16

Rates of change

Objectives

- To recognise **relationships** between variables.
- To calculate **average rates of change**.
- To estimate **gradients of tangents** to curves.
- To estimate **instantaneous rates of change**.
- To apply the estimation and calculation of rates of change to solving problems.

Throughout this book, we have been looking at situations where there is a relationship between two variables. We have developed and applied the idea that one variable, say y, is a function of another variable, say x.

Furthermore, we have represented such relationships graphically, and investigated key features such as axis intercepts, turning points and asymptotes.

This graphical representation can also be used to see how the relationship is changing.

In applications, how the relationship is changing is of critical importance in establishing how accurately a given rule models the relationship between the variables in question. For example, if x increases, does y also increase, or does it decrease, or remain unaltered? And, if it does change, does it do so consistently, quickly, slowly, indefinitely, etc.?

This chapter serves as an introduction to the ideas of calculus, which we begin to study more formally in the next chapter. In this chapter, we talk about rates of change informally, based on our intuition. Our study of calculus will enable us to talk about these ideas more precisely.

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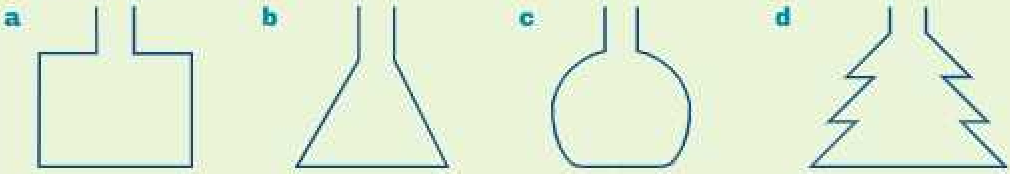
16A Recognising relationships

In previous chapters, we have studied polynomial, exponential, logarithmic and circular functions, and have seen that many real-life situations may be modelled by these functions.

In this first section, we look at several real-life situations involving two variables, and investigate the form of the relationships between the variables through graphs. The algebraic relationship is not established.

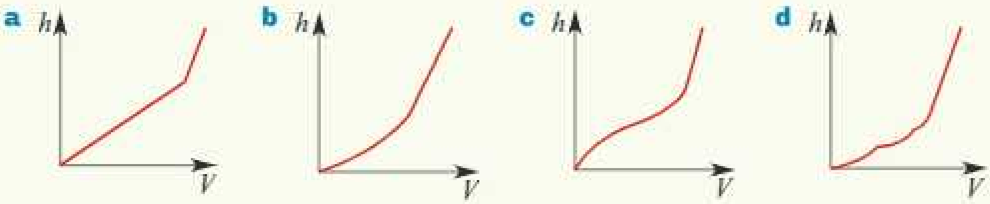
Example 1



Water is being poured steadily into each of these vessels.

Draw a graph that shows the relationship between the height of the water (h) and the volume (V) that has been poured in.

Solution



Example 2

A particle travels in a straight line.

The graph shows the distance, D metres, of the particle from a fixed point O over a period of 20 minutes.

Describe the motion of the particle.

Solution

The particle is initially 5 m from O. It travels away from O for 3 minutes at a constant speed of $\frac{5}{3}$ m/min. It then remains stationary at a distance of 10 m from O for 4 minutes, before returning to O at a speed which is gradually decreasing so that it comes to rest at O at time $t = 20$ minutes.

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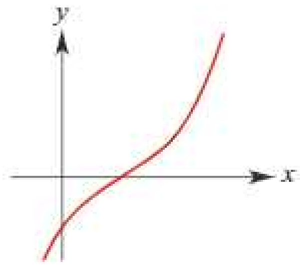
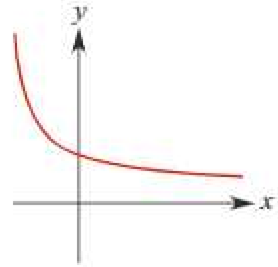
By examining the graph representing a function, it can be determined whether the rate of change is positive, negative or neither.



Consider the graph of a function $y = f(x)$.

If the graph shows that y is increasing as x increases, then we can say that the **rate of change** of y with respect to x is **positive**. (The graph 'slopes upwards'.)



If the graph shows that y is decreasing as x increases, then we can say that the **rate of change** of y with respect to x is **negative**. (The graph 'slopes downwards'.)

If y remains the same value as x changes, the corresponding graph is a horizontal line, and we say that the rate of change of y with respect to x is **zero**.

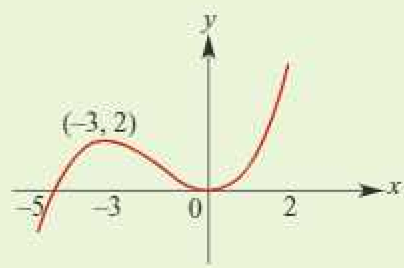
All of this is consistent with the gradient (rate of change) of a linear function, which was discussed in Chapter 2.

Example 3

For the graph shown on the right for $x \in [-5, 2]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is negative

b the rate of change of y with respect to x is positive.



Solution

a The rate of change of y with respect to x is negative for $x \in (-3, 0)$.

b The rate of change of y with respect to x is positive for $x \in [-5, -3) \cup (0, 2]$.

Summary 16A

Consider the graph of a function with rule $y = f(x)$.

- If the graph shows that y is increasing as x increases over some interval, then the **rate of change** of y with respect to x is **positive** for that interval.
- If the graph shows that y is decreasing as x increases over some interval, then the **rate of change** of y with respect to x is **negative** for that interval.
- If y remains the same value as x changes, the corresponding graph is a horizontal line and the rate of change of y with respect to x is **zero**.

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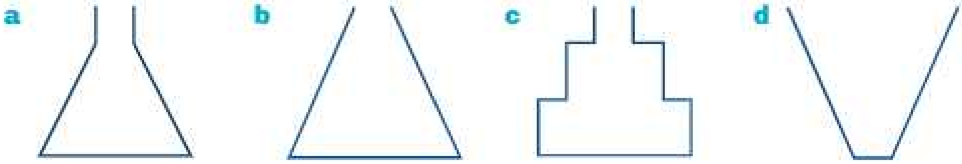
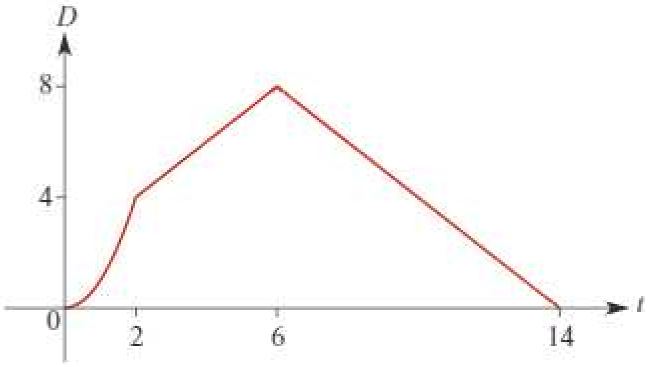
Exercise 16A

Example 1 **1** Water is being poured steadily into these vessels:



For each of the vessels, draw a graph of the height, h, of water in the vessel and the volume, V, that has been poured in. Label the horizontal axis V and the vertical axis h.

Example 2 **2** A particle travels in a straight line. The graph shows the distance, D metres, of the particle from a fixed point O on the line over a period of 14 minutes. Describe the motion of the particle.

