6

Polynomials

Objectives

- To add, subtract and multiply polynomials.
- To **divide polynomials**.
- To use the **remainder theorem**, **factor theorem** and **rational-root theorem** to identify the linear factors of cubic and quartic polynomials.
- To solve **equations** and **inequalities** involving cubic and quartic polynomials.
- To recognise and sketch the graphs of **cubic and quartic functions**.
- To find the rules for given cubic graphs.
- To apply cubic functions to solving problems.
- To use the **bisection method** to solve polynomial equations numerically.

In Chapter 3 we looked at polynomial functions of degree 2, or quadratics.

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

A polynomial function of degree 4 is called a **quartic function**. The general rule for such a function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In Chapter 3 it was shown that all quadratic functions can be written in 'turning point form' and that the graph of a quadratic has one basic form, the parabola.

This is not true of cubic or quartic functions. There is a range of different graph 'shapes' for cubic and quartic functions, depending on the values of the coefficients (a, b, c, d and e).

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6A The language of polynomials

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where n is a natural number or zero, and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.

- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Note: The constant function $P(x) = 0$ is called the **zero polynomial**; its degree is undefined.

Example 1

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 2

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, find the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$P(1) = 2(1)^4 - (1)^3 + 2c + 6$
$= 2 - 1 + 2c + 6$
$= 7 + 2c$

Since $P(1) = 21$,

$7 + 2c = 21$
$c = 7$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

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b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.	First find $Q(-1)$ and $Q(2)$ in terms of a and b.
$Q(-1) = 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20$ $= 2 + 1 + a - b + 20$ $= 23 + a - b$	
$Q(2) = 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20$ $= 128 - 8 + 4a + 2b + 20$ $= 140 + 4a + 2b$	
Since $Q(-1) = Q(2) = 0$, this gives $23 + a - b = 0$ (1) $140 + 4a + 2b = 0$ (2)	Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.
Divide (2) by 2: $70 + 2a + b = 0$ (3) Add (1) and (3): $93 + 3a = 0$ $a = -31$ Substitute in (1): $b = -8$ Hence $a = -31$ and $b = -8$.	

The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

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Example 3

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a
$$\begin{aligned} & P(x) + Q(x) \\ &= x^3 - 6x + 3 + x^2 - 3x + 1 \\ &= x^3 + x^2 - 6x - 3x + 3 + 1 \\ &= x^3 + x^2 - 9x + 4 \end{aligned}$$

b
$$\begin{aligned} & P(x) - Q(x) \\ &= x^3 - 6x + 3 - (x^2 - 3x + 1) \\ &= x^3 - 6x + 3 - x^2 + 3x - 1 \\ &= x^3 - x^2 - 6x + 3x + 3 - 1 \\ &= x^3 - x^2 - 3x + 2 \end{aligned}$$

c
$$\begin{aligned} P(x)Q(x) &= (x^3 - 6x + 3)(x^2 - 3x + 1) \\ &= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1) \\ &= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3 \\ &= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3 \\ &= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3 \end{aligned}$$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\} \qquad \text{(provided } f + g \neq 0\text{)}$$
$$\deg(f \times g) = \deg(f) + \deg(g)$$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 4

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Find the values of b, c and r.

Solution

$$\begin{aligned} & (x - 2)(x^2 + bx + c) + r \\ &= x(x^2 + bx + c) - 2(x^2 + bx + c) + r \\ &= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r \\ &= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r \end{aligned}$$

Explanation

We first expand the brackets of

$$(x - 2)(x^2 + bx + c) + r$$

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We have

$x^3 + 3x^2 + 2x + 1 = x^3 + (b-2)x^2 + (c-2b)x - 2c + r$

for all real numbers x. This implies

$$b - 2 = 3 \qquad \therefore\ b = 5$$
$$c - 2b = 2 \qquad \therefore\ c = 2b + 2 = 12$$
$$-2c + r = 1 \qquad \therefore\ r = 2c + 1 = 25$$

We now equate coefficients: first the coefficients of x^2, then the coefficients of x, and finally the constants.

Hence $b = 5$, $c = 12$ and $r = 25$. This means that

$$P(x) = (x-2)(x^2 + 5x + 12) + 25$$

Substitute the values for b, c and r into $(x-2)(x^2 + bx + c) + r$.

Summary 6A

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where n is a natural number or zero, and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$. The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Skill-sheet

Exercise 6A

Example 1

1 Let $P(x) = x^3 - 3x^2 - 2x + 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

2 Let $P(x) = 8x^3 - 4x^2 - 2x + 1$. Find:

a $P\left(\frac{1}{2}\right)$ **b** $P\left(-\frac{1}{2}\right)$

3 Let $P(x) = x^3 + 4x^2 - 2x + 6$. Find:

a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 2

4 **a** Let $P(x) = x^3 + 5x^2 - ax - 20$. If $P(2) = 0$, find the value of a.

b Let $P(x) = 2x^3 + ax^2 - 5x - 7$. If $P(3) = 68$, find the value of a.

c Let $P(x) = x^4 + x^3 - 2x + c$. If $P(1) = 6$, find the value of c.

d Let $P(x) = 3x^6 - 5x^3 + ax^2 + bx + 10$. If $P(-1) = P(2) = 0$, find the values of a and b.

e Let $P(x) = x^5 - 3x^4 + ax^3 + bx^2 + 24x - 36$. If $P(3) = P(1) = 0$, find the values of a and b.

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Example 3

5 Let $f(x) = x^3 - 2x^2 + x$, $g(x) = 2 - 3x$ and $h(x) = x^2 + x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$
d $3f(x)$ **e** $f(x)g(x)$ **f** $g(x)h(x)$
g $f(x) + g(x) + h(x)$ **h** $f(x)h(x)$

6 Expand each of the following products and collect like terms:

a $(x-2)(x^2-2x+3)$ **b** $(x-4)(x^2-2x+3)$ **c** $(x-1)(2x^2-3x-4)$
d $(x-2)(x^2+bx+c)$ **e** $(2x+1)(x^2-4x-3)$

Example 4

7 It is known that $x^3 - 7x^2 + 4x + 12 = (x+1)(x^2+bx+c)$ for all values of x, for suitable values of b and c.

a Expand $(x+1)(x^2+bx+c)$ and collect like terms.
b Find b and c by equating coefficients.
c Hence write $x^3 - 7x^2 + 4x + 12$ as a product of three linear factors.

8 Let $x^2 + 6x - 2 = (x-b)^2 + c$. Find the values of b and c so that this is true for all x.

6B Division of polynomials

In order to sketch the graphs of many cubic and quartic functions (as well as higher degree polynomials) it is often necessary to find the x-axis intercepts. As with quadratics, finding x-axis intercepts can be done by factorising and then solving the resulting equation using the null factor theorem.

All cubics will have at least one x-axis intercept. Some will have two and others three.

We shall first look at the techniques for dividing one polynomial by another. One process for division of polynomials is exactly the same as the long division process for numbers.

The long-division algorithm for positive integers

We show the process for $274 \div 13$.

$$\begin{array}{r} 21 \\ 13 \overline{)\,274} \\ \underline{26} \\ 14 \\ \underline{13} \\ 1 \end{array}$$

We have

$$274 = 13 \times 21 + 1$$

In this example:

- 274 is the **dividend**
- 13 is the **divisor**
- 21 is the **quotient**
- 1 is the **remainder**.

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Division of positive integers

When we divide the number p by d we obtain two integers, q the quotient and r the remainder, such that

$$p = dq + r \quad \text{and} \quad 0 \le r < d$$

For example, dividing 27 by 4 gives

$$27 = 4 \times 6 + 3$$

Note: If $r = 0$, then d is a **factor** of p. For example, $24 = 4 \times 6$.

The long-division algorithm for polynomials

The process for dividing a polynomial by a linear polynomial follows very similar steps. For example, $(x^2 + 7x + 11) \div (x - 2)$ gives

$x + 9$	
$x - 2 \,\overline{)\, x^2 + 7x + 11}$	Divide x^2 by x. This gives x.
$\underline{x^2 - 2x}$	Multiply $x - 2$ by x and subtract from $x^2 + 7x + 11$.
$9x + 11$	This leaves $9x + 11$. Now x into $9x$ goes 9 times.
$\underline{9x - 18}$	Multiply $x - 2$ by 9 and subtract from $9x + 11$.
29	This leaves 29 remainder.

Thus $(x^2 + 7x + 11) \div (x - 2) = x + 9$ with remainder 29. We write

$$x^2 + 7x + 11 = (x - 2)(x + 9) + 29$$

We can see in this example that $x - 2$ is *not* a factor of $x^2 + 7x + 11$. We can also write the result as

$$\frac{x^2 + 7x + 11}{x - 2} = x + 9 + \frac{29}{x - 2}$$

In this example:

- $x^2 + 7x + 11$ is the dividend
- $x - 2$ is the divisor
- $x + 9$ is the quotient
- 29 is the remainder.

Division of polynomials

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

Note: If $R(x) = 0$, then $D(x)$ is a **factor** of $P(x)$. For example, let $P(x) = x^2 + 6x + 8$ and $D(x) = x + 2$. Then $P(x) = (x + 2)(x + 4) = D(x)(x + 4) + 0$.

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Example 5

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r}
x^2 - x - 12 \\
x+2 \overline{\big) x^3 + x^2 - 14x - 24} \\
\underline{x^3 + 2x^2 \qquad\qquad} \\
-x^2 - 14x - 24 \\
\underline{-x^2 - 2x \qquad} \\
-12x - 24 \\
\underline{-12x - 24} \\
0
\end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

Example 6

Divide $3x^3 + x - 3$ by $x - 2$.

Solution

$$\begin{array}{r}
3x^2 + 6x + 13 \\
x-2 \overline{\big) 3x^3 + 0x^2 + x - 3} \\
\underline{3x^3 - 6x^2 \qquad\qquad} \\
6x^2 + x - 3 \\
\underline{6x^2 - 12x \qquad} \\
13x - 3 \\
\underline{13x - 26} \\
23
\end{array}$$

Explanation

Here there is no term in x^2, however we can rewrite the polynomial as $3x^3 + 0x^2 + x - 3$.

- Divide x, from $x - 2$, into $3x^3$ to get $3x^2$.
- Multiply $3x^2$ by $x - 2$ to give $3x^3 - 6x^2$.
- Subtract from $3x^3 + 0x^2 + x - 3$, leaving $6x^2 + x - 3$.
- Now divide x, from $x - 2$, into $6x^2$ to get $6x$.
- Multiply $6x$ by $x - 2$ to give $6x^2 - 12x$.
- Subtract from $6x^2 + x - 3$, leaving $13x - 3$.
- Divide x into $13x$ to get 13.
- Multiply 13 by $x - 2$ to give $13x - 26$.
- Subtract from $13x - 3$, leaving remainder of 23.

From this example, we have

$$3x^3 + x - 3 = (x - 2)(3x^2 + 6x + 13) + 23$$

Alternatively, we can write

$$\frac{3x^3 + x - 3}{x - 2} = 3x^2 + 6x + 13 + \frac{23}{x - 2}$$

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Equating coefficients to divide

We will briefly outline how to carry out divisions by equating coefficients as shown in the first section of this chapter.

To divide $x^3 - 7x^2 + 5x - 4$ by $x - 3$, first write the identity

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 + bx + c) + r$$

We first find b, then c and finally r by equating coefficients of the left-hand side and right-hand side of this identity.

x^2 term Left-hand side: $-7x^2$. Right-hand side: $-3x^2 + bx^2 = (-3 + b)x^2$.
Therefore $-3 + b = -7$. Hence $b = -4$.

x term Left-hand side: $5x$. Right-hand side: $12x + cx = (12 + c)x$.
Therefore $12 + c = 5$. Hence $c = -7$.

constant term Left-hand side: -4. Right-hand side: $21 + r$.
Therefore $21 + r = -4$. Hence $r = -25$.

So we can write

$$x^3 - 7x^2 + 5x - 4 = (x - 3)(x^2 - 4x - 7) - 25$$

We do the following example using this method. You can see how the long division has the same arithmetic steps.

Example 7

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$\begin{array}{r|l} & \tfrac{3}{2}x^2 + \tfrac{1}{4}x - \tfrac{5}{8} \\ \hline 2x+1 & 3x^3 + 2x^2 - x - 2 \\ & 3x^3 + \tfrac{3}{2}x^2 \\ \hline & \tfrac{1}{2}x^2 - x - 2 \\ & \tfrac{1}{2}x^2 + \tfrac{1}{4}x \\ \hline & -\tfrac{5}{4}x - 2 \\ & -\tfrac{5}{4}x - \tfrac{5}{8} \\ \hline & -1\tfrac{3}{8} \end{array}$$

Alternative

For the alternative method, write the identity
$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:
$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:
$2 = a + 2b$. Therefore $b = \frac{1}{2}(2 - \frac{3}{2}) = \frac{1}{4}$.

Equate coefficients of x:
$-1 = 2c + b$. Therefore $c = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8}$.

Equate constant terms:
$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

Note: The Interactive Textbook includes an online appendix on polynomials that describes another method for division (called **synthetic division**).

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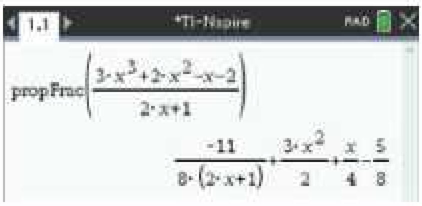
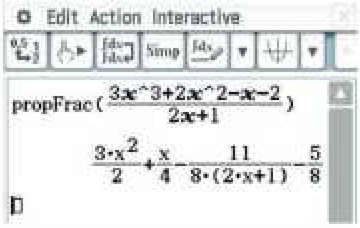
Using the TI-Nspire

Use **propFrac()** from menu > **Algebra** > **Fraction Tools** > **Proper Fraction** as shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, select the fraction template from the Math1 keyboard.
- Enter the expression $\dfrac{3x^3 + 2x^2 - x - 2}{2x + 1}$.
- Highlight the expression and go to **Interactive** > **Transformation** > **Fraction** > **propFrac**.



Dividing by a non-linear polynomial

To divide by a non-linear polynomial, use the same technique as for a linear polynomial.

Example 8

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$
\begin{array}{r l}
 & 3x - 2 \\
x^2 + 0x - 1 \,\big) & 3x^3 - 2x^2 + 3x - 4 \\
 & \underline{3x^3 + 0x^2 - 3x\phantom{{}-4}} \\
 & -2x^2 + 6x - 4 \\
 & \underline{-2x^2 + 0x + 2} \\
 & 6x - 6
\end{array}
$$

$\therefore 3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

Summary 6B

- When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
- Two methods for dividing polynomials are long division and equating coefficients. (A third method is presented in the polynomials appendix in the Interactive Textbook.)

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Exercise 6B

Example 5

1 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + x^2 - 2x + 3,\ x - 1$
b $2x^3 + x^2 - 4x + 3,\ x + 1$
c $3x^3 - 4x^2 + 2x + 1,\ x + 2$
d $2x^3 - 3x^2 + x - 2,\ x - 3$

Example 6

2 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 3x - 4,\ x + 1$
b $2x^3 + 17x + 15,\ x + 4$
c $x^3 + 4x^2 + 2,\ x + 3$
d $x^3 - 3x^2 + 6x,\ x - 2$

3 For each of the following, divide the polynomial by the accompanying linear expression and hence show that the linear expression is a factor of the polynomial:

a $x^3 - x^2 + 3x + 5,\ x + 1$
b $2x^3 + 6x^2 - 14x - 24,\ x + 4$
c $x^3 - 5x^2 + 18,\ x - 3$
d $3x^3 - 7x^2 - 4x + 12,\ x - 2$

4 Find the quotient and remainder when the first polynomial is divided by the second:

a $x^3 + 2x^2 - 3x + 1,\ x + 2$
b $x^3 - 3x^2 + 5x - 4,\ x - 5$
c $2x^3 - x^2 - 3x - 7,\ x + 1$
d $5x^3 - 3x + 7,\ x - 4$

Example 7

5 For each of the following, divide the polynomial by the accompanying linear expression:

a $x^3 + 6x^2 + 8x + 11,\ 2x + 5$
b $2x^3 + 5x^2 - 4x - 5,\ 2x + 1$

6 For each of the following, divide the cubic polynomial by the linear polynomial: CAS

a $2x^3 + 3x^2 - 32x + 15,\ 2x - 1$
b $x^3 - 3x^2 + 1,\ 3x - 1$

7 a Write $\dfrac{x^3 + 2x^2 + 5x + 1}{x - 1}$ in the form $P(x) + \dfrac{a}{x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

b Write $\dfrac{2x^3 - 2x^2 + 5x + 3}{2x - 1}$ in the form $P(x) + \dfrac{a}{2x - 1}$, where $P(x)$ is a quadratic expression and a is a real number.

