4

A gallery of graphs

Objectives

- To recognise the rules of a number of common algebraic relations:
  - $y = x^{-1}$ (rectangular hyperbola)
  - $y = x^{-2}$
  - $y^2 = x$
  - $y = \sqrt{x}$
  - $x^2 + y^2 = 1$ (circle).
- To be able to sketch the graphs of these relations.
- To be able to sketch the graphs of simple transformations of these relations.
- To find the key features of the graphs of these relations.
- To determine the rules of relations of these types given sufficient information.

In Chapter 2, we looked at linear graphs, sketching them and determining their rules given sufficient information. All linear graphs can be considered as transformations of $y = x$. The features we concentrated on for linear graphs were the x-axis intercept, the y-axis intercept and the gradient.

In Chapter 3, we considered quadratics written in 'turning point form' and sketched their graphs by using transformations of the graph of the basic quadratic $y = x^2$. The features we concentrated on for graphs of quadratic polynomials were the x-axis intercepts, the y-axis intercept and the coordinates of the turning point (vertex).

In this chapter, we study some other common algebraic relations, and develop methods similar to those used in Chapter 3 to sketch the graphs of these relations. The relations in this chapter have different types of key features. For example, we introduce asymptotes for graphs of rectangular hyperbolas and graphs of the form $y = x^{-2}$, and the coordinates of the centre and the length of the radius are key features in the study of circles.

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4A Rectangular hyperbolas

Consider the rule

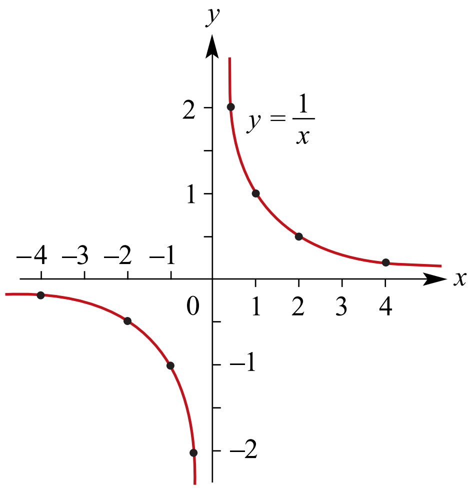
$$y = \frac{1}{x} = x^{-1} \qquad \text{for } x \neq 0$$

We can construct a table of values for $y = \frac{1}{x}$ for values of x between -4 and 4 as follows:

x	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{1}{2}$	-1	-2	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$

We can plot these points and then connect the dots to produce a continuous curve.
A graph of this type is an example of a **rectangular hyperbola**.

Note that y is undefined when $x = 0$, and that there is no x-value that will produce the value $y = 0$.



Asymptotes

There are two lines associated with this graph that help to describe its shape.

Horizontal asymptote

From the graph we see that, as x approaches infinity in either direction, the value of y approaches zero. The following notation will be used to state this:

- As $x \to \infty$, $y \to 0^+$. This is read: 'As x approaches infinity, y approaches 0 from the positive side.'
- As $x \to -\infty$, $y \to 0^-$. This is read: 'As x approaches negative infinity, y approaches 0 from the negative side.'

The graph approaches the x-axis (the line $y = 0$) but does not cross this line. The line $y = 0$ is a **horizontal asymptote**.

Vertical asymptote

As x approaches zero from either direction, the magnitude of y becomes very large. The following notation will be used to state this:

- As $x \to 0^+$, $y \to \infty$. This is read: 'As x approaches zero from the positive side, y approaches infinity.'
- As $x \to 0^-$, $y \to -\infty$. This is read: 'As x approaches zero from the negative side, y approaches negative infinity.'

The graph approaches the y-axis (the line $x = 0$) but does not cross this line. The line $x = 0$ is a **vertical asymptote**.

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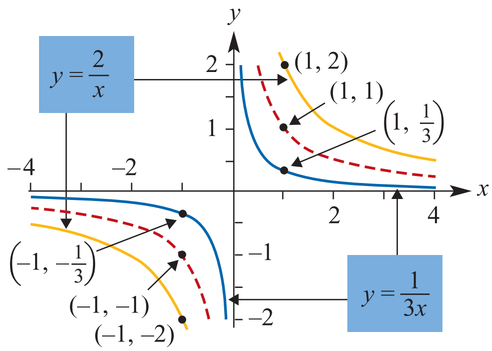
Dilations from an axis

The diagram on the right shows the graphs of

$$y = \frac{1}{x}, \quad y = \frac{2}{x} \quad \text{and} \quad y = \frac{1}{3x}$$

The asymptotes are the x-axis and the y-axis, and they have equations $y = 0$ and $x = 0$ respectively.

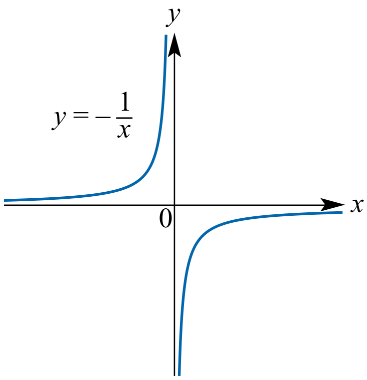


As can be seen from the diagram, the graphs of $y = \frac{2}{x}$ and $y = \frac{1}{3x}$ have the same 'shape' and asymptotes as the graph of $y = \frac{1}{x}$, but they have been 'stretched'.

The transformation that takes the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{2}{x}$ is called the **dilation** of factor 2 from the x-axis. For example, the point $(1, 1)$ on the graph of $y = \frac{1}{x}$ is taken to the point $(1, 2)$ on the graph of $y = \frac{2}{x}$. Dilations will be considered formally in Chapter 7.

Reflection in the x-axis

When the graph of $y = \frac{1}{x}$ is reflected in the x-axis, the result is the graph of $y = -\frac{1}{x}$.



The asymptotes are still the two axes, that is, the lines $x = 0$ and $y = 0$.

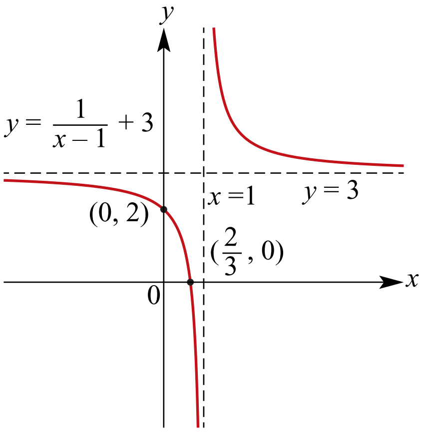
Similarly, $y = -\frac{2}{x}$ is the reflection of $y = \frac{2}{x}$ in the x-axis.

Reflecting in the y-axis gives the same result for these two graphs.

Translations

Now let us consider the graph of $y = \frac{1}{x-1} + 3$.

The basic graph of $y = \frac{1}{x}$ has been translated 1 unit to the right and 3 units up.



Asymptotes The equation of the vertical asymptote is now $x = 1$, and the equation of the horizontal asymptote is now $y = 3$.

Intercepts with the axes The graph now has x-axis and y-axis intercepts. These can be calculated in the usual way to add further detail to the graph.

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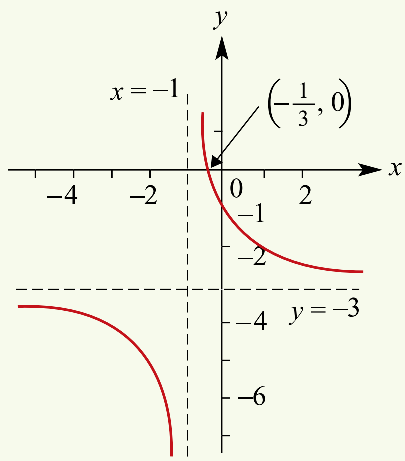
Sketching rectangular hyperbolas

Using dilations, reflections and translations, we are now able to sketch the graphs of all rectangular hyperbolas of the form $y = \dfrac{a}{x - h} + k$.

Example 1

Sketch the graph of $y = \dfrac{2}{x + 1} - 3$.

Solution



Explanation

The graph of $y = \dfrac{2}{x}$ has been translated 1 unit to the left and 3 units down. The asymptotes have equations $x = -1$ and $y = -3$.

When $x = 0$, $y = \dfrac{2}{0 + 1} - 3 = -1$.

$\therefore$ the y-axis intercept is -1.