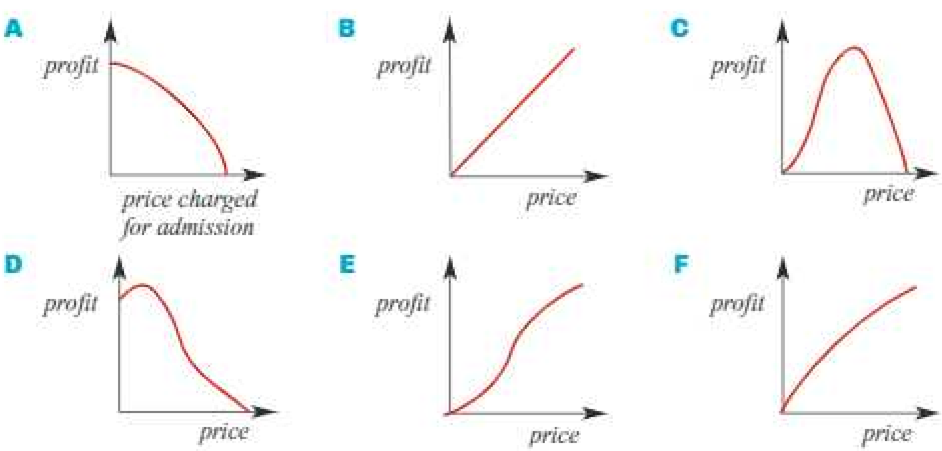


For Questions 3–6, there may be more than one correct answer. Your written explanations are an important part of the exercise.

3 The manager of a theatre wishes to know what effect changing the price of admission will have on the profit she makes.

a Which one of the following graphs would show the effect of change?

b Explain your choice, including comments on scales and axes and what the point of intersection of the axes represents.



4 Sketch a graph to show how the height of a person might vary with age.

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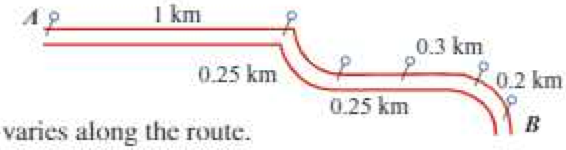
5 A motorist starts a journey at the point marked *A* on a country road, drives 2 km along the route shown, and stops at the point marked *B*. He is able to drive at 100 km/h, but must slow down at corners.



a Explain briefly how the car's speed varies along the route.

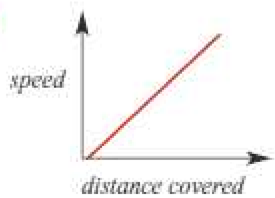
b Sketch a graph showing how the car's speed varies along the route. (Label the vertical axis *car's speed* and the horizontal axis *distance from A*.)

6 An athlete is a competitor in a 10 000 m race. Below are some graphs which could show the relationship between the speed of the runner and the distance covered.

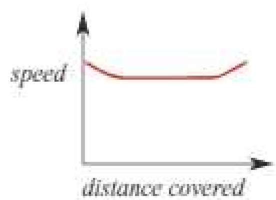
a Explain the meaning of each graph in words.

b Which graph is the most realistic for a winning athlete? If you do not think any of these graphs are realistic, draw your own and explain it fully.

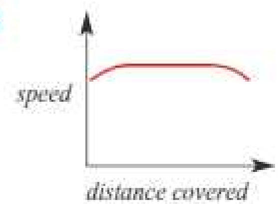
A



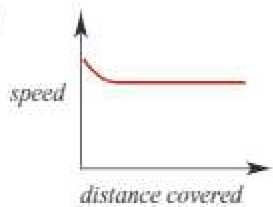
B



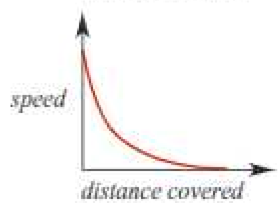
C



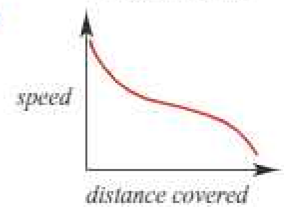
D



E



F

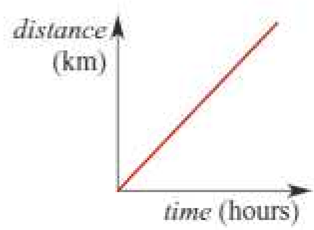


7 A sprinter covers 100 metres at a constant speed of 10 m/s. Sketch:

a the distance–time graph

b the speed–time graph

8 The graph relating the distance a car travels to the time taken is a straight line as shown. The graph shows that the car is



A speeding up

B slowing down

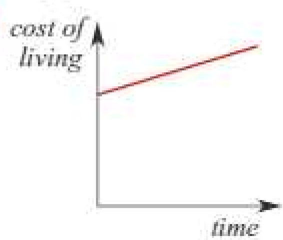
C travelling uphill

D travelling at a constant speed

E stationary

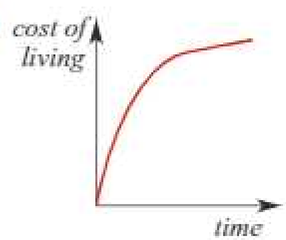
9 Which one of these graphs best represents the rate of increase in the cost of living slowing down?

A



B

C



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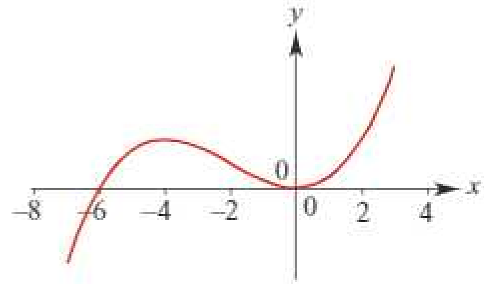
Example 3 **10** For the graph shown on the right for $x \in [-7, 3]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is negative

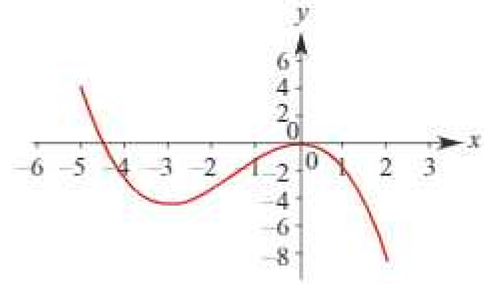
b the rate of change of y with respect to x is positive.



11 For the graph shown on the right for $x \in [-5, 2]$, use interval notation to describe the set of values of x for which:

a the rate of change of y with respect to x is positive

b the rate of change of y with respect to x is negative.



16B Constant rate of change

Any function that is linear will have a **constant rate of change**. That rate of change is simply the gradient of the graph and, given appropriate information, the rate can be calculated from the graph of the function or simply read from the rule of the function if it is stated.

Example 4

A car travels from Copahunga to Charlegum, which is a distance of 150 km, in 2 hours (120 minutes). Assuming the car travels at a constant speed, draw a distance–time graph and calculate the speed.

Solution

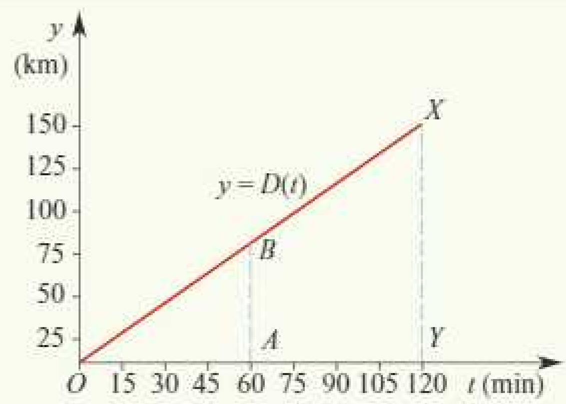
We denote the distance function by D. The graph of $y = D(t)$ is shown.

The rule of the function may be written:

$$D(t) = \frac{150}{120}t = \frac{5}{4}t$$

Note that

$$\frac{XY}{YO} = \frac{BA}{AO} = \frac{5}{4}$$



The gradient of the graph gives the speed. Therefore, the speed of the car is $\frac{5}{4}$ kilometres per minute. This speed may be expressed in kilometres per hour (km/h):

$$\text{Speed} = \frac{5}{4} \times 60 = 75 \text{ km/h}$$

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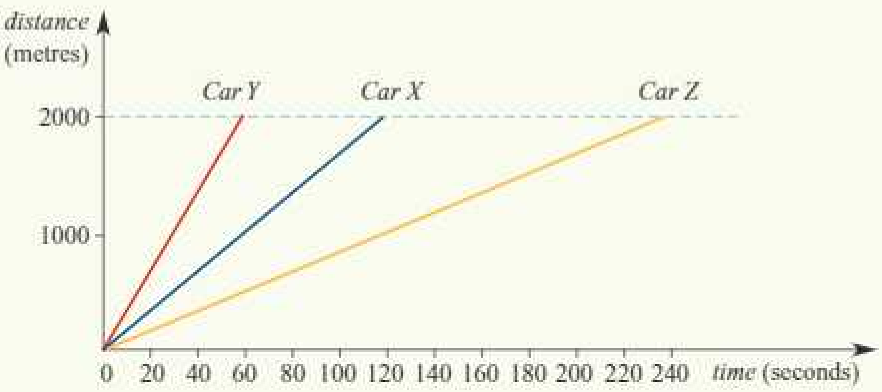
Example 5

Three cars are driven over a 2-kilometre straight track. They are all to go from point A to point B. Each car travels with constant speed. It is not a race as:

- the speed of car Y is twice that of car X
- the speed of car Z is half that of car X.