Example 8

8 For each of the following, divide the polynomial $P(x)$ by the polynomial $D(x)$:

a $P(x) = 2x^3 - 6x^2 - 4x + 12,\ D(x) = x^2 - 2$
b $P(x) = x^3 - 6x^2 + x - 8,\ D(x) = x^2 + 1$
c $P(x) = 2x^3 - 6x^2 - 4x + 54,\ D(x) = x^2 - 2$
d $P(x) = x^4 - 2x^3 - 7x^2 + 7x + 5,\ D(x) = x^2 + 2x - 1$

9 For each of the following, divide the polynomial $P(x)$ by the polynomial $D(x)$: CAS

a $P(x) = x^4 - x^3 + 7x + 2,\ D(x) = x^2 + 2x - 1$
b $P(x) = 2x^4 + x^3 + 13x + 10,\ D(x) = 2x^2 - x + 4$

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6C Factorisation of polynomials

You can factorise some cubic polynomials using techniques that you already know.

- Some cubics can be factorised by first taking out a common factor. For example:

$$\begin{aligned}2x^3 - 14x^2 + 24x &= 2x(x^2 - 7x + 12)\\ &= 2x(x-3)(x-4)\end{aligned}$$

- Some cubics can be factorised by using grouping of terms. For example:

$$\begin{aligned}x^3 - 3x^2 - 2x + 6 &= x^2(x-3) - 2(x-3)\\ &= (x^2 - 2)(x-3)\\ &= (x+\sqrt{2})(x-\sqrt{2})(x-3)\end{aligned}$$

We will apply these techniques to solving cubic equations in Section 6D. In this section we develop more general techniques for factorising cubic polynomials.

Remainder theorem

Since the aim of factorising a cubic is usually to solve an equation or to find the x-axis intercepts of a graph, the first step is to establish whether a particular linear expression is a factor of the given cubic or not. It is possible to do this without actually doing the division process.

Let $P(x) = x^3 + 3x^2 + 2x + 1$.

Divide $P(x)$ by $x - 2$:

$$\begin{array}{r|l} & x^2 + 5x + 12 \\ x-2 & x^3 + 3x^2 + 2x + 1 \\ & x^3 - 2x^2 \\ \hline & \quad 5x^2 + 2x + 1 \\ & \quad 5x^2 - 10x \\ \hline & \qquad 12x + 1 \\ & \qquad 12x - 24 \\ \hline & \qquad\qquad 25 \end{array}$$

The remainder is 25.

Now evaluate $P(2)$:

$$\begin{aligned}P(2) &= (2)^3 + 3(2)^2 + 2(2) + 1\\ &= 8 + 12 + 4 + 1\\ &= 25\end{aligned}$$

The example suggests that, when $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$. This is in fact true, and the result is called the **remainder theorem**.

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The remainder theorem can be proved as follows. Suppose that, when the polynomial $P(x)$ is divided by $x-\alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x-\alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$\therefore \quad P(\alpha) = (\alpha-\alpha)Q(\alpha) + R \qquad \therefore \quad R = P(\alpha)$$

i.e. the remainder when $P(x)$ is divided by $x-\alpha$ is equal to $P(\alpha)$. We therefore have

$$P(x) = (x-\alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Example 9

Use the remainder theorem to find the value of the remainder when:

a $P(x) = x^3 - 3x^2 + 2x + 6$ is divided by $x - 2$

b $P(x) = x^3 - 2x + 4$ is divided by $2x + 1$.

Solution

a $P(2) = (2)^3 - 3(2)^2 + 2(2) + 6$
$= 8 - 12 + 4 + 6$
$= 6$

The remainder is 6.

b $P\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 - 2\left(-\frac{1}{2}\right) + 4$
$= -\frac{1}{8} + 1 + 4$
$= \frac{39}{8}$

The remainder is $\frac{39}{8}$.

Explanation

We apply the remainder theorem by evaluating $P(2)$.

We apply the remainder theorem by evaluating $P(-\frac{1}{2})$.

Note: It is not necessary to perform polynomial division to find the remainder.

Example 10

When $P(x) = x^3 + 2x + a$ is divided by $x - 2$, the remainder is 4. Find the value of a.

Solution

$P(2) = 8 + 4 + a = 4$

Therefore $a = -8$.

Explanation

We apply the remainder theorem to form a linear equation in a.

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Factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

Example 11

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $\quad P(x) = x^3 - 4x^2 + x + 6$

Then $\quad P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$= 0$

Therefore $x + 1$ is a factor of $P(x)$, by the factor theorem.

Divide $P(x)$ by $x + 1$ to find the other factor:

$$
\begin{array}{r}
x^2 - 5x + 6 \\
x + 1 \overline{\smash{)}\, x^3 - 4x^2 + x + 6} \\
\underline{x^3 + x^2 } \\
-5x^2 + x + 6 \\
\underline{-5x^2 - 5x } \\
6x + 6 \\
\underline{6x + 6} \\
0
\end{array}
$$

Hence

$$
\begin{aligned}
x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) \\
&= (x + 1)(x - 3)(x - 2)
\end{aligned}
$$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Alternative

The division can also be performed using the method of equating coefficients.

Once we have shown that $x + 1$ is a factor, we know that we can write

$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$

Equating constant terms gives $c = 6$.

Equating coefficients of x^2 gives $1 + b = -4$, and so $b = -5$.

Hence

$$
\begin{aligned}
x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) \\
&= (x + 1)(x - 3)(x - 2)
\end{aligned}
$$

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Thinking about the numbers involved in the process of factorisation gives us a way of searching for factors.

For example, consider the polynomial $x^3 - 2x^2 - 5x + 6$. Assume that this polynomial has a linear factor $x - \alpha$, where α is an integer. Then we can write

$$\begin{aligned} x^3 - 2x^2 - 5x + 6 &= (x - \alpha)(x^2 + bx + c) \\ &= x^3 - (\alpha - b)x^2 - (\alpha b - c)x - \alpha c \end{aligned}$$

By considering the constant term, it can be seen that $\alpha c = -6$. Therefore α divides 6. (Since α is an integer, it follows that b and c are too.)

Thus only the factors of 6 need be considered (i.e. $\pm 1, \pm 2, \pm 3, \pm 6$).

Try these in turn until a value for α makes $P(\alpha) = 0$. This process is completed in the following example.

Example 12

Factorise $x^3 - 2x^2 - 5x + 6$.

Solution

$P(1) = 1 - 2 - 5 + 6 = 0$

$\therefore x - 1$ is a factor.

Now divide to find the other factors:

$$\begin{array}{r|l} & x^2 - x - 6 \\ \hline x - 1 & x^3 - 2x^2 - 5x + 6 \\ & x^3 - x^2 \\ \hline & \quad -x^2 - 5x + 6 \\ & \quad -x^2 + x \\ \hline & \qquad -6x + 6 \\ & \qquad -6x + 6 \\ \hline & \qquad\qquad 0 \end{array}$$

$$\begin{aligned} \therefore x^3 - 2x^2 - 5x + 6 &= (x - 1)(x^2 - x - 6) \\ &= (x - 1)(x - 3)(x + 2) \end{aligned}$$

Explanation

The factors of 6 are $\pm 1, \pm 2, \pm 3, \pm 6$.

We evaluate the first option, $P(1)$, which in fact equals 0. If $P(1)$ did not equal 0, we would try the other factors of 6 in turn until a zero result is found.

Note: For some cubics, the quadratic factor may not be able to be factorised. Such a cubic has only one linear factor. We consider the consequences for its graph in Section 6F.

Using the TI-Nspire

Use **factor()** from [menu] > **Algebra** > **Factor** to factorise the expression $x^3 - 2x^2 - 5x + 6$.

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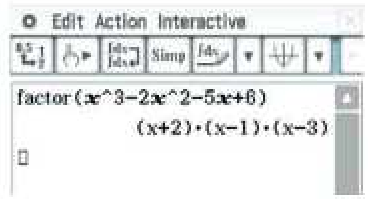
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $x^3 - 2x^2 - 5x + 6$.
- Highlight the expression and go to **Interactive** > **Transformation** > **factor**.



Rational-root theorem

Consider the cubic polynomial

$$P(x) = 2x^3 - x^2 - x - 3$$

We can easily show that

$$P(1) \neq 0, \quad P(-1) \neq 0, \quad P(3) \neq 0 \quad \text{and} \quad P(-3) \neq 0$$

Hence the equation $P(x) = 0$ has no solution that is an integer.

Does it have a rational solution, that is, a fraction for a solution?

The **rational-root theorem** helps us with this question.

The theorem tells us that, if $\beta x + \alpha$ is a factor of $2x^3 - x^2 - x - 3$, where α and β are integers with highest common factor 1 (i.e. α and β are relatively prime), then β must divide 2 and α must divide -3.

Therefore, if $-\dfrac{\alpha}{\beta}$ is a solution of the equation $P(x) = 0$ (where α and β are relatively prime), then β divides 2 and α divides -3. So the only value of β that needs to be considered is 2, and $\alpha = \pm 3$ or $\alpha = \pm 1$.

We can test these through the factor theorem. That is, check $P\left(\pm\dfrac{1}{2}\right)$ and $P\left(\pm\dfrac{3}{2}\right)$. We find

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) - 3 \\ &= 2 \times \frac{27}{8} - \frac{9}{4} - \frac{3}{2} - 3 \\ &= 0 \end{aligned}$$

We have found that $2x - 3$ is a factor.

Dividing through we find that

$$2x^3 - x^2 - x - 3 = (2x - 3)(x^2 + x + 1)$$

We can show that $x^2 + x + 1$ has no linear factors by showing that the discriminant of this quadratic is negative.

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Example 13

Factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Solution

$$P(1) = 8 \neq 0, \qquad P(-1) = -2 \neq 0,$$
$$P(5) = 580 \neq 0, \qquad P(-5) = -190 \neq 0,$$
$$P\left(-\frac{5}{3}\right) = 0$$

Therefore $3x + 5$ is a factor.

Dividing gives

$$\begin{aligned} P(x) &= 3x^3 + 8x^2 + 2x - 5 \\ &= (3x+5)(x^2+x-1) \end{aligned}$$

We complete the square for $x^2 + x - 1$ to factorise:

$$\begin{aligned} x^2 + x - 1 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 1 \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \\ &= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) \end{aligned}$$

Hence

$$P(x) = (3x+5)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Explanation

The only possible integer solutions are ± 5 or ± 1. So there are no integer solutions. We now use the rational-root theorem.

If $-\frac{\alpha}{\beta}$ is a solution, the only value of β that needs to be considered is 3 and $\alpha = \pm 5$ or $\alpha = \pm 1$.

Here is the complete statement of the theorem:

Rational-root theorem

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime).

If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

Note: A proof of the rational-root theorem is given in the polynomials appendix in the Interactive Textbook.

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Special cases: sums and differences of cubes

Example 14

Factorise $x^3 - 27$.

Solution

Let $P(x) = x^3 - 27$

Then $P(3) = 27 - 27 = 0$

Thus $x - 3$ is a factor.

Divide to find the other factor:

$$
\begin{array}{r|l}
 & x^2 + 3x + 9 \\
x - 3 & x^3 + 0x^2 + 0x - 27 \\
 & x^3 - 3x^2 \\
 & \quad 3x^2 + 0x - 27 \\
 & \quad 3x^2 - 9x \\
 & \qquad 9x - 27 \\
 & \qquad 9x - 27 \\
 & \qquad\quad 0
\end{array}
$$

Hence

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$

Alternative

The division can also be performed using the method of equating coefficients.

Let $x^3 - 27 = (x - 3)(x^2 + bx + c)$.

Equating constant terms gives $c = 9$.

Equating coefficients of x^2 gives $-3 + b = 0$, and so $b = 3$.

Hence $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.

In general, if $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 15

Factorise $8x^3 + 64$.

Solution

$$
\begin{aligned}
8x^3 + 64 &= (2x)^3 + (4)^3 \\
&= (2x + 4)(4x^2 - 8x + 16)
\end{aligned}
$$

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Summary 6C

- **Remainder theorem**
  When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- **Factor theorem**
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the process.
- **Rational-root theorem** Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Skill-sheet

Exercise 6C

Example 9

1 Without dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 - x^2 - 3x + 1,\ x - 1$
b $x^3 - 3x^2 + 4x - 1,\ x + 2$
c $2x^3 - 2x^2 + 3x + 1,\ x - 2$
d $x^3 - 2x + 3,\ x + 1$
e $x^3 + 2x - 5,\ x - 2$
f $2x^3 + 3x^2 + 3x - 2,\ x + 2$
g $6 - 5x + 9x^2 + 10x^3,\ 2x + 3$
h $10x^3 - 3x^2 + 4x - 1,\ 2x + 1$
i $108x^3 - 27x^2 - 1,\ 3x + 1$

Example 10

2 Find the value of a for each of the following:

a $x^3 + ax^2 + 3x - 5$ has remainder -3 when divided by $x - 2$
b $x^3 + x^2 - 2ax + a^2$ has remainder 8 when divided by $x - 2$
c $x^3 - 3x^2 + ax + 5$ has remainder 17 when divided by $x - 3$
d $x^3 + x^2 + ax + 8$ has remainder 0 when divided by $x - 1$

Example 11

3 Without dividing, show that the first polynomial is exactly divisible by the second polynomial:

a $x^3 - x^2 + x - 1,\ x - 1$
b $x^3 + 3x^2 - x - 3,\ x - 1$
c $2x^3 - 3x^2 - 11x + 6,\ x + 2$
d $2x^3 - 13x^2 + 27x - 18,\ 2x - 3$

4 Find the value of m if the first polynomial is exactly divisible by the second:

a $x^3 - 4x^2 + x + m,\ x - 3$
b $2x^3 - 3x^2 - (m + 1)x - 30,\ x - 5$
c $x^3 - (m + 1)x^2 - x + 30,\ x + 3$

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Example 12

5 Factorise each of the following:

a $2x^3 + x^2 - 2x - 1$
b $x^3 + 3x^2 + 3x + 1$
c $6x^3 - 13x^2 + 13x - 6$
d $x^3 - 21x + 20$
e $2x^3 + 3x^2 - 1$
f $x^3 - x^2 - x + 1$
g $4x^3 + 3x - 38$
h $4x^3 + 4x^2 - 11x - 6$

6 Find the remainder when $(1 + x)^4$ is divided by $x + 2$.

Example 13

7 Use the rational-root theorem to help factorise each of the following cubic polynomials:

a $2x^3 - 7x^2 + 16x - 15$
b $2x^3 - 3x^2 + 8x + 5$
c $2x^3 - 3x^2 - 12x - 5$
d $2x^3 - x^2 - 8x - 3$

Example 15

8 Factorise each of the following:

a $x^3 - 1$
b $x^3 + 64$
c $27x^3 - 1$
d $64x^3 - 125$
e $1 - 125x^3$
f $8 + 27x^3$
g $64m^3 - 27n^3$
h $27b^3 + 8a^3$

9 Factorise each of the following:

a $x^3 + x^2 - x + 2$
b $3x^3 - 7x^2 + 4$
c $x^3 - 4x^2 + x + 6$
d $6x^3 + 17x^2 - 4x - 3$

6C

10 Find the values of a and b and factorise the polynomial $P(x) = x^3 + ax^2 - x + b$, given that $P(x)$ is divisible by $x - 1$ and $x + 3$.

11 **a** Show that $x - a$ is a factor of $x^n - a^n$, for any constant a and any natural number n.

b Find conditions (if any) on n that are required in order that:

i $x + a$ is a factor of $x^n + a^n$

ii $x + a$ is a factor of $x^n - a^n$.

12 The polynomial $P(x)$ has a remainder of 2 when divided by $x - 1$ and a remainder of 3 when divided by $x - 2$. The remainder when $P(x)$ is divided by $(x - 1)(x - 2)$ is $ax + b$, i.e. $P(x)$ can be written as $P(x) = (x - 1)(x - 2)Q(x) + ax + b$.

a Find the values of a and b.

b **i** Given that $P(x)$ is a cubic polynomial with coefficient of x^3 being 1, and -1 is a solution of the equation $P(x) = 0$, find $P(x)$.

ii Show that the equation $P(x) = 0$ has no other real solutions.

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6D Solving cubic equations

In order to solve a cubic equation, the first step is often to factorise. We start with an example of a cubic already written in factorised form.

Example 16

Solve $(x-2)(x+1)(x+3)=0$.