When $y = 0$,

$$0 = \frac{2}{x+1} - 3$$

$$3 = \frac{2}{x+1}$$

$$3(x+1) = 2$$

$$x = -\frac{1}{3}$$

$\therefore$ the x-axis intercept is $-\frac{1}{3}$.

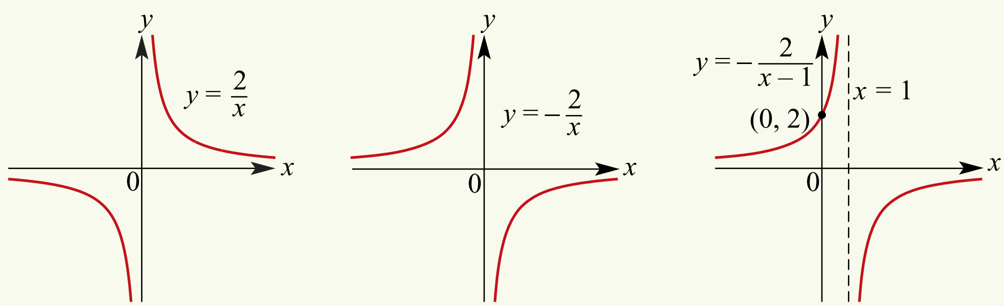
Example 2

Sketch the graph of $y = \dfrac{-2}{x - 1}$.

Solution

The graph of $y = -\dfrac{2}{x}$ is obtained from the graph of $y = \dfrac{2}{x}$ by reflection in the x-axis.

This graph is then translated 1 unit to the right to obtain the graph of $y = \dfrac{-2}{x - 1}$.



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Summary 4A

- For $a > 0$, a dilation of factor a from the x-axis transforms the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{a}{x}$.
- A reflection in the x-axis transforms the graph of $y = \frac{a}{x}$ to the graph of $y = -\frac{a}{x}$.
- For $h, k \geq 0$, a translation of h to the right and k upwards transforms the graph of $y = \frac{a}{x}$ to the graph of $y = \frac{a}{x-h} + k$.
- A rectangular hyperbola with rule of the form $y = \frac{a}{x-h} + k$ has:
  - vertical asymptote $x = h$
  - horizontal asymptote $y = k$.

Exercise 4A

Example 1
Example 2

1 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{x}$ **b** $y = \frac{2}{x}$ **c** $y = \frac{1}{2x}$

d $y = \frac{-3}{x}$ **e** $y = \frac{1}{x} + 2$ **f** $y = \frac{1}{x} - 3$

g $y = \frac{2}{x} - 4$ **h** $y = \frac{-1}{2x} + 5$ **i** $y = \frac{1}{x-1}$

j $y = \frac{-1}{x+2}$ **k** $y = \frac{1}{x+1} + 3$ **l** $y = \frac{-2}{x-3} - 4$

2 Write down the equations of the asymptotes for each of the graphs in Question 1.

3 **a** We can write $y = \frac{1}{3x+6}$ as $y = \frac{1}{3(x+2)}$.

Sketch the graph of $y = \frac{1}{3x}$ and hence the graph of $y = \frac{1}{3x+6}$.

b We can write $y = \frac{3}{2x+4}$ as $y = \frac{3}{2(x+2)}$.

Sketch the graph of $y = \frac{3}{2x}$ and hence the graph of $y = \frac{3}{2x+4}$.

c We can write $y = \frac{-1}{2x+4}$ as $y = -\frac{1}{2(x+2)}$.

Sketch the graph of $y = -\frac{1}{2x}$ and hence the graph of $y = \frac{-1}{2x+4}$.

d We can write $y = \frac{1}{2x+1}$ as $y = \frac{1}{2(x+\frac{1}{2})}$.

Sketch the graph of $y = \frac{1}{2x}$ and hence the graph of $y = \frac{1}{2x+1}$.

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4 Sketch the graphs of the following, showing all important features of the graphs:

a $y = \frac{1}{3x+1}$ **b** $y = \frac{1}{3x+1} - 1$ **c** $y = \frac{-1}{3x+1} - 1$ **d** $y = \frac{-2}{3x+1}$

e $y = \frac{-2}{3x+1} - 4$ **f** $y = \frac{-2}{3x+1} + 3$ **g** $y = \frac{2}{3x+2} - 1$ **h** $y = \frac{3}{3x+4} - 1$

5 Show that $\frac{x+3}{x-1} = \frac{4}{x-1} + 1$ and hence sketch the graph of $y = \frac{x+3}{x-1}$.

6 Show that $\frac{2x+3}{x+1} = \frac{1}{x+1} + 2$ and hence sketch the graph of $y = \frac{2x+3}{x+1}$.

7 Show that $\frac{3-2x}{x-2} = -\frac{1}{x-2} - 2$ and hence sketch the graph of $y = \frac{3-2x}{x-2}$.

8 Sketch the graphs of the following with the help of your calculator: CAS

a $y = \frac{4}{3x-2} - 1$ **b** $y = \frac{3x+2}{2x-2}$ **c** $y = \frac{5x+2}{3x-2}$

4B The truncus

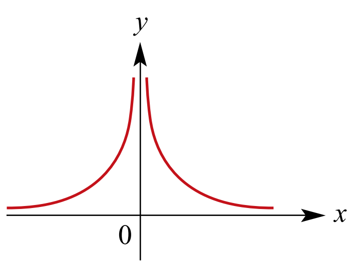
Now consider the rule

$$y = \frac{1}{x^2} = x^{-2} \qquad \text{for } x \neq 0$$

We can construct a table of values for x between −4 and 4 as follows:

x	−4	−3	−2	−1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4
y	$\frac{1}{16}$	$\frac{1}{9}$	$\frac{1}{4}$	1	4	4	1	$\frac{1}{4}$	$\frac{1}{9}$	$\frac{1}{16}$

We can plot these points and then connect the dots to produce a continuous curve. A graph of this shape is sometimes called a **truncus**.



Note that y is undefined when $x = 0$, and that there is no x-value that will produce a negative value of y.

Note: As $x \to \infty$, $y \to 0^+$. As $x \to -\infty$, $y \to 0^+$.
As $x \to 0^+$, $y \to \infty$. As $x \to 0^-$, $y \to \infty$.

The graph of $y = \frac{1}{x^2}$ has asymptotes $y = 0$ and $x = 0$.

The transformations considered in the previous section can be applied to the graph of $y = \frac{1}{x^2}$. All graphs of the form

$$y = \frac{a}{(x-h)^2} + k$$

will have the same basic 'truncus' shape. The **asymptotes** will be the lines $y = k$ and $x = h$.

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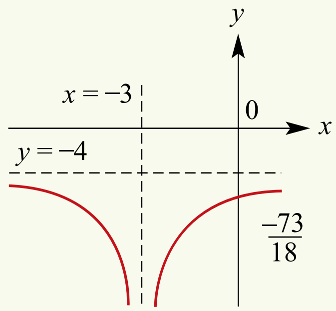
Example 3

Sketch the graph of $y = \frac{-1}{2(x+3)^2} - 4$.

Solution



Explanation

The graph of $y = \frac{-1}{2x^2}$ is translated 3 units to the left and 4 units down.

The y-axis intercept is found by putting $x = 0$; it is $\frac{-73}{18}$.

Exercise 4B

Example 3

1 Sketch the graphs of the following, showing all important features:

a $y = \frac{1}{(x+3)^2}$ **b** $y = \frac{1}{x^2} - 4$ **c** $y = \frac{-1}{(x-2)^2}$

d $y = \frac{1}{(x-1)^2} + 3$ **e** $y = \frac{1}{2(x+3)^2} - 4$ **f** $y = \frac{-2}{(x-2)^2} + 1$

g $y = \frac{3}{(x+3)^2} - 6$ **h** $y = \frac{-1}{(x-4)^2} + 2$

2 Write down the equations of the asymptotes for each of the graphs in Question 1.

3 **a** We can write $y = \frac{1}{(3x+6)^2}$ as $y = \frac{1}{9(x+2)^2}$.

Sketch the graph of $y = \frac{1}{9x^2}$ and hence the graph of $y = \frac{1}{(3x+6)^2}$

b We can write $y = \frac{3}{(2x+4)^2}$ as $y = \frac{3}{4(x+2)^2}$.

Sketch the graph of $y = \frac{3}{4x^2}$ and hence the graph of $y = \frac{3}{(2x+4)^2}$.

c We can write $y = \frac{-1}{(2x+4)^2}$ as $y = -\frac{1}{4(x+2)^2}$.

