7

Transformations

Objectives

- To introduce a notation for considering **transformations** of the plane, including translations, reflections in an axis and dilations from an axis.
- To determine the rule for the image of the graph of a relation after a transformation has been applied.
- To determine a sequence of transformations that will take the graph of a relation to its image, where the rule for the image is known.

We have studied transformations in the earlier chapters of this book without any systematic consideration of transformations of the points of the plane in general. In particular, we have looked at translations, reflections in an axis and dilations from an axis applied to the graphs of various functions and relations.

In this chapter, we define these transformations in general and develop techniques for both applying and identifying transformations.

We state the following result as an example of the relationship between the transformation of points and the graph-sketching techniques we have used earlier in this book. Similar results hold for reflections in an axis and dilations from an axis.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Note: Transformations of the plane with rules of the form $(x, y) \to (ax + by, cx + dy)$ can also be studied using 2×2 matrices. This approach is covered in an alternative version of this chapter available in the Interactive Textbook.

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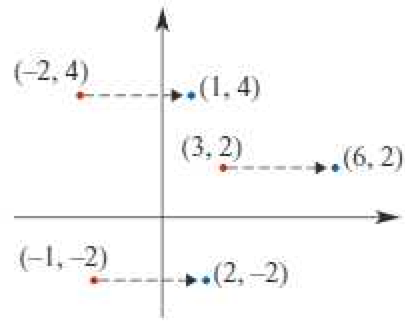
7A Translations of functions

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{ (x, y) : x, y \in \mathbb{R} \}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$. This translation is a transformation of the plane. Each point in the plane is mapped to a unique second point. Furthermore, every point in the plane is an image of another point under this translation.

Notation

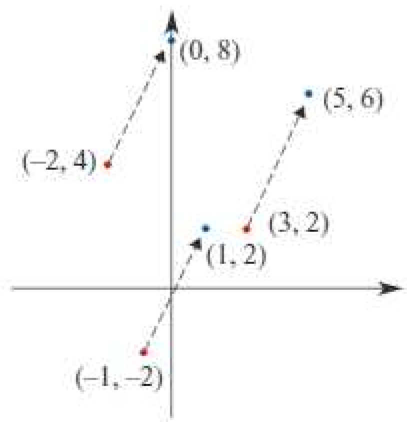
The translation 3 units to the right can be written $(x, y) \rightarrow (x + 3, y)$. This reads as '$(x, y)$ maps to $(x + 3, y)$'.



For example, $(-1, -2) \rightarrow (-1 + 3, -2)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written for this translation:

$$x' = x + 3 \quad \text{and} \quad y' = y$$



As another example, consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up). This can be described by the rule $(x, y) \rightarrow (x + 2, y + 4)$.

For example, $(3, 2) \rightarrow (3 + 2, 2 + 4)$.

The following can be written for this translation:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

$$(x, y) \rightarrow (x + h, y + k)$$

or $\quad x' = x + h \quad \text{and} \quad y' = y + k$

where h and k are positive numbers.

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A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

$$(x, y) \to (x - h, y - k)$$

or $\quad x' = x - h \quad$ and $\quad y' = y - k$

where h and k are positive numbers.

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a particular translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(a - h, b - k) \to (a, b)$ under this translation.)

Applying translations to sketch graphs

We look at a particular example.

Translate the set of points defined by the function

$$\{ (x, y) : y = x^2 \}$$

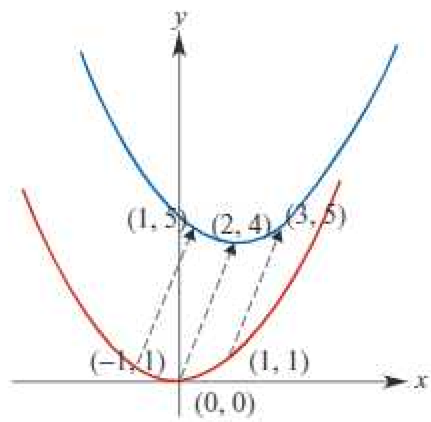
by the translation defined by the rule

$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.



This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.

Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$$\Leftrightarrow \quad f(a) = b$$

$$\Leftrightarrow \quad f(a + h - h) = b$$

$$\Leftrightarrow \quad f(a + h - h) = b + k - k$$

$$\Leftrightarrow \quad (a + h, b + k) \text{ is a point on the graph of } y - k = f(x - h)$$

Note: The double arrows indicate that the steps are reversible.

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Example 1

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \frac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution

Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.

Then $x' = x + 3$ and $y' = y - 2$.

Hence $x = x' - 3$ and $y = y' + 2$.

The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$

i.e. $y = \frac{1}{x}$ is mapped to

$$y' + 2 = \frac{1}{x' - 3}$$

Explanation

The rule is $(x, y) \to (x + 3, y - 2)$.

Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of

$$y = \frac{1}{x - 2}$$

note that it is of the form $y = f(x - 2)$ where $f(x) = \frac{1}{x}$. That is, the graph of $y = \frac{1}{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x - 2) = (x - 2)^2$$

$$f(x) = \sqrt{x} \qquad f(x - 2) = \sqrt{x - 2}$$

Summary 7A

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

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Exercise 7A

1 Find the image of the point $(-3, 4)$ after a mapping of a translation:

a of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b of 2 units in the negative direction of the x-axis and 4 units in the positive direction of the y-axis

c of 3 units in the negative direction of the x-axis and 2 units in the negative direction of the y-axis

d defined by the rule $(x, y) \to (x - 4, y + 5)$

e defined by the rule $(x, y) \to (x - 2, y - 1)$.

2 For each of the following, describe the translation that maps the first point to the second point:

a $(-1, 5)$ to $(-6, 8)$ **b** $(2, 4)$ to $(8, -11)$ **c** $(4, -5)$ to $(-8, 12)$

Example 1

3 In each of the following, find the rule for the image of the graph of $y = f(x)$ under the given translation:

a $f(x) = \dfrac{1}{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis

b $f(x) = \dfrac{1}{x^2}$ under a translation 4 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis

c $f(x) = x^2$ under a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

d $f(x) = x^3$ under a translation 4 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis

e $f(x) = \sqrt{x}$ under a translation 2 units in the positive direction of the x-axis and 1 unit in the negative direction of the y-axis.

