.

6 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \leq t \leq 24$.

a Sketch the graph of $D(t)$ for $0 \leq t \leq 24$.

b Find the values of t for which $D(t) \geq 8.5$.

c Boats which need a depth of w metres are permitted to enter the harbour only if the depth of the water at the entrance is at least w metres for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w which satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D metres, where $D = p + q\cos(rt)^\circ$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \leq t \leq 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

8 For each of the following, construct a formula involving a circular function which could be used to model the situation described:

a Water depths in a canal vary between a minimum of 3 metres and a maximum of 6 metres over a 24-hour period.

b At a certain town just south of the Arctic circle, the number of hours of daylight varies between 2 and 22 hours during a 365-day year.

Review

Chapter summary

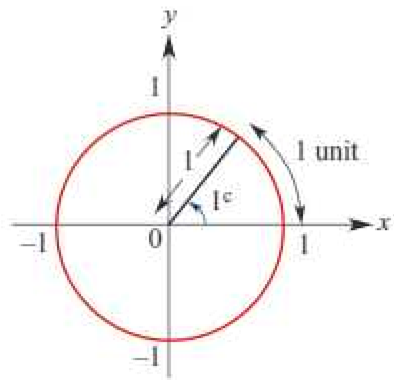
Assign-ment

Nrich

- **Definition of a radian**

  One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

  $$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



- **Sine and cosine**

  x-coordinate of $P(\theta)$ on unit circle:

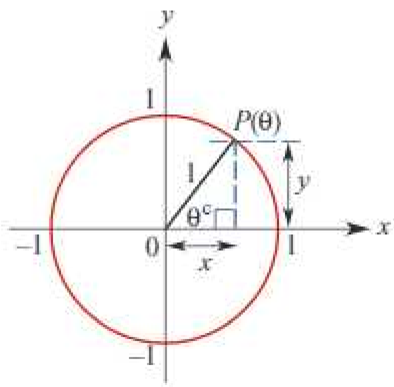
  $$x = \text{cosine } \theta, \quad \theta \in \mathbb{R}$$

  y-coordinate of $P(\theta)$ on unit circle:

  $$y = \text{sine } \theta, \quad \theta \in \mathbb{R}$$

  Abbreviated to

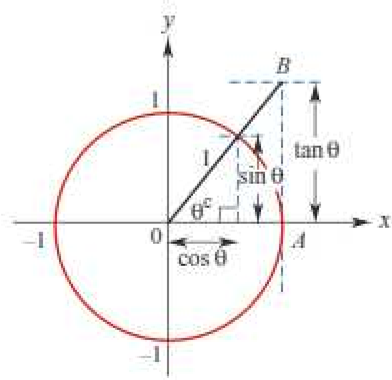
  $$x = \cos \theta$$
  $$y = \sin \theta$$



- **Tangent**

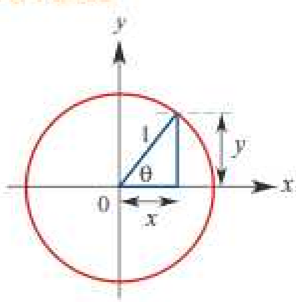
  If the tangent to the unit circle at A is drawn, then the y-coordinate of B is called tangent θ (abbreviated to tan θ). By using similar triangles:

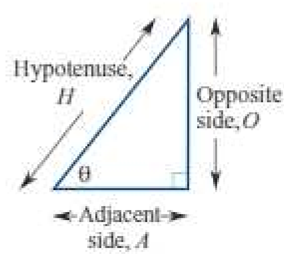
  $$\tan \theta = \frac{\sin \theta}{\cos \theta}$$



- **Circular functions and trigonometric ratios**

  $$\sin \theta = \frac{O}{H} = \frac{y}{1} = y$$
  $$\cos \theta = \frac{A}{H} = \frac{x}{1} = x$$
  $$\tan \theta = \frac{O}{A} = \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$





