3 Quadratics

Objectives

- To recognise and sketch the graphs of **quadratic polynomials**.
- To find the key features of the graph of a quadratic polynomial: axis intercepts, turning point and axis of symmetry.
- To determine the **maximum** or **minimum** value of a quadratic function.
- To **solve quadratic equations** by factorising, completing the square and using the general formula.
- To apply the **discriminant** to determine the nature and number of solutions of a quadratic equation.
- To apply quadratic functions to solving problems.

A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and $a_0, a_1, \ldots, a_n$ are numbers called **coefficients**.

The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient. For example:

- $y = 2x + 3$ is a polynomial function of degree 1
- $y = 2x^2 + 3x - 2$ is a polynomial function of degree 2.

This chapter deals with polynomials of degree 2. These are called **quadratic polynomials**.

The graph of a linear polynomial function, $y = mx + c$, is a straight line and the graph of a quadratic polynomial function, $y = ax^2 + bx + c$, $a \neq 0$, is a **parabola**. Polynomials of higher degree will be studied in Chapter 6.

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3A Expanding and collecting like terms

In order to sketch graphs of quadratics, we need to find the x-axis intercepts (if they exist), and to do this we need to solve quadratic equations. As an introduction to the methods of solving quadratic equations, the first two sections of this chapter review the basic algebraic processes of expansion and factorisation.

An algebraic expression is the sum of its **terms**. For example:

- The terms of the linear expression $3x - 1$ are $3x$ and -1.
- The terms of the quadratic expression $-2x^2 + 3x - 4$ are $-2x^2$, $3x$ and -4.

Example 1

Simplify $2(x - 5) - 3(x + 5)$ by first expanding.

Solution

$$\begin{aligned} 2(x-5) - 3(x+5) &= 2x - 10 - 3x - 15 \\ &= 2x - 3x - 10 - 15 \\ &= -x - 25 \end{aligned}$$

Explanation

Expand each bracket.

Collect like terms.

Example 2

Expand $2x(3x - 2) + 3x(x - 2)$.

Solution

$$\begin{aligned} 2x(3x-2) + 3x(x-2) &= 6x^2 - 4x + 3x^2 - 6x \\ &= 9x^2 - 10x \end{aligned}$$

For expansions of the type $(a + b)(c + d)$, proceed as follows:

$$\begin{aligned} (a+b)(c+d) &= a(c+d) + b(c+d) \\ &= ac + ad + bc + bd \end{aligned}$$

Example 3

Expand the following:

a $(x + 3)(2x - 3)$ **b** $(x - 3)(2x - 2\sqrt{2})$

Solution

a $(x + 3)(2x - 3)$

$= x(2x - 3) + 3(2x - 3)$

$= 2x^2 - 3x + 6x - 9$

$= 2x^2 + 3x - 9$

Explanation

Each term in the second pair of brackets is multiplied by each term in the first.

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b $(x - 3)(2x - 2\sqrt{2})$

$= x(2x - 2\sqrt{2}) - 3(2x - 2\sqrt{2})$

$= 2x^2 - 2\sqrt{2}x - 6x + 6\sqrt{2}$

$= 2x^2 - (2\sqrt{2} + 6)x + 6\sqrt{2}$

Be careful with negative signs.

You can also complete binomial expansions with a table; this emphasises the terms.

	x	-3
$2x$	$2x^2$	$-6x$
$-2\sqrt{2}$	$-2\sqrt{2}x$	$6\sqrt{2}$

You add the terms to complete the expansion.

Example 4

Expand $(2x - 1)(3x^2 + 2x + 4)$.

Solution

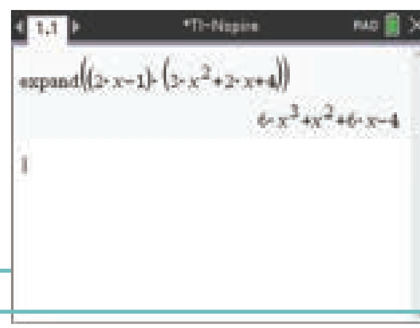
$$(2x - 1)(3x^2 + 2x + 4) = 2x(3x^2 + 2x + 4) - 1(3x^2 + 2x + 4)$$
$$= 6x^3 + 4x^2 + 8x - 3x^2 - 2x - 4$$
$$= 6x^3 + x^2 + 6x - 4$$

Using the TI-Nspire

To expand the expression

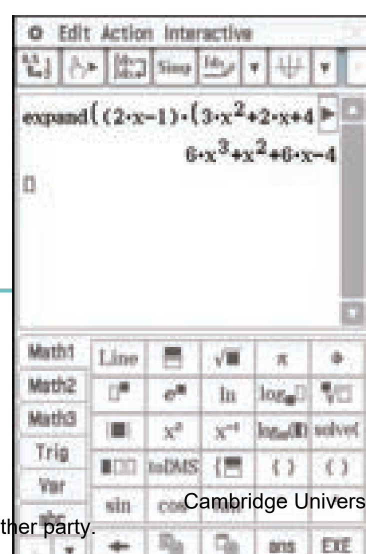
$$(2x - 1)(3x^2 + 2x + 4)$$

use (menu) > **Algebra** > **Expand**.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $(2x - 1)(3x^2 + 2x + 4)$.
- Highlight the expression and select **Interactive** > **Transformation** > **expand**.
- Tap OK.



Perfect squares

Consider the expansion of a perfect square, $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b)\\ &= a(a+b)+b(a+b)\\ &= a^2+ab+ab+b^2\\ &= a^2+2ab+b^2\end{aligned}$$

Thus the general result can be stated as:

$$(a+b)^2 = a^2+2ab+b^2$$

That is, to expand $(a+b)^2$ take the sum of the squares of the terms and add twice the product of the terms.

Example 5

Expand $(3x-2)^2$.

Solution

$$\begin{aligned}(3x-2)^2 &= (3x)^2+2(3x)(-2)+(-2)^2\\ &= 9x^2-12x+4\end{aligned}$$

Explanation

Use the expansion $(a+b)^2 = a^2+2ab+b^2$.
Here $a = 3x$ and $b = -2$.

Difference of two squares

Consider the expansion of $(a+b)(a-b)$:

$$\begin{aligned}(a+b)(a-b) &= a(a-b)+b(a-b)\\ &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned}$$

Thus the expansion of the difference of two squares has been obtained:

$$(a+b)(a-b) = a^2-b^2$$

Example 6

Expand:

a $(2x-4)(2x+4)$

b $(x-2\sqrt{7})(x+2\sqrt{7})$

Solution

a $$\begin{aligned}(2x-4)(2x+4) &= (2x)^2-(4)^2\\ &= 4x^2-16\end{aligned}$$

b $$\begin{aligned}(x-2\sqrt{7})(x+2\sqrt{7}) &= x^2-(2\sqrt{7})^2\\ &= x^2-28\end{aligned}$$

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Example 7

Expand $(2a - b + c)(2a - b - c)$.

Solution

$$\begin{aligned}(2a - b + c)(2a - b - c) &= ((2a - b) + c)((2a - b) - c)\\ &= (2a - b)^2 - c^2\\ &= 4a^2 - 4ab + b^2 - c^2\end{aligned}$$

Summary 3A

- A **polynomial function** has a rule of the type

$$y = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and $a_0, a_1, \ldots, a_n$ are numbers called **coefficients**.
- The **degree** of a polynomial is given by the value of n, the highest power of x with a non-zero coefficient.
- A polynomial function of degree 2 is called a **quadratic function**. The general rule is of the form $y = ax^2 + bx + c$, where $a \neq 0$. The graph of a quadratic function is called a **parabola**.
- General binomial expansion:

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= ac + ad + bc + bd\end{aligned}$$

- Perfect square expansion:

$$(a + b)^2 = a^2 + 2ab + b^2$$

- Difference of two squares expansion:

$$(a + b)(a - b) = a^2 - b^2$$

Exercise 3A

1 Expand each of the following:

a $2(x - 4)$ **b** $-2(x - 4)$ **c** $3(2x - 4)$
d $-3(4 - 2x)$ **e** $x(x - 1)$ **f** $2x(x - 5)$

2 Collect like terms in each of the following:

a $2x + 4x + 1$ **b** $2x - 6 + x$ **c** $3x + 1 - 2x$ **d** $-x + 2x - 3 + 4x$

Example 1 **3** Simplify each of the following by expanding and collecting like terms:

a $8(2x - 3) - 2(x + 4)$ **b** $2x(x - 4) - 3x$
c $4(2 - 3x) + 4(6 - x)$ **d** $4 - 3(5 - 2x)$

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Example 2 **4** Simplify each of the following by expanding and collecting like terms:

a $2x(x-4)-3x$ **b** $2x(x-5)+x(x-5)$ **c** $2x(-10-3x)$

d $3x(2-3x+2x^2)$ **e** $3x-2x(2-x)$ **f** $3(4x-2)-6x$

Example 3 **5** Simplify each of the following by expanding and collecting like terms:

a $(3x-7)(2x+4)$ **b** $(x-10)(x-12)$ **c** $(3x-1)(12x+4)$

d $(4x-5)(2x-3)$ **e** $(x-\sqrt{3})(x-2)$ **f** $(2x-\sqrt{5})(x+\sqrt{5})$

g $(3x-2\sqrt{7})(x+\sqrt{7})$ **h** $(5x-3)(x+2\sqrt{2})$ **i** $(\sqrt{5}x-3)(\sqrt{5}x-32\sqrt{2})$

Example 4 **6** Simplify each of the following by expanding and collecting like terms:

a $(2x-3)(3x^2+2x-4)$ **b** $(x-1)(x^2+x+1)$

c $(6-2x-3x^2)(4-2x)$ **d** $(5x-3)(x+2)-(2x-3)(x+3)$

e $(2x+3)(3x-2)-(4x+2)(4x-2)$

Example 5 **7** Simplify each of the following by expanding and collecting like terms:

a $(x-4)^2$ **b** $(2x-3)^2$ **c** $(6-2x)^2$

d $\left(x-\frac{1}{2}\right)^2$ **e** $(x-\sqrt{5})^2$ **f** $(x-2\sqrt{3})^2$

Example 6 **8** Simplify each of the following by expanding and collecting like terms:

a $(x-3)(x+3)$ **b** $(2x-4)(2x+4)$

c $(9x-11)(9x+11)$ **d** $(2x-3)(2x+3)$

e $(2x+5)(2x-5)$ **f** $(x-\sqrt{5})(x+\sqrt{5})$

g $(2x+3\sqrt{3})(2x-3\sqrt{3})$ **h** $(\sqrt{3}x-\sqrt{7})(\sqrt{3}x+\sqrt{7})$

Example 7 **9** Simplify each of the following by expanding and collecting like terms:

a $(x-y+z)(x-y-z)$ **b** $(2a-b+c)(2a-b-c)$

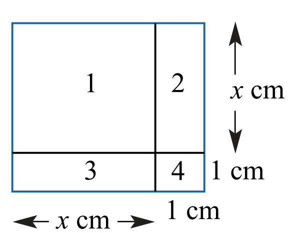
c $(3w-4z+u)(3w+4z-u)$ **d** $(2a-\sqrt{5}b+c)(2a+\sqrt{5}b+c)$

10 Find the area of each of the following by:

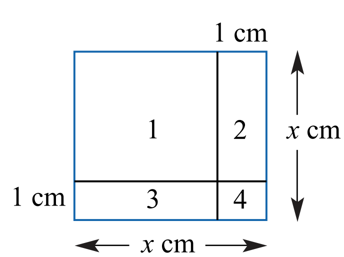
i adding the areas of the four 'non-overlapping' rectangles (two of which are squares)

ii multiplying length by width of the undivided square (boundary in blue).

a



b



11 Using your calculator, expand each of the following: CAS

a $(2x-\sqrt{3})(2x+\sqrt{3})$ **b** $(\sqrt{3}x-5)(\sqrt{7}x-2)$ **c** $(5-x-2x^2)(3-5x)$

d $(a-2b+c)(a-b-c)$ **e** $(a-2b+c)^2$ **f** $(a+b+c)(a^2-b^2)$

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3B Factorising

Four different types of factorisation will be considered.

Factorisation using common factors

If each term in an algebraic expression to be factorised contains a **common factor**, then this common factor is a factor of the entire expression. To find the other factor, divide each term by the common factor. The common factor is placed outside the brackets. This process is known as 'taking the common factor outside the brackets'.

Example 8

a Factorise $9x^2 + 81x$.

b Factorise $2a^2 - 8ax^2$.

Solution

a $9x^2 + 81x = 9x \times x + 9x \times 9$

$= 9x(x + 9)$

b $2a^2 - 8ax^2 = 2a \times a - 2a \times 4x^2$

$= 2a(a - 4x^2)$

Note: The answers can be checked by expanding.

Explanation

The common factor $9x$ is 'taken out' of the brackets.

The common factor $2a$ is 'taken out' of the brackets.

In general, take out as many common factors as possible.

Example 9

Factorise $7x^2y - 35xy^2$.

Solution

$7x^2y - 35xy^2 = 7xy(x - 5y)$

Explanation

The common factor $7xy$ is 'taken out' of the brackets.

Grouping of terms

This method can be used for expressions containing four terms.

Example 10

Factorise $x^3 + 4x^2 - 3x - 12$.

Solution

$x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) - (3x + 12)$

$= x^2(x + 4) - 3(x + 4)$

$= (x^2 - 3)(x + 4)$

Explanation

The terms in this expression can be grouped as shown.

The common factor $(x + 4)$ is 'taken out' of the brackets.

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Difference of two squares

You will recall the following identity from the previous section:

$$(a + b)(a - b) = a^2 - b^2$$

We can now use the result the other way in order to factorise:

$$a^2 - b^2 = (a + b)(a - b)$$

Example 11

a Factorise $3x^2 - 75$.

b Factorise $9x^2 - 36$.

Solution	Explanation
a $3x^2 - 75 = 3(x^2 - 25)$	First 'take out' the common factor 3.
$= 3(x + 5)(x - 5)$	Use the difference of squares identity.
b $9x^2 - 36 = 9(x^2 - 4)$	First 'take out' the common factor 9.
$= 9(x - 2)(x + 2)$	Use the difference of squares identity.

Example 12

Factorise $(x - y)^2 - 16y^2$.

Solution

$$\begin{aligned}(x - y)^2 - 16y^2 &= (x - y)^2 - (4y)^2 \\ &= (x - y + 4y)(x - y - 4y) \\ &= (x + 3y)(x - 5y)\end{aligned}$$

Explanation

Use the difference of squares identity

$$a^2 - b^2 = (a + b)(a - b)$$

with $a = (x - y)$ and $b = 4y$.

Factorising quadratic polynomials

A quadratic polynomial is an expression of the form $ax^2 + bx + c$ with $a \neq 0$. We have seen in the previous section that we can expand a product of two binomial factors to obtain a quadratic expression. For example:

$$\begin{aligned}(x + 2)(x - 4) &= x(x - 4) + 2(x - 4) \\ &= x^2 - 4x + 2x - 8 \\ &= x^2 - 2x - 8\end{aligned}$$

We want to be able to reverse this process. That is, we want to start from the expanded expression and obtain the factorised form. We have already done this for expressions that are differences of squares. We now turn our attention to the general case.

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Example 13

Factorise $x^2 - 2x - 8$.

Solution

Using the method described in the explanation opposite, we can factorise without any further setting out:

$$x^2 - 2x - 8 = (x - 4)(x + 2)$$

Alternatively, we can reverse the process we used for expanding:

$$\begin{aligned} x^2 - 2x - 8 &= x^2 - 4x + 2x - 8 \\ &= x(x - 4) + 2(x - 4) \\ &= (x - 4)(x + 2) \end{aligned}$$

Explanation

We want

$$\begin{aligned} x^2 - 2x - 8 &= (x + a)(x + b) \\ &= x^2 + (a + b)x + ab \end{aligned}$$

The values of a and b are such that $ab = -8$ and $a + b = -2$.

Values of a and b which satisfy these two conditions are $a = -4$ and $b = 2$.

A quadratic polynomial is called a **monic quadratic polynomial** if the coefficient of x^2 is 1. The quadratic polynomial $x^2 - 2x - 8$ factorised in the previous example is monic.

Factorising non-monic quadratic polynomials involves a slightly different approach. We need to consider all possible combinations of factors of the x^2 term and the constant term. The next example and the following discussion give two methods.

Example 14

Factorise $6x^2 - 13x - 15$.

Solution

There are several combinations of factors of $6x^2$ and -15 to consider. Only one combination is correct.

$$6x^2 - 13x - 15 = (6x + 5)(x - 3)$$

Factors of $6x^2$	Factors of -15	'Cross-products' add to give $-13x$
$6x$	$+5$	$+5x$
x	-3	$-18x$
		$-13x$

Here is a second method for factorising $6x^2 - 13x - 15$ which still requires some trial and error but is more systematic. It is the reverse process of expanding $(x - 3)(6x + 5)$.

We let

$$ax^2 + bx + c = (\alpha x + \gamma)(\beta x + \delta)$$

Expanding the right-hand side gives

$$ax^2 + bx + c = \alpha\beta x^2 + (\gamma\beta + \alpha\delta)x + \gamma\delta$$

Note that $ac = \alpha\beta\gamma\delta$ and $b = \gamma\beta + \alpha\delta$.