4 For $y = f(x) = \dfrac{1}{x}$, sketch the graph of each of the following, labelling asymptotes and axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$

d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

5 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1)$ **b** $y = f(x) + 1$ **c** $y = f(x + 3)$

d $y = f(x) - 3$ **e** $y = f(x + 1)$ **f** $y = f(x) - 1$

6 For $y = f(x) = x^2$, sketch the graph of each of the following, labelling axis intercepts:

a $y = f(x - 1) + 2$ **b** $y = f(x - 3) + 1$ **c** $y = f(x + 3) - 5$

d $y = f(x + 1) - 3$ **e** $y + 2 = f(x + 1)$ **f** $y = f(x - 5) - 1$

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7B Dilations and reflections

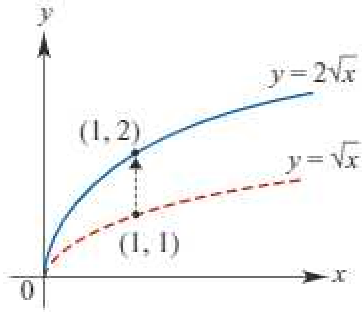
The same techniques can be applied to dilations from an axis and reflections.

Dilation from the x-axis

A dilation of factor 2 from the x-axis can be defined by the rule $(x, y) \to (x, 2y)$.

Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.



- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \frac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\frac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.

For b a positive constant, a dilation of factor b from the x-axis is described by the rule

$$(x, y) \to (x, by)$$

or $\quad x' = x \quad$ and $\quad y' = by$

For the graph of $y = f(x)$, the following two processes yield the same result:

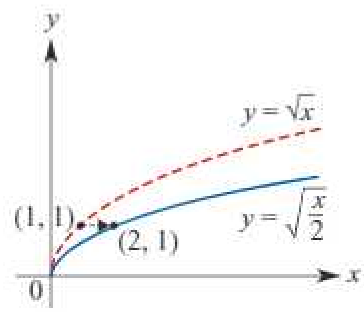
- Applying the dilation from the x-axis $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
- Replacing y with $\frac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.

Dilation from the y-axis

A dilation of factor 2 from the y-axis can be defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = 2x$ and $y' = y$, and thus $x = \frac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\frac{x'}{2}}$.



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For a a positive constant, a dilation of factor a from the y-axis is described by the rule

$$(x, y) \to (ax, y)$$

or $\quad x' = ax \quad$ and $\quad y' = y$

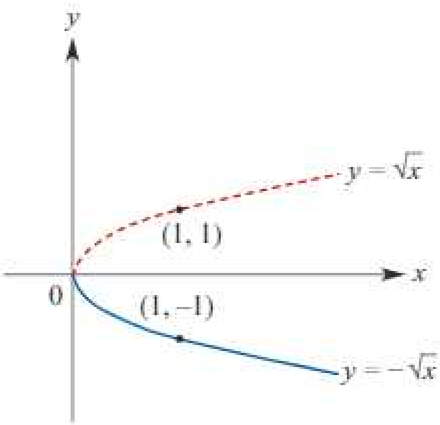
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation from the y-axis $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
- Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.

Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



A reflection in the x-axis is described by the rule

$$(x, y) \to (x, -y)$$

or $\quad x' = x \quad$ and $\quad y' = -y$

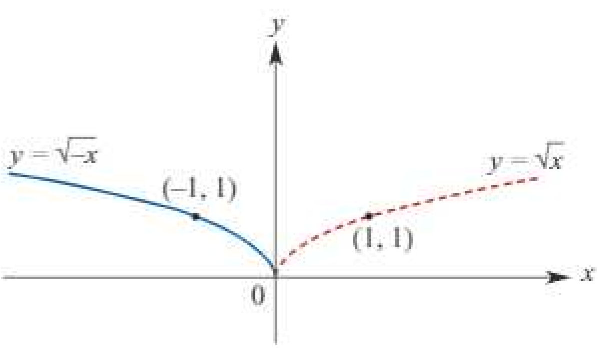
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the x-axis $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



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A reflection in the y-axis is described by the rule

$$(x, y) \to (-x, y)$$

or $\quad x' = -x \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the y-axis $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Example 2

Determine the rule of the image when the graph of $y = \dfrac{1}{x^2}$ is dilated by a factor of 4:

a from the y-axis **b** from the x-axis.

Solution

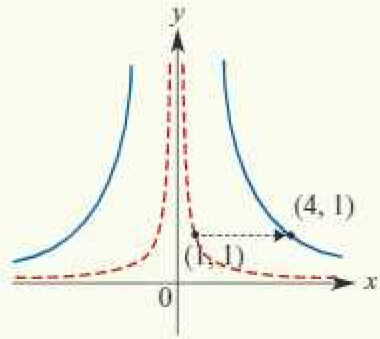
a $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x$, $y' = y$.

Rearranging gives $x = \dfrac{x'}{4}$, $y = y'$.

Therefore $y = \dfrac{1}{x^2}$ becomes $y' = \dfrac{1}{(\frac{x'}{4})^2}$.

The rule of the transformed function is $y = \dfrac{16}{x^2}$.



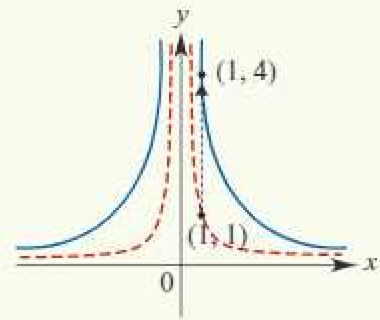
b $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 4y$.

Rearranging gives $x = x'$, $y = \dfrac{y'}{4}$.

Therefore $y = \dfrac{1}{x^2}$ becomes $\dfrac{y'}{4} = \dfrac{1}{(x')^2}$.

The rule of the transformed function is $y = \dfrac{4}{x^2}$.



Applying dilations and reflections to sketch graphs

In order to sketch the graph of $y = \sqrt{\dfrac{x}{2}}$, note that it is of the form $y = f\left(\dfrac{x}{2}\right)$ where $f(x) = \sqrt{x}$. This is the graph of $y = \sqrt{x}$ dilated by factor 2 from the y-axis.

Examples of other functions under this dilation are:

$$f(x) = x^2 \qquad f\left(\frac{x}{2}\right) = \left(\frac{x}{2}\right)^2 = \frac{x^2}{4}$$

$$f(x) = \frac{1}{x} \qquad f\left(\frac{x}{2}\right) = \frac{1}{\frac{x}{2}} = \frac{2}{x}$$

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It should be noted that each of these functions formed by a dilation of factor 2 from the y-axis can also be formed by a dilation from the x-axis. This result is not true in general, as will be seen when new functions are introduced in Chapters 13 and 14.