Sketch the graph of $y = -\frac{1}{4x^2}$ and hence the graph of $y = \frac{-1}{(2x+4)^2}$.

4 Sketch the graphs of the following with the help of your calculator: CAS

a $y = \frac{4}{(3x-2)^2} - 1$ **b** $y = \frac{2}{x^2 - 6x + 9} + 2$ **c** $y = \frac{6}{x^2 + 8x + 16} - 2$

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4C The graph of $y^2 = x$

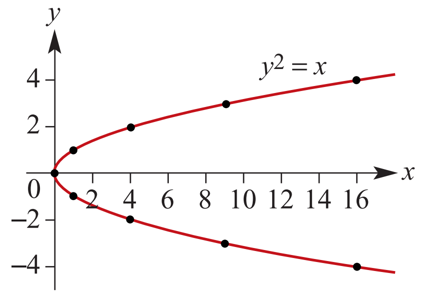
Next consider the rule $y^2 = x$. We can construct a table of values for y between −4 and 4 as follows:

y	−4	−3	−2	−1	0	1	2	3	4
x	16	9	4	1	0	1	4	9	16

We plot these points with y against x and then connect the dots to produce a continuous curve.

The graph of $y^2 = x$ is a parabola. It can be obtained from the graph of $y = x^2$ by a reflection in the line $y = x$.

The vertex of the parabola is at $(0, 0)$, and the axis of symmetry is the x-axis.



The transformations considered in the previous sections can be applied to the graph of $y^2 = x$. All graphs of the form

$$(y - k)^2 = a^2(x - h)$$

will have the same basic parabola shape. The vertex of the parabola will be at the point (h, k), and the axis of symmetry will be the line $y = k$.

Example 4

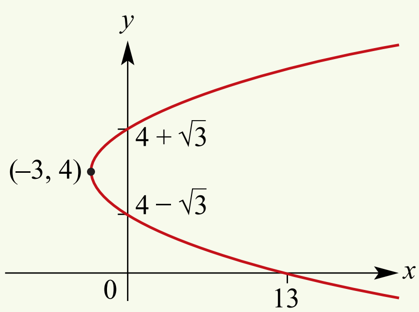
Sketch the graph of:

a $(y - 4)^2 = x + 3$

b $y^2 + 2y = 2x + 3$

Solution

a



Explanation

The graph of $y^2 = x$ is translated 3 units to the left and 4 units up.

The vertex has coordinates $(-3, 4)$.

When $x = 0$, $\quad (y - 4)^2 = 3$

$$y - 4 = \pm\sqrt{3}$$

$$y = 4 \pm \sqrt{3}$$

When $y = 0$, $\quad 16 = x + 3$

$$x = 13$$

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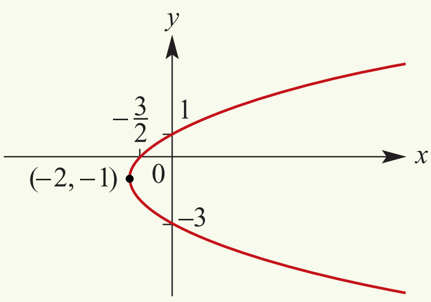
b Complete the square:

$$y^2 + 2y = 2x + 3$$
$$y^2 + 2y + 1 = 2x + 4$$
$$(y + 1)^2 = 2(x + 2)$$



The graph of $(y + 1)^2 = 2(x + 2)$ is obtained from the graph of $y^2 = x$ by a dilation of factor $\sqrt{2}$ from the x-axis and then a translation 2 units to the left and 1 unit down.

The vertex has coordinates $(-2, -1)$.

When $x = 0$, $(y + 1)^2 = 4$

$$y + 1 = \pm 2$$
$$y = -1 \pm 2$$
$$y = 1 \text{ or } y = -3$$

When $y = 0$, $1 = 2x + 4$

$$x = -\frac{3}{2}$$

Exercise 4C

Example 4

1 Sketch the graph of each of the following relations, showing all important features:

a $(y - 2)^2 = x - 3$ **b** $(y + 2)^2 = x + 4$ **c** $y^2 = 2x$
d $y^2 = 2(x + 5)$ **e** $(y - 4)^2 = 2(x + 3)$ **f** $(y + 4)^2 = 2x$
g $(y + 3)^2 = 2x - 4$ **h** $y^2 = \frac{x}{2}$ **i** $y^2 + 4y = 2x + 4$
j $y^2 + 6y - 2x + 3 = 0$ **k** $y^2 + y - x = 0$ **l** $y^2 + 7y - 5x + 3 = 0$
m $y^2 = -x$ **n** $y^2 + 2y - x = 0$

2 Sketch the graph of each of the following with the help of your calculator. (Instructions are given in the calculator appendices of the Interactive Textbook.)

CAS

a $y^2 = -4(x + 5)$ **b** $y^2 + 2y + 3x + 12 = 0$ **c** $(y - 5)^2 = 7 - 4x$

4D The graph of $y = \sqrt{x}$

The graph of the rule

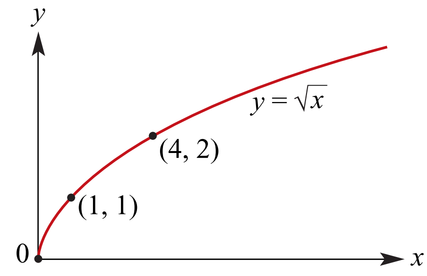
$$y = \sqrt{x} = x^{\frac{1}{2}} \qquad \text{for } x \geq 0$$

is the upper half of the parabola $y^2 = x$.

All graphs of the form

$$y = a\sqrt{x - h} + k$$

will have the same basic shape as the graph of $y = \sqrt{x}$.



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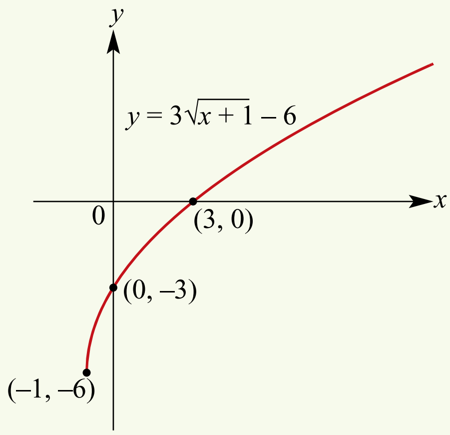
Example 5

Sketch the graph of $y = 3\sqrt{x+1} - 6$.

Solution



When $x = 0$, $y = -3$

When $y = 0$, $3\sqrt{x+1} - 6 = 0$

$$3\sqrt{x+1} = 6$$
$$\sqrt{x+1} = 2$$
$$x + 1 = 4$$
$$x = 3$$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 3 and then translating 1 unit to the left and 6 units down.

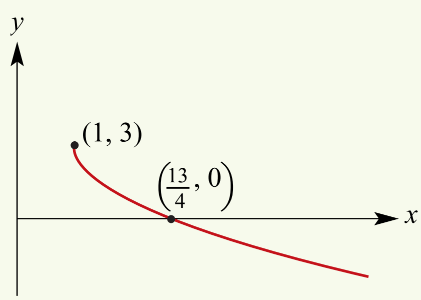
The rule is defined for $x \geq -1$.

The set of values the rule can take (the range) is all numbers greater than or equal to -6, i.e. $y \geq -6$.

Example 6

Sketch the graph of $y = -2\sqrt{x-1} + 3$.

Solution



When $y = 0$: $\quad -2\sqrt{x-1} + 3 = 0$

$$2\sqrt{x-1} = 3$$

Square both sides: $\quad 4(x-1) = 9$

Therefore $x = \frac{9}{4} + 1 = \frac{13}{4}$

Explanation

The graph is formed by dilating the graph of $y = \sqrt{x}$ from the x-axis by factor 2, reflecting this in the x-axis and then translating it 1 unit to the right and 3 units up.

The rule is defined for $x \geq 1$.

The set of values the rule can take (the range) is all numbers less than or equal to 3, i.e. $y \leq 3$.

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The graph of $y = \sqrt{-x}$

The rule

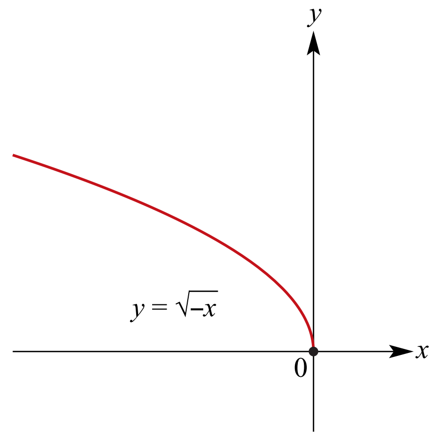
$$y = \sqrt{-x} \qquad \text{for } x \le 0$$

yields a graph which is the reflection of the graph of $y = \sqrt{x}$ in the y-axis.