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We now apply this to factorising $6x^2 - 13x - 15$.

First we look for two numbers that multiply together to give ac and add to give b. That is, we look for two numbers whose product is $6 \times (-15) = -90$ and whose sum is -13.

The two numbers are -18 and 5. We write:

$$\begin{aligned} 6x^2 - 13x - 15 &= 6x^2 - 18x + 5x - 15 \\ &= 6x(x - 3) + 5(x - 3) \\ &= (x - 3)(6x + 5) \end{aligned}$$

Example 15

Factorise $8x^2 + 2x - 15$.

Solution

$$\begin{aligned} 8x^2 + 2x - 15 &= 8x^2 + 12x - 10x - 15 \\ &= 4x(2x + 3) - 5(2x + 3) \\ &= (4x - 5)(2x + 3) \end{aligned}$$

Explanation

$ac = 8 \times (-15) = -120$ and $b = 2$.

The two numbers are 12 and -10.
So we write $2x = 12x - 10x$.

It is sometimes possible to take out a common factor first to simplify the factorisation.

Example 16

Factorise $2x^2 + 6x - 20$.

Solution

$$\begin{aligned} 2x^2 + 6x - 20 &= 2(x^2 + 3x - 10) \\ &= 2(x + 5)(x - 2) \end{aligned}$$

Explanation

The common factor 2 is 'taken out' first.

Example 17

Factorise $(x + 1)^2 - 2(x + 1) - 3$.

Solution

$$\begin{aligned} (x + 1)^2 - 2(x + 1) - 3 &= a^2 - 2a - 3 \\ &= (a - 3)(a + 1) \\ &= (x + 1 - 3)(x + 1 + 1) \\ &= (x - 2)(x + 2) \end{aligned}$$

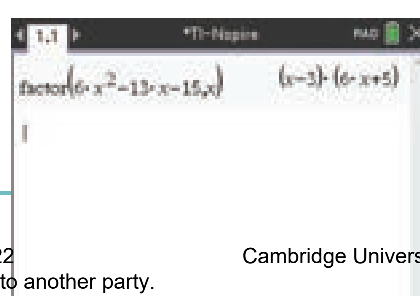
Explanation

The substitution $a = x + 1$ makes it easier to recognise the required factorisation.

Using the TI-Nspire

To factorise the expression $6x^2 - 13x - 15$, use menu > **Algebra** > **Factor**.

Note: Specify the variable as shown to factorise over the real numbers.



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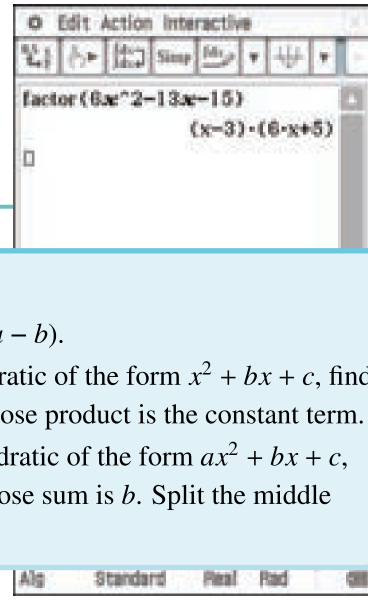
Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter the expression $6x^2 - 13x - 15$.
- Highlight the expression and select **Interactive** > **Transformation** > **factor**.



Summary 3B

- Difference of two squares identity: $a^2 - b^2 = (a + b)(a - b)$.
- Factorisation of monic quadratics: To factorise a quadratic of the form $x^2 + bx + c$, find two numbers whose sum is the coefficient of x and whose product is the constant term.
- Factorisation of general quadratics: To factorise a quadratic of the form $ax^2 + bx + c$, find two numbers e and f whose product is ac and whose sum is b. Split the middle term bx as $ex + fx$ and then factorise by grouping.

Exercise 3B

1 Factorise each of the following:

a $2x + 4$ **b** $4a - 8$ **c** $6 - 3x$
d $2x - 10$ **e** $18x + 12$ **f** $24 - 16x$

Example 8

2 Factorise:

a $4x^2 - 2xy$ **b** $8ax + 32xy$ **c** $6ab - 12b$
d $6xy + 14x^2y$ **e** $x^2 + 2x$ **f** $5x^2 - 15x$
g $-4x^2 - 16x$ **h** $7x + 49x^2$ **i** $2x - x^2$

Example 9

3 Factorise:

a $6x^3y^2 + 12y^2x^2$ **b** $7x^2y - 6y^2x$ **c** $8x^2y^2 + 6y^2x$

Example 10

4 Factorise:

a $x^3 + 5x^2 + x + 5$ **b** $xy + 2x + 3y + 6$ **c** $x^2y^2 - x^2 - y^2 + 1$
d $ax + ay + bx + by$ **e** $a^3 - 3a^2 + a - 3$ **f** $2ab - 12a - 5b + 30$
g $2x^2 - 2x + 5x - 5$ **h** $x^3 - 4x + 2x^2 - 8$ **i** $x^3 - bx^2 - a^2x + a^2b$

Example 11

5 Factorise:

a $x^2 - 36$ **b** $x^2 - 81$ **c** $x^2 - a^2$
d $4x^2 - 81$ **e** $9x^2 - 16$ **f** $25x^2 - y^2$
g $3x^2 - 48$ **h** $2x^2 - 98$ **i** $3ax^2 - 27a$
j $a^2 - 7$ **k** $2a^2 - 5$ **l** $x^2 - 12$

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Example 12

6 Factorise:

a $(x-2)^2-16$ **b** $25-(2+x)^2$ **c** $3(x+1)^2-12$

d $(x-2)^2-(x+3)^2$ **e** $(2x-3)^2-(2x+3)^2$ **f** $(2x-1)^2-(3x+6)^2$

Example 13

7 Factorise:

a $x^2-7x-18$ **b** $y^2-19y+48$ **c** $a^2-14a+24$

d $a^2+18a+81$ **e** $x^2-5x-24$ **f** $x^2-2x-120$

Example 14
Example 15

8 Factorise:

a $3x^2-7x+2$ **b** $6x^2+7x+2$ **c** $5x^2+23x+12$

d $2x^2+9x+4$ **e** $6x^2-19x+10$ **f** $6x^2-7x-3$

g $12x^2-17x+6$ **h** $5x^2-4x-12$ **i** $5x^3-16x^2+12x$

Example 16

9 Factorise:

a $3y^2-12y-36$ **b** $2x^2-18x+28$ **c** $4x^2-36x+72$

d $3x^2+15x+18$ **e** $ax^2+7ax+12a$ **f** $48x-24x^2+3x^3$

Example 17

10 Factorise:

a $(x-1)^2+4(x-1)+3$ **b** $2(x-1)^2+5(x-1)-3$ **c** $(2x+1)^2+7(2x+1)+12$

11 Factorise using your calculator: CAS

a $-4x^2+8x-3$ **b** $9x^2-18x+8$ **c** $6x^2+7x-20$

d $2x^2+11x-21$ **e** $2x^2+17x+21$ **f** $3a^2+4a-4$

3C Quadratic equations

This section looks at the solution of quadratic equations by simple factorisation. There are three steps to solving a quadratic equation by factorisation:

Step 1 Write the equation in the form $ax^2+bx+c=0$.

Step 2 Factorise the quadratic expression.

Step 3 Use the result that $mn=0$ implies $m=0$ or $n=0$ (or both); this is known as the **null factor theorem**.

For example, to solve the equation $x^2-x=12$:

$$x^2-x=12$$
$$x^2-x-12=0 \qquad \text{(Step 1)}$$
$$(x-4)(x+3)=0 \qquad \text{(Step 2)}$$
$$\therefore\ x-4=0 \quad \text{or} \quad x+3=0 \qquad \text{(Step 3)}$$
$$x=4 \quad \text{or} \quad x=-3$$

In the simplest cases, the first two steps may have been done already.

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Example 18

Solve $x^2 + 11x + 24 = 0$.

Solution

x^2	$+24$	$+11x$
x	$+3$	$+3x$
x	$+8$	$+8x$
		$+11x$

Factorising gives

$$x^2 + 11x + 24 = 0$$
$$(x + 3)(x + 8) = 0$$
$$\therefore \quad x + 3 = 0 \quad \text{or} \quad x + 8 = 0$$
$$x = -3 \quad \text{or} \quad x = -8$$

Explanation

The quadratic can also be factorised in the following way:

$$\begin{aligned} x^2 + 11x + 24 &= x^2 + 8x + 3x + 24 \\ &= x(x + 8) + 3(x + 8) \\ &= (x + 8)(x + 3) \end{aligned}$$

Note: We can check the answer for this example by substituting into the equation:

$$(-3)^2 + 11(-3) + 24 = 0$$
$$(-8)^2 + 11(-8) + 24 = 0$$

Example 19

Solve $2x^2 + 5x - 12 = 0$.

Solution

$2x^2$	-12	$+5x$
$2x$	-3	$-3x$
x	$+4$	$+8x$
		$+5x$

Factorising gives

$$2x^2 + 5x - 12 = 0$$
$$(2x - 3)(x + 4) = 0$$
$$\therefore \quad 2x - 3 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = \frac{3}{2} \quad \text{or} \quad x = -4$$

Explanation

The quadratic can also be factorised in the following way:

$$\begin{aligned} 2x^2 + 5x - 12 &= 2x^2 + 8x - 3x - 12 \\ &= 2x(x + 4) - 3(x + 4) \\ &= (2x - 3)(x + 4) \end{aligned}$$

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Applications of quadratic equations

Problems involving the solution of quadratic equations arise in many situations. We will meet more such problems in Section 3L.

Example 20

The perimeter of a rectangle is 20 cm and its area is 24 cm^2. Calculate the length and width of the rectangle.

Solution

Let x cm be the length of the rectangle and y cm the width.

Then $2(x + y) = 20$ and thus $y = 10 - x$.

The area is 24 cm^2 and therefore $xy = x(10 - x) = 24$.

i.e. $10x - x^2 = 24$

$x^2 - 10x + 24 = 0$

$(x - 6)(x - 4) = 0$

Thus the length is 6 cm or 4 cm. The width is 4 cm or 6 cm.

Summary 3C

To solve a quadratic equation by factorisation:

Step 1 Write the equation in the form $ax^2 + bx + c = 0$.

Step 2 Factorise the quadratic polynomial.

Step 3 Use the result that $mn = 0$ implies $m = 0$ or $n = 0$ (or both).

Skill-sheet

Exercise 3C

1 Solve each of the following for x:

a $(x - 2)(x - 3) = 0$ **b** $x(2x - 4) = 0$ **c** $(x - 4)(2x - 6) = 0$
d $(3 - x)(x - 4) = 0$ **e** $(2x - 6)(x + 4) = 0$ **f** $2x(x - 1) = 0$
g $(5 - 2x)(6 - x) = 0$ **h** $x^2 = 16$

CAS

2 Use your calculator to solve each of the following equations. Give answers correct to two decimal places.

a $x^2 - 4x - 3 = 0$ **b** $2x^2 - 4x - 3 = 0$ **c** $-2x^2 - 4x + 3 = 0$

Example 18

3 Solve for x in each of the following:

a $x^2 - x - 72 = 0$ **b** $x^2 - 6x + 8 = 0$ **c** $x^2 - 8x - 33 = 0$
d $x(x + 12) = 64$ **e** $x^2 + 5x - 14 = 0$ **f** $x^2 = 5x + 24$

Example 19

4 Solve for x in each of the following:

a $2x^2 + 5x + 3 = 0$ **b** $4x^2 - 8x + 3 = 0$ **c** $6x^2 + 13x + 6 = 0$

d $2x^2 - x = 6$ **e** $6x^2 + 15 = 23x$ **f** $2x^2 - 3x - 9 = 0$

g $10x^2 - 11x + 3 = 0$ **h** $12x^2 + x = 6$ **i** $4x^2 + 1 = 4x$

j $x(x + 4) = 5$ **k** $\frac{1}{7}x^2 = \frac{3}{7}x$ **l** $5x^2 = 11x - 2$

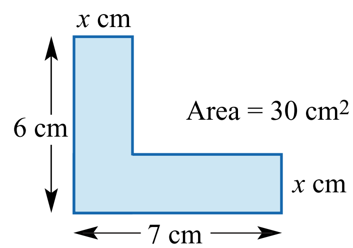
CAS

5 Use your calculator to solve each of the following equations:

a $2x^2 + 7x + 3 = 0$ **b** $15a^2 - 43a + 30 = 0$ **c** $18x^2 - 15x - 12 = 0$

d $2x + \frac{1}{x} = 3$ **e** $4x^2 - 12x = 7$ **f** $\frac{2x}{2x + 3} = \frac{2x - 5}{x - 2}$

6 Calculate the value of x.



7 The bending moment, M, of a simple beam used in bridge construction is given by the formula

$$M = \frac{w\ell}{2}x - \frac{w}{2}x^2$$

If $\ell = 13$ m, $w = 16$ kg/m and $M = 288$ kg m, calculate the value of x.

8 The height, h metres, reached by a projectile after t seconds travelling vertically upwards is given by the formula $h = 70t - 16t^2$. Calculate t if h is 76 metres.

9 A polygon with n sides has $\frac{n(n - 3)}{2}$ diagonals. How many sides has a polygon with 65 diagonals?

10 For a particular electric train, the tractive 'resistance' R at speed v km/h is given by $R = 1.6 + 0.03v + 0.003v^2$. Find v when the tractive resistance is 10.6.

Example 20

11 The perimeter of a rectangle is 16 cm and its area is 12 cm^2. Calculate the length and width of the rectangle.

12 The altitude of a triangle is 1 cm shorter than the base. If the area of the triangle is 15 cm^2, calculate the altitude.

13 Tickets for a concert are available at two prices. The more expensive ticket is \$30 more than the cheaper one. Find the cost of each type of ticket if a group can buy 10 more of the cheaper tickets than the expensive ones for \$1800.

14 The members of a club hire a bus for \$2100. Seven members withdraw from the club and the remaining members have to pay \$10 more each to cover the cost. How many members originally agreed to go on the bus?

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3D Graphing quadratics

A quadratic polynomial function is defined by the general rule

$$y = ax^2 + bx + c$$

where a, b and c are constants and $a \neq 0$. This is called **polynomial form**.

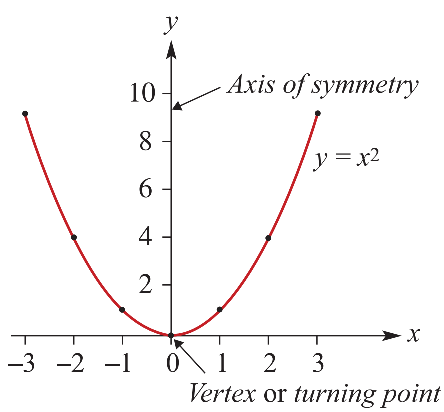
The parabola $y = x^2$

The simplest quadratic function is $y = x^2$. If a table of values is constructed for $y = x^2$ for $-3 \leq x \leq 3$, these points can be plotted and then connected to produce a continuous curve.

x	−3	−2	−1	0	1	2	3
y	9	4	1	0	1	4	9

Features of the graph of $y = x^2$:

- The graph is called a **parabola**.
- The possible y-values are all positive real numbers and 0. (This is called the **range** of the quadratic and is discussed in a more general context in Chapter 5.)
- The graph is symmetrical about the y-axis. The line about which the graph is symmetrical is called the **axis of symmetry**.
- The graph has a **vertex** or **turning point** at the origin $(0, 0)$.
- The minimum value of y is 0 and it occurs at the turning point.



Transformations of $y = x^2$

By a process called **completing the square** (to be discussed in Section 3E), all quadratics in polynomial form $y = ax^2 + bx + c$ may be transposed into what will be called the **turning point form**:

$$y = a(x - h)^2 + k$$

We first consider the effect of changing the value of a for our basic graph of $y = x^2$.

We then consider the effect of changing h and k for graphs of the form $y = ax^2$. Graphs of the form $y = a(x - h)^2 + k$ are formed by **translating** the graph of $y = ax^2$. The graph of $y = a(x - h)^2 + k$ is exactly the same shape as $y = ax^2$. All of these graphs are indeed congruent to $y = ax^2$ and each other.

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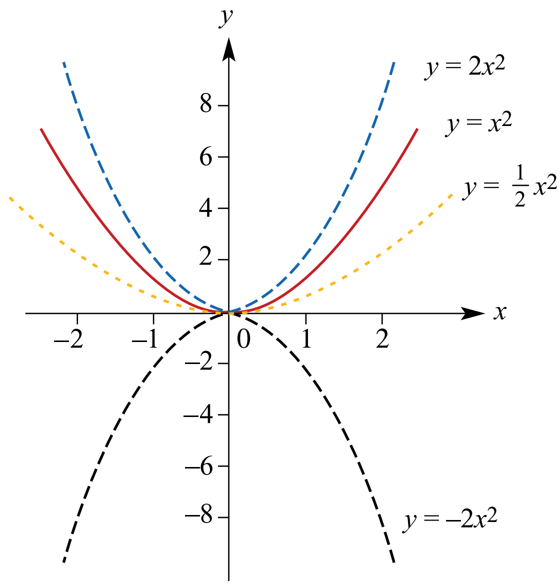
Graphs of $y = ax^2$

We first consider graphs of the form $y = ax^2$. In this case both $h = 0$ and $k = 0$. In the basic graph of $y = x^2$, the value of a is 1.