- For the graph of $y = \sqrt{\frac{x}{2}}$, we can write $y = \frac{1}{\sqrt{2}}\sqrt{x} = \frac{1}{\sqrt{2}}f(x)$, where $f(x) = \sqrt{x}$. That is, it is formed by a dilation of factor $\frac{1}{\sqrt{2}}$ from the x-axis.
- For the graph of $y = \frac{x^2}{4}$, we can write $y = \frac{1}{4}x^2 = \frac{1}{4}f(x)$, where $f(x) = x^2$. That is, it is formed by a dilation of factor $\frac{1}{4}$ from the x-axis.
- For the graph of $y = \frac{2}{x}$, we can write $y = 2f(x)$, where $f(x) = \frac{1}{x}$. That is, it is formed by a dilation of factor 2 from the x-axis.

Summary 7B

For the graph of $y = f(x)$, we have the following four pairs of equivalent processes:

1. - Applying the **dilation from the x-axis** $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
   - Replacing y with $\frac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.
2. - Applying the **dilation from the y-axis** $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
   - Replacing x with $\frac{x}{a}$ in the equation to obtain $y = f\left(\frac{x}{a}\right)$ and graphing the result.
3. - Applying the **reflection in the x-axis** $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
   - Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.
4. - Applying the **reflection in the y-axis** $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
   - Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Exercise 7B

1 Find the image of the point $(-2, -3)$ after:

a a reflection in the x-axis
b a reflection in the y-axis
c a dilation of factor 4 from the x-axis
d a dilation of factor 4 from the y-axis.

Example 2

2 Write down the equation of the image obtained when the graph of each of the functions below is transformed by:

i a dilation of factor $\frac{1}{2}$ from the y-axis
ii a dilation of factor 5 from the y-axis
iii a dilation of factor $\frac{2}{3}$ from the x-axis
iv a dilation of factor 4 from the x-axis
v a reflection in the x-axis
vi a reflection in the y-axis.

a $y = x^2$ **b** $y = \frac{1}{x^2}$ **c** $y = \frac{1}{x}$ **d** $y = \sqrt{x}$

3 Sketch the graph of each of the following:

a $y = 3\sqrt{x}$ **b** $y = -\frac{1}{x}$ **c** $y = \frac{3}{x}$ **d** $y = \frac{1}{2x}$ **e** $y = \sqrt{3x}$ **f** $y = \frac{3}{2x}$

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7C Combinations of transformations

In this section, we look at sequences of transformations. For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 3

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

The image has equation $\dfrac{y'}{-2} = \sqrt{x'}$ and hence $y' = -2\sqrt{x'}$.

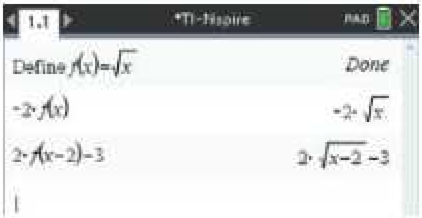
b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

The image has equation $\dfrac{y' + 3}{2} = \sqrt{x' - 2}$ and hence $y' = 2\sqrt{x' - 2} - 3$.

Using the TI-Nspire

Use (menu) > **Actions** > **Define** to define the function $f(x) = \sqrt{x}$. Complete as shown.



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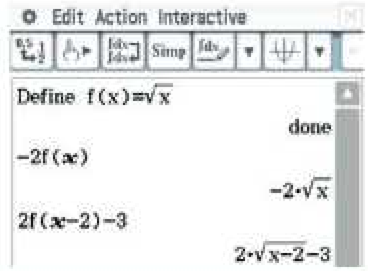
Using the Casio ClassPad

Use **Interactive** > **Define** to define the function $f(x) = \sqrt{x}$, and then complete as shown.

Note: The symbol $\sqrt{}$ is found in Math1.



Summary 7C

Given a sequence of transformations, we can find the rule for transforming points of the plane. For example, the sequence

- a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
- followed by a reflection in the y-axis

can be described by the rule $(x, y) \to (x + 2, y - 3) \to (-x - 2, y - 3)$.

Skill-sheet

Exercise 7C

1 Find the image of the point $(2, -1)$ under each sequence of transformations:

a a translation of 3 units in the positive direction of the x-axis followed by a dilation of factor 2 from the x-axis

b a reflection in the x-axis followed by a dilation of factor 3 from the y-axis

c a dilation of factor 3 from the x-axis followed by a translation of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Example 3

2 Find the equation of the image of the graph of $y = \sqrt{x}$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 from the x-axis

b a translation of 3 units in the negative direction of the x-axis followed by a reflection in the x-axis

c a reflection in the x-axis followed by a dilation of factor 3 from the x-axis

d a reflection in the x-axis followed by a dilation of factor 2 from the y-axis

e a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

3 Repeat Question 2 for $y = \dfrac{1}{x}$.

4 Repeat Question 2 for $y = x^{\frac{1}{3}}$.

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5 Find the equation of the image of the graph of $y = x^2$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the positive direction of the x-axis followed by a dilation of factor 3 from the y-axis

b a dilation of factor 3 from the y-axis followed by a translation of 2 units in the positive direction of the x-axis.

6 Find the equation of the image of the graph of $y = \frac{1}{x}$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the positive direction of the x-axis followed by a reflection in the y-axis

b a reflection in the y-axis followed by a translation of 2 units in the positive direction of the x-axis.

7 Find the equation of the image of the graph of $y = 2(x - 2)^2 + 3$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis

b a reflection in the x-axis followed by a translation of 3 units in the positive direction of the y-axis

c a reflection in the y-axis followed by a dilation of factor 3 from the y-axis.

8 Find the equation of the image of the graph of $y = \frac{2}{x - 4} + 5$ when each of the following sequences of transformations have been applied:

a a translation of 2 units in the negative direction of the x-axis followed by dilation of factor 3 from the y-axis

b a dilation of factor 3 from the y-axis followed by a translation of 2 units in the negative direction of the x-axis.

9 The graph of $y = x^2$ is transformed by: CAS

- a translation of a units in the positive direction of the x-axis
- followed by a dilation of factor k from the x-axis.

a If the image of the graph passes through the points $(1, 1)$ and $(2, 4)$, find the possible values of a and k.

b If the image of the graph passes through the points $(1, 1)$ and $(2, 2)$, find the possible values of a and k.