All graphs of the form

$$y = a\sqrt{-(x-h)} + k$$

will have the same basic shape as the graph of $y = \sqrt{-x}$.

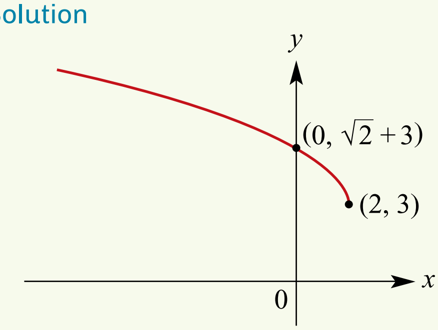


Example 7

Sketch the graph of $y = \sqrt{2-x} + 3$.

Note: $\sqrt{2-x} = \sqrt{-(x-2)}$

Solution



When $x = 0$, $y = \sqrt{2} + 3$.

Explanation

We can write the rule as

$$y = \sqrt{-(x-2)} + 3$$

The rule is defined for $x \le 2$. The set of values the rule can take (the range) is all numbers greater than or equal to 3, i.e. $y \ge 3$.

Summary 4D

- All graphs of the form $y = a\sqrt{x-h} + k$ will have the same basic shape as the graph of $y = \sqrt{x}$. The graph will have endpoint (h, k).
- The graph of $y = \sqrt{-x}$ is the reflection in the y-axis of the graph of $y = \sqrt{x}$.

Exercise 4D

Example 5

1 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = 2\sqrt{x} + 3$ **b** $y = \sqrt{x-2} + 3$ **c** $y = \sqrt{x-2} - 3$

d $y = \sqrt{x+2} + 1$ **e** $y = -\sqrt{x+2} + 3$ **f** $y = 2\sqrt{x+2} - 3$

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Example 6
Example 7

2 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = -\sqrt{x-2} + 3$ **b** $y = \sqrt{-(x-4)} - 2$ **c** $y = -2\sqrt{-(x+4)} - 1$

d $y = 2\sqrt{3-x}$ **e** $y = -2\sqrt{3-x}$ **f** $y = 4\sqrt{3-x} - 4$

3 For each of the following rules, sketch the corresponding graph, giving the axis intercepts when they exist, the set of x-values for which the rule is defined and the set of y-values which the rule takes:

a $y = \sqrt{3x}$ **b** $y = \sqrt{3(x-1)}$ **c** $y = -\sqrt{2x}$

d $y = \sqrt{2(3-x)}$ **e** $y = -2\sqrt{4(2-x)}$ **f** $y = 4\sqrt{2(3-x)} - 4$

4 Sketch the graph of each of the following with the help of your calculator: CAS

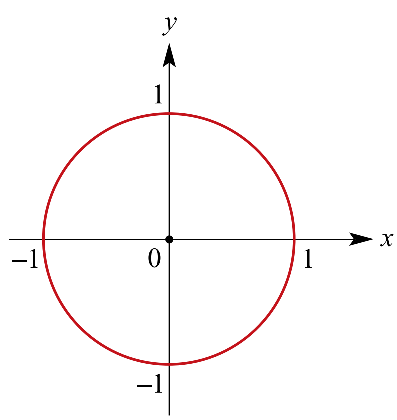
a $y = \sqrt{-4(x+5)}$ **b** $y = -\sqrt{5-x} - 3$ **c** $y + 6 = 2\sqrt{2x-4}$

4E Circles*

Consider a circle in the coordinate plane with centre the origin and radius r. If $P(x, y)$ is a point on the circle, its distance from the origin is r and so by Pythagoras' theorem $x^2 + y^2 = r^2$.

Conversely, if a point $P(x, y)$ in the plane satisfies the equation $x^2 + y^2 = r^2$, its distance from the origin is r, so it lies on a circle with centre the origin and radius r.

To the right is the graph of the circle with equation $x^2 + y^2 = 1$.



All circles can be considered as being transformations of this basic graph.

As has been seen with other graphs, the basic graph may be translated horizontally and vertically.

> The equation for a circle is
>
> $$(x-h)^2 + (y-k)^2 = r^2$$
>
> where the **centre** of the circle is the point (h, k) and the **radius** is r.

* Circles are listed in the 10A Victorian Curriculum, but not explicitly listed in the VCAA study design for Mathematical Methods Units 1 & 2. This section is included to provide valuable background for the study of functions and relations in this subject and in Specialist Mathematics.

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If the radius and the coordinates of the centre of the circle are given, the equation of the circle can be determined.

Example 8

Write down the equation of the circle with centre $(-3, 5)$ and radius 2.

Solution

If the radius is 2 and the centre is the point $(-3, 5)$, then the equation will be

$$(x - (-3))^2 + (y - 5)^2 = 4$$
$$(x + 3)^2 + (y - 5)^2 = 4$$

If the equation of the circle is given, the radius and the centre of the circle can be determined and the graph sketched.

Example 9

Find the centre and radius of the circle $(x - 1)^2 + (y - 2)^2 = 4$.

Solution

The equation $(x - 1)^2 + (y - 2)^2 = 4$ defines a circle of radius 2 with centre at $(1, 2)$.

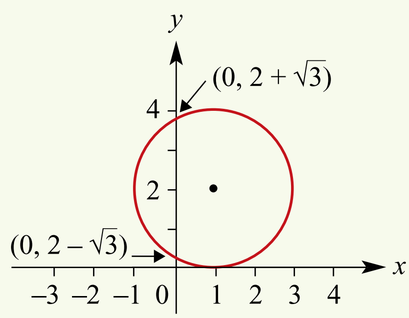
Explanation

We can sketch the circle with a little extra work.

When $x = 0$,

$$1 + (y - 2)^2 = 4$$
$$(y - 2)^2 = 3$$

Hence $y = 2 \pm \sqrt{3}$



Example 10

Sketch the graph of the circle $(x + 1)^2 + (y + 4)^2 = 9$.

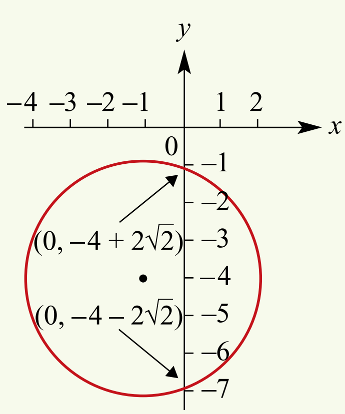
Solution

When $x = 0$,

$$1 + (y + 4)^2 = 9$$
$$(y + 4)^2 = 8$$

Hence $y = -4 \pm \sqrt{8}$

$$= -4 \pm 2\sqrt{2}$$



Explanation

The circle has radius 3 and centre $(-1, -4)$.

The y-axis intercepts can be found in the usual way.

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The equation of a circle may not always be written in the form $(x - h)^2 + (y - k)^2 = r^2$.

Expanding the general equation of a circle gives

$$\begin{aligned}(x - h)^2 + (y - k)^2 &= r^2\\ x^2 - 2hx + h^2 + y^2 - 2ky + k^2 &= r^2\\ x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 &= 0\end{aligned}$$

Let $c = h^2 + k^2 - r^2$. Then we obtain an alternative form for the equation of a circle:

The **general form** for the equation of a circle is

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

You will note that there is some similarity with the general form of a straight line, $ax + by + c = 0$.

Notice that in the general form of the circle equation, the coefficients of x^2 and y^2 are both 1 and there is no xy term.

In order to sketch a circle with equation expressed in this form, the equation can be converted to the 'centre–radius' form by completing the square for both x and y.

Example 11

Find the radius and the coordinates of the centre of the circle with equation

$$x^2 + y^2 - 6x + 4y - 12 = 0$$

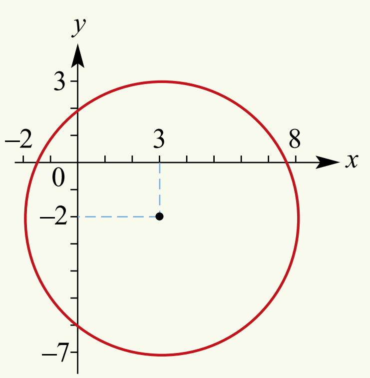
and hence sketch the graph.