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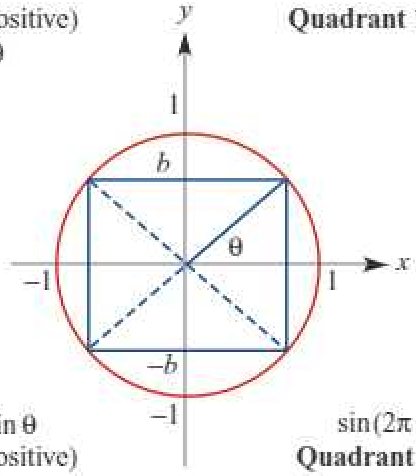
Review

Symmetry properties of circular functions

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



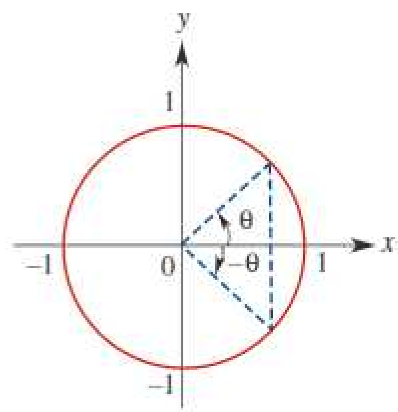
$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Further symmetry properties

Negative angles:

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = -\sin\theta$$
$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \quad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

Pythagorean identity

$$\cos^2\theta + \sin^2\theta = 1$$

Exact values of circular functions

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undef

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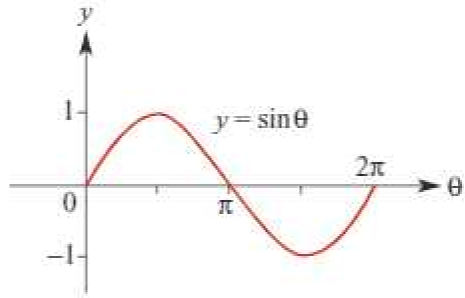
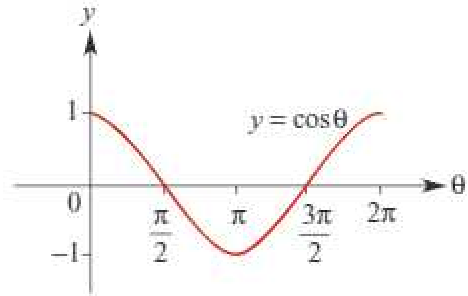
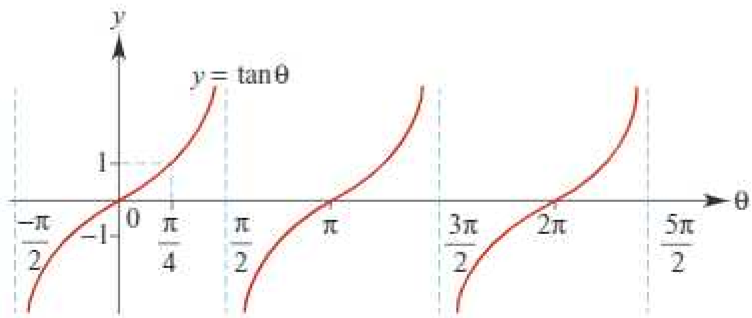
- **Graphs of circular functions**



Amplitude = 1
Period = 2π

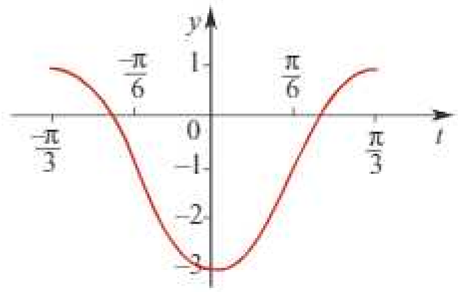


Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

- Graphs of circular functions of the type
  $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$
  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$
  - Amplitude, $a = 2$
  - Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
  - The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.