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7D Determining transformations

The method that has been used to find the effect of transformations can be reversed to determine the sequence of transformations used to take a graph to its image.

For example, to find a sequence of transformations which maps $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution of Example 3 **a**:

- $y = \sqrt{x}$ maps to $\frac{y'}{-2} = \sqrt{x'}$.
- We can write $x = x'$ and $y = \frac{y'}{-2}$, and so $x' = x$ and $y' = -2y$.
- The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection if you recognise the form of the image. For the combinations of transformations in this course, it is often simpler to do this.

Example 4

a Find a sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = 2(x-2)^2 + 3$.

b Find a sequence of transformations which takes the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{5x-2}$.

Solution

a **By inspection**

By inspection, it is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

By the method

$y = x^2$ maps to $y' = 2(x'-2)^2 + 3$. Rearranging this equation gives

$$\frac{y'-3}{2} = (x'-2)^2$$

We choose to write $y = \frac{y'-3}{2}$ and $x = x' - 2$.

Solving for x' and y' gives

$$x' = x + 2 \quad \text{and} \quad y' = 2y + 3$$

The transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b We have $y' = \sqrt{5x'-2}$ and $y = \sqrt{x}$. We choose to write $y = y'$ and $x = 5x' - 2$. Hence

$$x' = \frac{x+2}{5} = \frac{x}{5} + \frac{2}{5} \quad \text{and} \quad y' = y$$

The transformation is a dilation of factor $\frac{1}{5}$ from the y-axis followed by a translation of $\frac{2}{5}$ units in the positive direction of the x-axis.

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Example 5

a Find a sequence of transformations which takes the graph of $y = \frac{3}{(x-1)^2} + 6$ to the graph of $y = \frac{1}{x^2}$.

b Find a sequence of transformations which takes the graph of $y = (5x-1)^2 + 6$ to the graph of $y = x^2$.

Solution

a Write $\frac{y-6}{3} = \frac{1}{(x-1)^2}$ and $y' = \frac{1}{(x')^2}$. The points (x, y) satisfying $\frac{y-6}{3} = \frac{1}{(x-1)^2}$ are mapped to the points (x', y') satisfying $y' = \frac{1}{(x')^2}$.

Hence we choose to write

$$y' = \frac{y-6}{3} \quad \text{and} \quad x' = x - 1$$

One transformation is a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis followed by a dilation of factor $\frac{1}{3}$ from the x-axis.

b Write $y - 6 = (5x-1)^2$ and $y' = (x')^2$. The points (x, y) satisfying $y - 6 = (5x-1)^2$ are mapped to the points (x', y') satisfying $y' = (x')^2$.

Hence we choose to write

$$y' = y - 6 \quad \text{and} \quad x' = 5x - 1$$

One transformation is a dilation of factor 5 from the y-axis followed by a translation of 6 units in the negative direction of the y-axis and 1 unit in the negative direction of the x-axis.

We note that the transformations we found are far from being the only possible answers. In fact there are infinitely many choices.

Summary 7D

The notation developed in this chapter can be used to help find the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x' - 3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 from the x-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x' - 3)$. The one we chose is conventional.

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Skill-sheet

Exercise 7D

Example 4

1 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 2(x-1)^2 + 3$ **ii** $y = -(x+1)^2 + 2$ **iii** $y = (2x+1)^2 - 2$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{2}{x+3}$ **ii** $y = \frac{1}{x+3} + 2$ **iii** $y = \frac{1}{x-3} - 2$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x+3} + 2$ **ii** $y = 2\sqrt{3x}$ **iii** $y = -\sqrt{x} + 2$

Example 5

2 **a** Find a sequence of transformations that takes the graph of $y = \frac{5}{(x-3)^2} - 7$ to the graph of $y = \frac{1}{x^2}$.

b Find a sequence of transformations that takes the graph of $y = (3x+2)^2 + 5$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = -3(3x+1)^2 + 7$ to the graph of $y = x^2$.

d Find a sequence of transformations that takes the graph of $y = 2\sqrt{4-x}$ to the graph of $y = \sqrt{x}$.

e Find a sequence of transformations that takes the graph of $y = 2\sqrt{4-x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

3 **a** Find a rule for a transformation that takes the graph of $y = 2x + 7$ to the graph of $y = 3x + 2$.

b Find a rule for a transformation that takes the graph of $y = \frac{1}{(x-2)^2} - 1$ to the graph of $y = \frac{3}{(x-5)^2} + 4$.

c Find a rule for a transformation that takes the graph of $y = (x+2)^2 + 4$ to the graph of $y = (3x-2)^2 - 5$.

d Find a rule for a transformation that takes the graph of $y = 2\sqrt{3-x}$ to the graph of $y = 5\sqrt{x-6}$.

e Find a rule for a transformation that takes the graph of $y = 2\sqrt{2-x} + 3$ to the graph of $y = -5\sqrt{x} + 6$.

4 **a** By equating coefficients, find a and b such that $2x^3 - 12x^2 + 24x - 13 = 2(x-a)^3 + b$ for all $x \in \mathbb{R}$.

b Hence, describe a sequence of transformations that takes the graph of $y = x^3$ to the graph of $y = 2x^3 - 12x^2 + 24x - 13$.

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7E Transformations of graphs of functions

In this section, we further explore the effect of transformations on the graphs of functions.

Applying transformation rules to graphs

The following table summarises what we have established in previous sections. The rule for each transformation is given and the rule for the image of the graph of $y = f(x)$.

Mapping	Rule	The graph of $y = f(x)$ maps to
Reflection in the x-axis	$(x, y) \to (x, -y)$	$y = -f(x)$
Reflection in the y-axis	$(x, y) \to (-x, y)$	$y = f(-x)$
Dilation of factor a from the y-axis	$(x, y) \to (ax, y)$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$(x, y) \to (x, by)$	$y = bf(x)$
Translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis	$(x, y) \to (x + h, y + k)$	$y - k = f(x - h)$

Note that each transformation in this table has a rule of the form

$$(x, y) \to (ax + h, by + k), \qquad \text{where } a \neq 0 \text{ and } b \neq 0$$

Moreover, any sequence of these transformations will also have a rule of this form.

Example 6

A transformation is defined by the rule $(x, y) \to \left(\frac{1}{3}x, 2y\right)$.

Find the equation of the image of the graph of $y = x^2 + 2x + 3$ under this transformation.