The following graphs are shown on the same set of axes:

$y = x^2$
$y = 2x^2$ $(a = 2)$
$y = \frac{1}{2}x^2$ $(a = \frac{1}{2})$
$y = -2x^2$ $(a = -2)$



If $a > 1$, the graph is 'narrower'. If $0 < a < 1$, the graph is 'broader'. The transformation which produces the graph of $y = 2x^2$ from the graph of $y = x^2$ is called a **dilation of factor 2 from the *x*-axis**.

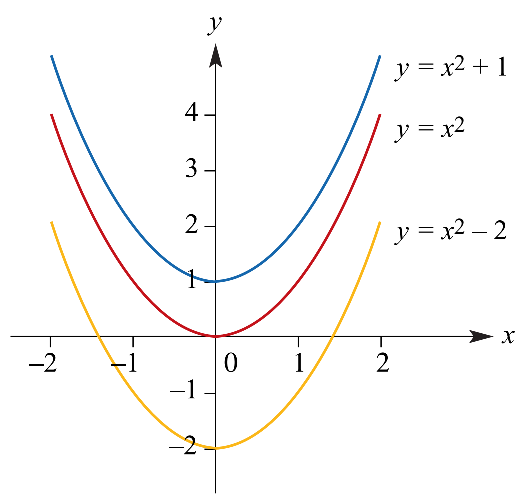
When a is negative, the graph is reflected in the x-axis. The transformation which produces the graph of $y = -x^2$ from the graph of $y = x^2$ is called a **reflection in the *x*-axis**.

Graphs of $y = x^2 + k$

On this set of axes are the graphs of

$y = x^2$
$y = x^2 - 2$ $(k = -2)$
$y = x^2 + 1$ $(k = 1)$

As can be seen, changing k moves the basic graph of $y = x^2$ in a vertical direction.



- When $k = -2$ the graph is **translated** 2 units in the negative direction of the y-axis. The vertex is now $(0, -2)$ and the range is now all real numbers greater than or equal to -2.
- When $k = 1$ the graph is **translated** 1 unit in the positive direction of the y-axis. The vertex is now $(0, 1)$ and the range is now all real numbers greater than or equal to 1.

All other features of the graph are unchanged. The axis of symmetry is still the y-axis.

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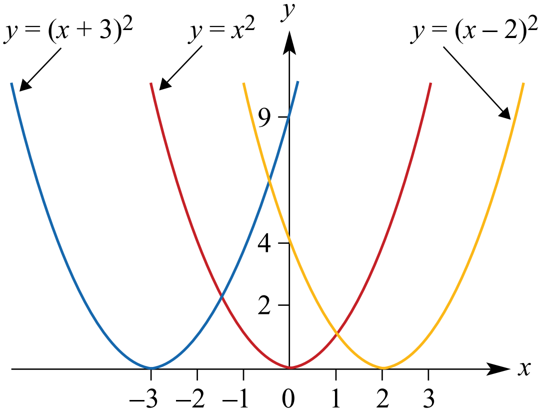
Graphs of $y = (x - h)^2$

On this set of axes are the graphs of

$$y = x^2$$
$$y = (x-2)^2 \qquad (h = 2)$$
$$y = (x+3)^2 \qquad (h = -3)$$



As can be seen, changing h moves the graph in a horizontal direction.

- When $h = 2$ the graph is **translated** 2 units in the positive direction of the x-axis. The vertex is now $(2, 0)$ and the axis of symmetry is now the line $x = 2$.
- When $h = -3$ the graph is **translated** 3 units in the negative direction of the x-axis. The vertex is now $(-3, 0)$ and the axis of symmetry is now the line $x = -3$.

In both cases, the range is unchanged and is still all non-negative real numbers.

Examples of transformations

By combining dilations, reflections and translations, we can sketch the graph of any quadratic expressed in the form $y = a(x-h)^2 + k$:

- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive numbers, then the graph of $y = a(x-h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 21

Sketch the graph of $y = x^2 - 3$.

Solution

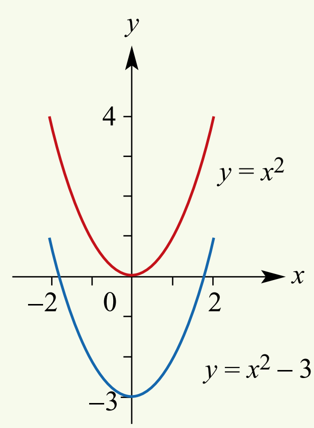
The graph of $y = x^2 - 3$ is obtained from the graph of $y = x^2$ by translating 3 units in the negative direction of the y-axis.

The vertex is now at $(0, -3)$. The axis of symmetry is the line with equation $x = 0$.

To find the x-axis intercepts, let $y = 0$:

$$0 = x^2 - 3$$
$$x^2 = 3$$
$$\therefore \quad x = \pm\sqrt{3}$$

Hence the x-axis intercepts are $\pm\sqrt{3}$.



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Example 22

Sketch the graph of $y = -(x+1)^2$.

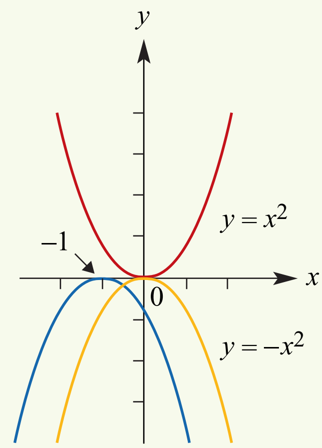
Solution

The graph of $y = -(x+1)^2$ is obtained from the graph of $y = x^2$ by a reflection in the x-axis followed by a translation of 1 unit in the negative direction of the x-axis.

The vertex is now at $(-1, 0)$.

The axis of symmetry is the line with equation $x = -1$.

The x-axis intercept is -1.



Example 23

Sketch the graph of $y = 2(x-1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

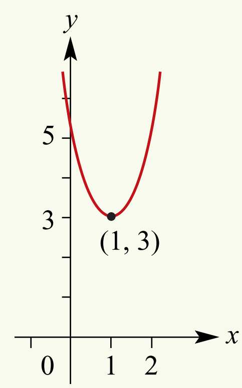
The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range will be $y \geq 3$.

To add further detail to our graph, we can find the axis intercepts:



y-axis intercept

When $x = 0$, $y = 2(0-1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$\begin{aligned} 0 &= 2(x-1)^2 + 3 \\ -3 &= 2(x-1)^2 \\ -\tfrac{3}{2} &= (x-1)^2 \end{aligned}$$

As the square root of a negative number is not a real number, this equation has no real solutions.

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Example 24

Sketch the graph of $y = -(x + 1)^2 + 4$.

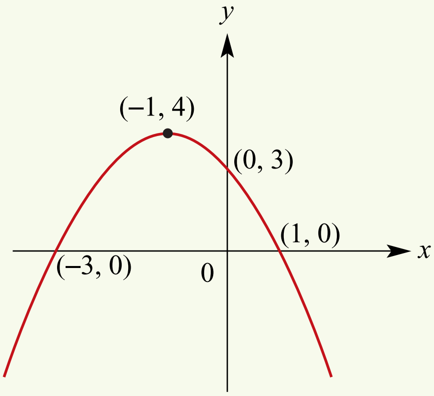
Solution

The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.

When $x = 0$, $y = -(0 + 1)^2 + 4 = 3$.
$\therefore$ the y-axis intercept is 3.

When $y = 0$,
$$\begin{aligned} -(x + 1)^2 + 4 &= 0 \\ (x + 1)^2 &= 4 \\ x + 1 &= \pm 2 \\ x &= \pm 2 - 1 \end{aligned}$$

$\therefore$ the x-axis intercepts are 1 and -3.



Summary 3D

- The graph of $y = x^2$ is called a parabola. The vertex (or turning point) is the point $(0, 0)$ and the axis of symmetry is the y-axis.
- The graph of $y = -x^2$ is the reflection of the graph of $y = x^2$ in the x-axis.
- For $y = ax^2$ and $a > 1$, the graph is 'narrower' than the graph of $y = x^2$.
- For $y = ax^2$ and $0 < a < 1$, the graph is 'broader' than the graph of $y = x^2$.
- All quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x - h)^2 + k$.
- The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$.
  - The vertex (or turning point) is the point (h, k).
  - The axis of symmetry is $x = h$.
  - If h and k are positive numbers, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
  - Similar results hold for different combinations of h and k positive and negative.

Exercise 3D

Sketch the graph of each of the following quadratics by first finding:

i the coordinates of the turning point
ii the axis of symmetry
iii the x-axis intercepts (if any).

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Example 21

1 **a** $y = x^2 - 4$ **b** $y = x^2 + 2$ **c** $y = -x^2 + 3$
d $y = -2x^2 + 5$ **e** $y = -x^2 + 4$ **f** $y = 3x^2 - 9$

Example 22

2 **a** $y = (x-2)^2$ **b** $y = (x+3)^2$ **c** $y = -(x+1)^2$ **d** $y = -\frac{1}{2}(x-4)^2$

Example 23
Example 24

3 **a** $y = (x-2)^2 + 1$ **b** $y = (x-2)^2 - 1$ **c** $y = (x-1)^2 + 2$
d $y = (x+1)^2 - 1$ **e** $y = -(x-3)^2 + 1$ **f** $y = (x+2)^2 - 4$
g $y = 2(x+2)^2 - 18$ **h** $y = -3(x-4)^2 + 3$ **i** $y = -\frac{1}{2}(x+5)^2 - 2$
j $y = 3(x+2)^2 - 12$ **k** $y = -4(x-2)^2 + 8$ **l** $y = \frac{1}{3}(x-1)^2 - 3$

3E Completing the square and turning points

To sketch the graph of a quadratic using the techniques from the previous section, the quadratic must be expressed in **turning point form**. This can be done using two different but related methods: by completing the square and by using the equation of the axis of symmetry.

Completing the square

To transpose a quadratic in polynomial form we can **complete the square**.

Consider the expansion of a perfect square:

$$(x+a)^2 = x^2 + 2ax + a^2$$

The last term of the expansion is the square of half the coefficient of the middle term.

Now consider the quadratic polynomial

$$x^2 + 2x - 3$$

This is not a perfect square. However, by adding and subtracting a new term, we can form a perfect square as part of a new expression for the same polynomial.

We have that

$$x^2 + 2x + 1 = (x+1)^2$$

which is a perfect square. In order to keep our original quadratic 'intact', we both add and subtract the 'correct' new term. For example:

$$x^2 + 2x - 3 = (x^2 + 2x + 1) - 1 - 3$$

This can now be simplified to

$$(x+1)^2 - 4$$

Hence the quadratic $y = x^2 + 2x - 3$ is expressed in turning point form as $y = (x+1)^2 - 4$, and so the **vertex** (turning point) of its graph is the point with coordinates $(-1, -4)$.

In the above example, the coefficient of x^2 was 1. If the coefficient is not 1, this coefficient must first be 'factored out' before proceeding to complete the square.

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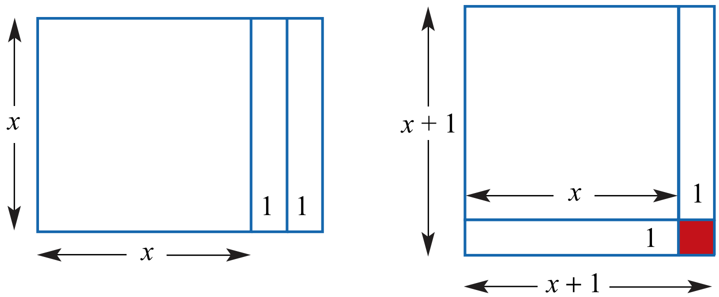
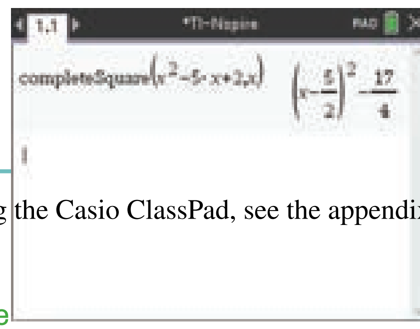
A geometric representation of completing the square

Completing the square for $x^2 + 2x$ is represented in the following diagrams. The diagram on the left shows $x^2 + 2x$. The small rectangle to the right is moved to the 'base' of the x by x square. The red square of area 1 unit is added. Thus $x^2 + 2x + 1 = (x + 1)^2$.



Using the TI-Nspire

Use (menu) > **Algebra** > **Complete the Square** to rearrange the expression $x^2 - 5x + 2$.



Note: For instructions on completing the square using the Casio ClassPad, see the appendix in the Interactive Textbook.

Solving equations by completing the square

The process of completing the square can also be used for the solution of equations.

Example 25

Solve each of the following equations for x by first completing the square:

a $x^2 - 3x + 1 = 0$ **b** $2x^2 - 3x - 1 = 0$

Solution

a Completing the square:

$$x^2 - 3x + 1 = 0$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 1 = 0$$

$$\left(x - \frac{3}{2}\right)^2 - \frac{5}{4} = 0$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{5}{4}$$

Therefore $x - \frac{3}{2} = \pm\frac{\sqrt{5}}{2}$

and so $x = \frac{3}{2} \pm \frac{\sqrt{5}}{2} = \frac{3 \pm \sqrt{5}}{2}$

Explanation

$\frac{1}{2} \times (-3) = -\frac{3}{2}$

We add and subtract $\left(-\frac{3}{2}\right)^2 = \frac{9}{4}$ on the left-hand side of the equation.

This gives an equivalent expression to the expression of the left-hand side.

Solve the equation as shown.

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b Completing the square:

$$2x^2 - 3x - 1 = 0$$

$$2\left(x^2 - \frac{3}{2}x - \frac{1}{2}\right) = 0$$

$$x^2 - \frac{3}{2}x + \left(\frac{3}{4}\right)^2 - \left(\frac{3}{4}\right)^2 - \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 = \frac{17}{16}$$

Therefore $x - \frac{3}{4} = \pm\frac{\sqrt{17}}{4}$

and so $x = \frac{3}{4} \pm \frac{\sqrt{17}}{4} = \frac{3 \pm \sqrt{17}}{4}$

Divide both sides by 2 before completing the square.

$\frac{1}{2} \times \left(-\frac{3}{2}\right) = -\frac{3}{4}$

We add and subtract $\left(-\frac{3}{4}\right)^2 = \frac{9}{16}$ on the left-hand side of the equation.

Sketching the graph of a quadratic polynomial after completing the square

Completing the square enables the quadratic rule to be written in turning point form. We have seen that this can be used to sketch the graphs of quadratic polynomials.

Example 26

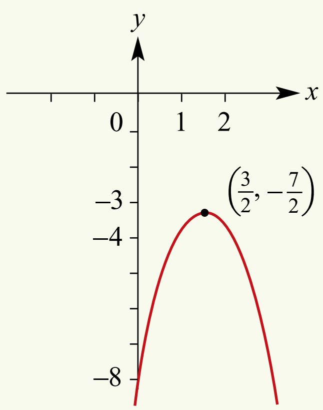
Find the coordinates of the vertex by completing the square and hence sketch the graph of $y = -2x^2 + 6x - 8$.

Solution

Take out -2 as a common factor and then complete the square:

$$\begin{aligned} y &= -2x^2 + 6x - 8 \\ &= -2(x^2 - 3x + 4) \\ &= -2\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 4\right) \\ &= -2\left(\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 4\right) \\ &= -2\left(\left(x - \frac{3}{2}\right)^2 + \frac{7}{4}\right) \end{aligned}$$

$$\therefore \quad y = -2\left(x - \frac{3}{2}\right)^2 - \frac{7}{2}$$



Therefore the vertex is $\left(\frac{3}{2}, -\frac{7}{2}\right)$ and the axis of symmetry is $x = \frac{3}{2}$.

The y-axis intercept is -8.

The graph has maximum value of $-\frac{7}{2}$, and so there are no x-axis intercepts.

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The equation for the axis of symmetry of a parabola

We first complete the square for $y = ax^2 + bx + c$:

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right) \qquad \text{completing the square} \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

Axis of symmetry of a parabola

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\frac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\frac{b}{2a}$. Substitute this value into the quadratic polynomial to find the y-coordinate of the turning point.

Example 27

Use the axis of symmetry to find the turning point of the graph and hence express in turning point form:

a $y = x^2 - 4x + 3$ **b** $y = -2x^2 + 12x - 7$

Solution

a The x-coordinate of the turning point is 2.

When $x = 2$, $y = 4 - 8 + 3 = -1$.

The coordinates of the turning point are $(2, -1)$.

Hence the equation is $y = (x - 2)^2 - 1$.

Explanation

Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\frac{-4}{2}\right) = 2$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.

Solution

b The x-coordinate of the turning point is 3.

When $x = 3$, $y = -2 \times 3^2 + 12 \times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$.

Hence the equation is $y = -2(x - 3)^2 + 11$.

Explanation

Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\frac{12}{-4}\right) = 3$.

For the turning point form $y = a(x - h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

Summary 3E

- Quadratic equations can be solved by completing the square. This method allows us to deal with all quadratic equations, even though some have no solutions.