Illustrate this situation with a distance–time graph. Assume that car X travels at 1 km/min.

Solution



Note: The gradient of the graph for car X is $\frac{2000}{120} = 16\frac{2}{3}$

The gradient of the graph for car Y is $\frac{2000}{60} = 33\frac{1}{3}$

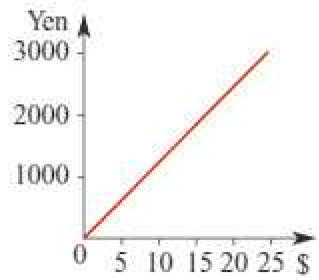
The gradient of the graph for car Z is $\frac{2000}{240} = 8\frac{1}{3}$

An object whose motion can be described by a linear distance–time graph is travelling at a constant speed equal to the gradient of the linear graph.

There are many other examples in which a real-life situation is usefully modelled by a straight-line graph in such a way that the gradient of the graph is meaningful.

In all these situations, the gradient of the straight-line graph represents a **rate**.

For example, the graph shown on the right represents the conversion between dollars and yen. The gradient of the graph gives the exchange rate of dollars to yen.



Summary 16B

A linear function $f(x) = mx + c$ has a constant rate of change, m.

Examples include an object travelling with constant speed and currency exchange rates.

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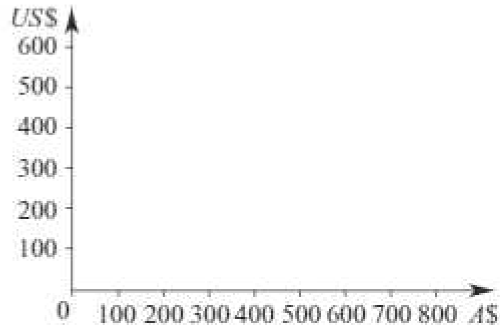
Exercise 16B

Example 4 **1** A car travels from Bombay to Pune, a distance of 200 km, at a constant speed. The journey takes 150 minutes. Draw a distance–time graph and calculate the speed.

Example 5 **2** Two cars are driven in a straight line for 5 kilometres. They both go from point A to point B and start at the same time. (They are actually driving parallel to each other.) Each car travels with constant speed. The speed of car Y is twice that of car X, and the speed of car X is 40 km/h. Illustrate this with a distance–time graph.

3 The exchange rate for the Australian dollar in terms of the American dollar was A\$1 = US\$0.75.
Draw a straight-line graph that illustrates this relationship. The axes should be as shown.



4 Find the speed for each of the following (assuming constant speed):

a distance travelled 120 km, time taken 2 hours
b distance travelled 60 m, time taken 20 seconds
c distance travelled 8000 m, time taken 20 minutes
d distance travelled 200 km, time taken 5 hours 40 minutes
e distance travelled 6542 m, time taken 5 minutes 20 seconds

5 Find the rate of flow from the following taps in litres per minute:

a a tap which fills a 40-litre drum in 5 minutes
b a tap which fills a 600-litre tank in 12 minutes
c a tap which takes 17 minutes 20 seconds to fill a 180-litre tank

6 Water comes out of a tap at the rate of 15 litres per minute.

a Copy and complete this table showing the amount which has come out at time t:

Time in minutes, t	0	0.5	1	1.5	2	3	4	5
Amount in litres, A	0							

b Draw a graph from the table.

7 A worker is paid \$200 for 13 hours work. What is their rate of pay per hour?

8 A spherical balloon is blown up so that its volume is increasing by 8 cm^3 every second. Sketch a graph to show how the volume of the balloon changes with time.

9 Two cars start together and travel with constant speed over a 1-kilometre straight track. Car 1 has speed 60 km/h, and car 2 travels at three-quarters of this speed. Illustrate this situation with distance–time graphs for both cars on the one set of axes.

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16C Average rate of change

Many moving objects do not travel with constant speed. For example, the speedometer of a car being driven in city traffic rarely stays still for long.

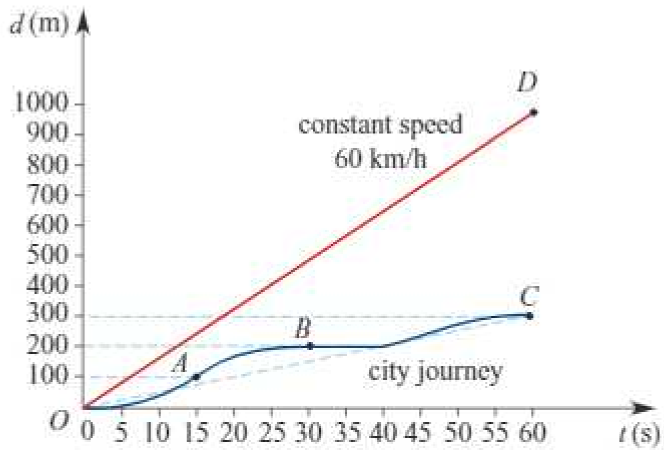
Similarly, not all functions are linear, and so not all functions have a constant rate of change. For a function that is non-linear, the rate of change of the function varies, and may in fact be different for every different point on the graph of the function.

Average speed

We will use a distance–time graph to illustrate the idea of average speed.

The graph below shows the motion of two cars both travelling in a straight line away from a fixed point O, where d is the distance travelled (in metres) at time t (in seconds).

The straight-line graph through D shows a constant speed of 60 km/h. By comparison, the graph through points A, B and C shows a motorist travelling at varying speeds. The motorist accelerates to reach 60 km/h at A before slowing for the lights at B, where there is a 10-second standstill. There is then another short burst of speed before another standstill at C.



Although we do not know the actual speed of the car travelling in the city at any particular time (other than when it is stationary), we can work out the average speed of the car over the full 60 seconds.

The average speed is given by

$$\frac{\text{distance travelled}}{\text{time taken}} = \frac{300}{60}$$

which gives an average speed of 5 metres per second. The average speed is the gradient of the line OC.

The average speed may also be calculated for any given time interval. For example, for the time interval from $t = 15$ to $t = 30$, the average speed is given by the gradient of the line joining points A and B. This is $\frac{100}{15} = 6\frac{2}{3}$ metres per second.

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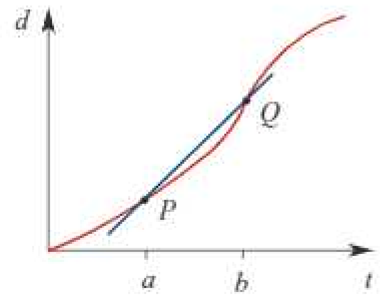
In general:

$$\text{average speed} = \frac{\text{total distance travelled}}{\text{total time taken}}$$

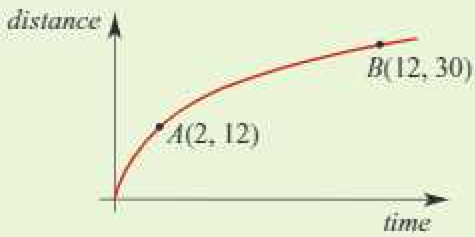
So the average speed of an object for $a \le t \le b$ is given by the gradient of the line passing through points P and Q.



Example 6

The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown.

Find the average speed of the object in m/s over the interval from $t = 2$ to $t = 12$.