Solution

Let $(x, y) \to (x', y')$. Then $x' = \frac{1}{3}x$ and $y' = 2y$.

Thus $x = 3x'$ and $y = \frac{y'}{2}$.

The graph of $y = x^2 + 2x + 3$ is mapped to $\frac{y'}{2} = (3x')^2 + 2(3x') + 3$.

Hence the image has equation $y = 18x^2 + 12x + 6$.

Reflection in the line $y = x$

In our study of inverse functions in Section 5G, we have considered the reflection in the line $y = x$. This transformation has the rule

$$(x, y) \to (y, x)$$

The graph of $y = f(x)$ maps to $x = f(y)$, which is not necessarily the graph of a function.

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Example 7

A transformation is defined by the rule

$$(x, y) \to (-3y + 1, 2x + 2)$$

Find the image of the straight line with equation $y = 2x + 5$ under this transformation.

Solution

Let $(x, y) \to (x', y')$. Then $x' = -3y + 1$ and $y' = 2x + 2$.

So $x = \frac{y'}{2} - 1$ and $y = -\frac{x'}{3} + \frac{1}{3}$.

The graph of $y = 2x + 5$ is mapped to

$$-\frac{x'}{3} + \frac{1}{3} = 2\left(\frac{y'}{2} - 1\right) + 5$$

Rearranging this equation gives $y' = -\frac{x'}{3} - \frac{8}{3}$.

Hence the image is the straight line with equation $y = -\frac{x}{3} - \frac{8}{3}$.

Transformations of the graphs of quadratic functions

For any two quadratic functions f_1 and f_2, there is a sequence of transformations that takes the graph of $y = f_1(x)$ to the graph of $y = f_2(x)$. This is illustrated in the next example.

Example 8

Find a sequence of transformations that takes the graph of the quadratic function with rule $f_1(x) = x^2 + 2x + 6$ to the graph of the quadratic function with rule $f_2(x) = 2x^2 + 8x + 5$.

Solution

Complete the square for each of the rules:

$$f_1(x) = (x + 1)^2 + 5 \quad \text{and} \quad f_2(x) = 2(x + 2)^2 - 3$$

To find a sequence of transformations, we write

$$y = (x + 1)^2 + 5 \quad \text{and} \quad y' = 2(x' + 2)^2 - 3$$

Rearranging gives

$$y - 5 = (x + 1)^2 \quad \text{and} \quad \frac{y' + 3}{2} = (x' + 2)^2$$

We choose to write $y - 5 = \frac{y' + 3}{2}$ and $x + 1 = x' + 2$.

Solving for x' and y' gives

$$x' = x - 1 \quad \text{and} \quad y' = 2y - 13$$

The transformation is a dilation of factor 2 from the x-axis followed by a translation of 1 unit in the negative direction of the x-axis and 13 units in the negative direction of the y-axis.

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Using function notation

We have seen throughout this chapter how the effect of a transformation may be described using function notation. For example, if the graph of $y = f(x)$ is translated 2 units in the positive direction of the x-axis, then its image is the graph of $y = f(x - 2)$.

Example 9

Let $f(x) = x^2$.

a Write the equation for the graph of $y = 4f(x + 3)$.

b Give a sequence of transformations that takes the graph of $y = f(x)$ to the graph of $y = 4f(x + 3)$.

Solution

a $y = 4(x + 3)^2$

b The graph of $y = x^2$ maps to $y' = 4(x' + 3)^2$. Rearranging gives $\frac{y'}{4} = (x' + 3)^2$.

We choose to write $y = \frac{y'}{4}$ and $x = x' + 3$. This gives $x' = x - 3$ and $y' = 4y$.

The transformation is a dilation of factor 4 from the x-axis followed by a translation of 3 units in the negative direction of the x-axis.

The image of two points

We can determine the rule for a transformation of the form $(x, y) \to (ax + h, by + k)$ if we are given the images of two points in the plane. (The two points must not lie on the same horizontal or vertical line.)

Example 10

The points $(1, 1)$ and $(3, 9)$ lie on the graph of $y = x^2$. A transformation with a rule of the form $(x, y) \to (ax + h, by + k)$ maps $(1, 1) \to (5, 4)$ and $(3, 9) \to (11, 12)$.

a Find the values of a, h, b and k for this transformation.

b Find the image of $y = x^2$ under this transformation.

Solution

a Given $(1, 1) \to (5, 4)$ and $(3, 9) \to (11, 12)$, we obtain four equations:

$$a + h = 5 \quad (1)$$
$$b + k = 4 \quad (2)$$
$$3a + h = 11 \quad (3)$$
$$9b + k = 12 \quad (4)$$

From equations (1) and (3), we have $a = 3$ and $h = 2$.
From equations (2) and (4), we have $b = 1$ and $k = 3$.

Hence the transformation has rule $(x, y) \to (3x + 2, y + 3)$.

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b Let $(x, y) \to (x', y')$. Then $x' = 3x + 2$ and $y' = y + 3$.

Hence $x = \dfrac{x' - 2}{3}$ and $y = y' - 3$.

The graph of $y = x^2$ is mapped to $y' = \dfrac{(x' - 2)^2}{9} + 3$.

Function notation for transformations

We can consider a transformation of the plane as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$. In this section, we have been considering transformations of the form

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (ax + h, by + k)$$

and $$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (ay + h, bx + k)$$

Another important class of transformations are those of the form

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (ax + by, cx + dy)$$

These are called **linear transformations**, and they are considered Questions 20 and 21 of the exercise.

Exercise 7E

Example 6

1 A transformation is defined by the rule $(x, y) \to (3x, -2y)$. Find the equation of the image of the graph of $y = x^2 + x + 2$ under this transformation.

2 A transformation is defined by the rule $(x, y) \to (4x, -2y)$. Find the equation of the image of the graph of $y = x^3 + 2x$ under this transformation.

Example 7

3 A transformation is defined by the rule $(x, y) \to (2y, -3x)$. Find the image of the straight line with equation $y = 2x + 3$ under this transformation.

4 A transformation is defined by the rule $(x, y) \to (4y, -2x)$. Find the image of the straight line with equation $y = -2x + 4$ under this transformation.