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- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the brackets.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\frac{b}{2a}$.
- To convert the quadratic function $y = ax^2 + bx + c$ into turning point form using the axis of symmetry:
  1. The x-coordinate h of the vertex of the parabola is $-\frac{b}{2a}$.
  2. Find the y-coordinate k of the vertex by substituting in $y = ax^2 + bx + c$.
  3. Substitute these values for h and k in $y = a(x - h)^2 + k$.

Skill-sheet

Exercise 3E

1 Expand each of the following:

a $(x-1)^2$ **b** $(x+2)^2$ **c** $(x-3)^2$ **d** $(-x+3)^2$

e $(-x-2)^2$ **f** $(x-5)^2$ **g** $\left(x-\frac{1}{2}\right)^2$ **h** $\left(x-\frac{3}{2}\right)^2$

2 Factorise each of the following:

a x^2-4x+4 **b** $x^2-12x+36$ **c** $-x^2+4x-4$ **d** $2x^2-8x+8$

e $-2x^2+12x-18$ **f** $x^2-x+\frac{1}{4}$ **g** $x^2-3x+\frac{9}{4}$ **h** $x^2+5x+\frac{25}{4}$

Example 25

3 Solve each of the following equations for x by first completing the square:

a $x^2-2x-1=0$ **b** $x^2-4x-2=0$ **c** $x^2-6x+2=0$

d $x^2-5x+2=0$ **e** $2x^2-4x+1=0$ **f** $3x^2-5x-2=0$

g $x^2+2x+k=0$ **h** $kx^2+2x+k=0$ **i** $x^2-3kx+1=0$

Example 26

4 Express each of the following in the form $y = a(x-h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y=x^2-2x+3$ **b** $y=x^2+4x+1$ **c** $y=x^2-3x+1$

5 Express each of the following in the form $y = a(x-h)^2 + k$ by completing the square. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y=2x^2-2x-5$ **b** $y=4x^2+8x+8$ **c** $y=3x^2-6x-4$

Example 27

6 Express each of the following in the form $y = a(x-h)^2 + k$ using the axis of symmetry. Hence state the coordinates of the turning point and sketch the graph in each case.

a $y=x^2-8x+12$ **b** $y=x^2-x-2$ **c** $y=2x^2+4x-2$

d $y=-x^2+4x+1$ **e** $y=-2x^2-12x-12$ **f** $y=3x^2-6x+12$

7 Use your calculator to complete the square for each expression: CAS

a $2x^2+8x+3$ **b** $3x^2+12x-2$ **c** $3x^2-5kx-7$

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3F Graphing quadratics in polynomial form

It is not always essential to convert a quadratic to turning point form in order to sketch its graph. We can sometimes find the x- and y-axis intercepts and the axis of symmetry from polynomial form by other methods and use these details to sketch the graph.

Step 1 Find the y-axis intercept

Let $x = 0$. For the general quadratic $y = ax^2 + bx + c$, this gives

$$y = a(0)^2 + b(0) + c$$
$$y = c$$

Hence the y-axis intercept is always equal to c.

Step 2 Find the x-axis intercepts

Let $y = 0$. In general, this gives

$$0 = ax^2 + bx + c$$

In order to solve such an equation it is necessary to factorise the right-hand side and then use the **null factor theorem**.

Step 3 Find the equation of the axis of symmetry

Once the x-axis intercepts have been found, the equation of the axis of symmetry can be found by using the symmetry properties of the parabola. The axis of symmetry is the perpendicular bisector of the line segment joining the x-axis intercepts.

Step 4 Find the coordinates of the turning point

The axis of symmetry gives the x-coordinate of the turning point. Substitute this into the quadratic polynomial to obtain the y-coordinate.

Example 28

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 - 4x$.

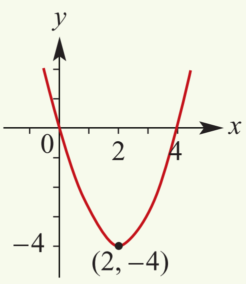
Solution

Step 1 $c = 0$. Therefore the y-axis intercept is 0.

Step 2 Let $y = 0$. Then

$$0 = x^2 - 4x$$
$$0 = x(x - 4)$$
$$\therefore \quad x = 0 \text{ or } x = 4$$

The x-axis intercepts are 0 and 4.



Step 3 The axis of symmetry is the line with equation $x = \dfrac{0 + 4}{2}$, that is, $x = 2$.

Step 4 When $x = 2$, $y = (2)^2 - 4(2) = -4$. The turning point has coordinates $(2, -4)$.

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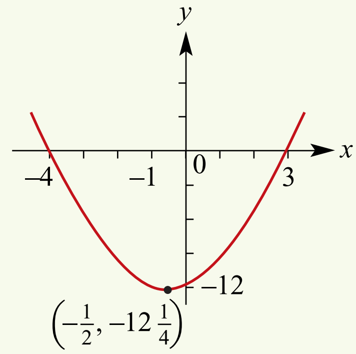
Example 29

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.



Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x + 4)(x - 3)$$
$$\therefore \quad x = -4 \text{ or } x = 3$$

The x-axis intercepts are -4 and 3.

Step 3 The axis of symmetry is the line with equation

$$x = \frac{-4 + 3}{2} = -\frac{1}{2}$$

Step 4 When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

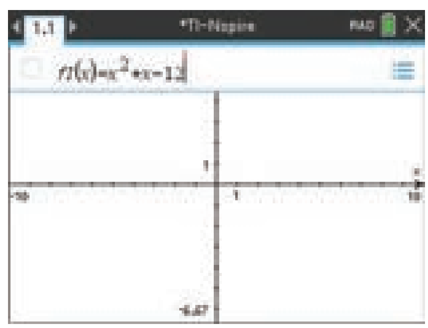
$$= -12\tfrac{1}{4}$$

The turning point has coordinates $(-\frac{1}{2}, -12\frac{1}{4})$.

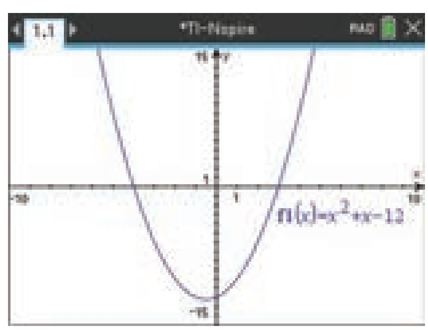
Using the TI-Nspire

To graph the quadratic function with rule $y = x^2 + x - 12$:

- Enter the rule in the entry line of a **Graphs** application as shown, and press enter.



- Using menu > **Window/Zoom** > **Window Settings**, select the window settings $-10 \le x \le 10$ and $-15 \le y \le 15$ to obtain the graph shown.



Note: You can also double click on the end values to change the window settings.

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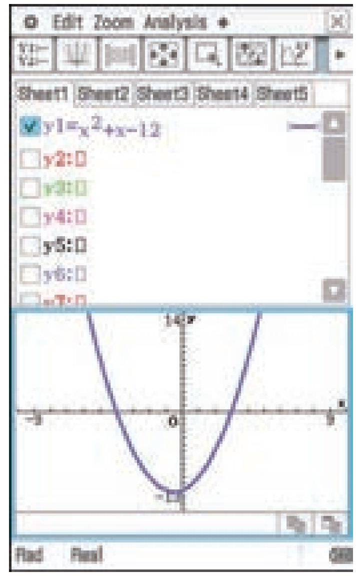
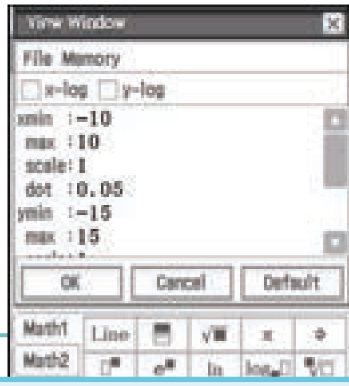
Using the Casio ClassPad

- Open the menu Menu; select **Graph & Table**.
- Type the expression $x^2 + x - 12$ in $y1$.
- Tick the box and tap the graph icon.
- It may be necessary to change the view window by using and the settings shown below.





Summary 3F

Steps for sketching the graph of a quadratic function given in polynomial form:

Step 1 Find the y-axis intercept.

Step 2 Find the x-axis intercepts.

Step 3 Find the equation of the axis of symmetry.

Step 4 Find the coordinates of the turning point.

Exercise 3F

1 **a** A parabola has x-axis intercepts 4 and 10. State the x-coordinate of the vertex.

b A parabola has x-axis intercepts 6 and 8. State the x-coordinate of the vertex.

c A parabola has x-axis intercepts -6 and 8. State the x-coordinate of the vertex.

2 **a** A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at 6. Find the other x-axis intercept.

b A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at -4. Find the other x-axis intercept.

c A parabola has vertex $(2, -6)$ and one of the x-axis intercepts is at the origin. Find the other x-axis intercept.

Example 28

3 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 - 1$ **b** $y = x^2 + 6x$ **c** $y = 25 - x^2$ **d** $y = x^2 - 4$

e $y = 2x^2 + 3x$ **f** $y = 2x^2 - 4x$ **g** $y = -2x^2 - 3x$ **h** $y = x^2 + 1$

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Example 29

4 Sketch each of the following parabolas, clearly showing the axis intercepts and the turning point:

a $y = x^2 + 3x - 10$ **b** $y = x^2 - 5x + 4$ **c** $y = x^2 + 2x - 3$

d $y = x^2 + 4x + 3$ **e** $y = 2x^2 - x - 1$ **f** $y = 6 - x - x^2$

g $y = -x^2 - 5x - 6$ **h** $y = x^2 - 5x - 24$

5 Use your calculator to help sketch each of the following parabolas, clearly showing the axis intercepts and the turning point: CAS

a $y = x^2 - 6x - 7$ **b** $y = 5x^2 - 7x$ **c** $y = x^2 - x - 42$

3G Solving quadratic inequalities

In Chapter 1 we looked at solving linear inequalities. The situation is a little more complex for quadratic inequalities. We suggest one possible approach.

To solve a quadratic inequality (for example, $x^2 + x - 12 > 0$):

Step 1 Solve the corresponding equation (for example, $x^2 + x - 12 = 0$).

Step 2 Sketch the graph of the quadratic polynomial (for example, $y = x^2 + x - 12$).

Step 3 Use the graph to determine the set of x-values which satisfy the inequality.

Example 30

Solve $x^2 + x - 12 > 0$.

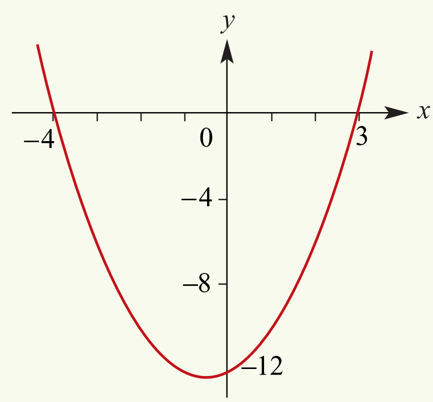
Solution

Step 1 Solve the equation

$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$\therefore \quad x = -4 \quad \text{or} \quad x = 3$$

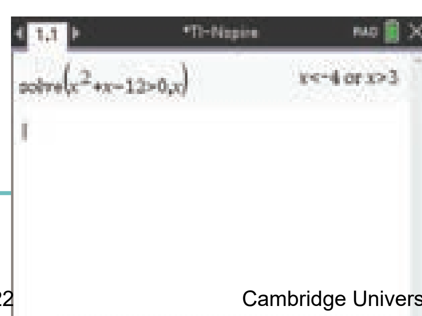
Step 2 Sketch the graph of the quadratic $y = x^2 + x - 12$.

Step 3 From the graph it can be seen that $x^2 + x - 12 > 0$ when $x < -4$ or $x > 3$.



Using the TI-Nspire

The calculator may be used to solve quadratic inequalities.



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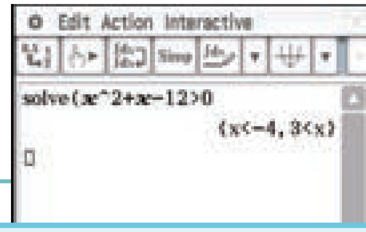
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select solve(from the Math3 keyboard.
- Enter the inequality $x^2 + x - 12 > 0$. (The inequality symbols can be found in the Math3 keyboard.)
- Tap EXE.



Summary 3G

When solving quadratic inequalities of the form $ax^2 + bx + c \le 0$ (or with $\ge$, $>$ or $<$), it is best to sketch the graph of $y = ax^2 + bx + c$.

Skill-sheet

Exercise 3G

Example 30

1 **a** Solve the equation $x^2 - 2x - 8 = 0$. **b** Sketch the graph of $y = x^2 - 2x - 8$.
c Solve the inequality $x^2 - 2x - 8 \le 0$. **d** Solve the inequality $x^2 - 2x - 8 > 0$.

2 Solve each of the following inequalities:
a $(x - 3)(x + 2) \ge 0$ **b** $(x + 4)(x + 3) < 0$ **c** $(2x - 1)(x + 4) \le 0$
d $(x - 6)(2x - 4) > 0$ **e** $(2x - 6)(2x - 4) < 0$ **f** $(7 - 2x)(2x - 3) \ge 0$
g $(2x + 7)(2x - 4) < 0$ **h** $(3x + 6)(2x - 5) \le 0$ **i** $(5 - 2x)(5 + x) < 0$

3 Solve each of the following inequalities:
a $(5 - x)(5 + x) < 0$ **b** $4 - 9y^2 \ge 0$ **c** $16 - y^2 < 0$
d $36 - 25x^2 \ge 0$ **e** $1 - 16y^2 \le 0$ **f** $25 - 36y^2 < 0$

4 Solve each of the following inequalities:
a $x^2 + 2x - 8 \ge 0$ **b** $x^2 - 5x - 24 < 0$ **c** $x^2 - 4x - 12 \le 0$
d $2x^2 - 3x - 9 > 0$ **e** $6x^2 + 13x < -6$ **f** $-x^2 - 5x - 6 \ge 0$
g $12x^2 + x > 6$ **h** $10x^2 - 11x \le -3$ **i** $x(x - 1) \le 20$
j $4 + 5p - p^2 \ge 0$ **k** $3 + 2y - y^2 < 0$ **l** $x^2 + 3x \ge -2$

5 Solve each of the following inequalities:
a $x^2 + 3x - 5 \ge 0$ **b** $x^2 - 5x + 2 < 0$ **c** $2x^2 - 3x - 1 \le 0$
d $8 - 3x - x^2 > 0$ **e** $2x^2 + 7x + 1 < 0$ **f** $2x^2 - 8x + 5 \ge 0$

6 Explain why $(x - 3)^2 \ge 0$ for all x.

7 Explain why $-(x - 1)^2 \le 0$ for all x.

8 Complete the square for $x^2 + 2x + 7$ and hence show that $x^2 + 2x + 7 \ge 6$ for all x.

9 Complete the square for $-x^2 - 2x - 7$ and hence show that $-x^2 - 2x - 7 \le -6$ for all x.

10 Use your calculator to solve each of the following inequalities: CAS
a $(3x - 5)(x + 6) \ge -30$ **b** $(x - 2)^2 < (2x + 1)^2$ **c** $(x + 2)^2 \ge 1$

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3H The general quadratic formula

Not all quadratics can be factorised by inspection, and it is often difficult to find the x-axis intercepts this way. There is a general formula for finding the solutions of a quadratic equation in polynomial form. This formula comes from 'completing the square' for the general quadratic.

In Section 3E we showed that

$$\begin{aligned} y &= ax^2 + bx + c \\ &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c \end{aligned}$$

We can use this to solve the general quadratic equation:

$$\begin{aligned} ax^2 + bx + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c &= 0 \\ a\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2}{4a} - c \end{aligned}$$

Now divide both sides by a:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$\therefore \quad x = -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note: The quadratic formula provides an alternative method for solving quadratic equations to 'completing the square', but it is probably not as useful for curve sketching as 'completing the square', which gives the turning point coordinates directly.

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation

$$x = -\frac{b}{2a}$$

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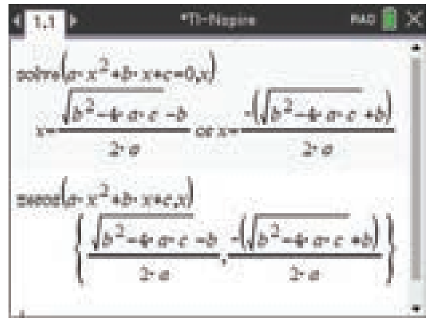
Also, from the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

This will be further explored in the next section.

A CAS calculator gives the result shown opposite.



Example 31

Solve each of the following equations for x by using the quadratic formula:

a $x^2 - x - 1 = 0$ **b** $x^2 - 2kx - 3 = 0$

Solution

a $x^2 - x - 1 = 0$

Here $a = 1$, $b = -1$ and $c = -1$.

The formula gives

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times (-1)}}{2 \times 1}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

b $x^2 - 2kx - 3 = 0$

Here $a = 1$, $b = -2k$ and $c = -3$.