Solution

$$\begin{aligned}\text{Average speed} &= \frac{\text{total distance travelled}}{\text{total time taken}}\\ &= \frac{30-12}{12-2}\\ &= \frac{18}{10}\\ &= 1.8 \text{ m/s}\end{aligned}$$

Average rate of change for a function

The line which passes through two points on a curve is called a **secant**.

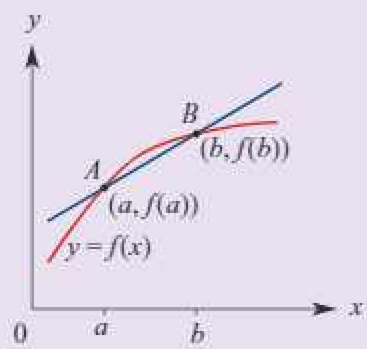
The line segment joining two points on a curve is called a **chord**.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through $A(a, f(a))$ and $B(b, f(b))$ (secant AB).

That is,

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



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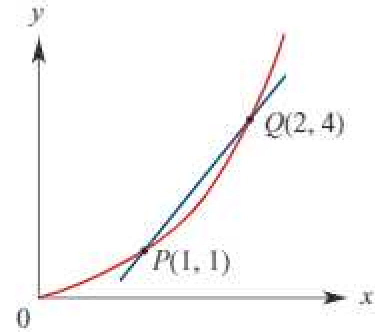
For example, for the function with the graph shown, the average rate of change of y with respect to x over the interval $[1, 2]$ is given by the gradient of the secant PQ:

$$\text{gradient} = \frac{4-1}{2-1} = 3$$



Example 7

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

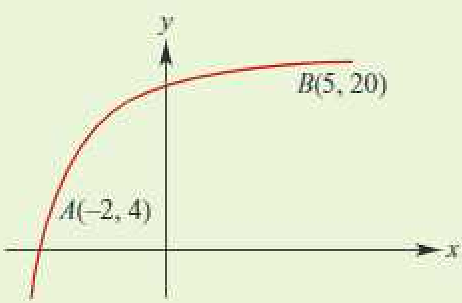
$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

$$\text{Average rate of change} = \frac{20-4}{5-1} = 4$$

Example 8

Find the average rate of change of the function depicted in the graph for the interval $[-2, 5]$.



Solution

Average rate of change for the interval $[-2, 5]$

$$= \frac{20-4}{5-(-2)}$$

$$= \frac{16}{7}$$

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Example 9

The air temperature, T°C, at a weather station on a particular evening is modelled by the equation $T = \dfrac{600}{t^2 + 2t + 30}$, where t is the time in hours after 6 p.m.

a Find the temperature at 6 p.m.

b Find the temperature at midnight.

c Find the average rate of change of the air temperature from 6 p.m. until midnight.

Solution

a At 6 p.m., $t = 0$. Hence

$$T = \frac{600}{(0)^2 + 2(0) + 30} = 20^\circ\text{C}$$

b At midnight, $t = 6$. Hence

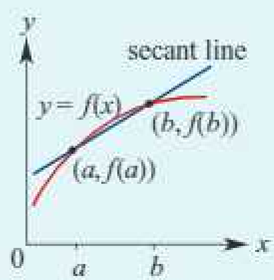
$$T = \frac{600}{(6)^2 + 2(6) + 30} = \frac{100}{13} = 7.69^\circ\text{C} \quad \text{(correct to two decimal places)}$$

c Average rate of change of temperature $= \dfrac{\frac{100}{13} - 20}{6 - 0} = -\dfrac{80}{39} = -2.05^\circ\text{C}$ per hour

Summary 16C

- The line which passes through two points on a curve is called a **secant**.
- The line segment joining two points on a curve is called a **chord**.
- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.
  That is,

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$

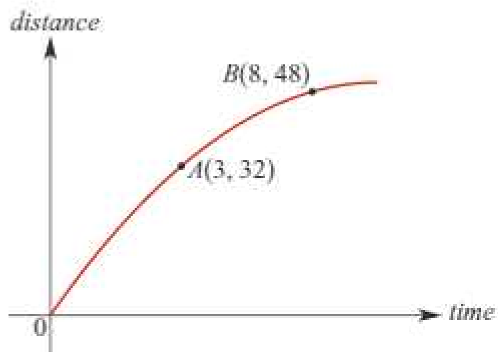


Skill-sheet

Exercise 16C

Example 6

1 The graph of distance travelled (metres) against time (seconds) for the motion of an object is shown. Find the average speed of the object in m/s over the interval from $t = 3$ to $t = 8$.



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Example 7 **2** For each function, find the average rate of change over the stated interval:

a $f(x) = 2x + 5, \quad x \in [0, 3]$

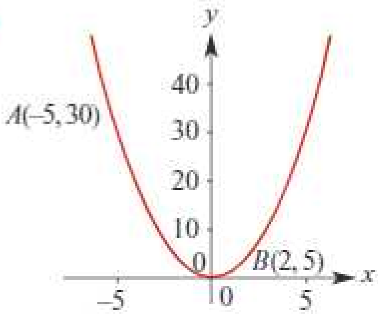
b $f(x) = 3x^2 + 4x - 2, \quad x \in [-1, 2]$

c $f(x) = \dfrac{2}{x-3} + 4, \quad x \in [4, 7]$

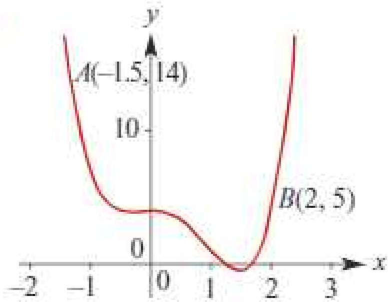
d $f(x) = \sqrt{5 - x}, \quad x \in [0, 4]$

Example 8 **3** Find the average rate of change of y with respect to x from point A to point B for each of the following graphs:

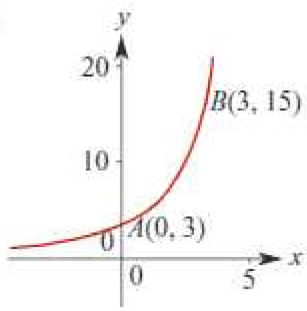
a



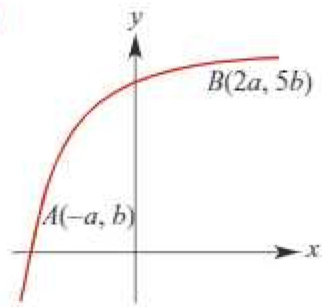
b



c



d



4 The distance (in metres) from a point O of an object t seconds after it starts to move in a straight line is given by the function $S(t) = t^3 + t^2 - 2t + 2, t \geq 0$. Find the average rate of change with respect to time of the distance of the object from O:

a in the first 2 seconds

b in the next 2 seconds.

5 A person invests \$2000 dollars, which increases in value by 7% per year for three years.

a Calculate the value of the investment after three years.

b Calculate the average rate of change in the value of the investment over that time.

Example 9 **6** The depth, d cm, of water in a bath tub t minutes after the tap is turned on is modelled by the function $d(t) = \dfrac{-300}{t+6} + 50, t \geq 0$. Find the average rate of change of the depth of the water with respect to time over the first 10 minutes after the tap is turned on.