5 For each of the following, find the image of the straight line with equation $y = -2x + 6$ under the transformation with the given rule:

a $(x, y) \to (-2(y - 1), x + 2)$

b $(x, y) \to (-2y - 1, x + 2)$

c $(x, y) \to (-2x - 2, 3y + 2)$

Example 8

6 Find a sequence of transformations that takes the graph of the quadratic function with rule $f_1(x) = x^2 + 4x + 12$ to the graph of the quadratic function with rule $f_2(x) = 3x^2 + 6x + 5$.

7 Find a sequence of transformations that takes the graph of the quadratic function with rule $f_1(x) = -x^2 + 6x + 8$ to the graph of the quadratic function with rule $f_2(x) = 2x^2 + 8x + 5$.

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8 **a** The transformation with rule $(x, y) \to (3x - 1, y + 2)$ is applied to the graph of $y = \dfrac{2}{x - 3}$. Determine the equation of the image.

b Describe a sequence of transformations that takes the graph of $y = \dfrac{2}{x - 3}$ to the graph of $y = \dfrac{4}{3x - 9} + 6$.

Example 9 **9** Let $f(x) = x^2$.

a Write the equation for the graph of $y = -5f(2x + 3)$.

b Give a sequence of transformations that takes the graph of $y = f(x)$ to the graph of $y = -5f(2x + 3)$.

10 Let $g(x) = \dfrac{1}{x^2}$.

a Write the equation for the graph of $y = -2g(x - 3) + 4$.

b Give a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = -2g(x - 3) + 4$.

Example 10 **11** A transformation of the plane with rule $(x, y) \to (ax + h, by + k)$ maps $(2, 5) \to (-3, 6)$ and $(3, 6) \to (-4, 7)$.

a Find the values of a, h, b and k.

b Find the image of the graph of $y = x^2$ under this transformation.

12 A transformation of the plane with rule $(x, y) \to (ax + h, by + k)$ maps $(1, 3) \to (-2, -3)$ and $(2, 4) \to (-3, 11)$.

a Find the values of a, h, b and k.

b Find the image of the graph of $y = x^2 + 2x$ under this transformation.

13 A transformation of the plane with rule $(x, y) \to (ax + h, by + k)$ maps $(1, -2) \to (-4, 5)$ and $(3, 4) \to (18, 5)$.

a Find the values of a, h, b and k.

b Find the image of the graph of $y = x^2$ under this transformation.

14 The graph of $y = x^2$ is transformed by a dilation of factor a from the y-axis followed by a translation of a units in the negative direction of the x-axis, where $a > 0$. The original graph and its image intersect at the point $(-1, 1)$.

a Find the value of a.

b Find the coordinates of the second point of intersection of the two graphs.

15 The graph of $y = x^2$ is translated by a units in the positive direction of the x-axis and a^2 units in the positive direction of the y-axis, where $a > 0$. The line $y = x$ is tangent to the image.

a Find the value of a.

b Find the coordinates of the point where the line $y = x$ touches the image.

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16 The graph of $y = \sqrt{x}$ is reflected in the y-axis and then dilated by a factor of a from the y-axis, where $a > 0$. The line $x + y = 1$ is tangent to the image.

a Find the value of a.

b Find the coordinates of the point where the line $x + y = 1$ touches the image.

17 **a** Given that the graph of $y = f(x)$ has x-axis intercepts at $(-4, 0)$ and $(2, 0)$, find the x-axis intercepts of the graph of $y = 3f(x - 4)$.

b Given that the graph of $y = g(x)$ has x-axis intercepts at $(-3, 0)$ and $(6, 0)$, find the x-axis intercepts of the graph of $y = 2g(3x)$.

c Given that the graph of $y = h(x)$ has x-axis intercepts at $(-2, 0)$ and $(8, 0)$, find the x-axis intercepts of the graph of $y = 3h(2(x - 4))$.

18 Let $f(x) = x^3$. The line $y = 3x - 2$ touches the graph of $y = f(x)$ at the point $(1, 1)$. Consider the transformation with rule $(x, y) \to (3x + 2, 3y)$.

a Find the equation of the image of the line $y = 3x - 2$ under this transformation.

b Find the equation of the image of the graph of $y = f(x)$ under this transformation.

c Give the coordinates of the point where the image of the line $y = 3x - 2$ touches the image of the graph of $y = f(x)$.

19 Let $f\colon \mathbb{R} \setminus \{\frac{2}{5}\} \to \mathbb{R}$, $f(x) = \dfrac{1}{5x - 2}$.

a Find f^{-1}.

b Find the translation that maps the graph of f to the graph of f^{-1}.

20 For the transformations considered in this book, two different points in the plane will map to two different image points. More formally, we say that a transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is **one-to-one** if $(x_1, y_1) \neq (x_2, y_2)$ implies $T(x_1, y_1) \neq T(x_2, y_2)$.

Now consider a linear transformation

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (ax + by, cx + dy)$$

where $a, b, c, d \in \mathbb{R}$. Prove that T is one-to-one if and only if $ad - bc \neq 0$.

Hint: Prove that if T is not one-to-one, then $ad - bc = 0$. Then prove that if $ad - bc = 0$, then T is not one-to-one. (Note that 'T is not one-to-one' means that there exist $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ such that $(x_1, y_1) \neq (x_2, y_2)$ and $T(x_1, y_1) = T(x_2, y_2)$.)

21 If a linear transformation T is one-to-one, then it has an **inverse transformation**

$$T^{-1}\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T^{-1}(x, y) = (z, w) \text{ if } T(z, w) = (x, y)$$

For example, if $T(3, 4) = (6, 2)$, then $T^{-1}(6, 2) = (3, 4)$.

a Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (2x + 3, -4y)$. Find T^{-1}.

b Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (3 - x, -4y)$. Find T^{-1}.

c Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (\frac{1}{2}x + 3, -2y + 5)$. Find T^{-1}.

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Review

Chapter summary

Assign-ment

Nrich

In the following table, the rule for each transformation is given and the rule for the image of the graph of $y = f(x)$.