- Solutions of trigonometric equations of the type $\sin x^\circ = a$ and $\cos x^\circ = a$

  e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
  First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
  Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

$$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

Technology-free questions

1 Convert each of the following to radian measure in terms of π:

a 330° **b** 810° **c** 1080° **d** 1035° **e** 135°
f 405° **g** 390° **h** 420° **i** 80°

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2 Convert each of the following to degree measure:

a $\frac{5\pi^c}{6}$ **b** $\frac{7\pi^c}{4}$ **c** $\frac{11\pi^c}{4}$ **d** $\frac{3\pi^c}{12}$ **e** $\frac{15\pi^c}{2}$

f $-\frac{3\pi^c}{4}$ **g** $-\frac{\pi^c}{4}$ **h** $-\frac{11\pi^c}{4}$ **i** $-\frac{23\pi^c}{4}$

3 Give exact values of each of the following:

a $\sin\left(\frac{11\pi}{4}\right)$ **b** $\cos\left(-\frac{7\pi}{4}\right)$ **c** $\sin\left(\frac{11\pi}{6}\right)$ **d** $\cos\left(-\frac{7\pi}{6}\right)$

e $\cos\left(\frac{13\pi}{6}\right)$ **f** $\sin\left(\frac{23\pi}{6}\right)$ **g** $\cos\left(-\frac{23}{3}\pi\right)$ **h** $\sin\left(-\frac{17}{4}\pi\right)$

4 State the amplitude and period of each of the following:

a $2\sin\left(\frac{\theta}{2}\right)$ **b** $-3\sin(4\theta)$ **c** $\frac{1}{2}\sin(3\theta)$

d $-3\cos(2x)$ **e** $-4\sin\left(\frac{x}{3}\right)$ **f** $\frac{2}{3}\sin\left(\frac{2x}{3}\right)$

5 Sketch the graph of each of the following (showing one cycle):

a $y = 2\sin(2x)$ **b** $y = -3\cos\left(\frac{x}{3}\right)$ **c** $y = -2\sin(3x)$

d $y = 2\sin\left(\frac{x}{3}\right)$ **e** $y = \sin\left(x - \frac{\pi}{4}\right)$ **f** $y = \sin\left(x + \frac{2\pi}{3}\right)$

g $y = 2\cos\left(x - \frac{5\pi}{6}\right)$ **h** $y = -3\cos\left(x + \frac{\pi}{6}\right)$

6 Solve each of the following equations for θ:

a $\sin\theta = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$ **b** $\sin(2\theta) = -\frac{\sqrt{3}}{2},\ \theta \in [-\pi, \pi]$

c $\sin\left(\theta - \frac{\pi}{3}\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$ **d** $\sin\left(\theta + \frac{\pi}{3}\right) = -1,\ \theta \in [0, 2\pi]$

e $\sin\left(\frac{\pi}{3} - \theta\right) = -\frac{1}{2},\ \theta \in [0, 2\pi]$

Multiple-choice questions

1 In a right-angled triangle, the two shorter side lengths are 3 cm and 4 cm. To the nearest degree, the magnitude of the smallest angle is

A 1° **B** 23° **C** 37° **D** 53° **E** 92°

2 The minimum value of $3 - 10\cos(2x)$ is

A −13 **B** −17 **C** −23 **D** −7 **E** −10

3 The range of the function $f\colon [0, 2\pi] \to \mathbb{R},\ f(x) = 4\sin\left(2x - \frac{\pi}{2}\right)$ is

A $\mathbb{R}$ **B** $[0, 4]$ **C** $[-4, 0]$ **D** $[0, 8]$ **E** $[-4, 4]$

4 The period of the graph of $y = 3\sin\left(\frac{1}{2}x - \pi\right) + 4$ is

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

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Review

5 The graph of $y = \sin x$ is dilated by factor $\frac{1}{2}$ from the y-axis and translated $\frac{\pi}{4}$ units in the positive direction of the x-axis. The equation of the image is