Solution

By completing the square for both x and y we have

$$\begin{aligned}x^2 + y^2 - 6x + 4y - 12 &= 0\\ (x^2 - 6x + 9) - 9 + (y^2 + 4y + 4) - 4 - 12 &= 0\\ (x^2 - 6x + 9) + (y^2 + 4y + 4) &= 25\\ (x - 3)^2 + (y + 2)^2 &= 5^2\end{aligned}$$

The radius is 5 and the centre is at (3, −2).



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Semicircles

Transposing the general equation of the circle $x^2 + y^2 = r^2$ to make y the subject, we have

$$y^2 = r^2 - x^2$$
$$y = \pm\sqrt{r^2 - x^2}$$

We can now consider two separate rules

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

which correspond to the top half and bottom half of the circle respectively.

Similarly, solving for x will give you the semicircles to the left and right of the y-axis:

$$x = \pm\sqrt{r^2 - y^2}$$

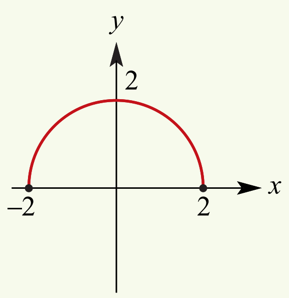
Example 12

Sketch the graphs of:

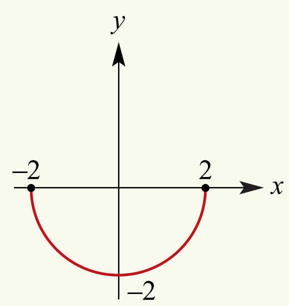
a $y = +\sqrt{4 - x^2}$
b $y = -\sqrt{4 - x^2}$
c $x = -\sqrt{4 - y^2}$
d $x = +\sqrt{4 - y^2}$

Solution

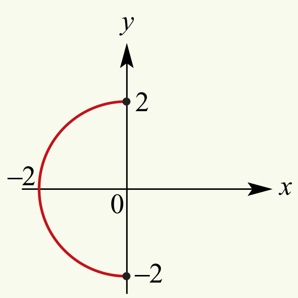
a $y = +\sqrt{4 - x^2}$



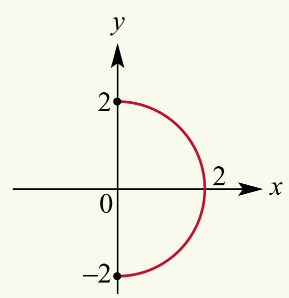
b $y = -\sqrt{4 - x^2}$



c $x = -\sqrt{4 - y^2}$



d $x = +\sqrt{4 - y^2}$



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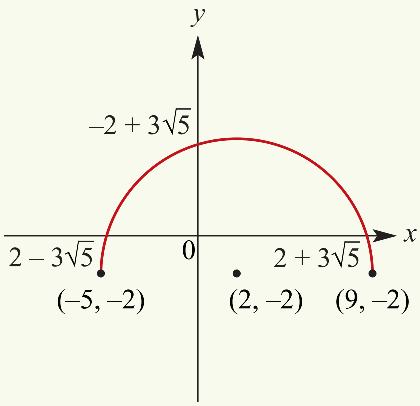
Example 13

Sketch the graph of $y = -2 + \sqrt{49 - (x-2)^2}$.

Solution



When $x = 0$,

$$y = -2 + \sqrt{45}$$
$$= -2 + 3\sqrt{5}$$

When $y = 0$,

$$-2 + \sqrt{49 - (x-2)^2} = 0$$
$$\sqrt{49 - (x-2)^2} = 2$$
$$49 - (x-2)^2 = 4$$
$$(x-2)^2 = 45$$
$$x = 2 \pm 3\sqrt{5}$$

Explanation

It is a semicircle of the circle

$$(x-2)^2 + (y+2)^2 = 49$$

The centre is at the point $(2, -2)$ and the radius is 7.

It is the semicircle $y = \sqrt{49 - x^2}$ translated 2 units to the right and 2 units down.

In the usual way, we find the x-axis intercepts and the y-axis intercept.

Summary 4E

- The equation of a circle with centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

- The **general form** for the equation of a circle is

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

- The two separate rules for semicircles with their base on the x-axis are

$$y = +\sqrt{r^2 - x^2} \quad \text{and} \quad y = -\sqrt{r^2 - x^2}$$

  They correspond to the top half and bottom half of the circle respectively.
- The two separate rules for semicircles with their base on the y-axis are

$$x = +\sqrt{r^2 - y^2} \quad \text{and} \quad x = -\sqrt{r^2 - y^2}$$

  They correspond to the right half and left half of the circle respectively.

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Skill-sheet

Exercise 4E

Example 8

1 Write down the equation of each of the following circles, with centre at $C(h, k)$ and radius r:

a $C(0, 0),\ r = 3$ **b** $C(0, 0),\ r = 4$ **c** $C(1, 3),\ r = 5$

d $C(2, -4),\ r = 3$ **e** $C(-3, 4),\ r = \frac{5}{2}$ **f** $C(-5, -6),\ r = 4.6$

Example 9

2 Find the centre, C, and the radius, r, of the following circles:

a $(x - 1)^2 + (y - 3)^2 = 4$ **b** $(x - 2)^2 + (y + 4)^2 = 5$

c $(x + 3)^2 + (y - 2)^2 = 9$ **d** $(x + 5)^2 + (y - 4)^2 = 8$

Example 10

3 Sketch the graphs of each of the following:

a $x^2 + y^2 = 64$ **b** $x^2 + (y - 4)^2 = 9$

c $(x + 2)^2 + y^2 = 25$ **d** $(x + 1)^2 + (y - 4)^2 - 169 = 0$

e $(2x - 3)^2 + (2y - 5)^2 = 36$ **f** $(x + 5)^2 + (y - 5)^2 = 36$

Example 11

4 Find the centre, C, and the radius, r, of the following circles:

a $x^2 + y^2 - 6y - 16 = 0$ **b** $x^2 + y^2 - 8x + 12y + 10 = 0$

c $x^2 + y^2 - 6x + 4y + 9 = 0$ **d** $x^2 + y^2 + 4x - 6y - 12 = 0$

e $x^2 + y^2 - 8x + 4y + 1 = 0$ **f** $x^2 + y^2 - x + 4y + 2 = 0$

Example 12

5 Sketch the graphs of each of the following:

a $y = +\sqrt{9 - x^2}$ **b** $x = +\sqrt{9 - y^2}$ **c** $y = -\sqrt{16 - x^2}$

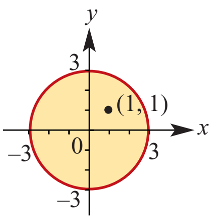
d $y = -\sqrt{25 - x^2}$ **e** $x = -\sqrt{49 - y^2}$ **f** $x = \sqrt{\frac{25}{4} - y^2}$

Example 13

6 Sketch the graphs of each of the following:

a $y = \sqrt{36 - (x - 2)^2}$ **b** $y - 2 = \sqrt{4 - (x + 2)^2}$

7 The graph of $x^2 + y^2 \le 9$ is as shown. Note that $(1, 1)$ satisfies $1^2 + 1^2 \le 9$. The coordinates of every point in the shaded region satisfy the inequality.



Sketch the graphs of each of the following. Use a dotted line to indicate that the boundary is not included.

a $x^2 + y^2 \le 4$ **b** $x^2 + y^2 > 1$ **c** $x^2 + y^2 \le 5$

d $x^2 + y^2 > 9$ **e** $x^2 + y^2 \ge 6$ **f** $x^2 + y^2 < 8$

8 Sketch the graphs of each of the following with the help of your calculator. (Instructions are given in the calculator appendices of the Interactive Textbook.) CAS

a $(x - 3)^2 + (y - 1)^2 = 1$ **b** $(x - 5)^2 + (y - 2)^2 = 4$

c $(x - 2)^2 + y^2 = 2$ **d** $x^2 + y^2 - 6y - 16 = 0$

e $x^2 + y^2 + 4x - 6y - 3 = 0$ **f** $x^2 + y^2 - 8x + 22y + 27 = 0$

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4F Determining rules

In Chapters 2 and 3 we looked at some sufficient conditions for determining the rules for straight lines and parabolas. For straight lines these included:

- the coordinates of two points
- the gradient and a point.

For parabolas these included:

- the coordinates of three points
- the coordinates of the vertex and the coordinates of one other point.