Mapping	Rule	The graph of $y = f(x)$ maps to
Reflection in the x-axis	$(x, y) \to (x, -y)$	$y = -f(x)$
Reflection in the y-axis	$(x, y) \to (-x, y)$	$y = f(-x)$
Dilation of factor a from the y-axis	$(x, y) \to (ax, y)$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$(x, y) \to (x, by)$	$y = bf(x)$
Translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis	$(x, y) \to (x + h, y + k)$	$y - k = f(x - h)$
Reflection in the line $y = x$	$(x, y) \to (y, x)$	$x = f(y)$

Technology-free questions

1 Find the image of the point $(-1, 3)$ under each of the following transformations:

- **a** dilation of factor 4 from the x-axis
- **b** dilation of factor 3 from the y-axis
- **c** reflection in the x-axis
- **d** reflection in the y-axis
- **e** reflection in the line $y = x$

2 Sketch the graph of each of the following, labelling asymptotes and axis intercepts:

- **a** $y = \frac{1}{x} + 3$
- **b** $y = \frac{1}{x^2} - 3$
- **c** $y = \frac{1}{(x+2)^2}$
- **d** $y = \sqrt{x - 2}$
- **e** $y = \frac{1}{x - 1}$
- **f** $y = \frac{1}{x} - 4$
- **g** $y = \frac{1}{x + 2}$
- **h** $y = \frac{1}{x - 3}$
- **i** $f(x) = \frac{1}{(x-3)^2}$
- **j** $f(x) = \frac{1}{(x+4)^2}$
- **k** $f(x) = \frac{1}{x - 1} + 1$
- **l** $f(x) = \frac{1}{x - 2} + 2$

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3 Sketch the graph of each of the following, stating the equations of asymptotes, the axis intercepts and the range of each function:

a $y = \frac{1}{x^2} + 1$ **b** $y = \frac{3}{x^2}$ **c** $y = \frac{1}{(x-1)^2}$ **d** $y = \frac{1}{x^2} - 4$

4 Consider the transformation of the plane consisting of a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

a Find the rule for this transformation in the form $(x, y) \to (x', y')$.

b Hence find x and y in terms of x' and y' respectively.

5 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^2$ to the graph of

i $y = 3(x+1)^2 + 2$ **ii** $y = -2(x+2)^2 + 3$ **iii** $y = (3x+1)^2 - 1$

b the graph of $y = \frac{1}{x}$ to the graph of

i $y = \frac{4}{x+2}$ **ii** $y = \frac{1}{x+6} - 12$ **iii** $y = \frac{4}{x-3} - 5$

c the graph of $y = \sqrt{x}$ to the graph of

i $y = \sqrt{x-4} + 2$ **ii** $y = 2\sqrt{2x}$ **iii** $y = -2\sqrt{x} + 3$

6 Consider the transformation of the plane consisting of a reflection in the x-axis followed by a dilation of factor 3 from the y-axis and then by a translation of 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis.

a Find the rule for this transformation in the form $(x, y) \to (x', y')$.

b Hence find x and y in terms of x' and y' respectively.

7 Let $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}$, $f(x) = \frac{1}{x-3}$. Write the equation for the graph of each of the following:

a $y = f(2x-1)$ **b** $y = f(2x) + 3$ **c** $y = f(2(x-4)) + 5$

Multiple-choice questions

1 The point $P(1, 7)$ is translated 3 units in the positive direction of the y-axis and then reflected in the x-axis. The coordinates of the final image of P are

A $(-1, 7)$ **B** $(-1, -7)$ **C** $(1, -10)$ **D** $(0, 7)$ **E** $(1, 0)$

2 The point $P(4, -3)$ lies on the graph of a function f. The graph of f is translated 4 units in the positive direction of the y-axis and then reflected in the y-axis. The coordinates of the final image of P are

A $(4, 3)$ **B** $(-4, -1)$ **C** $(-4, -7)$ **D** $(-4, 1)$ **E** $(4, -7)$

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3 A transformation of the plane is defined by $(x, y) \to (3x - 1, y + 2)$. If $(a, b) \to (8, 8)$, then

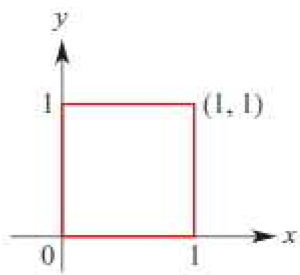
A $a = 3,\ b = 6$ **B** $a = -3,\ b = -6$ **C** $a = 7,\ b = 6$
D $a = 6,\ b = 3$ **E** $a = 23,\ b = 10$

4 A transformation of the plane is defined by $(x, y) \to (3x - 1, 2y + 2)$. If $(a, b) \to (a, b)$, then

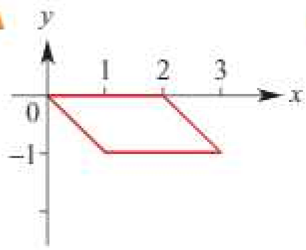
A $a = \frac{3}{2},\ b = -6$ **B** $a = \frac{1}{2},\ b = -2$ **C** $a = -1,\ b = 2$
D $a = 5,\ b = -2$ **E** $a = -\frac{1}{2},\ b = 2$

5 The square shown is subject to successive transformations. The first transformation has rule $(x, y) \to (-x, y)$ and the second transformation has rule $(x, y) \to (-y, -2x + y)$.

Which one of the following graphs shows the image of the square after these two transformations?



A



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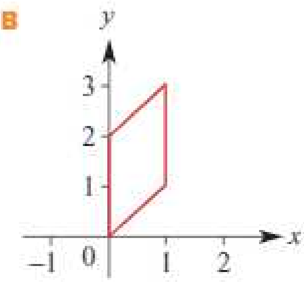

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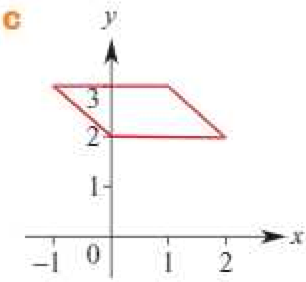

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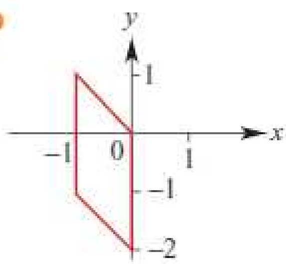

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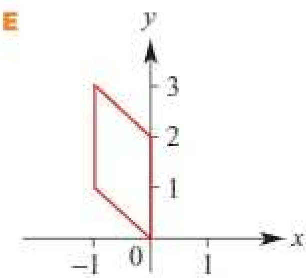

6 The graph of $y = g(x)$ is obtained by reflecting the graph of $y = f(x)$ in the x-axis. The relationship between f and g is given by

A $f(x) = g(x)$ **B** $f(x) = -g(x)$ **C** $f(x) = g(-x)$
D $f(x) = \frac{1}{g(x)}$ **E** $f(x) = 2g(x)$

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7 The translation that maps the graph of $y = x^2$ to the graph of $y = (x - 5)^2 - 2$ can be described by