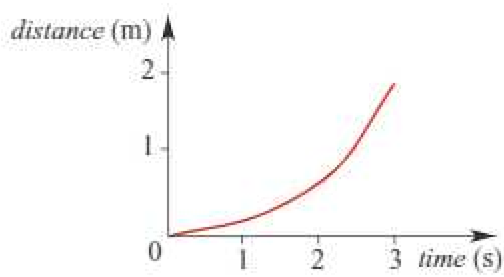
7 Using the information in the graph on the right, the average speed from $t = 0$ to $t = 3$ is

A 2 m/s

B 1 m/s

C $\dfrac{2}{3}$ m/s

D $1\frac{1}{2}$ m/s



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16D Instantaneous rate of change

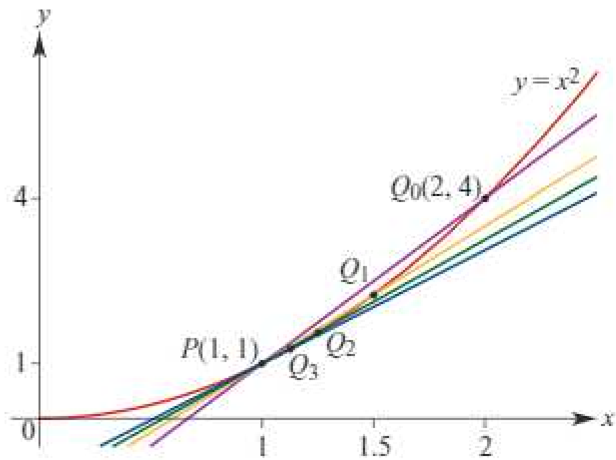
In the previous section, we investigated the average rate of change of a function over a stated interval. We saw that, in general (except for linear functions), the average rate of change of a function over different intervals in the domain of the function is not constant.

In this section, we investigate the idea of instantaneous rate of change.

Tangent line at a point

We talk about the idea of the tangent to a curve at a point informally here. It is a line which has 'the same slope' as the graph at this point. Of course, we don't know exactly what this means, but if we think of a very, very, ..., very small section of the curve around the point, we can consider it to be a line segment which can be extended out to a straight line. This straight line is what we call the 'tangent line' to the curve at the point.

We can illustrate this idea with a specific example. Part of the graph of $y = x^2$ is shown below. We will find the tangent line at the point $P(1, 1)$.



We start with the secant PQ_0 passing through $P(1, 1)$ and $Q_0(2, 4)$.

The gradient of PQ_0 is $\dfrac{4-1}{2-1} = 3$, and so the equation of the secant PQ_0 is $y = 3x - 2$.

The points $Q_1, Q_2, Q_3, \dots, Q_n, \dots$ on the curve $y = x^2$ are chosen so that they get closer and closer to P in the following way:

- The x-coordinate of Q_1 is $\frac{1}{2}(1 + 2) = \frac{3}{2}$.
- The x-coordinate of Q_2 is $\frac{1}{2}\left(1 + \frac{3}{2}\right) = \frac{5}{4}$.
- The x-coordinate of Q_3 is $\frac{1}{2}\left(1 + \frac{5}{4}\right) = \frac{9}{8}$.

We now look at the sequence of secants $PQ_0, PQ_1, PQ_2, PQ_3, \dots, PQ_n, \dots$. The following table shows the gradient and the equation for each secant.

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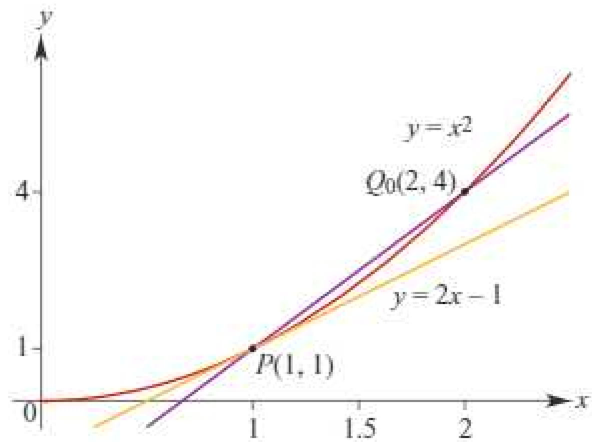
Secants of the curve $y = x^2$ through the point $P(1, 1)$

Step	Point on curve	Secant	Gradient	Equation of secant
0	$Q_0(2, 4)$	PQ_0	3	$y = 3x - 2$
1	$Q_1(\frac{3}{2}, \frac{9}{4})$	PQ_1	$\frac{5}{2}$	$y = \frac{5}{2}x - \frac{3}{2}$
2	$Q_2(\frac{5}{4}, \frac{25}{16})$	PQ_2	$\frac{9}{4}$	$y = \frac{9}{4}x - \frac{5}{4}$
3	$Q_3(\frac{9}{8}, \frac{81}{64})$	PQ_3	$\frac{17}{8}$	$y = \frac{17}{8}x - \frac{9}{8}$
n	$Q_n(1 + 2^{-n}, (1 + 2^{-n})^2)$	PQ_n	$2 + 2^{-n}$	$y = (2 + 2^{-n})x - (1 + 2^{-n})$

The sequence of gradients is $3, \frac{5}{2}, \frac{9}{4}, \frac{17}{8}, \dots, 2 + \frac{1}{2^n}, \dots$

We can see that the gradients get closer and closer to 2. This is particularly evident from the general gradient, $2 + \frac{1}{2^n}$, since as $n \to \infty$, $\frac{1}{2^n} \to 0$.

We can also see that the secants get closer and closer to the line with equation $y = 2x - 1$. This line is the **tangent line** at the point P, and the gradient of the tangent line is the **instantaneous rate of change** of y with respect to x at the point P. We define these two concepts in Chapters 17 and 18.



In the examples and exercises in this chapter, we only require approximations to the instantaneous rate of change.

Example 10

Estimate the instantaneous rate of change of y with respect to x at the point $P(2, 9)$ on the curve $y = x^3 + 1$ by considering the secant PQ, where $Q = (2.01, (2.01)^3 + 1)$.

Solution

$$\text{Gradient of } PQ = \frac{(2.01)^3 + 1 - 9}{2.01 - 2} = 12.0601$$

Note: An even better approximation can be made by choosing the points $P(2, 9)$ and $Q(2.001, (2.001)^3 + 1)$. Using the approach taken for $y = x^2$ in the discussion above, we would find that the instantaneous rate of change for this example is 12.

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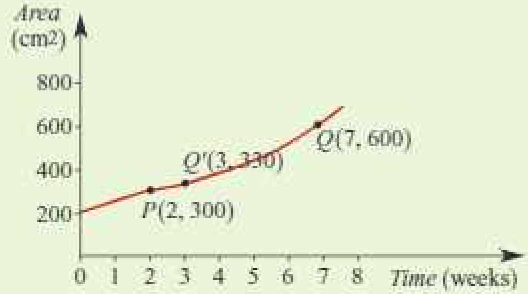
Example 11

The graph represents the area covered by a spreading plant. Area is measured in square centimetres and time in weeks.

a Find the gradient of the secant PQ.

b The point Q' has coordinates $(3, 330)$. Find the average rate of change of area with respect to time for the interval $[2, 3]$, and hence estimate the instantaneous rate of change of the area of the plant at $t = 2$.



Solution

a Gradient of $PQ = \dfrac{600 - 300}{7 - 2}$

$= \dfrac{300}{5}$

$= 60$

The average rate of change of area from $t = 2$ to $t = 7$ is 60 cm^2 per week.

b Gradient of $PQ' = \dfrac{330 - 300}{1} = 30$

$\therefore$ Gradient at P is approximately 30.

The instantaneous rate of change of the area of the plant with respect to time when $t = 2$ is approximately 30 cm^2 per week.

Example 12

Consider the curve $y = 2^x$.

a Using the secant through the points where $x = 3$ and $x = 3.1$, estimate the instantaneous rate of change of y with respect to x at the point where $x = 3$.

b Repeat for the points where $x = 3$ and $x = 3.001$.

Solution

a When $x = 3$, $y = 8$ and when $x = 3.1$, $y = 8.5742$ (correct to four decimal places).
The gradient of the line through $(3, 8)$ and $(3.1, 8.5742)$ is 5.7419.
Thus an estimate for the instantaneous rate of change of $y = 2^x$ at $x = 3$ is 5.742.

b When $x = 3.001$, $y = 8.005547$.
The gradient of the line through $(3, 8)$ and $(3.001, 8.005547)$ is 5.547.

Note: The true instantaneous rate of change of y with respect to x at $x = 3$ is 5.5452 (correct to four decimal places).

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Using the graph window of your calculator

The graph of $y = 0.5x^3 - 2x + 1$ is shown. We will investigate the gradient at the point $(0, 1)$.