Solution

Using the null factor theorem, $(x-2)(x+1)(x+3)=0$ implies

$$x-2=0 \quad \text{or} \quad x+1=0 \quad \text{or} \quad x+3=0$$

Thus the solutions are $x = 2, -1$ and -3.

Explanation

In this example, the cubic has already been factorised.

In the following example, a common factor of x is first taken out.

Example 17

Solve each of the following equations for x:

a $2x^3 - x^2 - x = 0$ **b** $x^3 + 2x^2 - 10x = 0$

Solution

a
$$2x^3 - x^2 - x = 0$$
$$x(2x^2 - x - 1) = 0$$
$$x(2x+1)(x-1) = 0$$
$$\therefore\ x = 0 \text{ or } x = -\tfrac{1}{2} \text{ or } x = 1$$

b
$$x^3 + 2x^2 - 10x = 0$$
$$x(x^2 + 2x - 10) = 0$$
$$x(x^2 + 2x + 1 - 11) = 0$$
$$x(x + 1 - \sqrt{11})(x + 1 + \sqrt{11}) = 0$$
$$\therefore\ x = 0 \text{ or } x = -1 + \sqrt{11} \text{ or } x = -1 - \sqrt{11}$$

In the following example, grouping is used to factorise.

Example 18

Solve each of the following equations for x:

a $x^3 - 4x^2 - 11x + 44 = 0$ **b** $x^3 - ax^2 - 11x + 11a = 0$

Solution

a $x^3 - 4x^2 - 11x + 44 = 0$

$x^2(x-4) - 11(x-4) = 0$

Therefore $(x-4)(x^2-11) = 0$

Hence $x = 4$ or $x = \pm\sqrt{11}$

b $x^3 - ax^2 - 11x + 11a = 0$

$x^2(x-a) - 11(x-a) = 0$

Therefore $(x-a)(x^2-11) = 0$

Hence $x = a$ or $x = \pm\sqrt{11}$

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In the following two examples, the factor theorem is used to find a linear factor.

Example 19

Solve $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

Let $P(x) = x^3 - 4x^2 - 11x + 30$

Then $P(1) = 1 - 4 - 11 + 30 \neq 0$

$P(-1) = -1 - 4 + 11 + 30 \neq 0$

$P(2) = 8 - 16 - 22 + 30 = 0$

$\therefore x - 2$ is a factor.

By division or inspection,

$$\begin{aligned} x^3 - 4x^2 - 11x + 30 &= (x-2)(x^2 - 2x - 15) \\ &= (x-2)(x-5)(x+3) \end{aligned}$$

$\therefore (x-2)(x-5)(x+3) = 0$

$\therefore x - 2 = 0$ or $x - 5 = 0$ or $x + 3 = 0$

$\therefore x = 2, 5$ or -3

Explanation

In this example we first identify a linear factor using the factor theorem.

The factorisation is completed using one of the methods given in the previous section.

Example 20

Solve $2x^3 - 5x^2 + 5x - 2 = 0$.

Solution

Let $P(x) = 2x^3 - 5x^2 + 5x - 2$

Then $P(1) = 2 - 5 + 5 - 2 = 0$

$\therefore x - 1$ is a factor.

By division or inspection,

$$2x^3 - 5x^2 + 5x - 2 = (x-1)(2x^2 - 3x + 2)$$

$\therefore (x-1)(2x^2 - 3x + 2) = 0$

$\therefore x = 1$

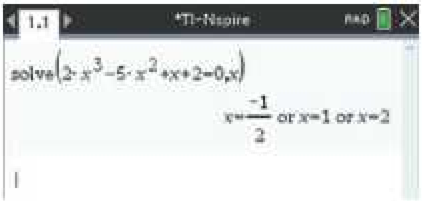
Explanation

First find a linear factor using the factor theorem. Then find the quadratic factor by division.

The discriminant of this quadratic is negative, so the quadratic cannot be factorised further. Hence there is only one linear factor and therefore only one solution.

Using the TI-Nspire

Use **solve( )** from (menu) > **Algebra** > **Solve** to solve the equation $2x^3 - 5x^2 + x + 2 = 0$.



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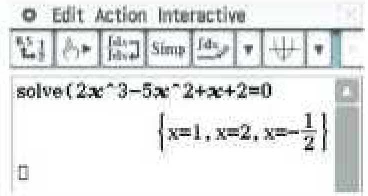
Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, select solve(from Math1 or Math3.
- Enter the equation $2x^3 - 5x^2 + x + 2 = 0$. Tap EXE.

Note: You can omit '= 0' here. When using solve(, it is not necessary to enter the right-hand side of the equation if it is zero.



Summary 6D

Cubic polynomial equations can be solved by first using an appropriate factorisation technique. Factorisation may involve:

- taking out a common factor
- using grouping of terms
- using the factor theorem
- polynomial division or equating coefficients
- sum or difference of two cubes
- using the quadratic formula to complete the factorisation.

Exercise 6D

Example 16

1 Solve each of the following:

a $(x-1)(x+2)(x-4) = 0$ **b** $(x-4)^2(x-6) = 0$

c $(2x-1)(x-3)(3x+2) = 0$ **d** $x(x+3)(2x-5) = 0$

Example 17

2 Solve each of the following:

a $x^3 - 2x^2 - 8x = 0$ **b** $x^3 + 2x^2 - 11x = 0$

c $x^3 - 3x^2 - 40x = 0$ **d** $x^3 + 2x^2 - 16x = 0$

Example 18

3 Use grouping to solve each of the following:

a $x^3 - x^2 + x - 1 = 0$ **b** $x^3 + x^2 + x + 1 = 0$

c $x^3 - 5x^2 - 10x + 50 = 0$ **d** $x^3 - ax^2 - 16x + 16a = 0$

Example 19

4 Solve each of the following:

a $x^3 - 19x + 30 = 0$ **b** $3x^3 - 4x^2 - 13x - 6 = 0$

c $x^3 - x^2 - 2x + 2 = 0$ **d** $5x^3 + 12x^2 - 36x - 16 = 0$

e $6x^3 - 5x^2 - 2x + 1 = 0$ **f** $2x^3 - 3x^2 - 29x - 30 = 0$

Example 20

5 Solve each of the following for x:

a $x^3 + x^2 - 24x + 36 = 0$ **b** $6x^3 + 13x^2 - 4 = 0$

c $x^3 - x^2 - 2x - 12 = 0$ **d** $2x^3 + 3x^2 + 7x + 6 = 0$

e $x^3 - x^2 - 5x - 3 = 0$ **f** $x^3 + x^2 - 11x - 3 = 0$

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6 Solve each of the following equations for x:

a $2x^3 = 16x$ **b** $2(x-1)^3 = 32$ **c** $x^3 + 8 = 0$

d $2x^3 + 250 = 0$ **e** $1000 = \dfrac{1}{x^3}$

7 Factorise each of the following cubic expressions using your CAS calculator:

a $2x^3 - 22x^2 - 250x + 2574$ **b** $2x^3 + 27x^2 + 52x - 33$

c $2x^3 - 9x^2 - 242x + 1089$ **d** $2x^3 + 51x^2 + 304x - 165$

6E Cubic functions of the form $f(x) = a(x-h)^3 + k$

In Chapter 3 we saw that all quadratic functions can be written in 'turning point form' and that the graphs of all quadratics have one basic form, the parabola. This is not true of cubic functions.

Let us first consider those cubics that are of the form

$$f(x) = a(x-h)^3 + k$$

The graphs of these functions can be formed by simple transformations of the graph of $f(x) = x^3$.

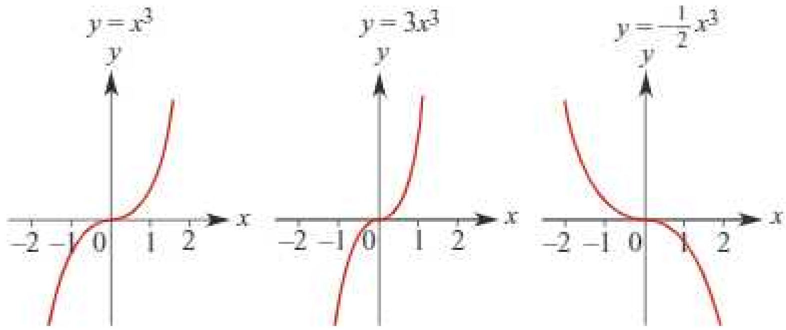
For example, the graph of $f(x) = (x-1)^3 + 3$ is obtained from the graph of $f(x) = x^3$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

Transformations of the graph of $f(x) = x^3$

Dilations from an axis and reflections in an axis

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is reflected in an axis. Note that reflecting in the x-axis and reflecting in the y-axis result in the same graph. This is because $(-x)^3 = -x^3$.

For example:



The implied **domain** of all cubics is $\mathbb{R}$ and the **range** is also $\mathbb{R}$.

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Point of inflection

The significant feature of the graph of a cubic of this form is the **point of inflection** (a point of zero gradient). This will be discussed fully in Chapter 18, but for the moment we note that it is the 'flat point' of the graph.

The point of inflection of $y = x^3$ is at the origin $(0, 0)$.

Vertical translations

By adding or subtracting a constant term to $y = x^3$, the graph moves either 'up' or 'down'.

The graph of $y = x^3 + k$ is the basic graph moved k units up (for $k > 0$). The point of inflection becomes $(0, k)$. In this case, the graph of $y = x^3$ is translated k units in the positive direction of the y-axis.

Horizontal translations

The graph of $y = (x - h)^3$ is simply the basic graph moved h units to the 'right' (for $h > 0$). The point of inflection is at $(h, 0)$. In this case, the graph of $y = x^3$ is translated h units in the positive direction of the x-axis.

General form

For the graph of a cubic function of the form

$$y = a(x - h)^3 + k$$

the point of inflection is at (h, k).

When sketching cubic graphs of the form $y = a(x - h)^3 + k$, first identify the point of inflection. To add further detail to the graph, find the x-axis and y-axis intercepts.

Example 21

Sketch the graph of the function $y = (x - 2)^3 + 4$.

Solution

The graph of $y = x^3$ is translated 2 units to the right and 4 units up.

Point of inflection is $(2, 4)$.

x-axis intercept:

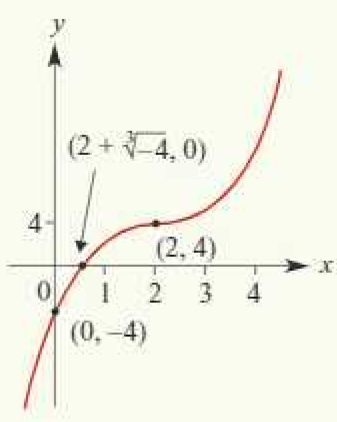
Let $y = 0$

$$0 = (x-2)^3 + 4$$
$$-4 = (x-2)^3$$
$$\sqrt[3]{-4} = x - 2$$
$$x = 2 + \sqrt[3]{-4}$$
$$\approx 0.413$$

y-axis intercept:

Let $x = 0$

$$y = (0-2)^3 + 4$$
$$y = -8 + 4$$
$$y = -4$$



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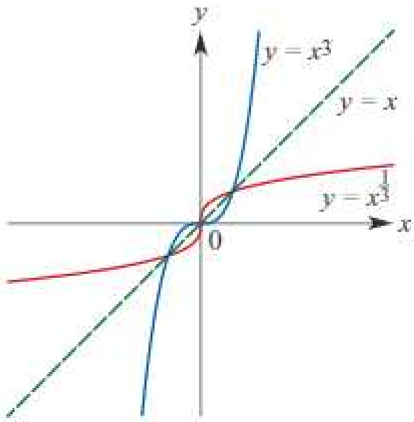
The function $f: \mathbb{R} \to \mathbb{R}, f(x) = x^{\frac{1}{3}}$

The functions with rules of the form $f(x) = a(x-h)^3 + k$ are one-to-one functions. Hence each of these functions has an inverse function.

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = x^{\frac{1}{3}}$.



The graphs of $y = x^3$ and $y = x^{\frac{1}{3}}$ are shown above.

The graph of $y = x^{\frac{1}{3}}$ is instantaneously vertical at $x = 0$. The graphs of $y = x^3$ and $y = x^{\frac{1}{3}}$ intersect at $(1, 1)$ and $(-1, -1)$.

Example 22

Sketch the graph of $y = (x-1)^{\frac{1}{3}} - 2$.

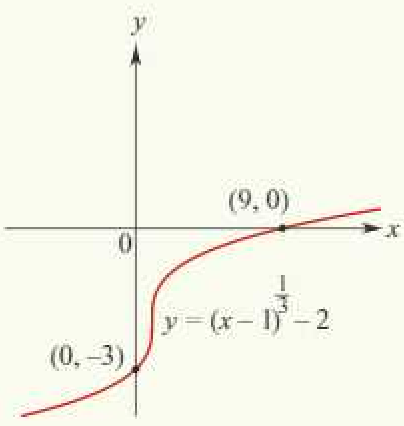
Solution

When $x = 0$,

$$y = (-1)^{\frac{1}{3}} - 2$$
$$= -1 - 2$$
$$= -3$$

When $y = 0$,

$$(x-1)^{\frac{1}{3}} - 2 = 0$$
$$(x-1)^{\frac{1}{3}} = 2$$
$$x - 1 = 2^3$$
$$x = 9$$



Explanation

Find the axis intercepts.

The graph of

$$y = (x-1)^{\frac{1}{3}} - 2$$

is the graph of $y = x^{\frac{1}{3}}$ translated 1 unit to the right and 2 units down.

We find the inverse function of a cubic of the form $y = a(x-h)^3 + k$ by using the technique introduced in Chapter 5.

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Example 23

Find the inverse function f^{-1} of $f: \mathbb{R} \to \mathbb{R}, f(x) = 2(x-4)^3 + 3$.

Solution

Interchange x and y:

$$x = 2(y-4)^3 + 3$$

Solve for y:

$$x - 3 = 2(y-4)^3$$

$$\frac{x-3}{2} = (y-4)^3$$

$$y - 4 = \left(\frac{x-3}{2}\right)^{\frac{1}{3}}$$

$$y = \left(\frac{x-3}{2}\right)^{\frac{1}{3}} + 4$$

Therefore $f^{-1}: \mathbb{R} \to \mathbb{R}, f^{-1}(x) = \left(\frac{x-3}{2}\right)^{\frac{1}{3}} + 4$

Explanation

Remember that $(x, y) \in f$ if and only if $(y, x) \in f^{-1}$.

The opposite operation to cubing is taking the cube root. That is, $\sqrt[3]{x} = x^{\frac{1}{3}}$

Summary 6E

- The graph of $y = a(x-h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants).
- The implied domain of all cubic functions is $\mathbb{R}$.
- The functions $f(x) = x^3$ and $f^{-1}(x) = x^{\frac{1}{3}}$ are inverse functions of each other.

Exercise 6E

Example 21

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x+2)^3 - 1$ **b** $y = (x-1)^3 - 1$ **c** $y = (x+3)^3 + 2$

d $y = (x-2)^3 + 5$ **e** $y = (x+2)^3 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^3 + 3$ **b** $y = 2(x-3)^3 + 2$ **c** $3y = x^3 - 5$

d $y = 3 - x^3$ **e** $y = (3-x)^3$ **f** $y = -2(x+1)^3 + 1$

g $y = \frac{1}{2}(x-3)^3 + 2$

Example 22

3 Sketch the graph of each of the following:

a $y = (x-1)^{\frac{1}{3}} - 2$ **b** $y = 2x^{\frac{1}{3}}$ **c** $y = 2(x-3)^{\frac{1}{3}} + 1$

d $y = 3(x+2)^{\frac{1}{3}} - 2$ **e** $y = -2(x-3)^{\frac{1}{3}} + 2$ **f** $y = -2(x+3)^{\frac{1}{3}} - 2$

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Example 23

4 Find the inverse function of each of the following functions:

a $f(x) = 2x^3 + 3$
b $f(x) = 3x^{\frac{1}{3}}$
c $f(x) = 2(x+1)^3 + 1$
d $f(x) = 2(x+3)^{\frac{1}{3}} - 2$
e $f(x) = -2(x-1)^{\frac{1}{3}} + 4$
f $f(x) = -2(x+2)^{\frac{1}{3}} - 1$

6F Graphs of factorised cubic functions

The general cubic function written in **polynomial form** is

$$y = ax^3 + bx^2 + cx + d$$

There is a variety of graph shapes for cubic functions, depending on the values of the coefficients. The graph of a cubic function is not necessarily a simple transformation (dilations, translations, reflections) of the graph of $y = x^3$.