A $(x, y) \to (x - 2, y + 5)$
B $(x, y) \to (x + 2, y + 5)$
C $(x, y) \to (x - 2, y - 5)$
D $(x, y) \to (x + 5, y - 2)$
E $(x, y) \to (x - 5, y - 2)$

8 The translation that maps the graph of $y = (x + 2)^2 + 8$ to the graph of $y = x^2$ can be described by

A $(x, y) \to (x - 8, y + 2)$
B $(x, y) \to (x + 2, y + 8)$
C $(x, y) \to (x - 2, y + 8)$
D $(x, y) \to (x + 2, y - 8)$
E $(x, y) \to (x - 2, y - 8)$

9 The graph of $y = \sqrt{x}$ is reflected in the y-axis and then translated by 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis. The image is the graph of

A $y = \sqrt{-x} - 2 - 3$
B $y = \sqrt{-x - 2} - 3$
C $y = \sqrt{x + 2} - 3$
D $y = \sqrt{x - 2} + 3$
E $y = \sqrt{-x + 2} + 3$

10 The graph of $y = \dfrac{1}{x^2}$ is reflected in the x-axis and then dilated by a factor of 2 from the x-axis. The image is the graph of

A $y = -\dfrac{2}{x^2}$
B $y = -\dfrac{1}{2x^2}$
C $y = \dfrac{2}{x^2}$
D $y = \dfrac{1}{2x^2}$
E $y = \dfrac{2}{(-x)^2}$

11 The graph of the relation $y^2 = x$ is transformed by a dilation of factor 2 from the x-axis followed by a dilation of factor $\dfrac{1}{3}$ from the y-axis. The equation of the image is

A $12y^2 = x$
B $y^2 = 12x$
C $18y^2 = x$
D $y^2 = 18x$
E $4y^2 = 3x$

12 Consider a transformation of the plane with a rule of the form $(x, y) \to (ax + h, by + k)$, where $a, b \in \mathbb{N}$ and $h, k \in \mathbb{R}$. Suppose that $(1, 2) \to (3, m)$ and $(m, 3) \to (5, 6)$ under this transformation, for some $m \in \mathbb{R}$. Then the largest possible value of $a + b + h + k$ is

A 1
B 3
C 8
D 11
E 13

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Review

Extended-response questions

1 Let $f(x) = x^2$.

a Find the value of k such that the line $y = x$ is tangent to the graph of $y = f(x) + k$.

b Find the value of h such that the line $y = x$ is tangent to the graph of $y = f(x - h)$.

2 **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 8$.

b The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Find the possible values of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quadratic with equation $y = ax^2 + bx$ has vertex with coordinates $(1, 8)$. Find the values of a and b.

3 A quadratic function g has rule $g(x) = x^2 + 4x - 6$.

a Find the value of k for which the equation $g(x) + k = 0$ has one solution.

b Find the values of h for which the equation $g(x - h) = 0$ has:

i two positive solutions

ii two negative solutions

iii one positive and one negative solution.

4 **a** Find the rule for the image of the parabola $y^2 = 4x$ reflected in the line $y = x$.

b The point $A(4, 4)$ lies on the parabola $y^2 = 4x$. Find the equation of the line which passes through $A(4, 4)$ and $B(1, 0)$ and then find the coordinates of the point C where this line meets the parabola again.

c Describe the image AC' of the line segment AC reflected in the line $y = x$.

d Illustrate the two parabolas and the line segments AC and AC' on the one set of axes.

5 **a** Find the dilation from the x-axis which takes the curve $y = x^2$ to the parabola with its vertex at the origin that passes through the point $(45, 9)$.

b State a rule which reflects this dilated parabola in the x-axis.

c State a rule which takes the reflected parabola of part **b** to a parabola with x-axis intercepts $(0, 0)$ and $(90, 0)$ and vertex $(45, 9)$.

d State a rule which takes the curve $y = x^2$ to the parabola defined in part **c**.

6 A cubic function f has rule $f(x) = (x - 3)(x + 4)(x - 5)$.

a Solve the equation $f(x - 2) = 0$.

b Solve the equation $f(x + 2) = 0$.

c It is known that the equation $f(x) + k = 0$ has a solution $x = 0$. Find the value of k and solve the equation $f(x) + k = 0$.

d The equation $f(x - h) = 0$ has a solution $x = 0$. Find the possible values of h.

e Find the values of h such that $f(x - h) = 0$ has only one positive solution.

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Review

7 **a** Find the image of the point with coordinates $(2, 6)$ under a reflection in the line $x = 3$.

b Find the rule for the following sequence of transformations:

- a translation of 3 units in the negative direction of the x-axis
- followed by a reflection in the y-axis
- followed by a translation of 3 units in the positive direction of the x-axis.

c Hence, give the rule for the transformation of reflection in the line $x = 3$.

d **i** Give a sequence of three transformations which would determine a reflection in the line $x = m$. (For convenience, assume that $m > 0$.)

ii Give the rule for the transformation of reflection in the line $x = m$.

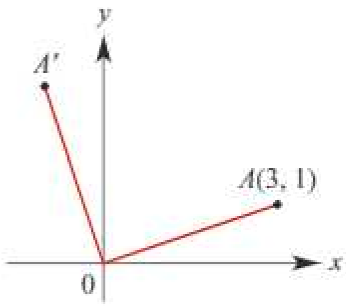
e **i** Give a sequence of three transformations which would determine a reflection in the line $y = n$. (For convenience, assume that $n > 0$.)

ii Give the rule for the transformation of reflection in the line $y = n$.

f Find the image of each of the following curves under a reflection in the line $x = 3$:

i $y = x - 3$ **ii** $y = x$ **iii** $y = x^2$ **iv** $y = (x - 3)^2$

8 **a** The point $A(3, 1)$ is rotated about the origin by $90°$ anticlockwise. Find the coordinates of A', the image of A.



b **i** State the gradient of the line OA.

ii State the gradient of the line OA'.

c Now consider a point $A(p, q)$.

i State the gradient of the line OA.

ii The point $A(p, q)$ is rotated about the origin by $90°$ anticlockwise. Find the coordinates of A', the image of A.

d Find the rule for the transformation of rotation about the origin by $90°$ anticlockwise.

e Find the image of each of the following curves under this transformation:

i $y = x$ **ii** $y = x^2$ **iii** $x^2 + y^2 = 1$ **iv** $y = \frac{1}{x}$

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