A $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

D $y = \sin\left(2x - \frac{\pi}{4}\right)$ **E** $y = \sin\left(2\left(x - \frac{\pi}{4}\right)\right)$

6 Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a\sin(bx) + c$, where a, b and c are positive constants. The period of the function f is

A a **B** b **C** $\frac{2\pi}{a}$ **D** $\frac{2\pi}{b}$ **E** $\frac{b}{2\pi}$

7 One cycle of the graph of $y = \tan(ax)$ has vertical asymptotes at $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{6}$. A possible value of a is

A 6 **B** π **C** $\frac{\pi}{6}$ **D** $\frac{1}{3}$ **E** 3

8 The equation $3\sin(x) + 1 = b$, where b is a positive real number, has one solution in the interval $[0, 2\pi]$. The value of b is

A 1 **B** 1.5 **C** 2 **D** 3 **E** 4

9 The number of solutions of the equation $b = a\sin x$, where $x \in [-2\pi, 2\pi]$ and a and b are positive real numbers with $a > b$, is

A 2 **B** 3 **C** 4 **D** 5 **E** 6

10 The depth of water, in metres, in a harbour at a certain point at time t hours is given by $D(t) = 8 + 2\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$. The depth of the water is first 9 m at

A $t = 0$ **B** $t = 1$ **C** $t = 2$ **D** $t = 3$ **E** $t = 4$

Extended-response questions

1 The number of hours of daylight at a point on the Antarctic Circle is modelled by the function $d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$, where t is the number of months which have elapsed since 1 January.

SAS

a **i** Find d on 21 June ($t \approx 5.7$). **ii** Find d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

2 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, B°C, outside the house at the same time is given by the rule $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

a Find the temperature inside the house at 8 a.m.

b Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.

c Sketch the graph of D for $0 \le t \le 24$.

d Determine when the inside temperature is less than the outside temperature.

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3 At a certain time of the year the depth of water, d metres, in the harbour at Bunk Island is given by the rule

$$d = 3 + 1.8\cos\left(\frac{\pi}{6}t\right)$$

where t is the time in hours after 3 a.m.

a Sketch the graph of d against t for a 24-hour period from 3 a.m. to 3 a.m.

b At what time(s) does high tide occur for $t \in [0, 24]$?

c At what time(s) does low tide occur for $t \in [0, 24]$?

A passenger ferry operates between Main Beach and Bunk Island. It takes 50 minutes to go from Main Beach to Bunk Island. The ferry only runs between the hours of 8 a.m. and 8 p.m., and is only able to enter the harbour at Bunk Island if the depth of water is at least 2 metres.

d What is the earliest time the ferry should leave Main Beach so that it arrives at Bunk Island and can immediately enter the harbour?

e The time to go from Bunk Island to Main Beach is also 50 minutes. The minimum time the ferry takes at Bunk Island harbour is 5 minutes. The minimum time at Main Beach is also 5 minutes.

i What is the latest time the ferry can leave Main Beach to complete a round trip in 105 minutes?

ii How many complete round trips can the ferry make in a day?

4 The depth of water, D metres, at the end of Brighton pier t hours after low tide is given by the rule $D = p - 2\cos(rt)$, where p and r are suitable constants.
At low tide ($t = 0$) the depth is 2 metres; at high tide, which occurs 8 hours later, the depth is 6 metres.

a Show that $r = \frac{\pi}{8}$ and $p = 4$.

b Sketch the graph of $D = 4 - 2\cos\left(\frac{\pi}{8}t\right)$ for $0 \le t \le 16$.

c If the first low tide occurs at 4 a.m., when will the next low tide occur?

d At what times will the depth be equal to 4 metres?

The poles that support the Brighton pier stand 7.5 metres above the sea bed.

e How much of a particular pole is exposed at:

i high tide **ii** 2 p.m.?

Over the years mussels have attached themselves to the pole. A particular mussel is attached 4 metres from the top of the pole so that some of the time it is exposed and some of the time it is covered by water.

f For how long will the mussel be covered by water during the time from one low tide to the next?

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