All cubics have at least one x-axis intercept. We have seen that cubic functions of the form $f(x) = a(x-h)^3 + k$ have only one x-axis intercept, but these are not the only cubic functions with one x-axis intercept. Some cubic functions have two and others have three.

The y-axis intercept is easily found by letting $x = 0$, and it is the point $(0, d)$.

When sketching the graphs of cubics which are not of the form $f(x) = a(x-h)^3 + k$, begin by finding the x-axis intercepts.

In the following example, the cubic is already in factorised form.

Example 24

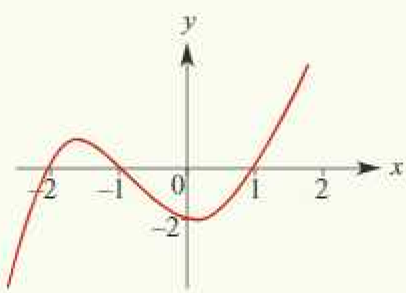
Sketch the graph of $y = (x-1)(x+2)(x+1)$. Do not give coordinates of turning points.

Solution

To find the x-axis intercepts, let $y = 0$.

Then $0 = (x-1)(x+2)(x+1)$
$\therefore x - 1 = 0$ or $x + 2 = 0$ or $x + 1 = 0$
$\therefore$ x-axis intercepts are 1, −1 and −2.

To find the y-axis intercept, let $x = 0$.
Then $y = (0-1)(0+2)(0+1) = -2$.
$\therefore$ y-axis intercept is −2.



Check the following by substituting values:

- When $x > 1$, $y > 0$.
- When $-1 < x < 1$, $y < 0$.
- When $-2 < x < -1$, $y > 0$.
- When $x < -2$, $y < 0$.

(Notice how the sign of the y-value changes from one side of an x-axis intercept to the other.) Finally, consider what happens to the graph 'beyond' the x-axis intercepts:

- For $x > 1$, $y > 0$ and as x increases y increases.
- For $x < -2$, $y < 0$ and as x decreases y decreases.

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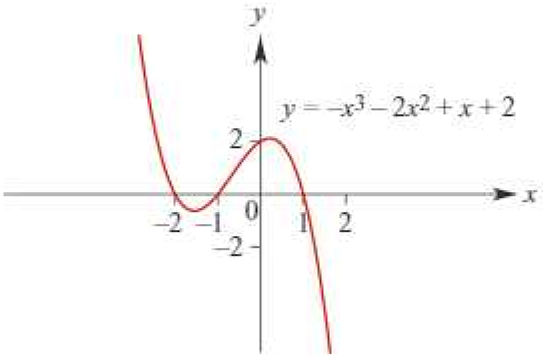
The polynomial form of the cubic in this example is $y = x^3 + 2x^2 - x - 2$. The coefficient of x^3 is positive. We now see what happens when the coefficient of x^3 is negative.

The graph of the cubic function

$$y = -x^3 - 2x^2 + x + 2$$

is the reflection in the x-axis of the graph of the cubic function considered in Example 24.



- When $x > 1$, $y < 0$.
- When $-1 < x < 1$, $y > 0$.
- When $-2 < x < -1$, $y < 0$.
- When $x < -2$, $y > 0$.

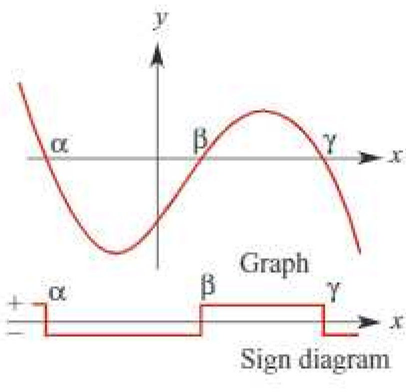
- For $x > 1$, $y < 0$ and as x increases y decreases.
- For $x < -2$, $y > 0$ and as x decreases y increases.

At this stage the location of the turning points is unspecified. However, it is important to note that, unlike quadratic graphs, the turning points are not symmetrically located between x-axis intercepts. How to determine the exact values of the coordinates of the turning points will be shown later in this book.

Sign diagrams

A sign diagram is a number-line diagram which shows when an expression is positive or negative.

The following is a sign diagram for a cubic function, the graph of which is also shown.



Using a sign diagram requires that the factors, and the x-axis intercepts, be found. The y-axis intercept and sign diagram can then be used to complete the graph.

This procedure is shown in Example 25.

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Example 25

Sketch the graph of $y = x^3 + 2x^2 - 5x - 6$.

Solution

Let $P(x) = x^3 + 2x^2 - 5x - 6$

Then $P(1) = 1 + 2 - 5 - 6 \neq 0$

$P(-1) = -1 + 2 + 5 - 6 = 0$

$\therefore$ $x + 1$ is a factor.

By division, $y = (x + 1)(x - 2)(x + 3)$.

$\therefore$ x-axis intercepts are -1, 2 and -3.

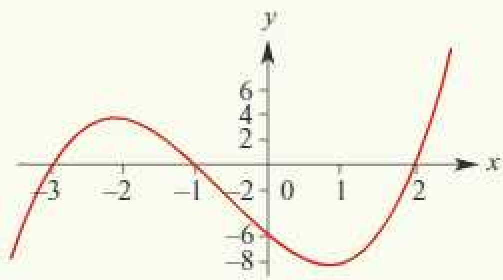
When $x < -3$, y is negative.

When $-3 < x < -1$, y is positive.

When $-1 < x < 2$, y is negative.

When $x > 2$, y is positive.

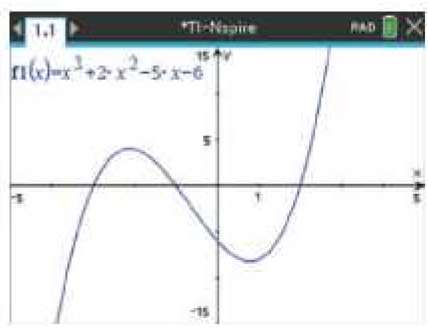
This gives the sign diagram.



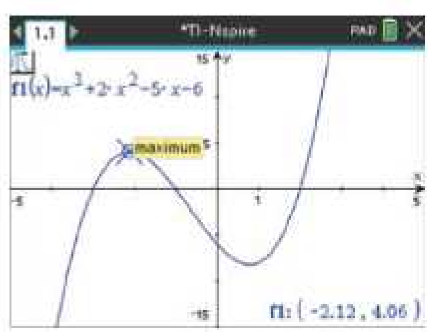
Using the TI-Nspire

To add detail to the graph, the coordinates of the turning points can be found with a CAS calculator.

- Enter $f1(x) = x^3 + 2x^2 - 5x - 6$ in a **Graphs** application.
- Choose a suitable window (menu > **Window/Zoom** > **Window Settings**).



- Use menu > **Analyze Graph** > **Maximum**.
- Move the cursor to the left of the point (lower bound), click, move to the right of the point (upper bound) and click to display the coordinates.
- Repeat for other points of interest.



Note: Alternatively, use menu > **Trace** > **Graph Trace** to find the coordinates of the two turning points. A label will appear on the screen to indicate that the calculator has found a local maximum or a local minimum.

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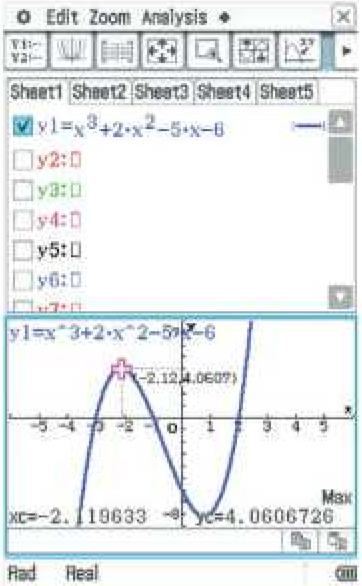
Using the Casio ClassPad

To add detail to the graph, the coordinates of the turning points can be found with a CAS calculator.

- Go to the menu, select **Graph & Table** and tap the cursor next to $y1$.
- Enter $x^3 + 2x^2 - 5x - 6$.
- Tick the box and select to produce the graph.
- Choose a suitable window using or a combination of **Zoom Out** and **Zoom Box**.
- Tap in the graph window to select it, then use **Analysis** > **G-Solve** > **Max** to find the local maximum and **Min** to find the local minimum.

Note: The maximum and minimum points must be visible on the screen before carrying out the analysis step.



Repeated factors

The polynomial function $f(x) = (x-1)^2(x+3)$ has a **repeated factor**. In this case $x - 1$ is repeated. Since the repeated factor is squared, it is easy to see that the sign of the y-value is the same 'close in' on either side of the corresponding x-axis intercept.

If the factorised cubic has a repeated factor and another linear factor, there are only two x-axis intercepts and the repeated factor corresponds to one of the turning points.

Example 26

Sketch the graph of $y = x^2(x-1)$.

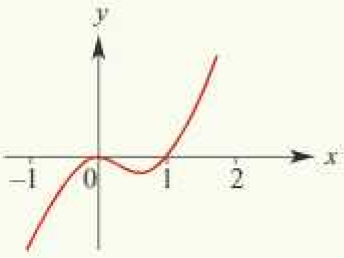
Solution

To find the x-axis intercepts, let $y = 0$.

Then $x^2(x-1) = 0$.

Thus the x-axis intercepts are at $x = 0$ and $x = 1$. Because the repeated factor is x^2, there is also a turning point at $x = 0$.

The y-axis intercept (letting $x = 0$) is at $y = 0$.



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Cubics with one x-axis intercept

Cubics of the form $y = (x - a)^3$ have only one x-axis intercept. Some other cubics also have only one x-axis intercept because, when they are factorised, they are found to have only one linear factor, with the remaining quadratic factor unable to be factorised further.

Example 27

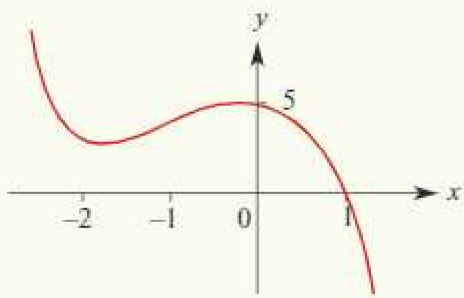
Sketch the graph of $y = -(x - 1)(x^2 + 4x + 5)$.

Solution

To find the x-axis intercept, let $y = 0$.

First, we note that the factor $x^2 + 4x + 5$ cannot be factorised further:

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4(1)(5) \\ &= -4\end{aligned}$$



∴ there are no further linear factors.

Hence, when solving the equation $-(x - 1)(x^2 + 4x + 5) = 0$, there is only one solution.

∴ x-axis intercept is $x = 1$.

To find the y-axis intercept, let $x = 0$. Then $y = -(0 - 1)(0^2 + 4(0) + 5) = 5$.

A CAS calculator can be used to find the turning points $(0, 5)$ and $(-1.82, 2.91)$, where the coordinates of the second point are given to two decimal places.

Note: At this stage of the course, you cannot determine all the features of the graph of a general cubic polynomial by hand. Further techniques for graphing cubics will be introduced in Chapter 18.

Summary 6F

- The graph of a cubic function can have one, two or three x-axis intercepts.
- If a cubic can be written as the product of three linear factors, $y = a(x-\alpha)(x-\beta)(x-\gamma)$, then its graph can be sketched by following these steps:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If the cubic has a repeated factor to the power 2, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 6F

Example 24

1 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

a $y = x(x-1)(x-3)$
b $y = (x-1)(x+1)(x+2)$
c $y = (2x-1)(x-2)(x+3)$
d $y = (x-1)(x-2)(x-3)$

Example 25

2 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

a $y = x^3 - 9x$
b $y = x^3 - 4x^2 - 3x + 18$
c $y = -x^3 + x^2 + 3x - 3$
d $y = 3x^3 - 4x^2 - 13x - 6$
e $y = 6x^3 - 5x^2 - 2x + 1$
f $y = 2x^3 - 9x^2 + 7x + 6$

Example 26

3 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.)

a $y = (x-1)(x-2)^2$
b $y = x^2(x-4)$
c $y = 2(x+1)^2(x-3)$
d $y = x^3 + x^2$
e $y = 4x^3 - 8x^2 + 5x - 1$
f $y = x^3 - 5x^2 + 7x - 3$

Example 27

CAS

4 Sketch the graph for each of the following and draw a sign diagram. Label your sketch graph showing the points of intersection with the axes. (Do not determine coordinates of turning points.) Use your calculator to help sketch each of them.

a $y = (x-1)(x^2+1)$ (**Note:** There is no turning point or 'flat point' of this cubic.)
b $y = (x^2+2)(x-4)$ (**Note:** There are two turning points.)

5 Sketch the graph for each of the following, using a CAS calculator to find the coordinates of axis intercepts and local maximum and local minimum values:

a $y = -4x^3 - 12x^2 + 37x - 15$
b $y = -4x^3 + 19x - 15$
c $y = -4x^3 + 0.8x^2 + 19.8x - 18$
d $y = 2x^3 + 11x^2 + 15x$
e $y = 2x^3 + 6x^2$
f $y = 2x^3 + 6x^2 + 6$

6 Show that the graph of f, where $f(x) = x^3 - x^2 - 5x - 3$, cuts the x-axis at one point and touches it at another. Find the values of x at these points.

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6G Solving cubic inequalities

As was done for quadratic inequalities, we can solve cubic inequalities by considering the graph of the corresponding polynomial.

Example 28

Find $\{x : x^3 + x^2 - 5x + 3 \le 0\}$.

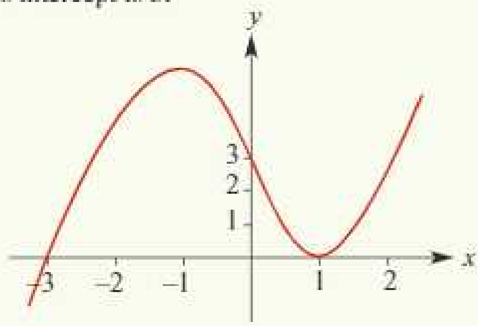
Solution

$P(x) = x^3 + x^2 - 5x + 3$

$P(1) = 1 + 1 - 5 + 3 = 0$

$\therefore x - 1$ is a factor. By division, $y = (x-1)^2(x+3)$.

There are only two x-axis intercepts, 1 and -3. The y-axis intercept is 3.



From the graph we can see that $y \le 0$ when $x \le -3$ or when $x = 1$.

$\therefore \quad \{x : x^3 + x^2 - 5x + 3 \le 0\} = (-\infty, -3] \cup \{1\}$

Explanation

- Use the factor theorem to find an initial linear factor.
- Complete the factorisation.
- Find the axis intercepts.
- Sketch the graph. (Note that this cubic has a repeated factor, and so $(1, 0)$ is a turning point as well an intercept.)
- Solve the inequality by inspecting the graph.
- Express the solution in formal set notation.

Summary 6G

Cubic inequalities can be solved by sketching the graph of the corresponding cubic function and analysing the graph.

Skill-sheet

Exercise 6G

Example 28

1 Solve the following cubic inequalities:

a $(x-1)(x+2)(x-3) \le 0$ **b** $(x+1)(x+2)(x-4) \ge 0$

c $(x-1)(x-2)^2 < 0$ **d** $x(x+2)(x-3) > 0$

e $(x-1)^3 + 8 \le 0$ **f** $x^3 - 1 \ge 0$

g $x^2(x-4) > 0$ **h** $(x+3)(x^2 + 2x + 5) \le 0$

2 Solve the following cubic inequalities. Begin by getting all of the terms on one side.

a $x^3 > 4x$ **b** $x^3 < 5x^2$ **c** $x^3 + 4x \le 4x^2$

d $x^3 > 9x$ **e** $x^3 - 6x^2 + x \ge 6$ **f** $2x^3 - 6x^2 - 4x < -12$

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6H Families of cubic polynomial functions

In Chapter 2 we considered the information that is necessary to determine the equation of a straight line. In Chapter 3 this was considered for quadratic functions, and in Chapter 4 for rectangular hyperbolas, circles and other types of relations.

Here are some examples of families of cubic polynomial functions:

$y = ax^3,\ a > 0$	The cubic graphs that are dilations from the x-axis of $y = x^3$.
$y = a(x-h)^3 + k,\ a \neq 0$	The cubic graphs that are translations of $y = ax^3$.
$y = a(x-2)(x+5)(x-4),\ a \neq 0$	The cubic graphs with x-axis intercepts 2, −5 and 4.
$y = ax^3 + bx^2 + cx,\ a \neq 0$	The cubic graphs that pass through the origin.

Recall that in this context we call a, b, c, h and k parameters.