The formula gives

$$x = \frac{-(-2k) \pm \sqrt{(-2k)^2 - 4 \times 1 \times (-3)}}{2 \times 1}$$

$$= \frac{2k \pm \sqrt{4k^2 + 12}}{2}$$

$$= k \pm \sqrt{k^2 + 3}$$

Explanation

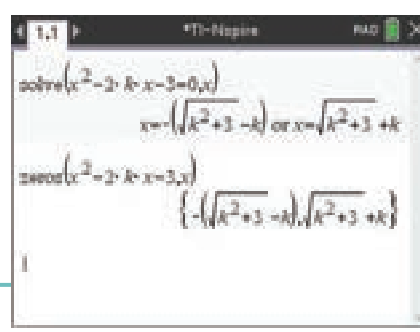
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that $k^2 + 3 \geq 0$ for all values of k, since $k^2 \geq 0$.

Using the TI-Nspire

- Use menu > **Algebra** > **Solve** to solve the equation $x^2 - 2kx - 3 = 0$ for x.
- Alternatively, use menu > **Algebra** > **Zeros**.

Note: You must use a multiplication sign between k and x.



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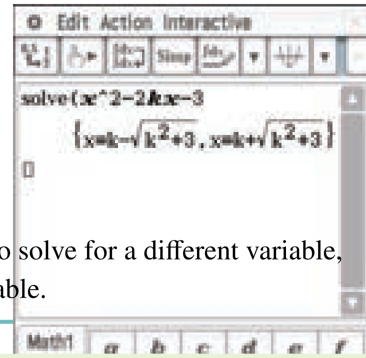
Using the Casio ClassPad

To solve the equation $x^2 - 2kx - 3 = 0$ for x:



- Select [solve(] from the [Math1] or [Math3] keyboard.
- Enter the expression $x^2 - 2kx - 3$, using the [Var] keyboard to enter the variable k.
- Tap [EXE].

Note: The default setting is to solve for the variable x. To solve for a different variable, follow the equation by a comma and then the variable.

Example 32

Sketch the graph of $y = -3x^2 - 12x - 7$. Use the quadratic formula to calculate the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

Axis of symmetry

$$x = -\frac{b}{2a} = -\left(\frac{-12}{2 \times (-3)}\right) = -2$$

Turning point

When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$.

The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

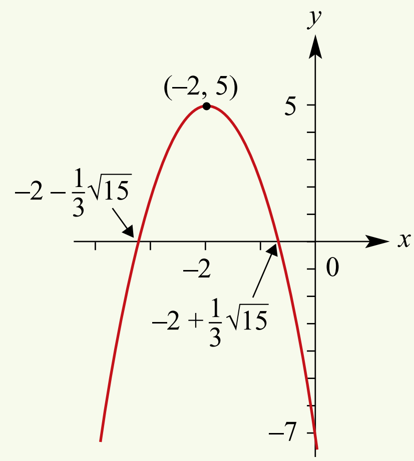
$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)}$$

$$= \frac{12 \pm \sqrt{60}}{-6}$$

$$= \frac{12 \pm 2\sqrt{15}}{-6}$$

$$= -2 \mp \tfrac{1}{3}\sqrt{15}$$

$$\approx -3.29 \text{ or } -0.71 \quad \text{(to two decimal places)}$$



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Summary 3H

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

Exercise 3H

1 For each of the following, the coefficients a, b and c of a quadratic $y = ax^2 + bx + c$ are given. Find:

i $b^2 - 4ac$ **ii** $\sqrt{b^2 - 4ac}$ in simplest surd form

a $a = 2$, $b = 4$ and $c = -3$
b $a = 1$, $b = 10$ and $c = 18$
c $a = 1$, $b = 10$ and $c = -18$
d $a = -1$, $b = 6$ and $c = 15$
e $a = 1$, $b = 9$ and $c = -27$

2 Simplify each of the following:

a $\frac{2 + 2\sqrt{5}}{2}$ **b** $\frac{9 - 3\sqrt{5}}{6}$ **c** $\frac{5 + 5\sqrt{5}}{10}$ **d** $\frac{6 + 12\sqrt{2}}{6}$

Example 31

3 Solve each of the following for x. Give exact answers.

a $x^2 + 6x = 4$
b $x^2 - 7x - 3 = 0$
c $2x^2 - 5x + 2 = 0$
d $2x^2 + 4x - 7 = 0$
e $2x^2 + 8x = 1$
f $5x^2 - 10x = 1$
g $-2x^2 + 4x - 1 = 0$
h $2x^2 + x = 3$
i $2.5x^2 + 3x + 0.3 = 0$
j $-0.6x^2 - 1.3x = 0.1$
k $2kx^2 - 4x + k = 0$
l $2(1 - k)x^2 - 4kx + k = 0$

Example 32

4 Sketch the graphs of the following parabolas. Use the quadratic formula to find the x-axis intercepts (if they exist) and the axis of symmetry and, hence, the turning point.

a $y = x^2 + 5x - 1$
b $y = 2x^2 - 3x - 1$
c $y = -x^2 - 3x + 1$
d $y + 4 = x^2 + 2x$
e $y = 4x^2 + 5x + 1$
f $y = -3x^2 + 4x - 2$
g $y = -x^2 + 5x + 6$
h $y = 4x^2 - 3x + 2$

CAS

5 Solve each of the following for x:

a $x^2 - kx - k = 0$, where $k > 0$
b $kx^2 - x + k = 0$, where $0 < k \leq \frac{1}{2}$
c $k^2x^2 - x + k^2 = 0$, where $0 < k \leq \frac{\sqrt{2}}{2}$

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3I The discriminant

In the previous section we found that the solutions to the quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression under the square root sign is called the **discriminant**. We write

$$\Delta = b^2 - 4ac$$

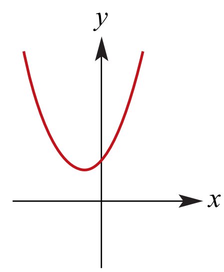
The number of x-axis intercepts

There are three different possibilities for the number of x-axis intercepts of a parabola:

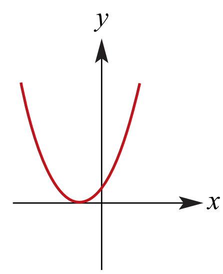
- zero – the graph is either all above or all below the x-axis
- one – the graph touches the x-axis and the turning point is the x-axis intercept
- two – the graph crosses the x-axis.

For a parabola $y = ax^2 + bx + c$, we can use the discriminant $\Delta = b^2 - 4ac$ to determine when each of these three situations occur.

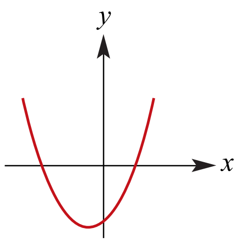
- If the discriminant $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has no solutions and the corresponding parabola will have no x-axis intercepts.



- If the discriminant $b^2 - 4ac = 0$, then the equation $ax^2 + bx + c = 0$ has one solution and the corresponding parabola will have one x-axis intercept. (We sometimes say the equation has two coincident solutions.)



- If the discriminant $b^2 - 4ac > 0$, then the equation $ax^2 + bx + c = 0$ has two solutions and the corresponding parabola will have two x-axis intercepts.



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Example 33

Find the discriminant of each of the following quadratics and state whether the graph of each crosses the x-axis, touches the x-axis or does not intersect the x-axis.

a $y = x^2 - 6x + 8$ **b** $y = x^2 - 8x + 16$ **c** $y = 2x^2 - 3x + 4$

Solution

a Discriminant $\Delta = b^2 - 4ac$

$= (-6)^2 - (4 \times 1 \times 8)$

$= 4$

As $\Delta > 0$, the graph intersects the x-axis at two distinct points, i.e. there are two distinct solutions of the equation $x^2 - 6x + 8 = 0$.

b $\Delta = b^2 - 4ac$

$= (-8)^2 - (4 \times 1 \times 16)$

$= 0$

As $\Delta = 0$, the graph touches the x-axis, i.e. there is one solution of the equation $x^2 - 8x + 16 = 0$.

c $\Delta = b^2 - 4ac$

$= (-3)^2 - (4 \times 2 \times 4)$

$= -23$

As $\Delta < 0$, the graph does not intersect the x-axis, i.e. there are no real solutions for the equation $2x^2 - 3x + 4 = 0$.

Example 34

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$\Delta = 0$

i.e. $4m^2 - 36 = 0$

$m^2 = 9$

$\therefore \quad m = \pm 3$

b For no solution:

$\Delta < 0$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$-3 < m < 3$

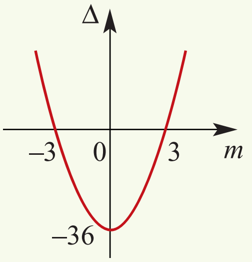
c For two distinct solutions:

$\Delta > 0$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$m > 3$ or $m < -3$



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The nature of the solutions of a quadratic equation

The discriminant can be used to assist in the identification of the particular type of solution for a quadratic equation $ax^2 + bx + c = 0$.

For a, b and c rational numbers:

- If $\Delta = b^2 - 4ac$ is a perfect square and $\Delta \neq 0$, then the quadratic equation has two rational solutions.
- If $\Delta = b^2 - 4ac = 0$, then the quadratic equation has one rational solution.
- If $\Delta = b^2 - 4ac$ is not a perfect square and $\Delta > 0$, then the quadratic equation has two irrational solutions.

Example 35

Show that the solutions of the equation $3x^2 + (m - 3)x - m = 0$ are rational for all rational values of m.

Solution

$$\begin{aligned}\Delta &= (m-3)^2 - 4 \times 3 \times (-m)\\ &= m^2 - 6m + 9 + 12m\\ &= m^2 + 6m + 9\\ &= (m+3)^2 \geq 0 \quad \text{for all } m\end{aligned}$$

Furthermore, Δ is a perfect square for all m.

Summary 3I

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$, where a, b and c are rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, there are two rational solutions.
- If $\Delta = 0$, there is one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, there are two irrational solutions.

Skill-sheet

Exercise 3I

1 Determine the discriminant of each of the following quadratics:

a $x^2 + 2x - 4$ **b** $x^2 + 2x + 4$ **c** $x^2 + 3x - 4$

d $2x^2 + 3x - 4$ **e** $-2x^2 + 3x + 4$

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Example 33

2 Without sketching the graphs of the following quadratics, determine whether they cross or touch the x-axis:

a $y = x^2 - 5x + 2$ **b** $y = -4x^2 + 2x - 1$ **c** $y = x^2 - 6x + 9$
d $y = 8 - 3x - 2x^2$ **e** $y = 3x^2 + 2x + 5$ **f** $y = -x^2 - x - 1$

3 By examining the discriminant, find the number of distinct solutions of:

a $x^2 + 8x + 7 = 0$ **b** $3x^2 + 8x + 7 = 0$ **c** $10x^2 - x - 3 = 0$
d $2x^2 + 8x - 7 = 0$ **e** $3x^2 - 8x - 7 = 0$ **f** $10x^2 - x + 3 = 0$

4 By examining the discriminant, state the nature and number of distinct solutions for each of the following:

a $9x^2 - 24x + 16 = 0$ **b** $-x^2 - 5x - 6 = 0$ **c** $x^2 - x - 4 = 0$
d $25x^2 - 20x + 4 = 0$ **e** $6x^2 - 3x - 2 = 0$ **f** $x^2 + 3x + 2 = 0$

Example 34

5 Find the values of m for which each of the following equations:

i has no solutions **ii** has one solution **iii** has two distinct solutions.

a $x^2 - 4mx + 20 = 0$ **b** $mx^2 - 3mx + 3 = 0$
c $5x^2 - 5mx - m = 0$ **d** $x^2 + 4mx - 4(m - 2) = 0$

Example 35

6 For m and n rational numbers show that $mx^2 + (2m + n)x + 2n = 0$ has rational solutions.

7 Find the values of p for which the equation $px^2 + 2(p + 2)x + p + 7 = 0$ has no solution.

8 Find the values of p for which the equation $(1 - 2p)x^2 + 8px - (2 + 8p) = 0$ has one solution.

9 Find the values of p for which:

a $px^2 - 6x + p = 0$ has one solution **b** $2x^2 - 4x + 3 = p$ has two solutions
c $3x^2 = 2x + p - 1$ has two solutions **d** $x^2 - 2x + 2 = p$ has two solutions.

10 Find the values of p for which the graph of $y = px^2 + 8x + p - 6$ crosses the x-axis.

11 Show that the equation $(p^2 + 1)x^2 + 2pqx + q^2 = 0$ has no real solution for any values of p and q $(q \neq 0)$.

12 **a** Find the discriminant of $x^2 + 4mx + 24m - 44$.
b Show the equation $x^2 + 4mx + 24m - 44 = 0$ has a solution for all values of m.

13 **a** Find the discriminant of $4mx^2 + 4(m - 1)x + m - 2$.
b Show the equation $4mx^2 + 4(m - 1)x + m - 2 = 0$ has a solution for all values of m.

14 Find the discriminant of the equation $4x^2 + (m - 4)x - m = 0$, where m is a rational number, and hence show that the equation has rational solution(s).

15 Find the discriminant of the equation $x^2 - (m + 2n)x + 2mn = 0$, where m and n are rational numbers, and hence show that the equation has rational solution(s).

16 If both a and c are positive, what can be said about the graph of $y = ax^2 + bx - c$?

17 If a is negative and c is positive, what can be said about the graph of $y = ax^2 + bx + c$?

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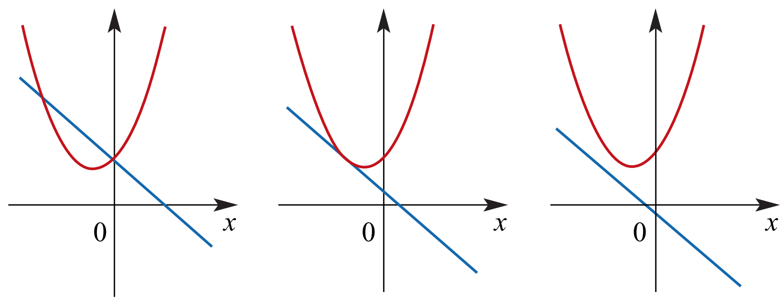
3J Solving simultaneous linear and quadratic equations

As discussed in Section 2H, when solving simultaneous linear equations we are actually finding the point of intersection of the two corresponding linear graphs.

If we wish to find the point or points of intersection between a straight line and a parabola, we can solve the equations simultaneously.

It should be noted that depending on whether the straight line intersects, touches or does not intersect the parabola we may get two, one or zero points of intersection.



Two points of intersection   One point of intersection   No point of intersection

If there is one point of intersection between the parabola and the straight line, then the line is a **tangent** to the parabola.

As we usually have the quadratic equation written with y as the subject, it is necessary to have the linear equation written with y as the subject. Then the linear expression for y can be substituted into the quadratic equation.

Example 36

Find the points of intersection of the line with equation $y = -2x + 4$ and the parabola with equation $y = x^2 - 8x + 12$.

Solution

At the point of intersection:

$$x^2 - 8x + 12 = -2x + 4$$
$$x^2 - 6x + 8 = 0$$
$$(x - 2)(x - 4) = 0$$

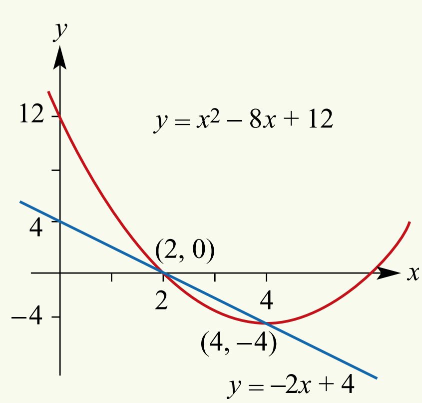
Hence $x = 2$ or $x = 4$.

When $x = 2$, $y = -2(2) + 4 = 0$.

When $x = 4$, $y = -2(4) + 4 = -4$.

Therefore the points of intersection are $(2, 0)$ and $(4, -4)$.

The result can be shown graphically.



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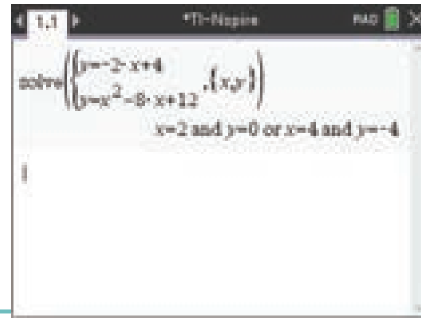
Using the TI-Nspire

To solve the simultaneous equations $y = -2x + 4$ and $y = x^2 - 8x + 12$:

- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Press (enter) to accept the default settings of two equations with variables x and y, and then complete the template as shown.

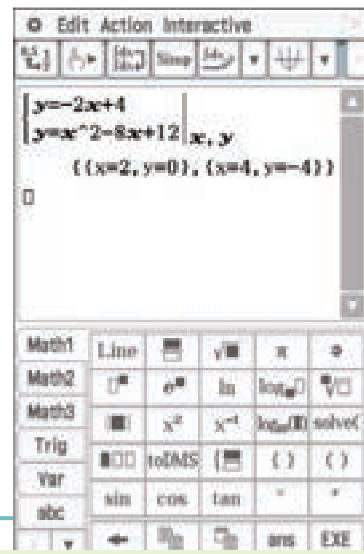


Using the Casio ClassPad

- In the Main $\sqrt{\alpha}$ screen, turn on the keyboard and select the simultaneous equations icon from (Math1).
- Enter the simultaneous equations

$$y = -2x + 4$$
$$y = x^2 - 8x + 12$$

  into the two lines, and enter x, y as the variables.
- Tap (EXE).