In this section we are looking at some sufficient conditions for determining the rules for the graphs of this chapter.

Example 14

a The rectangular hyperbola $y = \frac{a}{x} + 8$ passes through the point $(-2, 6)$. Find the value of a.

b The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(2, 7)$ and $(-1, 1)$. Find the values of a and k.

Solution

a When $x = -2$, $y = 6$. Hence

$$6 = \frac{a}{-2} + 8$$

$$-2 = \frac{a}{-2}$$

$$a = 4$$

The equation is $y = \frac{4}{x} + 8$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x - h} + k$$

In this case $h = 0$ and $k = 8$.

Solution

b When $x = 2$, $y = 7$. When $x = -1$, $y = 1$.
So we have the equations

$$7 = \frac{a}{2} + k \qquad (1)$$

$$1 = -a + k \qquad (2)$$

Subtract (2) from (1):

$$6 = \frac{a}{2} + a \qquad (3)$$

Multiply both sides of equation (3) by 2:

$$12 = a + 2a$$

$$a = 4$$

From equation (2): $k = 5$.

The equation is $y = \frac{4}{x} + 5$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = \frac{a}{x - h} + k$$

In this case $h = 0$ and the values of a and k are unknown.

Simultaneous equations need to be formed and then solved.

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Example 15

A graph which has rule $y = a\sqrt{x - h}$ passes through the points (4, 2) and (7, 4). Find the values of a and h.

Solution

When $x = 4$, $y = 2$. When $x = 7$, $y = 4$.

We have the equations

$$2 = a\sqrt{4 - h} \quad (1)$$

$$4 = a\sqrt{7 - h} \quad (2)$$

Divide (2) by (1):

$$2 = \frac{\sqrt{7 - h}}{\sqrt{4 - h}} \quad (3)$$

Multiply both sides of equation (3) by $\sqrt{4 - h}$:

$$2\sqrt{4 - h} = \sqrt{7 - h}$$

Square both sides of the equation:

$$\begin{aligned} 4(4 - h) &= 7 - h \\ 16 - 4h &= 7 - h \\ 3h &= 9 \\ h &= 3 \end{aligned}$$

Substitute in (1) to find $a = 2$.

The required equation is $y = 2\sqrt{x - 3}$.

Explanation

The general technique is to substitute the given values into the general equation

$$y = a\sqrt{x - h} + k$$

In this case $k = 0$ and the values of a and h are unknown.

Simultaneous equations need to be formed and then solved. Note that $h \neq 4$ from equation (1).

Example 16

Find the equation of the circle whose centre is at the point (1, −1) and which passes through the point (4, 3).

Solution

Let r be the length of the radius. Then

$$\begin{aligned} r &= \sqrt{(4 - 1)^2 + (3 - (-1))^2} \\ &= \sqrt{3^2 + 4^2} \\ &= 5 \end{aligned}$$

Hence the equation of the circle is

$$(x - 1)^2 + (y + 1)^2 = 25$$

Explanation

We use the centre–radius form for the equation of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

The centre is (1, −1). We need to find the radius.

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Skill-sheet

Exercise 4F

Questions marked with an asterisk (*) involve circles.

Example 14a

1 The rectangular hyperbola $y = \frac{a}{x} + 3$ passes through the point $(1, 8)$. Find the value of a.

2 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x - h} + k$$

has vertical asymptote $x = 3$, horizontal asymptote $y = 4$ and passes through the point $(0, 6)$. Find the values of a, h and k.

Example 14b

3 The rectangular hyperbola $y = \frac{a}{x} + k$ passes through the points $(1, 8)$ and $(-1, 7)$. Find the values of a and k.

4 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x - h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = -4$ and passes through the point $(0, 4)$. Find the values of a, h and k.

5 A graph which has rule $y = a\sqrt{x}$ passes through the point $(2, 8)$. Find the value of a.

Example 15

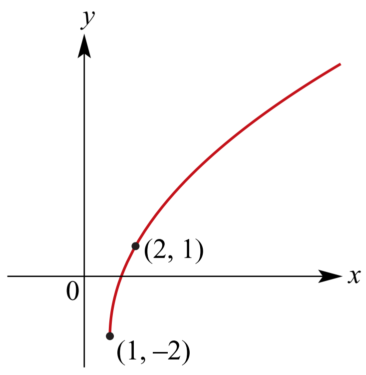
6 A graph which has rule $y = a\sqrt{x - h}$ passes through the points $(1, 2)$ and $(10, 4)$. Find the values of a and h.

7 A graph which has rule $y^2 = a(x - h)$ passes through the points $(9, 4)$ and $(6, -2)$. Find the values of a and h.

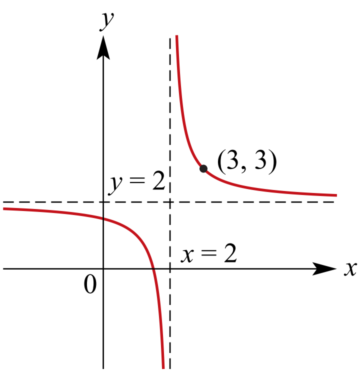
8 A graph which has rule $(y - k)^2 = -4(x - h)$ passes through the points $(4, 4)$ and $(-4, 8)$. Find the values of h and k.

9 Find the rule for each of the following graphs. The general form of the rule is given for each graph.

a $y = a\sqrt{x - h} + k$



b $y = \frac{a}{x - h} + k$



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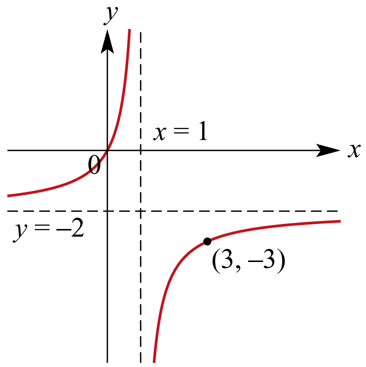
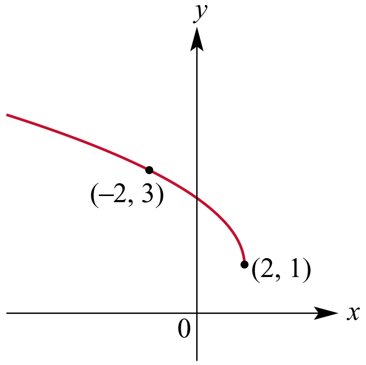
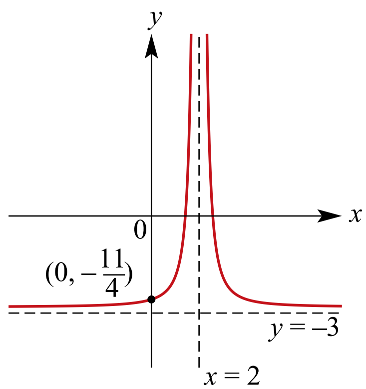
c $y = \dfrac{a}{x - h} + k$



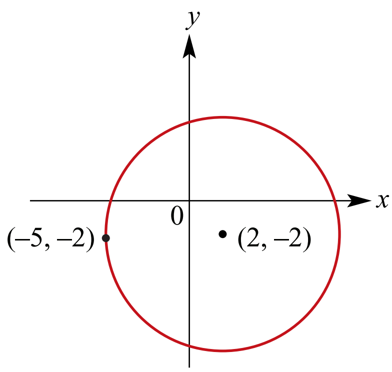
d $y = a\sqrt{h - x} + k$



e $y = \dfrac{a}{(x - h)^2} + k$



***f** $(x - h)^2 + (y - k)^2 = r^2$



Centre at (2, –2)

10 **a** A graph has equation of the form $y = \dfrac{a}{x - 2} + b$. Find the values of a and b if the graph passes through the points:

i (3, 10) and (10, 3) **ii** (3, −3) and (10, −10)

b A graph has equation of the form $y = 2\sqrt{x - h} + k$. Find the values of h and k if the graph passes through the points:

i (2, 3) and (10, 6) **ii** (2, −3) and (10, 0)

CAS

Example 16 ***11** For each of the following, find the equation of the circle:

a centre at the point (2, 1) and passes through the point (4, −3)

b centre at the point (−2, 3) and passes through the point (−3, 3)

c centre at the point (−2, 3) and passes through the point (2, 3)

d centre at the point (2, −3) and touches the x-axis

e centre on the line $y = 4$ and passes through the points (2, 0) and (6, 0)