Finding rules for cubic polynomial functions

The method used for finding the equation from the graph of a cubic will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x-h)^3 + k$ and the point of inflection (h, k) is given, then only one other point needs to be known in order to find the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

Example 29

a A cubic function has rule of the form $y = a(x-2)^3 + 2$. The point (3, 10) is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+2)(x-4)$. The point (5, 16) is on the graph of the function. Find the value of a.

Solution

a $y = a(x-2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$10 = a(3-2)^3 + 2$$
$$8 = a \times 1^3$$
$$a = 8$$

b $y = a(x-1)(x+2)(x-4)$

When $x = 5$, $y = 16$.

$$16 = a(5-1)(5+2)(5-4)$$
$$16 = 28a$$
$$a = \frac{4}{7}$$

Explanation

In each of these problems we substitute in the given values to find the unknown.

For part **a**, the coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

For part **b**, three x-axis intercepts are known and the coordinates of a fourth point are known.

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Example 30

A cubic function has rule of the form $f(x) = ax^3 + bx$. The points (1, 16) and (2, 30) are on the graph of the function. Find the values of a and b.

Solution

Since $f(1) = 16$ and $f(2) = 30$, we obtain the simultaneous equations

$$16 = a + b \qquad (1)$$

$$30 = a(2)^3 + 2b \qquad (2)$$

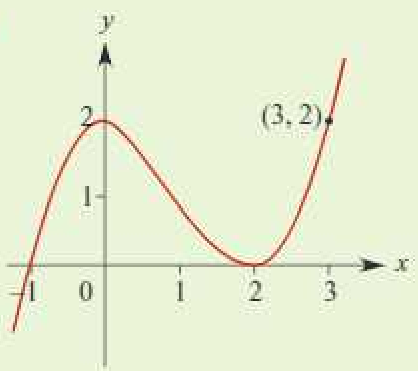
Multiply (1) by 2 and subtract from (2). This gives $-2 = 6a$ and hence $a = -\frac{1}{3}$.

Substitute in (1) to find $b = \frac{49}{3}$.

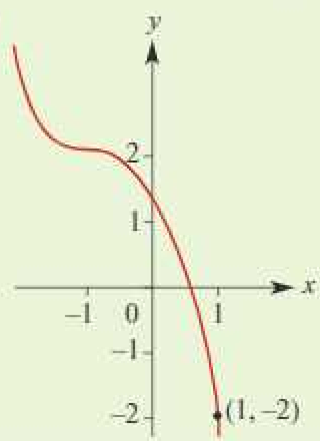
Example 31

Determine the rule for the cubic function shown in each of the following graphs:

a



b



Solution

a $y = a(x + 1)(x - 2)^2$

Put (3, 2) into the equation:

$$2 = a(4)(1)$$

$$\frac{1}{2} = a$$

The rule is $y = \frac{1}{2}(x + 1)(x - 2)^2$.

b $y = a(x + 1)^3 + 2$

To determine a, put the known point $(1, -2)$ into the equation:

$$-2 = a(2)^3 + 2$$

$$-4 = 8a$$

$$-\frac{1}{2} = a$$

The rule is $y = -\frac{1}{2}(x + 1)^3 + 2$.

Explanation

a The x-axis intercepts are -1 and 2, and the graph touches the x-axis at 2. So the cubic has a repeated factor $x - 2$.

Therefore the form of the rule appears to be $y = a(x + 1)(x - 2)^2$.

b This graph appears to be of the form $y = a(x - h)^3 + k$. The point of inflection is at $(-1, 2)$. Therefore $h = -1$ and $k = 2$.

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Example 32

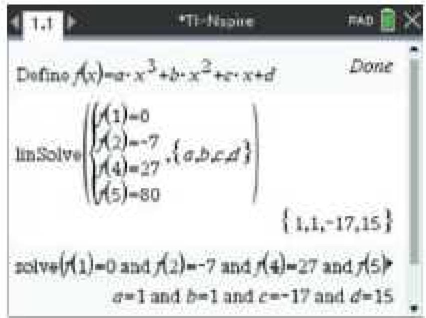
A cubic function f has rule $f(x) = ax^3 + bx^2 + cx + d$. If

$$f(1) = 0, \quad f(2) = -7, \quad f(4) = 27, \quad f(5) = 80$$

find the values of a, b, c and d.

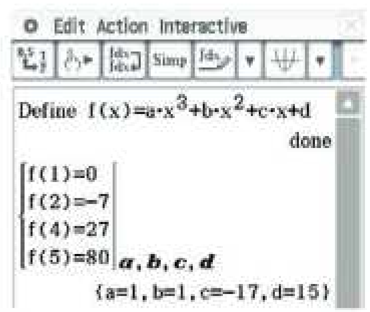
Using the TI-Nspire

- Define the function $f(x) = ax^3 + bx^2 + cx + d$.
- Use menu > **Algebra** > **Solve System of Equations** > **Solve System of Linear Equations**. Complete the pop-up screen and enter the equations as shown to find a, b, c and d.
- An alternative method is also shown.



Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter the expression $ax^3 + bx^2 + cx + d$. (Remember to enter the variables a, b, c and d using the Var keyboard.)
- Highlight the expression and go to **Interactive** > **Define**.
- In the Math1 keyboard, tap the simultaneous equations icon. Tap it twice more to expand for four simultaneous equations.



- Enter the known values $f(1) = 0$, $f(2) = -7$, $f(4) = 27$ and $f(5) = 80$ into the four lines and enter the variables a, b, c, d in the bottom right separated by commas. (You can enter the function name f using either the abc or the Math3 keyboard.)

Summary 6H

The rule of a cubic function can be determined if:

- the coordinates of four points on the graph are known
- the form of the function is known to be $f(x) = a(x - \alpha)^2(x - \beta)$, and α and β and the coordinates of one other point are known
- the form of the function is known to be $f(x) = a(x - h)^3 + k$, and the coordinates of the inflection point (h, k) and one other point are known.

There are other sets of information which can be used to determine the rule of a cubic function and more of these will be given in Chapter 18.

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Skill-sheet

Exercise 6H

Example 29a

1 **a** A cubic function has rule of the form $y = a(x - 3)^3 + 1$. The point $(4, 12)$ is on the graph of the function. Find the value of a.

Example 29b

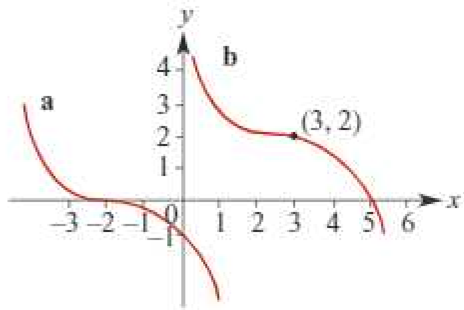
b A cubic function has rule of the form $y = a(x - 2)(x + 3)(x - 1)$. The point $(3, 24)$ is on the graph of the function. Find the value of a.

Example 30

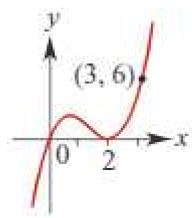
c A cubic function has rule of the form $y = ax^3 + bx$. The points $(1, 16)$ and $(2, 40)$ are on the graph of the function. Find the values of a and b.

Example 31

2 The graphs shown are similar to the basic curve $y = -x^3$. Find possible cubic functions which define each of the curves.



3 Find the equation of the cubic function for which the graph is shown.



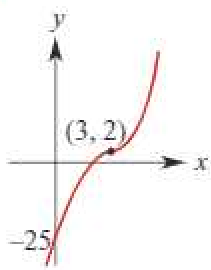
4 Find a cubic function whose graph touches the x-axis at $x = -4$, cuts it at the origin, and has a value 6 when $x = -3$.

5 The graph of a cubic function has x-axis intercepts 1, 3 and -1 and y-axis intercept -6. Find the rule for this cubic function.

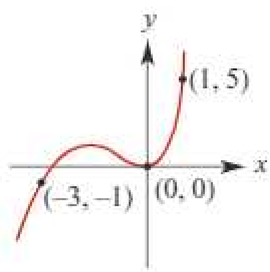
6 A cubic function f has rule $f(x) = (x - 3)(x^2 + a)$ and $f(6) = 216$. Find the value of a.

7 The graphs below have equations of the form shown. In each case, determine the equation.

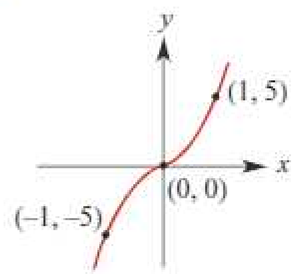
a $y = a(x - h)^3 + k$



b $y = ax^3 + bx^2$



c $y = ax^3$



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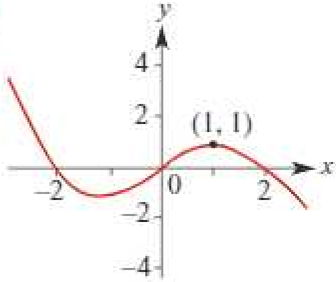
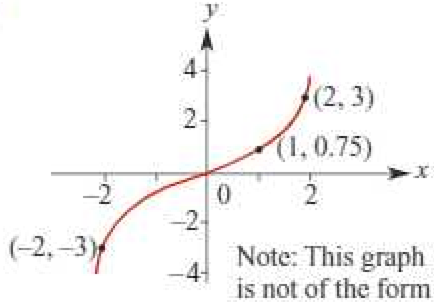
8 Find the expressions which define the following cubic curves:

a



b



Note: This graph is not of the form $y = a(x-h)^3 + k$.

Example 32

9 For each of the following, use a CAS calculator to find the values of a, b, c, d in the cubic equation $y = ax^3 + bx^2 + cx + d$, given that the following points lie on its graph:

a (0, 270), (1, 312), (2, 230), (3, 0)
b (−2, −406), (0, 26), (1, 50), (2, −22)
c (−2, −32), (2, 8), (3, 23), (8, 428)
d (1, −1), (2, 10), (3, 45), (4, 116)
e (−3, −74), (−2, −23), (−1, −2), (1, −2)
f (−3, −47), (−2, −15), (1, −3), (2, −7)
g (−4, 25), (−3, 7), (−2, 1), (1, −5)

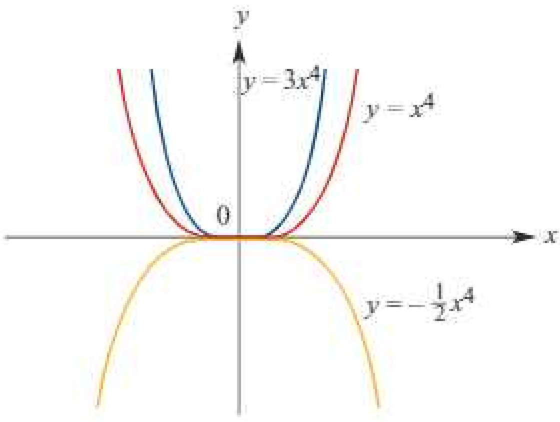
6I Quartic and other polynomial functions

In this section we look at polynomial functions of degree 4 and greater.

Quartic functions of the form $f(x) = a(x-h)^4 + k$

The graph of $f(x) = (x-1)^4 + 3$ is obtained from the graph of $y = x^4$ by a translation of 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

As with other graphs it has been seen that changing the value of a simply narrows or broadens the graph without changing its fundamental shape. Again, if $a < 0$, the graph is inverted.



The significant feature of the graph of a quartic of this form is the **turning point** (a point of zero gradient). The turning point of $y = x^4$ is at the origin (0, 0).

For the graph of a quartic function of the form

$$y = a(x-h)^4 + k$$

the turning point is at (h, k).

When sketching quartic graphs of the form $y = a(x-h)^4 + k$, first identify the turning point. To add further detail to the graph, find the x-axis and y-axis intercepts.

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Example 33

Sketch the graph of the function $y = (x - 2)^4 - 1$.

Solution

Turning point is $(2, -1)$.

x-axis intercepts:

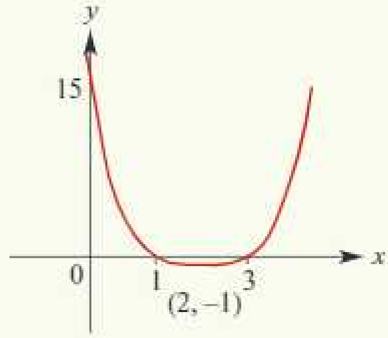
Let $y = 0$

$$0 = (x - 2)^4 - 1$$
$$1 = (x - 2)^4$$
$$\pm\sqrt[4]{1} = x - 2$$
$$x = 2 + 1 \text{ or } x = 2 - 1$$
$$x = 3 \text{ or } x = 1$$

y-axis intercept:

Let $x = 0$

$$y = (0 - 2)^4 - 1$$
$$= 16 - 1$$
$$= 15$$

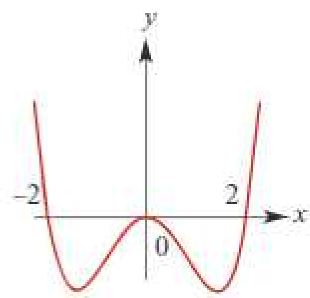


The implied **domain** of all quartics is $\mathbb{R}$, but unlike cubics the range is not $\mathbb{R}$.

Other quartic functions

The techniques for graphing quartic functions in general are very similar to those employed for cubic functions. A CAS calculator is to be used in the graphing of these functions. Great care needs to be taken in this process as it is easy to miss key points on the graph using these techniques.

The graph of $y = 2x^4 - 8x^2$ is shown.



Example 34

Solve each of the following equations for x:

a $x^4 - 8x = 0$ **b** $2x^4 - 8x^2 = 0$ **c** $x^4 - 2x^3 - 24x^2 = 0$

Solution

a $x^4 - 8x = 0$

Factorise to obtain

$x(x^3 - 8) = 0$

$\therefore x = 0 \text{ or } x^3 - 8 = 0$

Thus $x = 0$ or $x = 2$.

b $2x^4 - 8x^2 = 0$

Factorise to obtain

$2x^2(x^2 - 4) = 0$

$\therefore 2x^2 = 0 \text{ or } x^2 - 4 = 0$

Thus $x = 0$ or $x = 2$ or $x = -2$.

c $x^4 - 2x^3 - 24x^2 = 0$

Factorise to obtain $x^2(x^2 - 2x - 24) = 0$

$\therefore x^2 = 0 \text{ or } x^2 - 2x - 24 = 0$

i.e. $x = 0$ or $(x - 6)(x + 4) = 0$

Thus $x = 0$ or $x = 6$ or $x = -4$.

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Odd and even polynomials

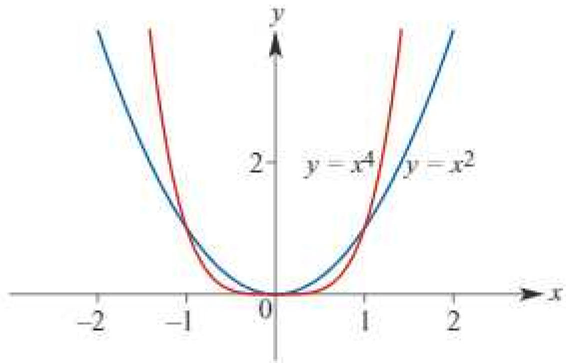
In this subsection we look briefly at odd and even polynomial functions. Knowing that a function is even or that it is odd is very helpful when sketching its graph.

- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis. That is, the graph appears the same after reflection in the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number. In this book we focus on the cases where r is a positive integer or $r \in \{-2, -1, \frac{1}{2}, \frac{1}{3}\}$.

Even-degree power functions

The functions with rules $f(x) = x^2$ and $f(x) = x^4$ are examples of even-degree power functions.



The following are properties of all even-degree power functions:

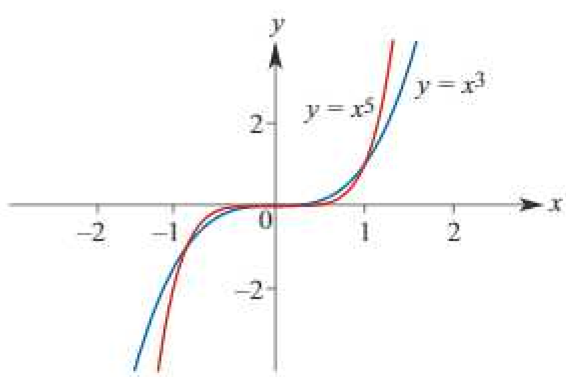
- $f(-x) = f(x)$ for all x
- $f(0) = 0$
- As $x \to \pm\infty$, $y \to \infty$.