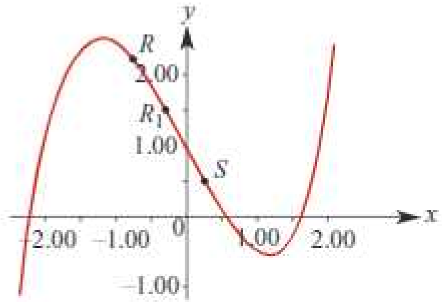
First find the gradient of the secant RS, where $R = (-0.75, 2.2891)$ and $S = (0.25, 0.5078)$:

gradient of $RS = -1.7813$

(The coordinates of R and S are given to four decimal places.)

Now consider another secant R_1S, where $R_1 = (-0.25, 1.4922)$ and $S = (0.25, 0.5078)$:

gradient of $R_1S = -1.9688$

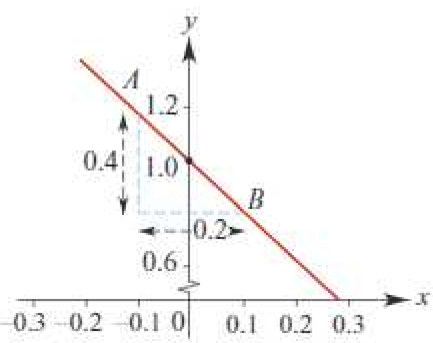


Using a calculator, zoom in on the graph near $x = 0$. As you zoom further in, this section of the curve appears increasingly linear. By assuming that this section of the curve is in fact linear and finding its gradient, we can approximate the gradient of the curve at $x = 0$.

This diagram shows a 'zoomed in' section of the graph around the point $(0, 1)$.

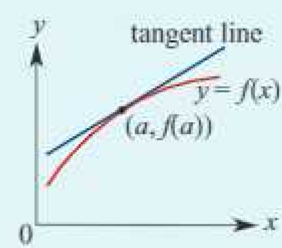
Consider the secant AB where $A = (-0.1, 1.1995)$ and $B = (0.1, 0.8005)$. The gradient of this line is approximately -2.

Therefore we make the approximation that the gradient of the curve $y = 0.5x^3 - 2x + 1$ at the point $(0, 1)$ is -2.



Summary 16D

For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



Exercise 16D

Example 10

1 By considering the secant through the points where $x = 1.2$ and $x = 1.3$, estimate the instantaneous rate of change of y with respect to x at the point on the curve $y = x^3 + x^2$ where $x = 1.3$.

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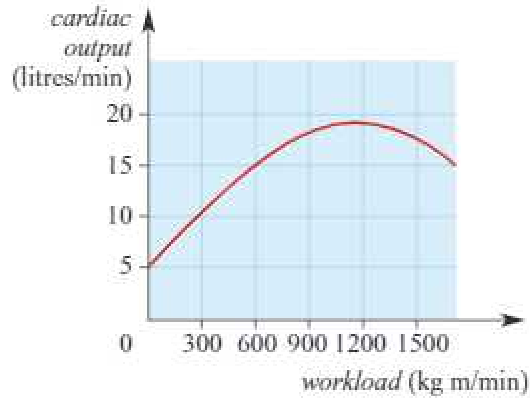
Example 11

2 Cardiac output is an important factor in athletic endurance. The graph shows a stress-test graph of cardiac output (measured in litres/min of blood) versus workload (measured in kg m/min).

a Estimate the average rate of change of cardiac output with respect to workload as the workload increases from 0 to 1200 kg m/min.

b Estimate the instantaneous rate of change of cardiac output with respect to workload at the point where the workload is 450 kg m/min.



Example 12

3 Let $y = 10^x$.

a Find the average rate at which y changes with respect to x over each of the following intervals:

i $[0, 1]$ **ii** $[0, 0.5]$ **iii** $[0, 0.1]$

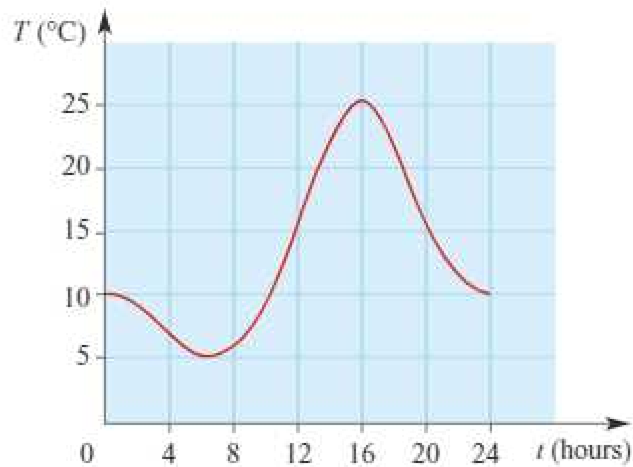
b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

4 Temperature (T°C) varies with time (t hours) over a 24-hour period, as illustrated in the graph.

a Estimate the maximum temperature and the time at which this occurs.

b The temperature rise between 10:00 and 14:00 is approximately linear. Estimate the rate at which the temperature is increasing in this period.

c Estimate the instantaneous rate of change of temperature at $t = 20$.



5 By considering the secant through the points at which $x = 1.2$ and $x = 1.4$, estimate the instantaneous rate of change of y with respect to x of the curve $y = \frac{1}{x}$ at $x = 1.2$.

6 Draw the graph of $y = \sqrt{16 - x^2}$, $-4 \le x \le 4$. Use an appropriate technique to find an estimate of the instantaneous rate of change of y with respect to x at the points:

a $x = 0$ **b** $x = 2$ **c** $x = 3$

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7 It is known that the straight line $y = 4x - 4$ touches the curve $y = x^2$ at the point $(2, 4)$. Sketch the graphs of both of these functions on the one set of axes.
Find the instantaneous rate of change of y with respect to x at the point at $(2, 4)$ on the curve $y = x^2$.

8 Water is being collected in a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 3t^2 + 4t + 2$.

- **a** Find the average rate of change of volume with respect to time between times $t = 1$ and $t = 3$.
- **b** Find an estimate for the instantaneous rate of change of volume with respect to time at $t = 1$.

9 A population of bacteria is growing. The population, P million, after time t minutes is given by $P = 3 \times 2^t$.

- **a** Find the average rate of change of population between times $t = 2$ and $t = 4$.
- **b** Find an estimate for the instantaneous rate of change of population with respect to time at $t = 2$.

10 Water is flowing out of a water tank. The volume, V cubic metres, of water in the tank after t minutes is given by $V = 5 \times 10^5 - 10^2 \times 2^t,\ 0 \le t \le 12$.

- **a** Find the average rate of change of volume with respect to time between times $t = 0$ and $t = 5$.
- **b** Find an estimate for the instantaneous rate of change of volume with respect to time when $t = 6$.
- **c** Find an estimate for the rate of change of volume when $t = 12$.

11 Use the technique of Examples 10 and 12 to estimate the instantaneous rate of change of y with respect to x for each of the following at the stated point:

- **a** $y = x^3 + 2x^2,\quad (1, 3)$
- **b** $y = 2x^3 + 3x,\quad (1, 5)$
- **c** $y = -x^3 + 3x^2 + 2x,\quad (2, 8)$
- **d** $y = 2x^3 - 3x^2 - x + 2,\quad (3, 26)$

12 The volume, V, of a cube with edge length x is given by $V = x^3$.

- **a** Find the average rate at which the volume of the cube changes with respect to x, as x increases from $x = 2$ to $x = 4$.
- **b** Find an estimate for the instantaneous rate at which V changes with respect to x when $x = 2$.

13 Let $y = 2x^2 - 1$.

- **a** Find the average rate at which y changes with respect to x over the interval $[1, 4]$.
- **b** Find an estimate for the instantaneous rate at which y changes with respect to x when $x = 1$.

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14 Let $y = \sin x$.

a Find the average rate at which y changes with respect to x over each of the following intervals:

i $\left[0, \frac{\pi}{2}\right]$ **ii** $\left[0, \frac{\pi}{4}\right]$ **iii** $[0, 0.5]$ **iv** $[0, 0.1]$

b Estimate the instantaneous rate of change of y with respect to x when $x = 0$.