Example 37

Prove that the straight line with the equation $y = 1 - x$ meets the parabola with the equation $y = x^2 - 3x + 2$ once only.

Solution

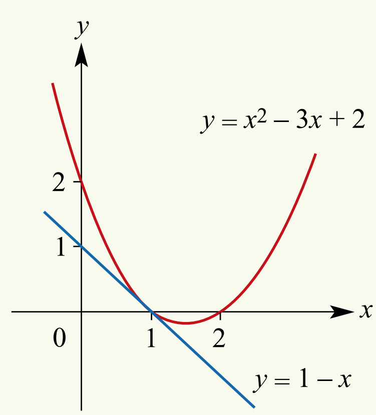
At the point of intersection:

$$x^2 - 3x + 2 = 1 - x$$
$$x^2 - 2x + 1 = 0$$
$$(x - 1)^2 = 0$$

Therefore $x = 1$ and $y = 1 - 1 = 0$.

The straight line just touches the parabola at $(1, 0)$.

This can be illustrated graphically.



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Summary 3J

To find the points of intersection of a straight line $y = mx + c_2$ and a parabola $y = ax^2 + bx + c_1$:

- Form the quadratic equation

$$ax^2 + bx + c_1 = mx + c_2$$

- Rearrange the equation so that the right-hand side is zero:

$$ax^2 + (b - m)x + (c_1 - c_2) = 0$$

- Solve the equation for x and substitute these x-values into the equation of the line to find the corresponding y-values.

The discriminant applied to the second equation, $ax^2 + (b - m)x + (c_1 - c_2) = 0$, can be used to determine the number of intersection points:

- If $\Delta > 0$, there are two intersection points.
- If $\Delta = 0$, there is one intersection point.
- If $\Delta < 0$, there are no intersection points.

Exercise 3J

Example 36

1 **a** Find the points of intersection of the line with equation $y = x - 2$ and the parabola with equation $y = x^2 - x - 6$.

b Find the points of intersection of the line with equation $x + y = 6$ and the parabola with equation $y = x^2$.

c Find the points of intersection of the line with equation $5x + 4y = 21$ and the parabola with equation $y = x^2$.

d Find the points of intersection of the line with equation $y = 2x + 1$ and the parabola with equation $y = x^2 - x + 3$.

2 Solve each of the following pairs of equations:

a $y = x^2 + 2x - 8$, $y = 2 - x$

b $y = x^2 - x - 3$, $y = 4x - 7$

c $y = x^2 + x - 5$, $y = -x - 2$

d $y = x^2 + 6x + 6$, $y = 2x + 3$

e $y = 6 - x - x^2$, $y = -2x - 2$

f $y = x^2 + x + 6$, $y = 6x + 8$

Example 37

3 Prove that, for each of the following pairs of equations, the straight line meets the parabola only once:

a $y = x^2 - 6x + 8$, $y = -2x + 4$

b $y = x^2 - 2x + 6$, $y = 4x - 3$

c $y = 2x^2 + 11x + 10$, $y = 3x + 2$

d $y = x^2 + 7x + 4$, $y = -x - 12$

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4 Solve each of the following pairs of equations:

a $y = x^2 - 6x$
$y = 8 + x$

b $y = 3x^2 + 9x$
$y = 32 - x$

c $y = 5x^2 + 9x$
$y = 12 - 2x$

d $y = -3x^2 + 32x$
$y = 32 - 3x$

e $y = 2x^2 - 12$
$y = 3(x - 4)$

f $y = 11x^2$
$y = 21 - 6x$

5 **a** Find the value of c such that $y = x + c$ is a tangent to the parabola $y = x^2 - x - 12$.
Hint: Consider the discriminant of the resulting quadratic.

b **i** Sketch the parabola with equation $y = -2x^2 - 6x + 2$.

ii Find the values of m for which the straight line $y = mx + 6$ is tangent to the parabola. Hint: Use the discriminant of the resulting quadratic.

6 Find the value(s) of a such that the line with equation $y = x$ is tangent to the parabola with equation $y = x^2 + ax + 1$.

7 Find the value of b such that the line with equation $y = -x$ is tangent to the parabola with equation $y = x^2 + x + b$.

8 Find the equation of the straight line(s) which pass through the point $(1, -2)$ and is (are) tangent to the parabola with equation $y = x^2$.

9 Solve each of the following pairs of simultaneous equations for x and y:

a $y = 4 - x^2$
$y = kx$

b $y = (x + 1)(x - 2)$
$y = kx$

c $y = 4 - x^2$
$y = kx + 5$, where $k \geq 2$ or $k \leq -2$

CAS

10 **a** Find the value of c such that the line with equation $y = 2x + c$ is tangent to the parabola with equation $y = x^2 + 3x$.

b Find the possible values of c such that the line with equation $y = 2x + c$ twice intersects the parabola with equation $y = x^2 + 3x$.

3K Families of quadratic polynomial functions

In Chapter 2 we considered the information that is necessary to determine the equation of a straight line and we also studied families of straight lines. In this section these two ideas are extended for our study of quadratic polynomials.

Here are some examples of families of quadratic polynomial functions:

$y = ax^2, \; a \neq 0$	The parabolas with their vertices at the origin.
$y = a(x - 2)^2 + 3, \; a \neq 0$	The parabolas with turning point at $(2, 3)$.
$y = a(x - 2)(x + 5), \; a \neq 0$	The parabolas with x-axis intercepts 2 and -5.
$y = a(x - h)(x - 2), \; a \neq 0$	The parabolas with x-axis intercept 2.
$y = ax^2 + bx, \; a \neq 0$ and $b \neq 0$	The parabolas with two x-axis intercepts, one of which is the origin.

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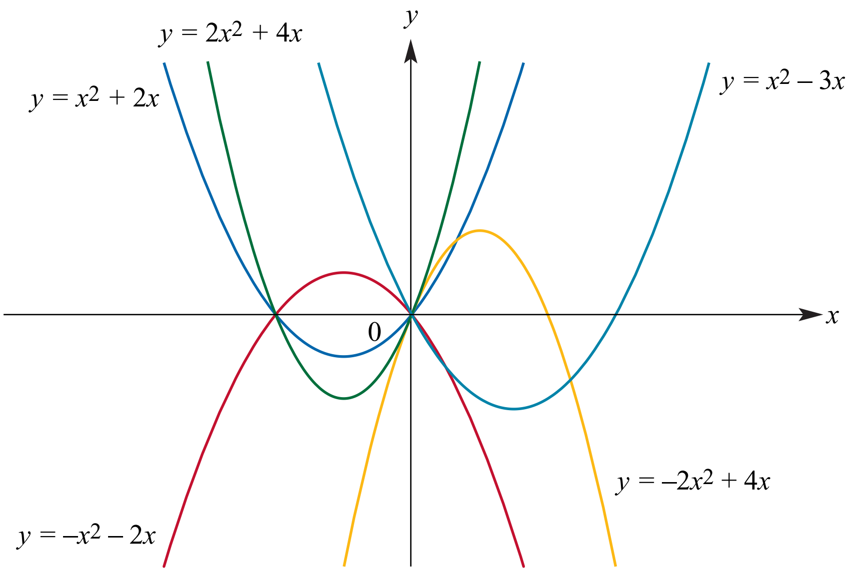
The letters a, b and h used to define these families are called parameters. Varying the values of the parameters produces different parabolas. For example, for $y = ax^2 + bx$ some possible curves are shown below.



Example 38

A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(1, 7)$ and $(2, 10)$, find the values of a and c.

Solution

When $x = 1$, $y = 7$ and when $x = 2$, $y = 10$.

$7 = a + c$ (1)

$10 = 4a + c$ (2)

Subtract (1) from (2):

$3 = 3a$

$a = 1$

Substitute in (1):

$7 = 1 + c$

$c = 6$

The equation is $y = x^2 + 6$.

Explanation

Substitute $x = 1$, $y = 7$ in the equation $y = ax^2 + c$ to obtain (1).

Substitute $x = 2$, $y = 10$ in the equation $y = ax^2 + c$ to obtain (2).

Example 39

A family of parabolas have rules of the form $y = ax^2 + bx + 2$, where $a \neq 0$.

a For a parabola in this family with its turning point on the x-axis, find a in terms of b.

b If the turning point is at $(4, 0)$, find the values of a and b.

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Solution

a The discriminant $\Delta = b^2 - 8a$.

We have $\Delta = 0$ and therefore $a = \frac{b^2}{8}$.

b We have $-\frac{b}{2a} = 4$, which implies $b = -8a$.

From part **a**, we have $a = \frac{b^2}{8}$.

Hence $a = \frac{64a^2}{8} = 8a^2$.

Thus $a(1 - 8a) = 0$ and, since $a \neq 0$, $a = \frac{1}{8}$.

Substituting for a in $b = -8a$ gives $b = -1$.

Explanation

The discriminant of $ax^2 + bx + c$ is $\Delta = b^2 - 4ac$. In this case $c = 2$.

The discriminant $\Delta = 0$ since the parabola touches the x-axis at its turning point.

The axis of symmetry has equation $x = -\frac{b}{2a}$.

Determining quadratic rules

At the beginning of this section we looked at different families of quadratic polynomial functions. We now consider three important such families which can be used as a basis for finding a quadratic rule from given information. These are certainly not the only useful forms. You will see others in the worked examples.

1 $y = a(x - e)(x - f)$ This can be used if two x-axis intercepts and the coordinates of one other point are known.

2 $y = a(x - h)^2 + k$ This can be used if the coordinates of the turning point and one other point are known.

3 $y = ax^2 + bx + c$ This can be used if the coordinates of three points on the parabola are known.

Example 40

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Find the rule for this parabola.

Solution

$y = a(x + 3)(x - 4)$

When $x = 1$, $y = 24$.

Therefore
$$24 = a(1 + 3)(1 - 4)$$
$$24 = -12a$$
$$a = -2$$

The rule is $y = -2(x + 3)(x - 4)$.

Explanation

Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

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Example 41

The coordinates of the turning point of a parabola are $(2, 6)$ and the parabola passes through the point $(3, 3)$. Find the rule for this parabola.

Solution

$y = a(x - 2)^2 + 6$

When $x = 3$, $y = 3$.

Therefore

$$3 = a(3 - 2)^2 + 6$$
$$3 = a + 6$$
$$a = -3$$

The rule is $y = -3(x - 2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x - h)^2 + k$.

Example 42

A parabola passes through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$. Find the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

$$4 = a + b + c \quad (1)$$
$$5 = c \quad (2)$$
$$10 = a - b + c \quad (3)$$

Substitute from equation (2) into equations (1) and (3):

$$-1 = a + b \quad (1')$$
$$5 = a - b \quad (3')$$

Add (1′) and (3′):

$$4 = 2a$$
$$a = 2$$

Using equation (1′), we obtain $b = -3$.

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore substitute values into the polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

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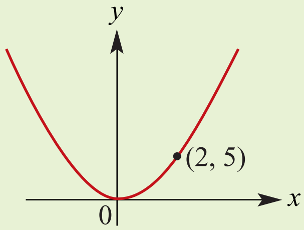
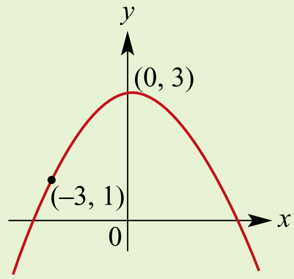
Example 43

Determine the quadratic rule for each of the following parabolas:

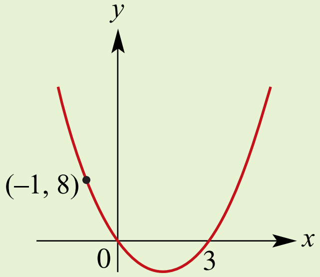
a



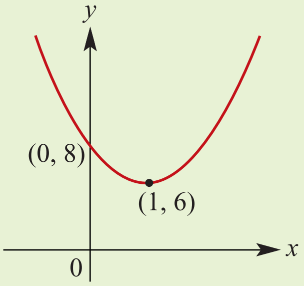
b



c



d



Solution

a This is of the form $y = ax^2$

For (2, 5): $5 = 4a$

$\therefore\ a = \dfrac{5}{4}$

Hence the rule is $y = \dfrac{5}{4}x^2$

b This is of the form $y = ax^2 + c$

For (0, 3): $3 = a(0) + c$

$\therefore\ c = 3$

For (−3, 1): $1 = a(-3)^2 + 3$

$1 = 9a + 3$

$\therefore\ a = -\dfrac{2}{9}$

Hence the rule is $y = -\dfrac{2}{9}x^2 + 3$

c This is of the form $y = ax(x - 3)$

For (−1, 8): $8 = -a(-1 - 3)$

$8 = 4a$

$\therefore\ a = 2$

Hence the rule is $y = 2x(x - 3)$

$y = 2x^2 - 6x$

d This is of the form $y = a(x - 1)^2 + 6$

For (0, 8): $8 = a + 6$

$\therefore\ a = 2$

Hence the rule is $y = 2(x - 1)^2 + 6$

$y = 2(x^2 - 2x + 1) + 6$

$y = 2x^2 - 4x + 8$

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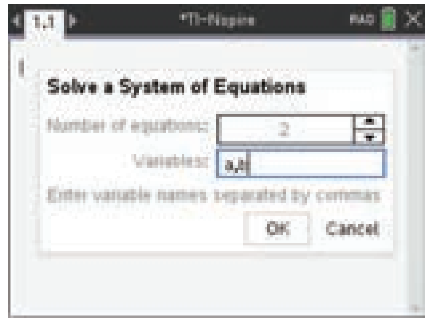
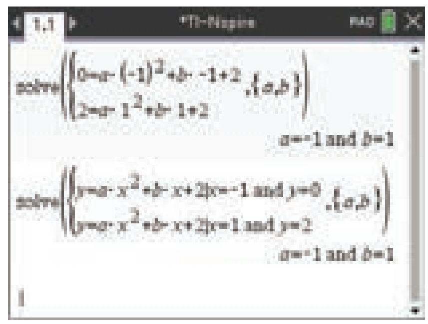
Using the TI-Nspire

The equation $y = ax^2 + bx + 2$ and the two points $(-1, 0)$ and $(1, 2)$ are used to generate equations in a and b. These equations are then solved simultaneously to find a and b.



You can either substitute the values for x, y prior to entering or substitute in the command line as shown.

Note: The 'with' or 'given' symbol | can be accessed using ctrl =.



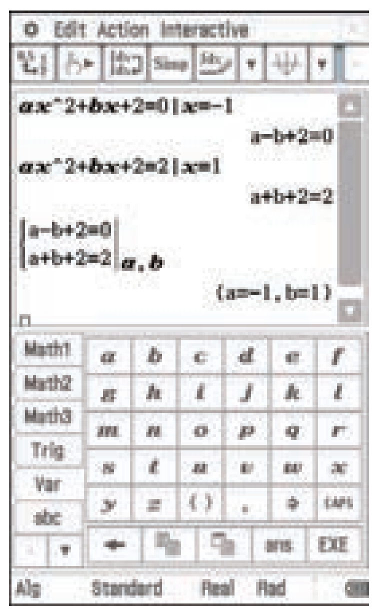
Using the Casio ClassPad

Substituting the points $(-1, 0)$ and $(1, 2)$ in the equation $y = ax^2 + bx + 2$ generates two equations in a and b.

These equations are then solved simultaneously to find a and b.

Note: To substitute $x = -1$ into $ax^2 + bx + 2 = 0$, use the symbol | found in Math3.

Remember to use the Var keyboard to enter the variables a and b.



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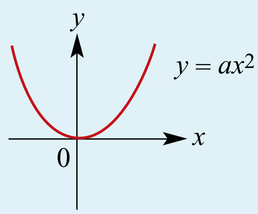
Summary 3K

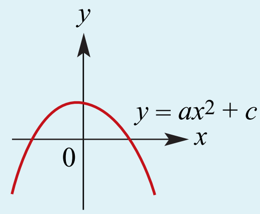
To find a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



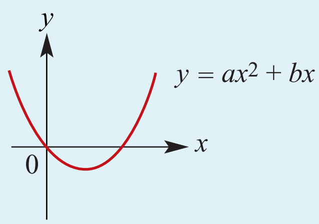
One point is needed to determine a.

ii



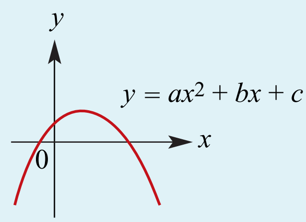
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Skill-sheet

Exercise 3K

Example 38

1 A family of parabolas have rules of the form $y = ax^2 + c$. For the parabola in this family that passes through the points $(-1, 2)$ and $(0, 6)$, find the values of a and c.

Example 39

2 A family of parabolas have rules of the form $y = ax^2 + bx + 4$, where $a \neq 0$.

a Find the discriminant of the quadratic polynomial $ax^2 + bx + 4$.

b For a parabola in this family with its turning point on the x-axis, find a in terms of b.

c If the turning point is at $(-4, 0)$, find the values of a and b.