Note that, if m and n are positive even integers with $m > n$, then:

- $x^m > x^n$ for $x > 1$ or $x < -1$
- $x^m < x^n$ for $-1 < x < 1$ with $x \neq 0$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

Odd-degree power functions

The functions with rules $f(x) = x^3$ and $f(x) = x^5$ are examples of odd-degree power functions.



The following are properties of all odd-degree power functions:

- $f(-x) = -f(x)$ for all x
- $f(0) = 0$
- As $x \to \infty$, $y \to \infty$ and as $x \to -\infty$, $y \to -\infty$.

Note that, if m and n are positive odd integers with $m > n$, then:

- $x^m > x^n$ for $x > 1$ or $-1 < x < 0$
- $x^m < x^n$ for $x < -1$ or $0 < x < 1$
- $x^m = x^n$ for $x = 1$ or $x = -1$ or $x = 0$.

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Example 35

State whether each of the following polynomials is even or odd:

a $f(x) = 6x^4 - 3x^2$

b $g(x) = 3x^5 - x^3 + x$

c $h(x) = x^6 - 3x^2 + 2$

d $m(x) = x^7 - 4x$

Solution

a $f(-x) = 6(-x)^4 - 3(-x)^2$
$= 6x^4 - 3x^2$
$= f(x)$

The function is even.

b $g(-x) = 3(-x)^5 - (-x)^3 + (-x)$
$= -3x^5 + x^3 - x$
$= -g(x)$

The function is odd.

c $h(-x) = (-x)^6 - 3(-x)^2 + 2$
$= x^6 - 3x^2 + 2$
$= h(x)$

The function is even.

d $m(-x) = (-x)^7 - 4(-x)$
$= -x^7 + 4x$
$= -m(x)$

The function is odd.

The results of the example are not surprising since:

- The sum of two even functions is even, and any constant multiple of an even function is even.
- The sum of two odd functions is odd, and any constant multiple of an odd function is odd.

Not every polynomial is even or odd. For example, the polynomial $f(x) = x^2 + x$ is neither.

Example 36

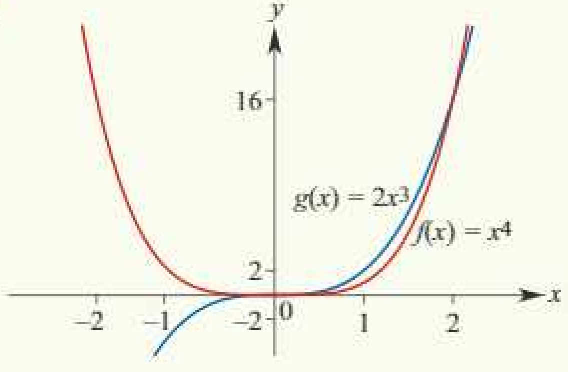
a On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 2x^3$.

b Solve the equation $f(x) = g(x)$.

c Solve the inequality $f(x) \le g(x)$.

Solution

a



b
$$f(x) = g(x)$$
$$x^4 = 2x^3$$
$$x^4 - 2x^3 = 0$$
$$x^3(x - 2) = 0$$
$$x = 0 \text{ or } x = 2$$

c $f(x) \le g(x)$

$x^4 \le 2x^3$

From the graphs and part **b**, we see that $f(x) \le g(x)$ for $x \in [0, 2]$.

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Summary 6I

- The graph of $y = a(x - h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (for h and k positive constants).
- The implied domain of all quartic functions is $\mathbb{R}$.
- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

Exercise 6I

Example 33

1 Using the method of horizontal and vertical translations, sketch the graph of each of the following:

a $y = (x + 2)^4 - 1$ **b** $y = (x - 1)^4 - 1$ **c** $y = (x + 3)^4 + 2$
d $y = (x - 2)^4 + 5$ **e** $y = (x + 2)^4 - 5$

2 Sketch the graphs of the following functions:

a $y = 2x^4 + 3$ **b** $y = 2(x - 3)^4 + 2$ **c** $y = x^4 - 16$
d $y = 16 - x^4$ **e** $y = (3 - x)^4$ **f** $y = -2(x + 1)^4 + 1$

Example 34

3 Solve each of the following equations for x:

a $x^4 - 27x = 0$ **b** $(x^2 - x - 2)(x^2 - 2x - 15) = 0$
c $x^4 + 8x = 0$ **d** $x^4 - 6x^3 = 0$
e $x^4 - 9x^2 = 0$ **f** $81 - x^4 = 0$
g $x^4 - 16x^2 = 0$ **h** $x^4 - 7x^3 + 12x^2 = 0$
i $x^4 - 9x^3 + 20x^2 = 0$ **j** $(x^2 - 4)(x^2 - 9) = 0$
k $(x - 4)(x^2 + 2x + 8) = 0$ **l** $(x + 4)(x^2 + 2x - 8) = 0$

CAS

4 Use a CAS calculator to help draw the graph of each of the following. Give x-axis intercepts and coordinates of turning points. (Values of coordinates of turning points to be given correct to two decimal places.)

a $y = x^4 - 125x$ **b** $y = (x^2 - x - 20)(x^2 - 2x - 24)$
c $y = x^4 + 27x$ **d** $y = x^4 - 4x^3$
e $y = x^4 - 25x^2$ **f** $y = 16 - x^4$
g $y = x^4 - 81x^2$ **h** $y = x^4 - 7x^3 + 12x^2$
i $y = x^4 - 9x^3 + 20x^2$ **j** $y = (x^2 - 16)(x^2 - 25)$
k $y = (x - 2)(x^2 + 2x + 10)$ **l** $y = (x + 4)(x^2 + 2x - 35)$

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Example 35

5 State whether each of the following polynomials is even or odd:

a $f(x) = 5x^6 - 3x^2$ **b** $g(x) = 7x^{11} - x^3 + 2x$

c $h(x) = x^4 - 3x^2 + 2$ **d** $m(x) = x^5 - 4x^3$

Example 36

6 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 2x^2$.

b Solve the equation $f(x) = g(x)$. **c** Solve the inequality $f(x) \leq g(x)$.

7 **a** On the one set of axes sketch the graphs of $f(x) = x^4$ and $g(x) = 9x^2$.

b Solve the equation $f(x) = g(x)$. **c** Solve the inequality $f(x) \leq g(x)$.

8 **a** On the one set of axes sketch the graphs of $f(x) = x^3$ and $g(x) = 4x$.

b Solve the equation $f(x) = g(x)$. **c** Solve the inequality $f(x) \leq g(x)$.

CAS

9 The graph of a quartic function can have zero, one, two, three or four x-axis intercepts. For each of the following, find the number of x-axis intercepts of the graph:

a $g(x) = x^4 + 2$ **b** $g(x) = (x^2 - 4)(x^2 + 1)$ **c** $g(x) = (x - 2)^2(x^2 + 1)$

d $g(x) = (x^2 - 4)(x^2 - 1)$ **e** $g(x) = x^2(x^2 - 4)$ **f** $g(x) = x^4 + x^2$

6J Applications of polynomial functions

In this section, we use a CAS calculator to find maximum and minimum values of restricted polynomial functions.

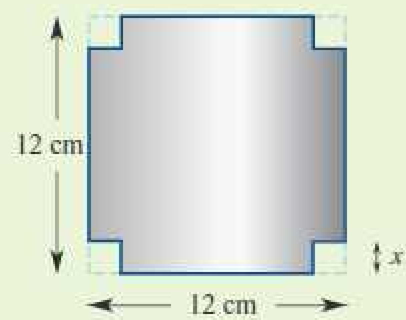
Example 37

A square sheet of tin measures 12 cm × 12 cm.

Four equal squares of edge x cm are cut out of the corners and the sides are turned up to form an open rectangular box. Find:

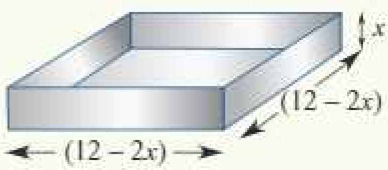
a the values of x for which the volume is 100 cm^3

b the maximum volume.



Solution

The figure shows how it is possible to form many open rectangular boxes with dimensions $12 - 2x$, $12 - 2x$ and x.



The volume of the box is

$$V = x(12 - 2x)^2, \quad 0 \leq x \leq 6$$

which is a cubic model.

We complete the solution using a CAS calculator as follows.

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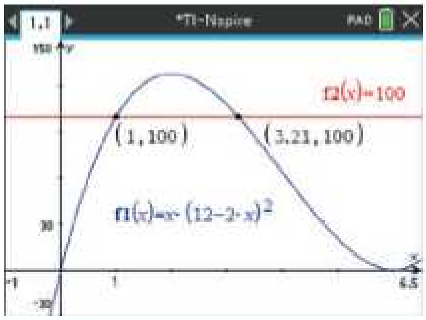
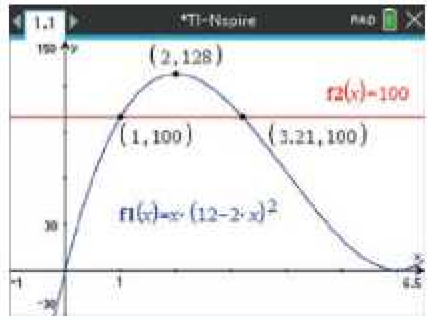
Using the TI-Nspire

Plot the graph of $V = x(12 - 2x)^2$.

a To find the values of x for which $V = 100$, plot the graph of $V = 100$ on the same screen and then find the intersection points using [menu] > **Geometry** > **Points & Lines** > **Intersection Point(s)**.

b To find the maximum volume, use [menu] > **Trace** > **Graph Trace** or [menu] > **Analyze Graph** > **Maximum**.



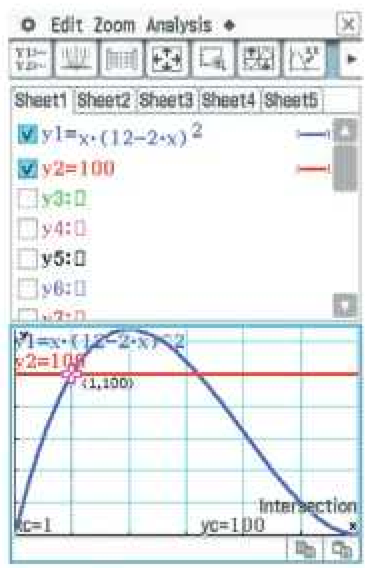


Using the Casio ClassPad

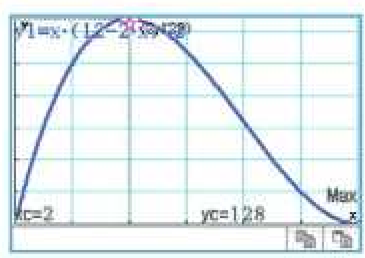
Plot the graph of $V = x(12 - 2x)^2$.

a To find the values of x for which $V = 100$:

- Plot the graph of $V = 100$ in the same window.
- To adjust your graph window use [icon]. Set $x_{\min} = 0$ and $x_{\max} = 6$. Then go to **Zoom Auto**. This will automatically adjust the window to the given domain. The grid appears when the y-scale is adjusted to 20 units.
- Select the graph window [icon] and go to **Analysis** > **G-Solve** > **Intersection**.
- Press the right arrow on the hard keyboard to find the other point of intersection.



b The maximum volume of the box may be found using **Analysis** > **G-Solve** > **Max**. (First remove the tick for $y2$ and redraw the graph.)

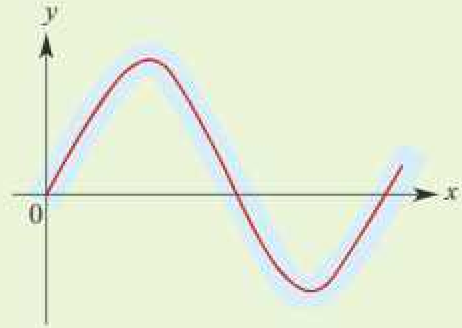


Example 38

It is found that 250 metres of the path of a stream can be modelled by a cubic function. The cubic passes through the points $(0, 0)$, $(100, 22)$, $(150, -10)$, $(200, -20)$.

a Find the equation of the cubic function.

b Find the maximum deviation of the graph from the x-axis for $x \in [0, 250]$.

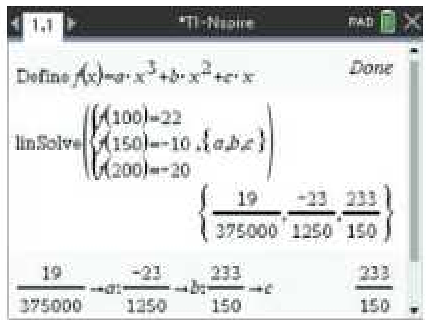


Using the TI-Nspire

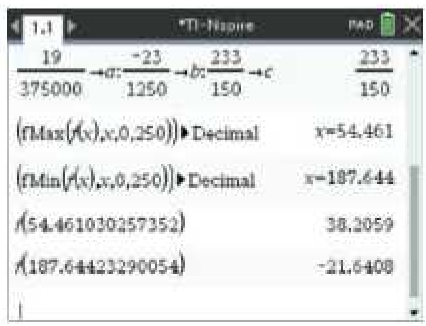
- Define $f(x) = ax^3 + bx^2 + cx$.
- Solve using the **Solve System of Linear Equations** command. Enter using the following function notation:

$$f(100) = 22, \quad f(150) = -10,$$
$$f(200) = -20$$

Proceed as shown in the first screen.



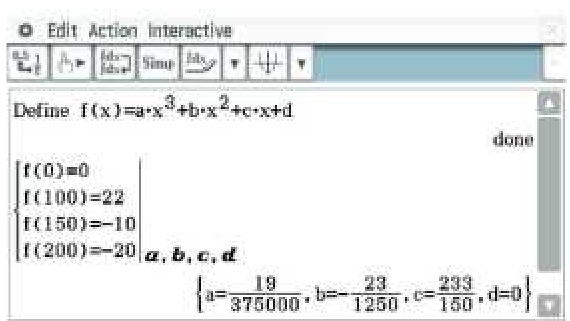
- Store these values as a, b and c respectively. (For the 'store' symbol →, press ctrl var.)
- Use **fMax( )** from menu > **Calculus** > **Function Maximum** to find where f obtains its maximum value.
- Use **fMin( )** from menu > **Calculus** > **Function Minimum** to find where f obtains its minimum value.



The maximum deviation is 38.21 metres.

Using the Casio ClassPad

- Define $f(x) = ax^3 + bx^2 + cx + d$.
- Enter the four equations shown as simultaneous equations with variables set as a, b, c, d.



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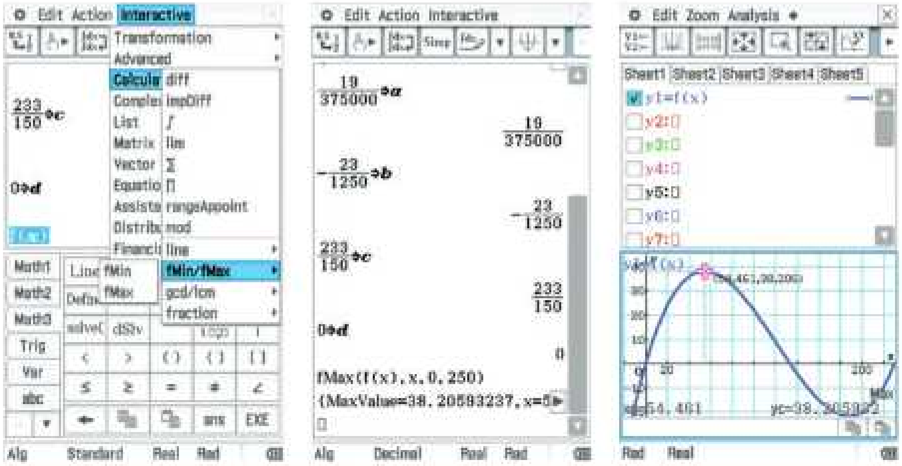
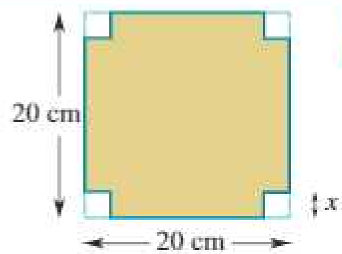
- Store the values found for a, b, c, d as shown below. (You can use **Edit** > **Copy** and **Edit** > **Paste**. The symbol ⇒ is found in Math1, Math2 and Math3.)
- The maximum value can be found in the main screen. Type $f(x)$, highlight it and go to **Interactive** > **Calculation** > **fMin/fMax**. Enter the required interval for x. Tap OK.
- Alternatively, find the maximum value in **Graph & Table**. Enter and graph $y1 = f(x)$ and then use **Analysis** > **G-Solve** > **Max**.