16E Position and average velocity

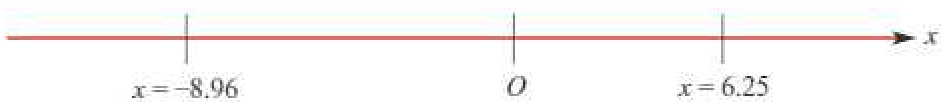
One of the key applications of rates of change is in the study of the motion of a particle.

In this section, we consider motion in a straight line. The study of motion in a straight line is continued in Chapter 18.

Position

The **position** of a particle is a specification of its location relative to a reference point.

Consider motion on a straight line with reference point O.



We say that position to the right of O is positive and to the left of O is negative.

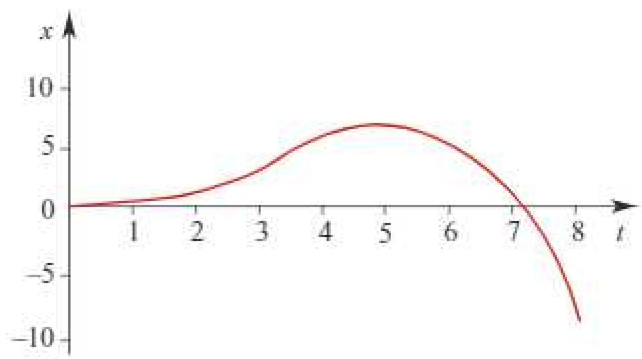
A particle is moving along the straight line. Let x metres denote its position relative to O at time t (where time is measured in seconds).

- At time $t = 0$, $x = 0$.
- At time $t = 5$, $x = 6.25$.
- At time $t = 8$, $x = -8.96$.

At $t = 0$, the particle starts from rest and moves to the right. At $t = 5$, the particle stops and moves back in the opposite direction. Its position–time graph is shown below.

Note that from $t = 0$ until $t = 7.1$, the position is positive, i.e. the particle is to the right of O.

For $t > 7.1$, the position is negative, i.e. the particle is to the left of O.



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Average velocity

The average velocity of a particle is the average rate of change in position with respect to time:

$$\textbf{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

For the moving particle we have been considering:

- At time $t = 0$, $x = 0$.
- At time $t = 5$, $x = 6.25$.
- At time $t = 8$, $x = -8.96$.

Therefore the particle's average velocity over the time interval $0 \le t \le 5$ is given by

$$\text{average velocity} = \frac{6.25}{5} = 1.25 \text{ metres per second (m/s)}$$

and over the time interval $5 \le t \le 8$,

$$\text{average velocity} = \frac{-8.96 - 6.25}{3} = -5.07 \text{ metres per second (m/s)}$$

Example 13

Let $s(t) = 6t - t^2$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval $[0, 1]$.

b Find the average velocity for the time interval $[6, 8]$.

Solution

a $s(0) = 0$ and $s(1) = 6 - 1^2 = 5$.

Therefore

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{5-0}{1-0} = 5\end{aligned}$$

The average velocity for $[0, 1]$ is 5 m/s.

b $s(6) = 0$ and $s(8) = 6 \times 8 - 8^2 = -16$.

Therefore

$$\begin{aligned}\text{average velocity} &= \frac{\text{change in position}}{\text{change in time}}\\ &= \frac{-16-0}{8-6} = -8\end{aligned}$$

The average velocity for $[6, 8]$ is -8 m/s.

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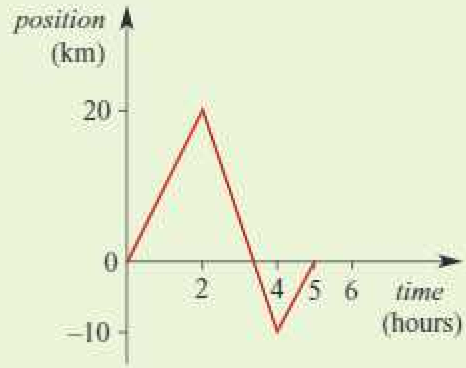
Velocity–time graphs

In the next two examples we look at the velocity–time graph obtained from a position–time graph. Each of the position–time graphs is the graph of a piecewise-defined function where each of the components is linear.

Example 14

The graph shown is the position–time graph for the bicycle trip of a boy who lives on a long straight road. The road runs north–south, and north is chosen to be the positive direction.



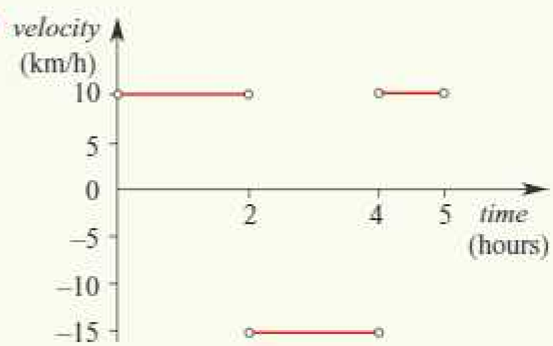
a Describe his trip.

b Draw the corresponding velocity–time graph.

Solution

a
- The boy heads north for 2 hours.
  His velocity for this period is $\frac{20-0}{2} = 10$ km/h.
- He then turns and rides south for 2 hours.
  His velocity for this period is $\frac{-10-20}{2} = -15$ km/h.
- He turns and rides north until he reaches home.
  His velocity for this period is $\frac{0-(-10)}{1} = 10$ km/h.

b



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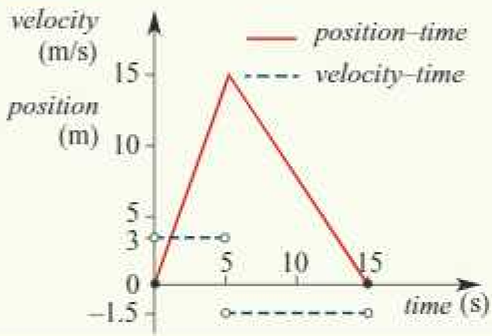
Example 15

A particle is moving in a straight line. It was initially at rest at a point O. It moves to the right of O with a constant velocity and reaches a point A, 15 metres from O, after 5 seconds. It then returns to O. The return trip takes 10 seconds. It stops at O.

On the one set of axes draw the position–time graph and the velocity–time graph for the motion.

Solution



Explanation

The gradient of the position–time graph for $0 < t < 5$ is 3.

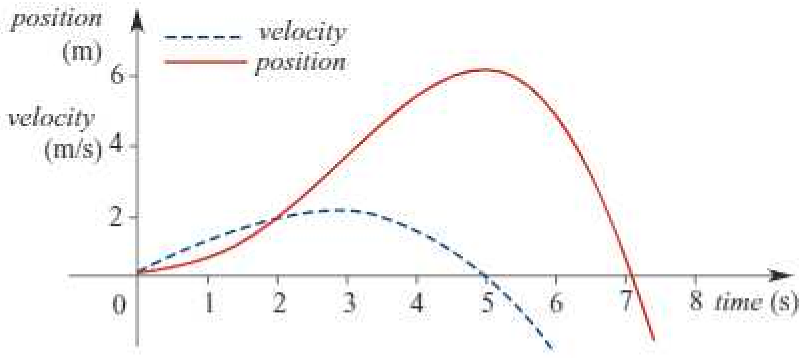
The gradient for $5 < t < 15$ is -1.5.

The gradient of the position–time graph determines the velocity–time graph.

Instantaneous velocity

Instantaneous velocity is the instantaneous rate of change in position with respect to time. It can be thought of as the gradient of the tangent to the position–time graph at a particular point. If we know an object's instantaneous velocity at every moment in time, we can sketch a velocity–time graph for an object moving with non-constant velocity.

In the graph shown below, the position–time graph is the same as the one from the start of this section.



This graph illustrates the relationship between position and velocity. The vertical axis is in both metres per second for velocity and metres for position. In the following, we use velocity to mean instantaneous velocity.