***12** Find the equations of the circles which touch the x-axis, have radius 5 and pass through the point (0, 8).

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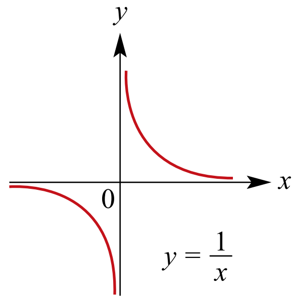
Review

Chapter summary

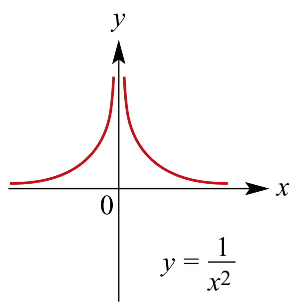
Assign-ment

Nrich

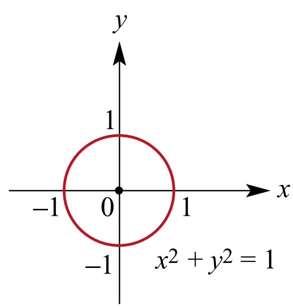
- The standard graphs:



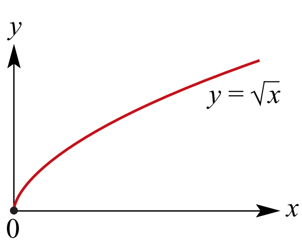
Rectangular hyperbola



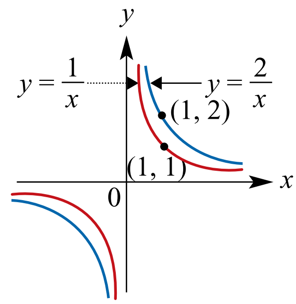
Truncus

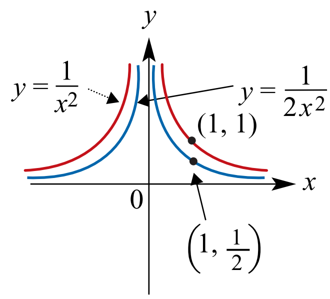


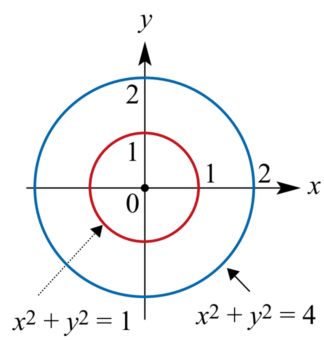
Circle

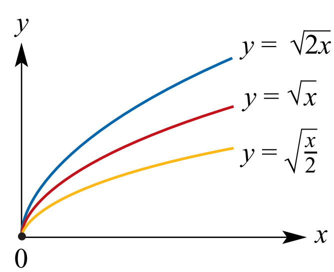


- Dilations of these graphs:









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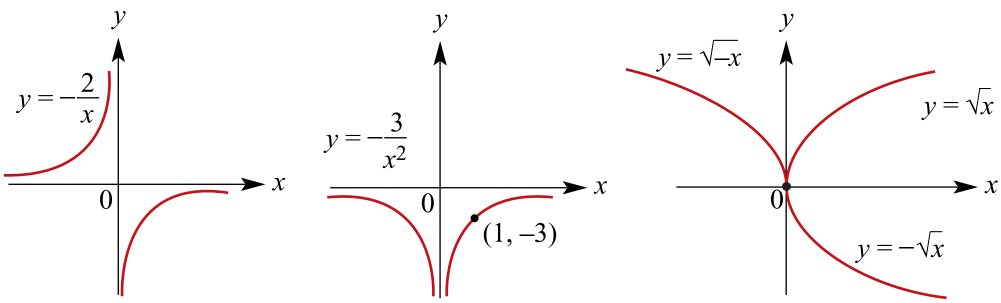
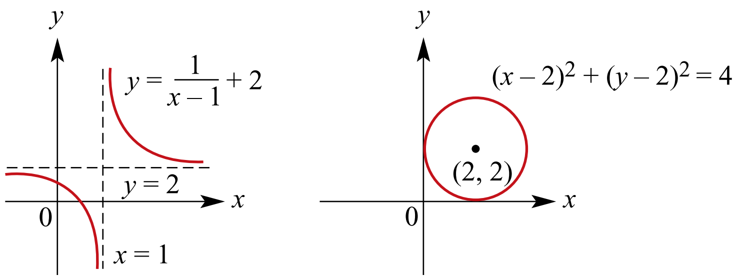
- Reflections in the axes:



- Translations of graphs:



- Equation for a circle with centre at (h, k) and radius r:

$$(x - h)^2 + (y - k)^2 = r^2$$

Alternative form:

$$x^2 + y^2 - 2hx - 2ky + c = 0$$

where $c = h^2 + k^2 - r^2$.

Technology-free questions

Questions marked with an asterisk (*) involve circles.

1 Sketch the graphs of each of the following:

a $y = \frac{-3}{x}$ **b** $y = \frac{2}{x^2}$ **c** $y = \frac{1}{x-1}$ **d** $y = \frac{2}{x+1} + 1$

e $y = \frac{-2}{x^2}$ **f** $y = \frac{-1}{x-1}$ **g** $y = \frac{4}{2-x} + 3$ **h** $y = \frac{-3}{x^2} + 1$

i $y = 2\sqrt{x} + 2$ **j** $y = 2\sqrt{x-3} + 2$ **k** $y = -2\sqrt{x+2} + 2$

l $y^2 = 4(x - 2)$ **m** $(y - 1)^2 = 16(x - 2)$ **n** $(y - 4)^2 = -4(x - 1)$

2 A rectangular hyperbola with rule of the form

$$y = \frac{a}{x-h} + k$$

has vertical asymptote $x = 2$, horizontal asymptote $y = 5$ and passes through the point $(0, 8)$. Find the values of a, h and k.

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3 A curve with a rule of the form

$$y = \frac{a}{(x-h)^2} + k$$

has vertical asymptote $x = -2$, horizontal asymptote $y = -5$ and passes through the point $(1, 4)$. Find the values of a, h and k.

4 A graph with a rule of the form $y = a\sqrt{x-h}$ passes through the points $(2, 6)$ and $(10, 10)$. Find the values of a and h.

5 A graph with a rule of the form $y^2 = a(x-h)$ passes through the points $(1, 3)$ and $(2, 5)$. Find the values of a and h.

6 For each of the following, write the equation in the form $(y-k)^2 = a(x-h)$ by completing the square, and state the coordinates of the vertex, the x-axis intercept and the y-axis intercepts of the graph of the equation:

a $y^2 + 4y = x + 2$ **b** $y^2 + 6y + 2x + 4 = 0$ **c** $2y^2 + 8y - 5x + 6 = 0$

7 Show that the line $x + 4y = 4$ touches the hyperbola $y = \frac{1}{x}$ at only one point.

8 Consider the hyperbola $y = \frac{12}{x}$. The points $P(2, 6)$ and $Q(-4, -3)$ lie on the hyperbola. The line PQ cuts the y-axis at B and the x-axis at A.

a Find the coordinates of points A and B.

b Prove that $PB = AQ$.

***9** By completing the square, write each of the following equations in the form $(x-a)^2 + (y-b)^2 = r^2$:

a $x^2 + y^2 - 6x + 4y - 12 = 0$ **b** $x^2 + y^2 - 3x + 5y - 4 = 0$

c $2x^2 + 2y^2 - x + y - 4 = 0$ **d** $x^2 + y^2 + 4x - 6y = 0$

e $x^2 + y^2 = 6(x + y)$ **f** $x^2 + y^2 = 4x - 6y$

***10** The equation of a circle is $x^2 + y^2 + 4x - 6y = 23$. Find the centre and radius.

***11** Find the length cut off on the x-axis and y-axis by the circle $x^2 + y^2 - 2x - 4y = 20$.

***12** Sketch the graphs of the following semicircles:

a $y = \sqrt{9 - x^2}$ **b** $y = -\sqrt{16 - (x+1)^2}$

c $y - 2 = -\sqrt{1 - x^2}$ **d** $y + 3 = \sqrt{4 - (x+2)^2}$

Multiple-choice questions

Questions marked with an asterisk (*) involve circles.