Exercise 6J

Example 37 **1** A square sheet of cardboard has edges of length 20 cm. Four equal squares of edge length x cm are cut out of the corners and the sides are turned up to form an open rectangular box.



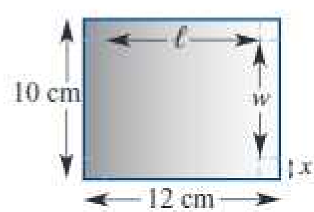
a Find the length of each edge of the base of the box in terms of x.

b Find the volume, V cm^3, of the box in terms of x.

c Find the volume of the box when $x = 5$.

d Find the values of x for which the volume is 500 cm^3.

2 A rectangular sheet of metal measuring 10 cm × 12 cm is to be used to construct an open rectangular tray. The tray will be constructed by cutting out four equal squares from the corners of the sheet as shown in the diagram.



a If the edge of each cut-out square is x cm, express ℓ and w in terms of x.

b Write down a rule for the volume, V cm^3, of the open tray in terms of x.

c Use a CAS calculator to help draw the graph of V against x for suitable values of x.

d Find the value of V when $x = 1$.

e Find the values of x for which $V = 50$.

f Find the maximum volume of the box and the value of x for which this occurs.

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3 The outside surface area of an open box is $75\ \text{cm}^2$. The base is a square with each edge x cm. Let h cm be the height of the box.

a Find the surface area of the box in terms of x and h.

b Hence, find h in terms of x.

c Find V in terms of x if $V\ \text{cm}^3$ is the volume of the box.

d Find V when:

i $x = 2$ **ii** $x = 5$ **iii** $x = 8$

e Given that $V = 59$ when $x = 4$, find the other value of x for which $V = 59$.

4 In an upright triangular prism, the triangular base has sides of length $5x$ cm, $12x$ cm and $13x$ cm. The height of the prism is h cm. The sum of the lengths of all of its edges is 180 cm.

a Find h in terms of x.

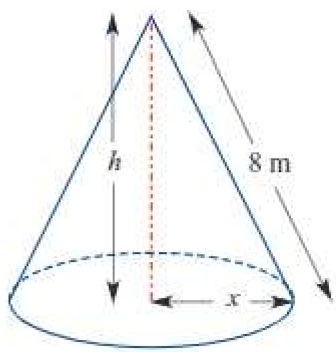
b Find V in terms of x, where $V\ \text{cm}^3$ is the volume of the prism.

c Find V when $x = 3$.

d Find the values of x for which $V = 1200$.

CAS

5 The diagram shows a conical heap of gravel. The slant height of the heap is 8 m, the radius of the base x m, and the height h m.



a Express x in terms of h.

b Construct a function which expresses V, the volume of the heap in m^3, in terms of h.

c Use a CAS calculator to help draw the graph of V against h.

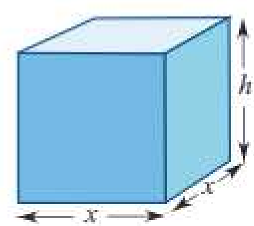
d State the domain for the function.

e Find the value of V when $h = 4$.

f Find the values of h for which $V = 150$.

g Find the maximum volume of the cone and the corresponding value of h.

6 The diagram shows a rectangular prism with a square cross-section. Measurements are in centimetres.



a If the sum of the dimensions, length plus width plus height, is 160 cm, express the height, h, in terms of x.

b Write down an expression for the volume, $V\ \text{cm}^3$, of the prism in terms of x.

c State the domain.

d Use a CAS calculator to help draw the graph of V against x.

e Find the value(s) of x for which $V = 50\,000$.

f Find the maximum volume of the prism.

6K The bisection method

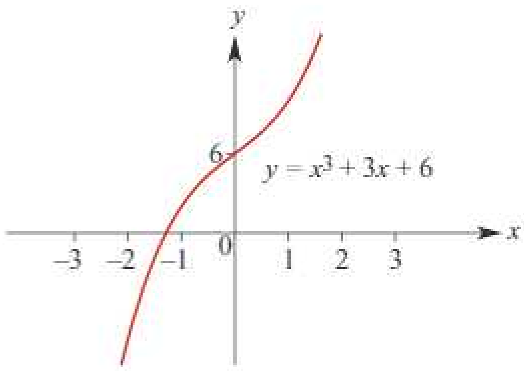
The **bisection method** is used for finding approximate solutions of equations of the form $f(x) = 0$. Here we restrict our attention to the case where f is a polynomial function.

With higher degree polynomials, the problem of finding exact solutions is more demanding, and it can be shown that beyond degree 4 there is no nice general formula to find all the solutions of a polynomial equation.

For the cubic function $f(x) = x^3 + 3x + 6$, the equation $f(x) = 0$ has only one real solution. The exact solution is

$$x = \frac{-1}{(-3 + \sqrt{10})^{1/3}} + (-3 + \sqrt{10})^{1/3}$$

Finding this by hand is beyond the requirements of this course, but we can find a numerical approximation to this solution using the bisection method.



We note that $f(-2) < 0$ and $f(-1) > 0$. This observation is central to using the bisection method.

In general, consider an equation $f(x) = 0$ that has one solution α in the interval $[a_1, b_1]$.

- The sign of $f(a_1)$ is the opposite of the sign of $f(b_1)$, because $f(\alpha) = 0$ and so the function must change from being positive to being negative or vice versa.
- Calculate $m_1 = \dfrac{a_1 + b_1}{2}$ and $f(m_1)$.
- Choose the interval $[a_1, m_1]$ if $f(a_1) \times f(m_1) < 0$, and the interval $[m_1, b_1]$ otherwise.
- The process is then repeated with the new interval, and then repeatedly until the required accuracy is reached.

Using a spreadsheet for the bisection method

We now return to the function $f(x) = x^3 + 3x + 6$ and finding the solution of the equation $x^3 + 3x + 6 = 0$.

Step 1 We start with the interval $[-2, -1]$, since we know the solution lies in this interval.

$f(-2) = -8 < 0$ and $f(-1) = 2 > 0$.

Let $m_1 = \dfrac{-2 + (-1)}{2} = -1.5$.

Since $f(-1.5) = -1.875 < 0$, we now know the solution is between -1.5 and -1.

Step 2 Choose -1.5 as the new left endpoint. Therefore the second interval is $[-1.5, -1]$.

Now $m_2 = \dfrac{-1.5 + (-1)}{2} = -1.25$ and $f(-1.25) = 0.296875 > 0$.

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Step 3 Choose −1.25 as the new right endpoint. Thus the third interval is [−1.5, −1.25].

Now $m_3 = \dfrac{-1.5 + (-1.25)}{2} = -1.375$ and $f(-1.375) = -0.724609 < 0$.

Step 4 Choose −1.375 as the new left endpoint. Thus the fourth interval is [−1.375, −1.25].

At this point we know that the solution is in the interval [−1.375, −1.25].

We continue with the spreadsheet as shown. This spreadsheet can be modified for other functions.

	Left endpoint a_i	Right endpoint b_i	Midpoint m_i	$f(m_i)$
Step 1	−2	−1	−1.5	−1.87500000
Step 2	−1.5	−1	−1.25	0.29687500
Step 3	−1.5	−1.25	−1.375	−0.72460938
Step 4	−1.375	−1.25	−1.3125	−0.19848633
Step 5	−1.3125	−1.25	−1.28125	0.05294800
Step 6	−1.3125	−1.28125	−1.296875	−0.07181931
Step 7	−1.296875	−1.28125	−1.2890625	−0.00919962
Step 8	−1.2890625	−1.28125	−1.28515625	0.02193302
Step 9	−1.2890625	−1.28515625	−1.28710938	0.00638143
Step 10	−1.2890625	−1.28710938	−1.28808594	−0.00140541
Step 11	−1.28808594	−1.28710938	−1.28759766	0.00248893

We conclude that the solution is −1.29 correct to two decimal places.

Whatever equation we are working with, if the starting interval is 1 unit, then the 'error' is at most $\frac{1}{2}$ after the next step and at most $(\frac{1}{2})^{n-1}$ after n steps.

Using pseudocode for the bisection method

Computers cannot directly understand instructions that are written in our natural language. We have to provide them with instructions in a **programming language**. The process of taking an algorithm and writing it in a programming language is called **coding**.

In Appendix A, we give an introduction to **pseudocode**. This is an informal notation for writing instructions that is closer to natural language. It makes no reference to any particular programming language. In order to actually implement the instructions on a computer, they must be translated into a specific programming language.

Note: You should work through Appendix A before starting this subsection. The Interactive Textbook also includes online appendices that provide an introduction to coding using the language *Python* and to coding using the TI-Nspire and the Casio ClassPad.

In the following example, we give an algorithm written in pseudocode for solving the equation $x^3 + 3x + 6 = 0$. The algorithm follows the same steps used to construct the spreadsheet above.

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Example 39

The equation $x^3 + 3x + 6 = 0$ has one real solution, which lies in the interval $[-2, -1]$. Using pseudocode, write an algorithm to find this solution correct to within 0.01.

Solution

```
define f(x):
  return x^3 + 3x + 6

a ← −2
b ← −1
m ← −1.5
while b − a > 2 × 0.01
    if f(a) × f(m) < 0 then
        b ← m
    else
        a ← m
    end if
    m ← (a + b)/2
    print a, m, b, f(a), f(m), f(b)
end while
print m
```

Explanation

We use the bisection method.

- Define the function $f(x) = x^3 + 3x + 6$.
- Assign initial values to the variables: the left endpoint a, the right endpoint b and the midpoint m.
- We use a while loop, since we don't know how many iterations will be required. We want to continue until $b - a \le 2 \times 0.01$.
  - Use an if-then block to update the value of the left endpoint a or the right endpoint b.
  - Then recalculate the value of the midpoint m.
  - At the end of each pass of the loop, print the values of a, m, b, $f(a)$, $f(m)$ and $f(b)$.
- After the while loop is complete, print the value of m, which is the approximate solution.

The following table shows the result of executing the algorithm from Example 39. The first row gives the initial values of a, m, b, $f(a)$, $f(m)$ and $f(b)$. The next rows give the values that are printed at the end of each pass of the while loop.

	a	m	b	$f(a)$	$f(m)$	$f(b)$
Initial	−2	−1.5	−1	−8	−1.875	2
Pass 1	−1.5	−1.25	−1	−1.875	0.296875	2
Pass 2	−1.5	−1.375	−1.25	−1.875	−0.724609	0.296875
Pass 3	−1.375	−1.3125	−1.25	−0.724609	−0.198486	0.296875
Pass 4	−1.3125	−1.28125	−1.25	−0.198486	0.052948	0.296875
Pass 5	−1.3125	−1.296875	−1.28125	−0.198486	−0.071819	0.052948
Pass 6	−1.296875	−1.289063	−1.28125	−0.071819	−0.009200	0.052948

After the 6th pass of the while loop, we have $b - a = 0.015625 \le 0.02$. So we exit the loop. The final printed value of m is -1.289063. You can achieve greater accuracy by changing the condition on the while loop.

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Exercise 6K

1 Use a spreadsheet with the bisection method to find approximate solutions for each of the following. The initial interval is given and the desired accuracy is stated.

a $x^3 - x - 1 = 0$	$[1, 2]$	2 decimal places
b $x^4 + x - 3 = 0$	$[1, 3]$	3 decimal places
c $x^3 - 5x + 4.2 = 0$	$[1, 2]$	3 decimal places
d $x^3 - 2x^2 + 2x - 5 = 0$	$[2, 3]$	3 decimal places
e $2x^4 - 3x^2 + 2x - 6 = 0$	$[-2, -1]$	2 decimal places

Example 39

2 The equation $-x^3 + 3x + 6 = 0$ has one real solution, which lies in the interval $[2, 3]$. Using pseudocode, write an algorithm to find this solution correct to within 0.001.

3 Let $f(x) = -x^3 - 3x + 6$. The equation $f(x) = 0$ has only one real solution.

a Find $f(1)$ and $f(2)$. Hence give an interval that contains the solution.

b Use the bisection method to determine an approximate solution to $f(x) = 0$, correct to two decimal places.

4 Show the results of the first three passes of the `while` loop when the pseudocode algorithm in Example 39 is adapted for each of the following equations. The initial values of the variables are given.

a $2x^3 + 6x^2 + 3 = 0$	Initial values: $a = -4,\ m = -3.5,\ b = -3$
b $2x^3 + 6x^2 + 3x + 5 = 0$	Initial values: $a = -3,\ m = -2.5,\ b = -2$
c $6x^3 + 6x^2 + 3x + 1 = 0$	Initial values: $a = -1,\ m = -0.5,\ b = 0$

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Review

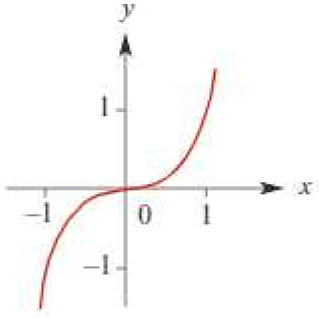
Chapter summary

Assign-ment

Nrich

- The sum, difference and product of two polynomials is a polynomial.
- Division of one polynomial by another does not always result in a polynomial.
- **Remainder theorem** When $P(x)$ is divided by $x - \alpha$, the remainder is equal to $P(\alpha)$.
  e.g. When $P(x) = x^3 + 3x^2 + 2x + 1$ is divided by $x - 2$, the remainder is
  $$P(2) = (2)^3 + 3(2)^2 + 2(2) + 1 = 25$$
- **Factor theorem** For a polynomial $P(x)$, if $P(\alpha) = 0$, then $x - \alpha$ is a factor. Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.
  e.g. For $P(x) = x^3 - 4x^2 + x + 6$,
  $$P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6 = 0$$
  and so $x + 1$ is a factor of $P(x)$.
- Sums and differences of cubes:
  $$x^3 - a^3 = (x - a)(x^2 + ax + a^2) \qquad \text{e.g.} \quad x^3 - 27 = (x - 3)(x^2 + 3x + 9)$$
  $$x^3 + a^3 = (x + a)(x^2 - ax + a^2) \qquad \text{e.g.} \quad 8x^3 + 64 = (2x)^3 + 4^3$$
  $$= (2x + 4)(4x^2 - 8x + 16)$$
- **Rational-root theorem** Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Steps for solving a cubic equation:
  - **i** If necessary, rearrange the equation so that the right-hand side is zero.
  - **ii** Factorise the cubic polynomial on the left-hand side by using the factor theorem and then dividing.
  - **iii** Use the null factor theorem to determine solutions.

  e.g. Solve $x^3 - 4x^2 - 11x + 30 = 0$.
  Since $P(2) = 8 - 16 - 22 + 30 = 0$, we know that $x - 2$ is a factor.
  Dividing $x - 2$ into $x^3 - 4x^2 - 11x + 30$ gives
  $$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$$
  $$= (x - 2)(x - 5)(x + 3)$$
  $$\therefore \quad (x - 2)(x - 5)(x + 3) = 0$$
  $$\therefore \quad x = 2, 5 \text{ or } -3$$
- The basic shape of the curve defined by $y = x^3$ is shown in the graph.



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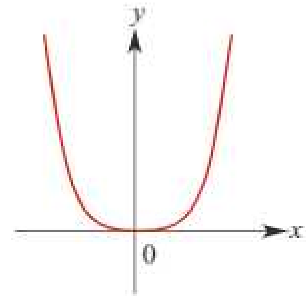
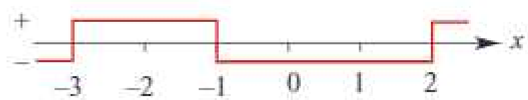
- The implied domain of all polynomial functions is $\mathbb{R}$.
- The functions $f(x) = x^3$ and $f^{-1}(x) = x^{\frac{1}{3}}$ are inverse functions of each other.
- The graph of $y = a(x - h)^3 + k$ has the same shape as $y = ax^3$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants). The point of inflection is at (h, k).
- The basic shape of the curve defined by $y = x^4$ is shown in the graph.



- The graph of $y = a(x - h)^4 + k$ has the same shape as $y = ax^4$ but is translated h units in the positive x-axis direction and k units in the positive y-axis direction (where h and k are positive constants). The turning point is at (h, k).
- Sign diagrams assist in sketching graphs of cubic functions.