Example 40

3 **a** A parabola has x-axis intercepts -2 and 6 and it passes through the point $(1, -30)$. Find the rule for this parabola.

Example 41

b The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(3, -46)$. Find the rule for this parabola.

Example 42

c A parabola passes through the points $(1, -2)$, $(0, -3)$ and $(-1, -6)$. Find the rule for this parabola.

4 A quadratic rule for a particular parabola is of the form $y = ax^2$. The parabola passes through the point with coordinates $(2, 8)$. Find the value of a.

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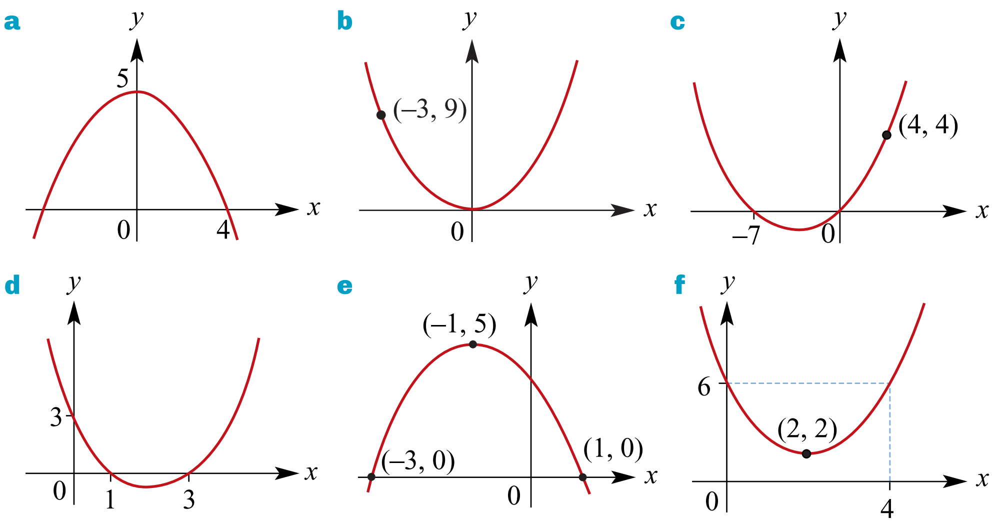
5 A quadratic rule for a particular parabola is of the form $y = ax^2 + bx$. The parabola passes through the point with coordinates $(-1, 4)$ and one of its x-axis intercepts is 6. Find the values of a and b.

6 A quadratic rule for a particular parabola is of the form $y = a(x - b)^2 + c$. The parabola has vertex $(1, 6)$ and passes through the point with coordinates $(2, 4)$. Find the values of a, b and c.

Example 43

7 Determine the equation of each of the following parabolas:

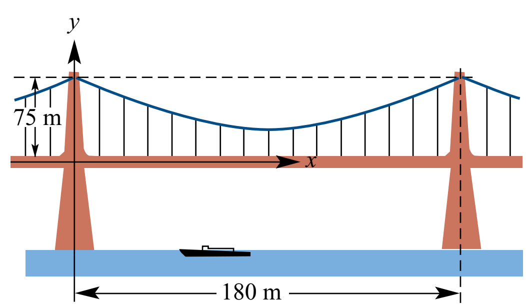


8 A parabola has vertex with coordinates $(-1, 3)$ and passes through the point with coordinates $(3, 8)$. Find the equation for the parabola.

9 A parabola has x-axis intercepts 6 and -3 and passes through the point $(1, 10)$. Find the equation of the parabola.

10 A parabola has vertex with coordinates $(-1, 3)$ and y-axis intercept 4. Find the equation for the parabola.

11 Assuming that the suspension cable shown in the diagram forms a parabola, find the rule which describes its shape. The minimum height of the cable above the roadway is 30 m.



12 A parabola has the same shape as $y = 2x^2$, but its turning point is $(1, -2)$. Write its equation.

13 A parabola has its vertex at $(1, -2)$ and passes through the point $(3, 2)$. Write its equation.

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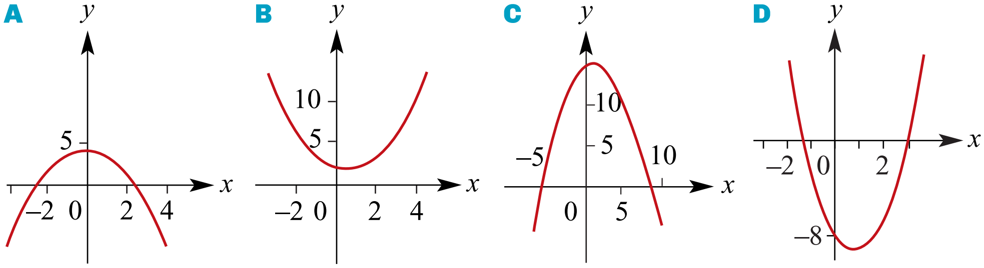
14 Which of the curves could be most nearly defined by each of the following?

a $y = \frac{1}{3}(x + 4)(8 - x)$ **b** $y = x^2 - x + 2$

c $y = -10 + 2(x - 1)^2$ **d** $y = \frac{1}{2}(9 - x^2)$



15 A family of parabolas satisfies the rule $y = ax^2 + 2x + a$.

a Express $ax^2 + 2x + a$ in the form $a(x + b)^2 + c$ for real numbers b and c.

b Give the coordinates of the turning point of the graph of $y = ax^2 + 2x + a$ in terms of a.

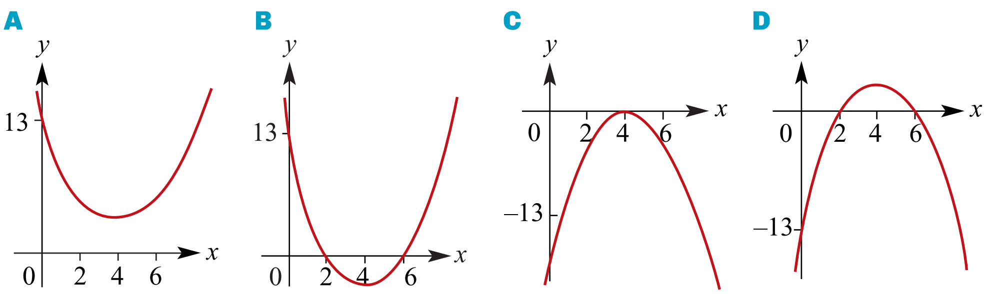
c For which values of a is $ax^2 + 2x + a$ a perfect square?

d For which values of a does the graph of $y = ax^2 + 2x + a$ have two x-axis intercepts?

16 A parabola has its vertex at $(2, 2)$ and passes through $(4, -6)$. Write its equation.

17 **a** Which of the graphs shown below could represent the equation $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?



18 Find the equation of the quadratic which passes through the points with coordinates:

a $(-2, -1), (1, 2), (3, -16)$ **b** $(-1, -2), (1, -4), (3, 10)$

19 Find the equation of the quadratic which passes through the points with coordinates: CAS

a $(-3, 5), (3, 20), (5, 57)$ **b** $(2, 3), (5, 36), (7, 78)$

20 The rate of rainfall during a storm t hours after it began was 3 mm per hour when $t = 5$, 6 mm per hour when $t = 9$ and 5 mm per hour when $t = 13$. Assuming that a quadratic model applies, find an expression for the rate of rainfall, r mm per hour, in terms of t.

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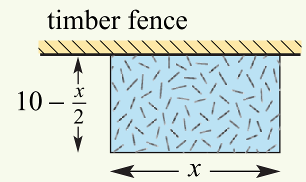
3L Quadratic models

In this section it is shown how quadratics can be used to solve worded problems, including problems which involve finding the maximum or minimum value of a quadratic polynomial that has been used to model a 'practical' situation.

Example 44

Jenny wishes to fence off a rectangular vegetable garden in her backyard. She has 20 m of fencing wire which she will use to fence three sides of the garden, with the existing timber fence forming the fourth side. Calculate the maximum area she can enclose.

Solution



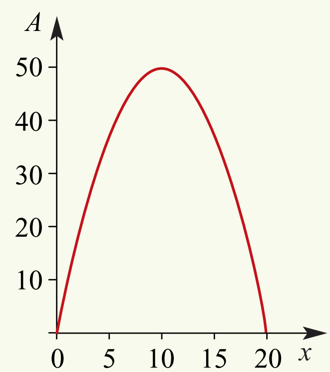
Let A = area of the rectangular garden

x = length of the garden

Then width $= \dfrac{20 - x}{2} = 10 - \dfrac{x}{2}$

Therefore

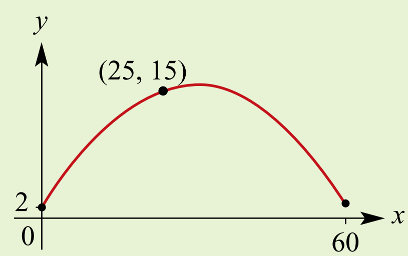
$$\begin{aligned} A &= x\left(10 - \frac{x}{2}\right) \\ &= 10x - \frac{x^2}{2} \\ &= -\tfrac{1}{2}(x^2 - 20x + 100 - 100) \quad \text{(completing the square)} \\ &= -\tfrac{1}{2}(x^2 - 20x + 100) + 50 \\ &= -\tfrac{1}{2}(x - 10)^2 + 50 \end{aligned}$$



Hence the maximum area is 50 m^2 when $x = 10$.

Example 45

A cricket ball is thrown by a fielder. It leaves his hand at a height of 2 metres above the ground and the wicketkeeper takes the ball 60 metres away again at a height of 2 metres. It is known that after the ball has gone 25 metres it is 15 metres above the ground. The path of the cricket ball is a parabola with equation $y = ax^2 + bx + c$.



a Find the values of a, b and c.

b Find the maximum height of the ball above the ground.

c Find the height of the ball when it is 5 metres horizontally before it hits the wicket-keeper's gloves.

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Solution

a The data can be used to obtain three equations:

$$2 = c \qquad (1)$$
$$15 = (25)^2a + 25b + c \qquad (2)$$
$$2 = (60)^2a + 60b + c \qquad (3)$$

Substitute equation (1) in equations (2) and (3):

$$13 = 625a + 25b \qquad (2')$$
$$0 = 3600a + 60b \qquad (3')$$

Simplify (3′) by dividing both sides by 60:

$$0 = 60a + b \qquad (3')$$

Multiply this by 25 and subtract from equation (2′):

$$13 = -875a$$
$$\therefore \quad a = -\frac{13}{875} \quad \text{and} \quad b = \frac{156}{175}$$

The path of the ball has equation

$$y = -\frac{13}{875}x^2 + \frac{156}{175}x + 2$$

b The maximum height occurs when $x = 30$ and $y = \frac{538}{35}$.

$\therefore$ maximum height is $\frac{538}{35}$ m.

c When $x = 55$, $y = \frac{213}{35}$.

$\therefore$ height of the ball is $\frac{213}{35}$ m.

Skill-sheet

Exercise 3L

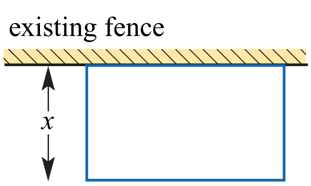
Example 44

1 A farmer has 60 m of fencing with which to construct three sides of a rectangular yard connected to an existing fence.

a If the width of the yard is x m and the area inside the yard is A m^2, write down the rule connecting A and x.

b Sketch the graph of A against x.

c Determine the maximum area that can be formed for the yard.



2 A rectangle has a perimeter of 20 m. Let x m be the length of one side. Find a formula for the area A of the rectangle in terms of x. Hence find the maximum area A.

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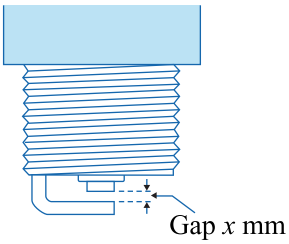
3 The efficiency rating, E, of a particular spark plug when the gap is set at x mm is said to be $400(x - x^2)$.



a Sketch the graph of E against x for $0 \le x \le 1$.

b What values of x give a zero efficiency rating?

c What value of x gives the maximum efficiency rating?

d Use the graph, or otherwise, to determine the values of x between which the efficiency rating is 70 or more.

4 A piece of wire 68 cm in length is bent into the shape of a rectangle.

a If x cm is the length of the rectangle and A cm^2 is the area enclosed by the rectangular shape, write down a formula which connects A and x.

b Sketch the graph of A against x for suitable x-values.

c Use your graph to determine the maximum area formed.

Example 45

5 A cricketer struck a cricket ball such that its height, d metres, after it had travelled x metres horizontally was given by the rule $d = 1 + \frac{3}{5}x - \frac{1}{50}x^2$, $x \ge 0$.

CAS

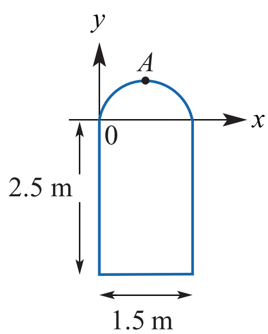
a Use a CAS calculator to graph d against x for values of x ranging from 0 to 30.

b **i** What was the maximum height reached by the ball?

ii If a fielder caught the ball when it was 2 m above the ground, how far was the ball from where it was hit?

iii At what height was the ball when it was struck?

6 An arch on the top of a door is parabolic in shape. The point A is 3.1 m above the bottom of the door. The equation $y = ax^2 + bx + c$ can be used to describe the arch. Find the values of a, b and c.



7 It is known that the daily spending of a government department follows a quadratic model. Let t be the number of days after 1 January and s be the spending in hundreds of thousands of dollars on a particular day, where $s = at^2 + bt + c$.

t	30	150	300
s	7.2	12.5	6

a Find the values of a, b and c.

b Sketch the graph for $0 \le t \le 360$. (Use a CAS calculator.)

c Find an estimate for the spending when:

i $t = 180$ **ii** $t = 350$

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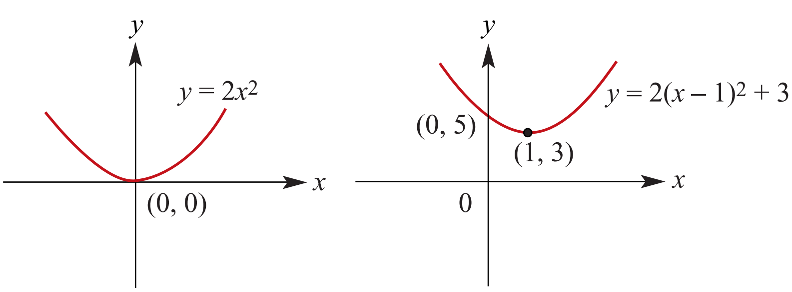
Chapter summary

Assign-ment

Nrich

- The general expression for a quadratic function is $y = ax^2 + bx + c$.
- Methods for **factorising**:
  - Taking out a common factor
    e.g. $9x^3 + 27x^2 = 9x^2(x + 3)$
  - Grouping of terms
    e.g. $x^3 + 4x^2 - 3x - 12 = (x^3 + 4x^2) - (3x + 12)$
    $= x^2(x + 4) - 3(x + 4)$
    $= (x^2 - 3)(x + 4)$
  - Difference of two squares: $x^2 - a^2 = (x + a)(x - a)$
    e.g. $16x^2 - 49 = (4x - 7)(4x + 7)$
  - Factorising quadratic expressions
    e.g. $x^2 + 2x - 8 = (x + 4)(x - 2)$
    $6x^2 - 13x - 15 = (6x + 5)(x - 3)$
- The graph of a quadratic function may be sketched by first expressing the rule in **turning point** form, $y = a(x - h)^2 + k$. The graph can then be obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis (for h, k positive).
  e.g. for $y = 2(x - 1)^2 + 3$



- A quadratic equation $ax^2 + bx + c = 0$ may be solved by:
  - Factorising
  - Completing the square
  - Using the general **quadratic formula** $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- The following steps can be used to sketch the graph of a quadratic function in **polynomial** form, $y = ax^2 + bx + c$:
  - If $a > 0$, the function has a minimum value.
  - If $a < 0$, the function has a maximum value.
  - The value of c gives the y-axis intercept.
  - The equation of the axis of symmetry is $x = -\dfrac{b}{2a}$.
  - The x-axis intercepts are determined by solving the equation $ax^2 + bx + c = 0$.

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Review

- The number of solutions of a quadratic equation $ax^2 + bx + c = 0$ can be found from the **discriminant** $\Delta = b^2 - 4ac$:
  - If $\Delta > 0$, the quadratic equation has two distinct solutions.
  - If $\Delta = 0$, the quadratic equation has one solution.
  - If $\Delta < 0$, the quadratic equation has no real solutions.
- To find a quadratic rule to fit given points, choose an appropriate form. For example:

$y = a(x - e)(x - f)$	This can be used if two x-axis intercepts and the coordinates of one other point are known.
$y = a(x - h)^2 + k$	This can be used if the coordinates of the turning point and one other point are known.
$y = ax^2 + bx + c$	This can be used if the coordinates of three points on the parabola are known.