1 For the rule $y = \frac{5}{x^2} + 3$, when $x = \frac{a}{2}$, $y =$

A $\frac{5}{4a^2} + 3$ **B** $\frac{12a^2 + 5}{4a^2}$ **C** $\frac{20a^2 + 3}{4a^2}$ **D** $\frac{12a^2 + 5}{a^2}$ **E** $\frac{20}{a^2} + 3$

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2 The equations of the asymptotes of the graph of $y = 5 - \frac{1}{3x - 5}$ are

A $x = 5,\ y = \frac{3}{5}$ **B** $y = 5,\ x = \frac{5}{3}$ **C** $x = 5,\ y = \frac{5}{3}$

D $y = 5,\ x = \frac{3}{5}$ **E** $x = 5,\ y = -\frac{5}{3}$

3 The equations of the asymptotes of the graph of $y = 5 + \frac{1}{(x - 2)^2}$ are

A $x = 2,\ y = 5$ **B** $x = -2,\ y = 5$ **C** $x = 5,\ y = 4$

D $x = 5,\ y = 2$ **E** $x = 4,\ y = 5$

4 For the rule $y = -2\sqrt{x} + 3$, where $x \geq 0$, the range of possible y-values is

A $y \geq 3$ **B** $y > -3$ **C** $y \geq -3$ **D** $y \leq 3$ **E** $y > 3$

5 The rule $(y - 2)^2 = -3x + 1$ is defined for

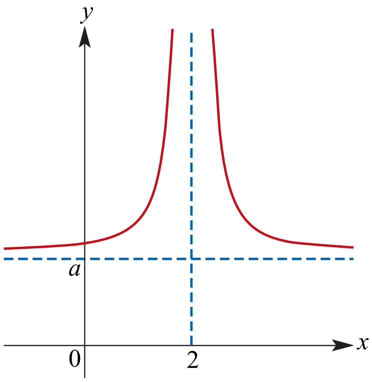
A $x \leq \frac{1}{3}$ **B** $x \geq 3$ **C** $x \leq 2$ **D** $x \geq \frac{1}{3}$ **E** $x \geq -\frac{1}{3}$

6 A possible rule for the graph shown opposite is

A $y - a = \frac{1}{x - 2}$ **B** $y + a = \frac{1}{(x + 2)^2}$

C $y - a = \frac{1}{(x + 2)^2}$ **D** $y - 2 = \frac{1}{(x - a)^2}$

E $y - a = \frac{1}{(x - 2)^2}$



7 The graph of $(y + a)^2 = -3(x - 2)$, where $a > 0$, has

A vertex at $(2, -a)$ and y-axis intercepts at $(0, -a - \sqrt{6})$ and $(0, -a + \sqrt{6})$

B vertex at $(-2, a)$ and y-axis intercepts at $(0, -a - \sqrt{3})$ and $(0, -a + \sqrt{3})$

C vertex at $(2, -a)$ and y-axis intercepts at $(0, a - \sqrt{6})$ and $(0, a + \sqrt{6})$

D vertex at $(-2, a)$ and y-axis intercepts at $(0, -a)$ and $(0, a)$

E vertex at $(a, 2)$ and y-axis intercepts at $(0, -a - \sqrt{6})$ and $(0, -a + \sqrt{6})$

8 The graph of $y - b = -\sqrt{a - x}$, where $a > 0$ and $b > 0$, has

A endpoint (a, b) and y-values $y \geq b$ **B** endpoint (b, a) and y-values $y \leq a$

C endpoint (a, b) and y-values $y \leq b$ **D** endpoint (a, b) and y-values $y \geq a + b$

E endpoint $(a, -b)$ and y-values $y \leq -b$

***9** The circle with equation $(x - a)^2 + (y - b)^2 = 36$ has its centre on the x-axis and passes through the point with coordinates $(6, 6)$. The values of a and b are

A $a = 0$ and $b = 6$ **B** $a = 0$ and $b = 0$ **C** $a = 2$ and $b = 0$

D $a = -6$ and $b = 0$ **E** $a = 6$ and $b = 0$

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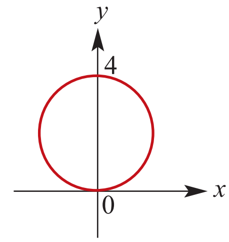
***10** If the y-axis is an axis of symmetry and the circle passes through the origin and $(0, 4)$, the equation of the circle shown is



A $x^2 + (y - 2)^2 = 4$ **B** $(x - 2)^2 + y^2 = 2$
C $(x + 2)^2 + y^2 = 4$ **D** $x^2 + (y + 2)^2 = 4$
E $x^2 + y^2 - 2 = 4$

***11** For the circle with equation $(x - 5)^2 + (y + 2)^2 = 9$, the coordinates of the centre and the radius are

A $(-5, 2)$ and 3 **B** $(-5, 2)$ and 9 **C** $(5, -2)$ and 9
D $(5, -2)$ and 3 **E** $(-2, 5)$ and 3

Extended-response questions

The following questions also involve techniques developed in Chapters 2 and 3. Questions marked with an asterisk (*) involve circles.

1 Consider the parabola with equation $(y - 2)^2 = 3x - 2$.

a Find the coordinates of the points of intersection of this parabola with the line $y = x$.
b Find, in terms of a, the coordinates of the points of intersection of this parabola with the line $y = x + a$.
c **i** Find the value of a for which the line $y = x + a$ just touches the parabola.
ii Find the values of a for which the line $y = x + a$ does not intersect the parabola.

2 **a** Show that the line $y = x + 1$ touches the parabola $y^2 = 4x$ at the point $(1, 2)$.
b Find the coordinates of the points of intersection, P and Q, of the line $y = x + 1$ and the parabola $y^2 = -4x$.
c Determine the distance between points P and Q.
d Determine the midpoint of the line segment PQ.

3 Consider the curve with equation $y = \sqrt{x - b} + c$. CAS

a Show that if the curve meets the line with equation $y = x$ at the point (a, a), then a satisfies the equation $a^2 - (2c + 1)a + c^2 + b = 0$.
b **i** If the line with equation $y = x$ is a tangent to the curve, show that $c = \dfrac{4b - 1}{4}$.
ii Sketch the graph of $y = \sqrt{x} - \frac{1}{4}$ and find the coordinates of the point on the graph at which the line with equation $y = x$ is a tangent.
c Find the values of k for which the line with equation $y = x + k$:
i meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ twice
ii meets the curve with equation $y = \sqrt{x} - \frac{1}{4}$ once
iii does not meet the curve with equation $y = \sqrt{x} - \frac{1}{4}$.

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4 For the curve with equation $y = \sqrt{x} - 1$ and the straight line with equation $y = kx$, find the values of k such that:

a the line meets the curve twice **b** the line meets the curve once.

5 The line $px + qy = 1$ touches the parabola $y^2 = 4a(a - x)$, where p, q and a are non-zero real numbers.

a Show that $a = \dfrac{p}{p^2 + q^2}$.

b For each of the following values of p and q, determine the value of a and the coordinates of the point where the line touches the parabola:

i $p = 1$ and $q = 1$ **ii** $p = 3$ and $q = 4$

Now assume that $a = 1$ and consider the parabola $y^2 = -4(x - 1)$.

c By using simultaneous equations in p and q, find the equation of the line $px + qy = 1$ that touches this parabola at:

i the point $(0, 2)$ **ii** the point $(-3, -4)$

6 The line $px + qy = 1$ touches the hyperbola $y = \dfrac{a}{x}$, where p, q and a are non-zero real numbers.

a Show that $a = \dfrac{1}{4pq}$.

b For each of the following values of p and q, determine the value of a and the coordinates of the point where the line touches the hyperbola:

i $p = 1$ and $q = 1$ **ii** $p = 2$ and $q = 2$

Now assume that $a = 1$ and consider the hyperbola $y = \dfrac{1}{x}$.

c By using simultaneous equations in p and q, find the equation of the line $px + qy = 1$ that touches this hyperbola at:

i the point $(2, \frac{1}{2})$ **ii** the point $(-2, -\frac{1}{2})$

***7** A line with equation $y = mx$ is tangent to the circle with centre $(10, 0)$ and radius 5 at the point $P(x, y)$.

a Find the equation of the circle.

b Show that the x-coordinate of the point P satisfies the equation $(1 + m^2)x^2 - 20x + 75 = 0$.

c Use the discriminant for this equation to find the two possible values of m.

d Find the coordinates of P. (There are two such points.)

e Find the distance of P from the origin.

***8** A circle has its centre at the origin and radius 4.

a Find the equation of this circle.

b Two lines which pass through the point $(8, 0)$ are tangents to this circle.

i Show that the equations of these tangents are of the form $y = mx - 8m$.

ii Use techniques similar to those used in Question 7 to find the possible values of m and, hence, the equations of the tangents.

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