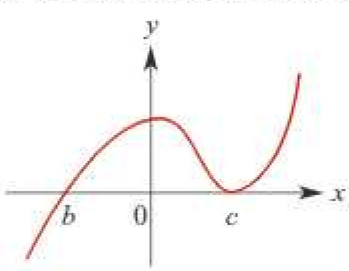
  e.g. $\quad y = x^3 + 2x^2 - 5x - 6$
  $\qquad = (x + 1)(x - 2)(x + 3)$

  When $x < -3$, y is negative.
  When $-3 < x < -1$, y is positive.
  When $-1 < x < 2$, y is negative.
  When $x > 2$, y is positive.



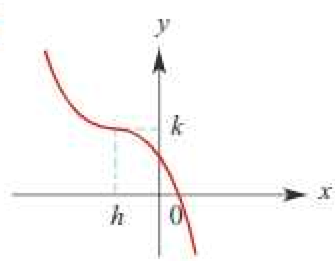
- Steps for sketching the graph of a cubic function $y = ax^3 + bx^2 + cx + d$:
  - **i** Use the factor theorem and division to determine the x-axis intercepts.
  - **ii** The y-axis intercept is d.
  - **iii** Draw a sign diagram.
- Finding equations for given cubic graphs:

i



Form: $y = a(x - b)(x - c)^2$
Assume b and c are known, substitute another known point to calculate a.

ii



Form: $y = a(x - h)^3 + k$
Substitute known values to determine a.

Alternatively, use the general form $y = ax^3 + bx^2 + cx + d$ and the known points to determine a, b, c and d.

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- A function f is **even** if $f(-x) = f(x)$. This means that the graph is symmetric about the y-axis.
- A function f is **odd** if $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin.
- A **power function** is a function f with rule $f(x) = x^r$ where r is a non-zero real number.

Technology-free questions

1 Sketch the graph of each of the following:

a $y = (x-1)^3 - 2$ **b** $y = (2x-1)^3 + 1$ **c** $y = 3(x-1)^3 - 1$
d $y = -3x^3$ **e** $y = -3x^3 + 1$ **f** $y = -3(x-2)^3 + 1$
g $y = 4(x+2)^3 - 3$ **h** $y = 1 - 3(x+2)^3$

2 Sketch the graph of each of the following:

a $y = (x-1)^4$ **b** $y = (2x-1)^4 + 1$ **c** $y = (x-1)^4 - 1$
d $y = -2x^4$ **e** $y = -3x^4 + 1$ **f** $y = -(x-2)^4 + 1$
g $y = 2(x+1)^4 - 3$ **h** $y = 1 - 2(x+2)^4$

3 Solve each of the following equations for x:

a $2x^3 + 3x^2 = 11x + 6$ **b** $x^2(5-2x) = 4$ **c** $x^3 + 4x + 12 = 7x^2$

4 **a** Use the factor theorem to show that $2x-3$ and $x+2$ are factors of $6x^3+5x^2-17x-6$. Find the other factor.

b Solve the equation $2x^3 - 3x^2 - 11x + 6 = 0$.

c Solve the equation $x^3 + x^2 - 11x - 3 = 8$.

d **i** Show that $3x - 1$ is a factor of $3x^3 + 2x^2 - 19x + 6$.

ii Find the factors of $3x^3 + 2x^2 - 19x + 6$.

5 Let $f(x) = x^3 - kx^2 + 2kx - k - 1$.

a Show that $f(x)$ is divisible by $x - 1$.

b Factorise $f(x)$.

6 Find the values of a and b for which $x^3 + ax^2 - 10x + b$ is divisible by $x^2 + x - 12$.

7 Draw a sign diagram for each of the following and hence sketch the graph:

a $y = (x+2)(3-x)(x+4)$ **b** $y = (x-2)(x+3)(x-4)$
c $y = 6x^3 + 13x^2 - 4$ **d** $y = x^3 + x^2 - 24x + 36$

8 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 4x^2 - 5x + 1,\ x + 6$ **b** $2x^3 - 3x^2 + 2x + 4,\ x - 2$
c $3x^3 + 2x + 4,\ 3x - 1$

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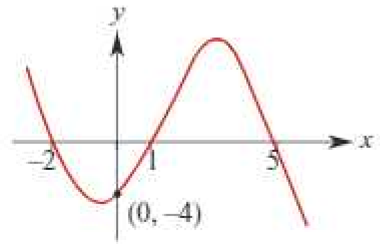
Review

9 Find the rule of the cubic for which the graph is shown.



10 Find a cubic function whose graph touches the x-axis at $x = -4$, passes through the origin and has a value of 10 when $x = 5$.

11 Let $f(x) = 2x^3 + ax^2 - bx + 3$. When $f(x)$ is divided by $x - 2$ the remainder is 15 and $f(1) = 0$.

a Calculate the values of a and b.

b Find the other two linear factors of $f(x)$.

12 Solve each of the following inequalities for x:

a $(x-3)^2(x+4) \le 0$

b $-(x+3)(x+4)(x-2) \ge 0$

c $x^3 - 4x^2 + x < -6$

13 For each of the following, find a sequence of transformations that takes the graph of $y = x^3$ to the graph of:

a $y = 2(x-1)^3 + 3$

b $y = -(x+1)^3 + 2$

c $y = (2x+1)^3 - 2$

Multiple-choice questions

1 If $P(x) = x^3 + 3x^2 + x - 3$, then $P(-2)$ equals

A 1 B −1 C −25 D 3 E −5

2 If $a > b > c$ and $P(x) = (x-a)^2(x-b)(x-c)$, then $P(x) < 0$ for $x \in$

A $(-\infty, a)$ B $(-\infty, b)$ C $(-\infty, c)$ D (c, b) E (b, a)

3 The image of the graph of $y = x^3$ under a dilation of factor 2 from the y-axis followed by a reflection in the y-axis and then a translation of 4 units in the negative direction of the y-axis is

A $y = -\dfrac{x^3}{8} - 4$

B $y = -\dfrac{x^3}{2} - 4$

C $y = -8x^3 - 4$

D $y = -\dfrac{x^3}{2} + 4$

E $y = \dfrac{x^3}{8} + 4$

4 The equation $x^3 + 5x - 10 = 0$ has only one solution. This solution lies between

A −2 and −1 B −1 and 0 C 0 and 1 D 1 and 2 E 2 and 8

5 Let $P(x) = x^4 + ax^2 - 4$. If $P(\sqrt{2}) = 0$ and $P(-\sqrt{2}) = 0$, the value of a is

A 0 B 2 C −2 D −3 E 3

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6 Let $P(x) = x^3 + ax^2 + bx - 9$. If $P(1) = 0$ and $P(-3) = 0$, the values of a and b are

A $a = 1,\ b = -3$ **B** $a = -1,\ b = 3$ **C** $a = 5,\ b = 3$
D $a = -5,\ b = -3$ **E** $a = 0,\ b = 0$

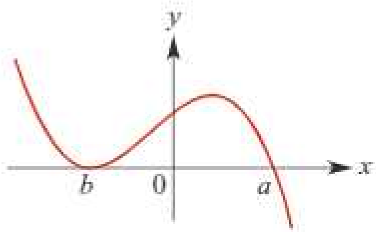
7 If $ax^3 + 2x^2 + 5$ is exactly divisible by $x + 1$, the value of a is

A 1 **B** 7 **C** -1 **D** 3 **E** -7

8 When the polynomial $P(x) = x^3 + 2x^2 - 5x + d$ is divided by $x - 2$, the remainder is 10. The value of d is

A 10 **B** 4 **C** -10 **D** -4 **E** 3

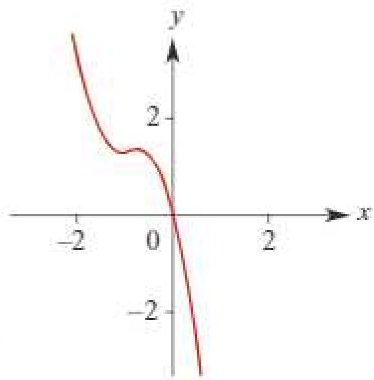
9 The diagram shows part of the graph of a polynomial function.



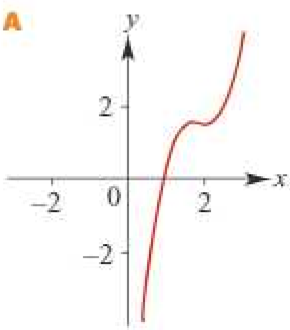
A possible equation for the rule of the function is

A $y = (x - b)^2(x - a)$ **B** $y = (x - a)^2(x - b)$ **C** $y = -(x + b)^2(x - a)$
D $y = (x - b)^2(a - x)$ **E** $y = (x + b)^2(a - x)$

10 The graph of $y = f(x)$ is shown on the right. Which one of the following could be the graph of $y = 1 - f(x)$?



A



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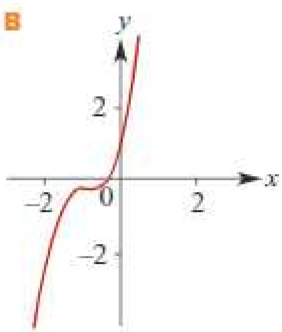

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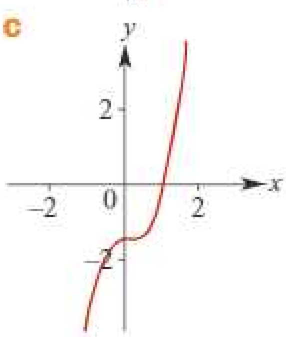

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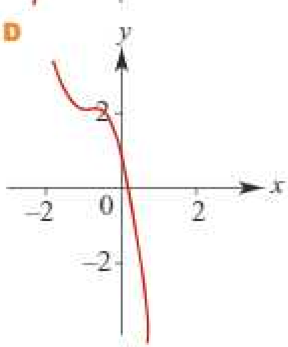
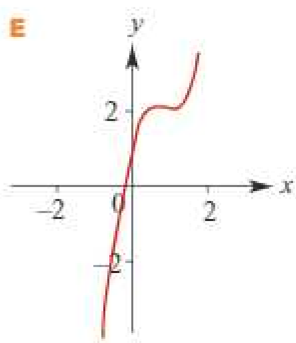


E



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Review

Extended-response questions

1 The volume of a cylinder is given by $V = \pi r^2 h$. It is given that $h + r = 6$.

a Write $V(r)$ in terms of r.

b State the values that r can have.

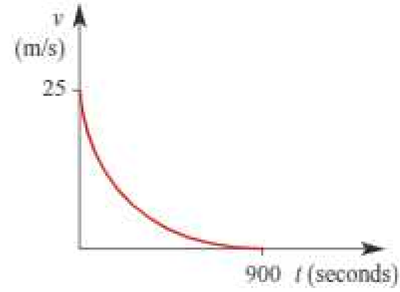
c Find $V(3)$.

d Find the values of r for which $V(r) = 27\pi$.

e Use your CAS calculator to find the maximum possible volume of the cylinder.

2 There is a proposal to provide a quicker, more efficient and more environmentally friendly system of inner-city public transport by using electric taxis. The proposal necessitates the installation of power sources at various locations as the taxis can only be driven for a limited time before requiring recharging.

The graph shows the speed v m/s that the taxi will maintain if it is driven at constant speed in such a way that it uses all its energy up in t seconds. The curve is a section of a parabola which touches the t-axis at $t = 900$. When $t = 0$, $v = 25$.



a Construct a rule for v in terms of t.

b If s metres is the distance that a taxi can travel before running out of electrical energy, write down a rule connecting s and t.

c Use a CAS calculator to help draw the graph of s against t.

d Originally the power sources were to be located at 2 km intervals. However there is a further proposal to place them at 3.5 km intervals. Is this new distance feasible?

e With the power sources at 2 km intervals, use your graph to determine approximately both the maximum and minimum speeds recommended for drivers. Explain your answer.

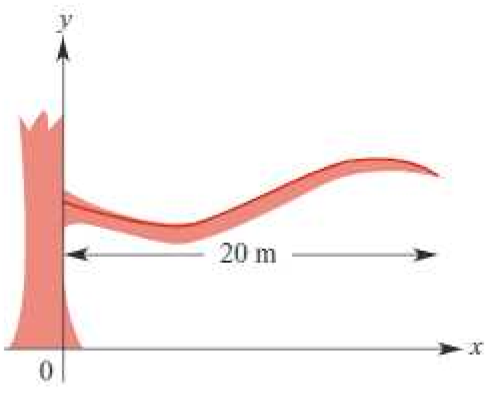
3 It is found that the shape of a branch of a eucalyptus tree can be modelled by a cubic function. The coordinates of several points on the branch are (0, 15.8), (10, 14.5), (15, 15.6), (20, 15).

a The rule for the function is of the form $y = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

b Find the coordinates of the point on the branch that is:

i closest to the ground

ii furthest from the ground.



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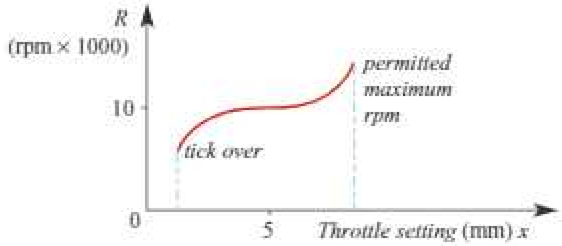
4 The figure shows part of a cubic graph that represents the relationship between the engine speed, $R \times 1000$ rpm, and the throttle setting, x mm from the closed position, for a new engine. It can be seen from the graph that the engine has a 'flat spot' where an increase in x has very little effect on R.



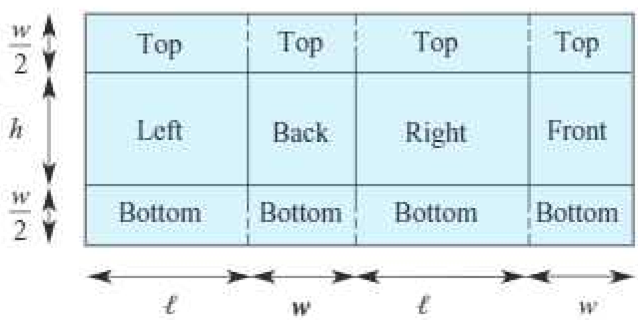
a Develop a cubic expression for R in terms of x of the form $R = a(x - h)^3 + k$.

b Find a if when the graph is extended it passes through the origin.

c In a proposed modification to the design, the 'flat spot' will occur when $x = 7$ mm. The speed of the engine in this case will be 12 000 rpm when $x = 7$ mm. Assuming that a cubic model still applies and that $R = 0$ when $x = 0$, write down an expression for R as a function of x.

CAS

5 A net for making a cardboard box with overlapping flaps is shown in the figure. The dotted lines represent cuts and the solid lines represent lines along which the cardboard is folded.



a If $\ell = 35$ cm, $w = 20$ cm and $h = 23$ cm, calculate the area of the net.

b If the area of the net is to remain constant at the value calculated in part **a** and $\ell = h$, write down an expression for V, the volume of the box in cm^3, as a function of ℓ. (The maximum volume of the box will occur when $\ell = h$).

c Use a CAS calculator to help draw the graph of V against ℓ.

d Find the value of ℓ when the volume of the box is:

i 14 000 cm^3

ii 10 litres = 10 000 cm^3

e Find the maximum volume of the box and the value of ℓ for which this occurs.

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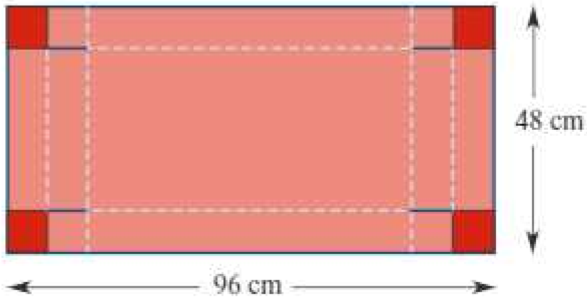
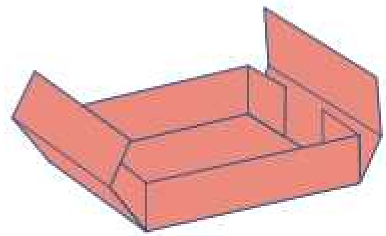
6 A reinforced box is made by cutting congruent squares of side length x cm from the four corners of a rectangular piece of cardboard that measures 48 cm by 96 cm. The flaps are folded up.





a Find an expression for V, the volume of the box formed.

b Plot a graph of V against x on your CAS calculator.

  i What is the domain of the function V?

  ii Using your CAS calculator, find the maximum volume of the box and the value of x for which this occurs (approximate values required).

c Find the volume of the box when $x = 10$.

d It is decided that $0 \le x \le 5$. Find the maximum volume possible.

e If $5 \le x \le 15$, what is the minimum volume of the box?

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