Technology-free questions

1 Express each of the following in the form $(ax + b)^2$:

a $x^2 + 9x + \frac{81}{4}$ **b** $x^2 + 18x + 81$ **c** $x^2 - \frac{4}{5}x + \frac{4}{25}$

d $x^2 + 2bx + b^2$ **e** $9x^2 - 6x + 1$ **f** $25x^2 + 20x + 4$

2 Expand each of the following products:

a $-3(x - 2)$ **b** $-a(x - a)$

c $(7a - b)(7a + b)$ **d** $(x + 3)(x - 4)$

e $(2x + 3)(x - 4)$ **f** $(x + y)(x - y)$

g $(a - b)(a^2 + ab + b^2)$ **h** $(2x + 2y)(3x + y)$

i $(3a + 1)(a - 2)$ **j** $(x + y)^2 - (x - y)^2$

k $u(v + 2) + 2v(1 - u)$ **l** $(3x + 2)(x - 4) + (4 - x)(6x - 1)$

3 Express each of the following as a product of factors:

a $4x - 8$ **b** $3x^2 + 8x$ **c** $24ax - 3x$

d $4 - x^2$ **e** $au + 2av + 3aw$ **f** $4a^2b^2 - 9a^4$

g $1 - 36x^2a^2$ **h** $x^2 + x - 12$ **i** $x^2 + x - 2$

j $2x^2 + 3x - 2$ **k** $6x^2 + 7x + 2$ **l** $3x^2 - 8x - 3$

m $3x^2 + x - 2$ **n** $6a^2 - a - 2$ **o** $6x^2 - 7x + 2$

4 Solve each of the following equations for x by first factorising:

a $x^2 - 2x - 15 = 0$ **b** $x^2 - 9x = 0$ **c** $2x^2 - 10x + 12 = 0$

d $x^2 - 24x - 25 = 0$ **e** $3x^2 + 15x + 18 = 0$ **f** $x^2 - 12x + 36 = 0$

g $2x^2 - 5x - 3 = 0$ **h** $12x^2 - 8x - 15 = 0$ **i** $5x^2 + 7x - 12 = 0$

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5 Sketch the graphs of each of the following:

a $y = 2x^2 + 3$ **b** $y = -2x^2 + 3$ **c** $y = 2(x-2)^2 + 3$
d $y = 2(x+2)^2 + 3$ **e** $y = 2(x-4)^2 - 3$ **f** $y = 9 - 4x^2$
g $y = 3(x-2)^2$ **h** $y = 2(2-x)^2 + 3$

6 Express in the form $y = a(x-h)^2 + k$ and hence sketch the graphs of the following:

a $y = x^2 - 4x - 5$ **b** $y = x^2 - 6x$ **c** $y = x^2 - 8x + 4$
d $y = 2x^2 + 8x - 4$ **e** $y = -3x^2 - 12x + 9$ **f** $y = -x^2 + 4x + 5$

7 For each of the following, find

i the axis intercepts
ii the axis of symmetry
iii the turning point

and hence sketch the graph:

a $y = x^2 - 7x + 6$ **b** $y = -x^2 - x + 12$ **c** $y = -x^2 + 5x + 14$
d $y = x^2 - 10x + 16$ **e** $y = 2x^2 + x - 15$ **f** $y = 6x^2 - 13x - 5$
g $y = 9x^2 - 16$ **h** $y = 4x^2 - 25$

8 Find the value(s) of p that will make the quadratic $(5p-1)x^2 - 4x + (2p-1)$ a perfect square.

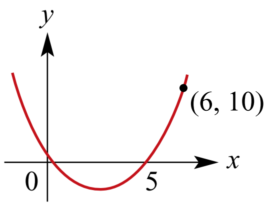
9 Solve the following quadratic inequalities:

a $x^2 > x$ **b** $(x+2)^2 \le 34$ **c** $3x^2 + 5x - 2 \le 0$
d $-2x^2 + 13x \ge 15$

10 Use the quadratic formula to solve each of the following:

a $x^2 + 6x + 3 = 0$ **b** $x^2 + 9x + 12 = 0$ **c** $x^2 - 4x + 2 = 0$
d $2x^2 + 7x + 2 = 0$ **e** $2x^2 + 7x + 4 = 0$ **f** $3x^2 + 9x - 1 = 0$

11 Find the equation of the quadratic, the graph of which is shown.



12 A parabola has the same shape as $y = 3x^2$ but its vertex is at $(5, 2)$. Find the equation corresponding to this parabola.

13 Find the values of m if $(2m-3)x^2 + (5m-1)x + (3m-2) = 0$ has two solutions.

14 Two numbers have a sum of 30. Find the maximum value of the product of such numbers.

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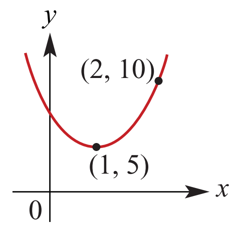
Review

15 Find the rule of the quadratic function which describes the graph.



16 Find the coordinates of the points of intersection of the graphs with equations:

a $y = 2x + 3$ and $y = x^2$

b $y = 8x + 11$ and $y = 2x^2$

c $y = 3x^2 + 7x$ and $y = 2$

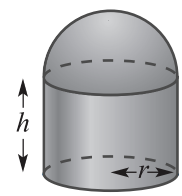
d $y = 2x^2$ and $y = 2 - 3x$

17 **a** A parabola has x-axis intercepts -4 and 1 and it passes through the point $(-1, -12)$. Find the rule for this parabola.

b The coordinates of the turning point of a parabola are $(-1, 3)$ and the parabola passes through the point $(1, -5)$. Find the rule for this parabola.

c A parabola passes through the points $(1, -3)$, $(0, -3)$ and $(-1, 1)$. Find the rule for this parabola.

18 The surface area, S, of a cylindrical tank with a hemispherical top is given by the formula $S = ar^2 + brh$, where $a = 9.42$ and $b = 6.28$. What is the radius of a tank of height 6 m which has a surface area of 125.6 m^2?



19 **a** For what value(s) of m does the equation $2x^2 + mx + 1 = 0$ have exactly one solution?

b For what values of m does the equation $x^2 - 4mx + 20 = 0$ have real solutions?

20 Consider the family of quadratics with rules of the form $y = x^2 + bx$, where b is a non-zero real number.

a Find the x-axis intercepts.

b Find the coordinates of the vertex of the parabola.

c **i** Find the coordinates of the points of intersection of the graph of $y = x^2 + bx$ with the line $y = x$, in terms of b.

ii For what value(s) of b is there one point of intersection?

iii For what value(s) of b are there two points of intersection?

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Multiple-choice questions

1 The linear factors of $12x^2 + 7x - 12$ are

A $4x - 3$ and $3x + 4$ **B** $3x - 4$ and $4x + 3$ **C** $3x - 2$ and $4x + 6$
D $3x + 2$ and $4x - 6$ **E** $6x + 4$ and $2x - 3$

2 The solutions of the equation $x^2 - 5x - 14 = 0$ are

A $x = -7$ only **B** $x = -7, x = 2$ **C** $x = -2, x = 7$
D $x = -2, x = -7$ **E** $x = -2$ only

3 For $y = 8 + 2x - x^2$, the maximum value of y is

A $-3\frac{1}{4}$ **B** $5\frac{1}{4}$ **C** 9 **D** $9\frac{1}{2}$ **E** 10

4 If the graph of $y = 2x^2 - kx + 3$ touches the x-axis, then the possible values of k are

A $k = 2$ or $k = -3$ **B** $k = 1$ **C** $k = -3$ or $k = -\frac{1}{2}$
D $k = 1$ or $k = 3$ **E** $k = 2\sqrt{6}$ or $k = -2\sqrt{6}$

5 The solutions of the equation $x^2 - 56 = x$ are

A $x = -8$ or 7 **B** $x = -7$ or 8 **C** $x = 7$ or 8
D $x = -9$ or 6 **E** $x = 9$ or -6

6 The value of the discriminant of $x^2 + 3x - 10$ is

A 5 **B** -5 **C** 49 **D** 7 **E** -2

7 The coordinates of the turning point of the graph with equation $y = 3x^2 + 6x - 1$ are

A $(\frac{1}{3}, -2)$ **B** $(-\frac{1}{3}, 2)$ **C** $(-\frac{1}{3}, -4)$ **D** $(1, -4)$ **E** $(-1, -4)$

8 The quadratic $5x^2 - 10x - 2$ in turning point form $a(x - h)^2 + k$, by completing the square, is

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$
D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

9 The value(s) of m that will give the equation $mx^2 + 6x - 3 = 0$ two solutions is (are)

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

10 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$
D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

11 The turning point of a quadratic with rule $y = x^2 - ax$ has coordinates

A $(0, a)$ **B** $\left(\frac{a}{2}, -\frac{a^2}{4}\right)$ **C** $(a, 0)$ **D** $\left(a, -\frac{a^2}{2}\right)$ **E** $\left(-\frac{a}{2}, -\frac{a^2}{4}\right)$

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12 The solution of the inequality $x^2 > b^2$, where $b < 0$, is

A $x > b$ or $x < -b$ **B** $x > b$ **C** $b < x < -b$

D $-b < x < b$ **E** $x < b$ or $x > -b$

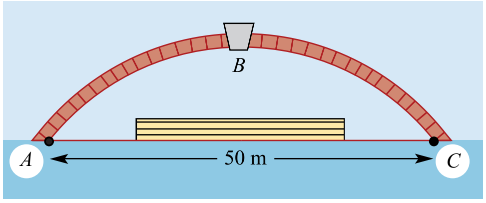
13 The quadratic equation $x^2 - 2ax + b = 0$, where a and b are positive constants, has one solution when

A $b = a$ and $a \neq 1$ **B** $b = \sqrt{a}$ and $b = -\sqrt{a}$ **C** $b = 1$ and $a \neq 1$

D $a = \sqrt{b}$ or $a = -\sqrt{b}$ **E** $b = a = 2$

Extended-response questions

1 The diagram shows a masonry arch bridge of span 50 m. The shape of the curve, ABC, is a parabola. The line AC is the water level and B is the highest point of the bridge.



a Taking A as the origin and the maximum height of the arch above the water level as 4.5 m, write down a formula for the curve of the arch where y is the height of the arch above AC and x is the horizontal distance from A.

b Calculate a table of values and accurately plot the graph of the curve.

c At what horizontal distance from A is the height of the arch above the water level equal to 3 m?

d What is the height of the arch at a horizontal distance from A of 12 m?

e A floating platform 20 m wide is towed under the bridge. What is the greatest height of the deck above water level if the platform is to be towed under the bridge with at least 30 cm horizontal clearance on either side?

2 A piece of wire 12 cm long is cut into two pieces. One piece is used to form a square shape and the other a rectangular shape in which the length is twice its width.

a If x cm is the side length of the square, write down the dimensions of the rectangle in terms of x.

b Formulate a rule for A, the combined area of the square and rectangle in cm^2, in terms of x.

c Determine the lengths of the two pieces if the sum of the areas is to be a minimum.

3 Water is pumped into an empty metal tank at a steady rate of 0.2 litres per minute. After 1 hour the depth of water in the tank is 5 cm; after 5 hours the depth is 10 cm.

a If the volume of water in the tank is V litres when the depth is x cm and there is a quadratic relationship between V and x, write down a rule which connects V and x.

b It is known that the maximum possible depth of water in the tank is 20 cm. For how long, from the beginning, can water be pumped into the tank at the same rate without overflowing?

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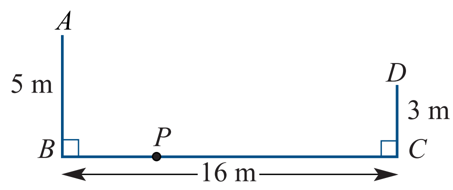
4 The point P is x m from B along the line BC.



a Find distance PA in terms of x.

b **i** Find distance PC in terms of x.

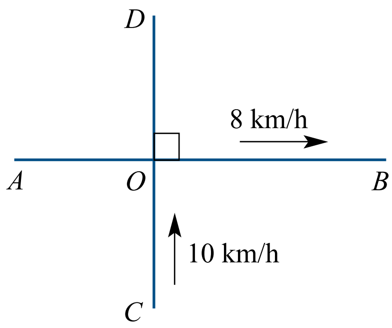
ii Find distance PD in terms of x.

c Find x if $PA = PD$.

d Find x if $PA = 2PD$. (Answer correct to three decimal places.)

e Find x if $PA = 3PD$. (Answer correct to three decimal places.)

5 AB and CD are crossroads. A jogger runs along road AB at a speed of 8 km/h and passes O at 1 p.m. Another runner is moving along road CD. The second runner is moving at 10 km/h and passes O at 1:30 p.m.



a Let y km be their distance apart t hours after 1 p.m.

i Find an expression for y in terms of t.

ii Plot the graph of y against t on a CAS calculator.

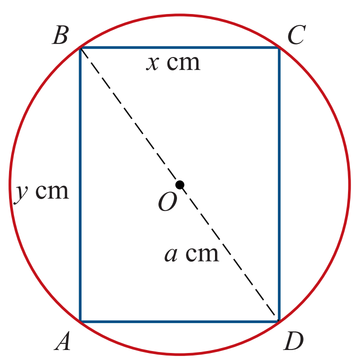
iii Find the time(s) when the runners are 4 km apart. (Use a CAS calculator.)

iv Find the time at which the runners are closest and their distance apart at this time.

b Find the exact value(s) of t for which:

i $y = 5$ **ii** $y = 6$

6 A rectangle of perimeter b cm is inscribed in a circle of radius a cm. The rectangle has width x cm and length y cm.



a Apply Pythagoras' theorem in triangle BCD to show that $x^2 + y^2 = 4a^2$.

b Form a second equation involving x, y and b.

c Eliminate y from these equations to form a quadratic equation in terms of x, a and b.

d As x, y and $2a$ are the sides of a triangle, $x + y > 2a$. Use this result and apply the discriminant to the quadratic equation formed in part **c** to show that the rectangle can be inscribed in the circle only if $4a < b \le 4\sqrt{2}a$.

e **i** If $a = 5$ and $b = 24$, find the values of x and y.

ii If $b = 4\sqrt{2}a$, find the values of x and y in terms of a.

f If $\frac{b}{a} = 5$, find the values of x and y in terms of a.

g Write a program to solve the quadratic equation found in part **c** for suitable choices of a and b and state the values of x and y. (Answers correct to two decimal places.)

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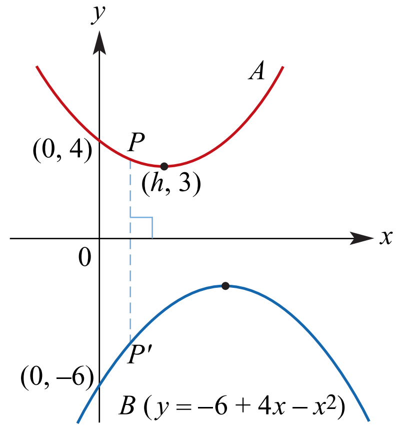
7 The equation of curve B is $y = -6 + 4x - x^2$.



a $(h, 3)$ is the vertex of a parabola A, with equation $y = x^2 + bx + c$. Find the values of b, c and h for $h > 0$.

b Let P be a point on curve A, and P' be a point on curve B such that PP' is perpendicular to the x-axis.

i The coordinates of P are $(x, x^2 + bx + c)$. State the coordinates of P' in terms of x.

ii Find the coordinates of M, the midpoint of PP', in terms of x.

iii Find the coordinates of M for $x = 0, 1, 2, 3, 4$.

iv Give the equation of the straight line on which all of these points lie. (This is called the locus of the midpoints.)

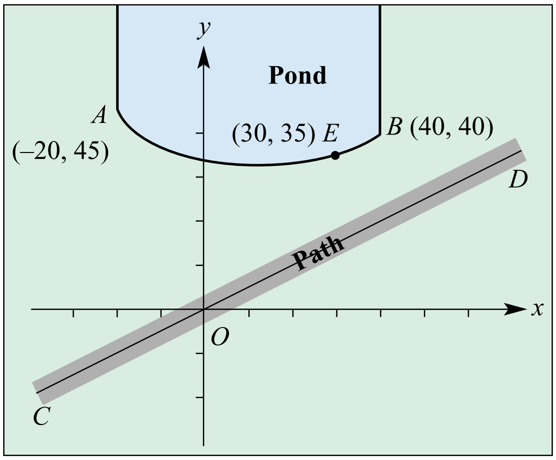
c Let d be the distance PP'.

i Express d in terms of x.

ii Sketch the graph of d against x.

iii Find the minimum value of d and the value of x for which this occurs.

8 A path cuts across a park. Its centreline can be described by the equation $y = \frac{x}{2}$, where the origin is at a point O in the park. The path starts at a point $C(-30, -15)$ and finishes at a point $D(60, 30)$.



a How long is the path?

One boundary of the pond in the park is parabolic in shape. The boundary passes through the points $A(-20, 45)$, $B(40, 40)$ and $E(30, 35)$. The equation of the parabola is of the form $y = ax^2 + bx + c$.

b **i** Find the equation of the parabola.

ii Find the coordinates of the vertex of the parabola.

c On the one set of axes sketch the graphs of $y = \frac{x}{2}$ and the parabola. (Use a CAS calculator to help.)

d Consider the rule $y = (ax^2 + bx + c) - \frac{1}{2}x$, where a, b and c have been determined in part **b i**.

i What does this expression determine?

ii Find the minimum value of this expression and the value of x for which this occurs.

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