We make the following observations from the graph:

- For $0 < t < 5$, the velocity is positive: the particle is travelling from left to right.
- For $t > 5$, the velocity is negative: the particle is travelling from right to left.
- For $t = 5$, the velocity is zero: the particle is instantaneously at rest.

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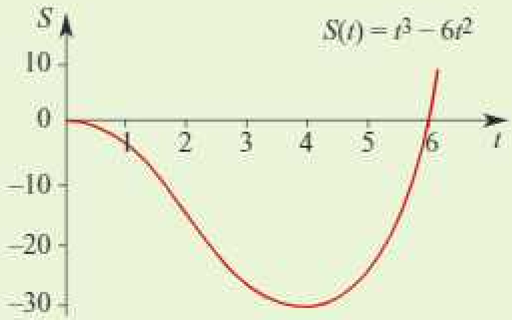
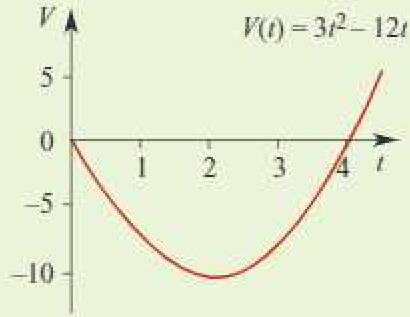
Example 16

The position of a particle moving in a straight line is given by the function $S(t) = t^3 - 6t^2$, $t \geq 0$. The graph of S against t is shown. The corresponding velocity–time graph is also shown. The function describing the velocity is $V(t) = 3t^2 - 12t$.





a Find the average velocity of the particle for the intervals:

i [3.5, 4.5] **ii** [3.9, 4.1] **iii** [3.99, 4.01]

b From part **a**, what is the instantaneous velocity when $t = 4$?

c **i** For what values of t is the velocity positive?

ii For what values of t is the velocity negative?

Solution

a **i** Average velocity $= \dfrac{S(4.5) - S(3.5)}{1} = \dfrac{-30.375 + 30.625}{1} = 0.25$

ii Average velocity $= \dfrac{S(4.1) - S(3.9)}{0.2} = \dfrac{-31.939 + 31.941}{0.2} = 0.01$

iii Average velocity $= \dfrac{S(4.01) - S(3.99)}{0.02} = \dfrac{-31.999399 + 31.999401}{0.02} = 0.0001$

b The results of part **a** suggest that the instantaneous velocity is zero when $t = 4$, and this is consistent with both graphs.

c **i** From the position–time graph, the velocity is positive for $t > 4$.

ii From the position–time graph, the velocity is negative for $0 < t < 4$.

Summary 16E

- The **position** of a particle moving along a straight line specifies its location relative to a reference point O.
- The average velocity of a particle is the average rate of change in position with respect to time:

$$\textbf{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

- Instantaneous velocity is the instantaneous rate of change in position with respect to time. It can be thought of as the gradient of the tangent to the position–time graph at a particular point.

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Exercise 16E

Example 13

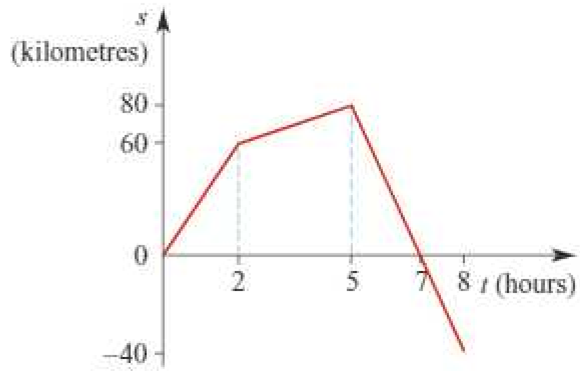
1 Let $s(t) = 6t - 2t^3$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval $[0, 1]$.

b Find the average velocity for the time interval $[0.8, 1]$.

Example 14

2 The following is the position–time graph for a train travelling on a straight track. Position is measured from the door of the ticket office at Jimbara station.



a What was the train's velocity over each of the following time intervals:

i $[0, 2]$

ii $[2, 5]$

iii $[5, 8]$

b Describe the train journey.

c Draw a velocity–time graph for the train's motion for the interval $[0, 8]$.

Example 15

3 The motion of a particle moving in a straight line is described by the following information:

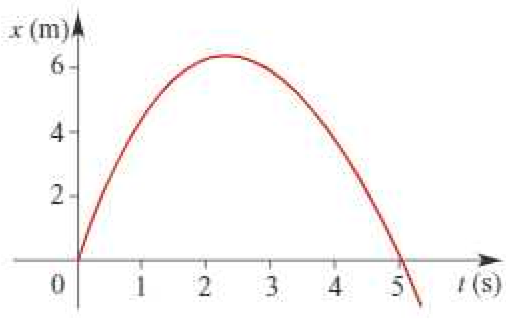
- For the time period $(0, 2)$, velocity $= -3$.
- For the time period $(2, 5)$, velocity $= 3$.
- For the time period $(5, 7)$, velocity $= 4$.

a Draw the velocity–time graph for the interval $[0, 7]$.

b Draw the position–time graph for the interval $[0, 7]$.

Example 16

4 A particle moves along a horizontal straight line. It starts from rest at a point O. The graph is the position–time graph for this motion.



a At what time is the instantaneous velocity zero?

b For which values of t is the instantaneous velocity positive?

c How far from O does the particle go to the right?

d How long does it take to return to O?

e Estimate the instantaneous velocity of the particle at $t = 1$.

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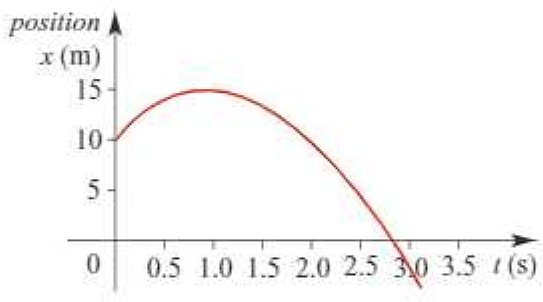
5 The table shows the distance, d metres, of a ball from its starting position at time t seconds after being thrown into the air.

t	0	1	2	3	4	5	6
d	0	25	40	45	40	25	0

Using the scales 2 cm = 1 second and 1 cm = 5 metres, draw the graph of d against t. From your graph find:

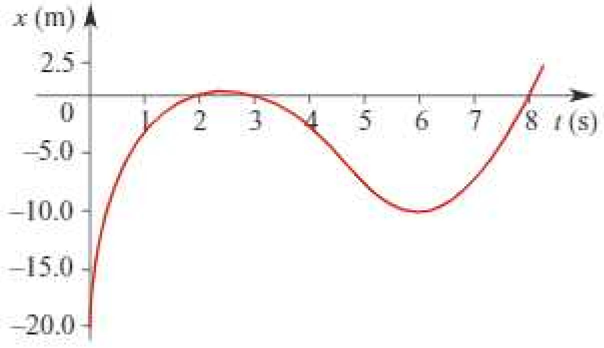
a when the ball returns to the starting point
b the average velocity of the ball from $t = 1$ to $t = 2$
c the average velocity of the ball from $t = 1$ to $t = 1.5$
d an estimate of the velocity of the ball when $t = 1$
e an estimate of the velocity of the ball when $t = 4$
f an estimate of the velocity of the ball when $t = 5$.

6 A stone is thrown vertically upwards from the edge of a platform which is 10 m above the ground. The position–time graph for the motion of the stone is shown. The motion of the stone is in a straight line and the reference point for position is taken as a point at ground level, directly below where the stone was thrown.



a From the graph, estimate the instantaneous velocity with which the stone is thrown.
b What is the maximum height reached by the stone?
c At what time does the stone reach its maximum height?
d At what time does the stone hit the ground?
e From the graph, estimate the instantaneous speed at which the stone hits the ground.

7 A particle is moving in a horizontal straight line. Position is measured from a point O. The particle starts at a point 20 m to the left of O. The position–time graph for the motion of the particle is as shown.



a At which times is the particle at O?
b For which values of t is the particle moving to the right?
c For which values of t is the particle stationary?

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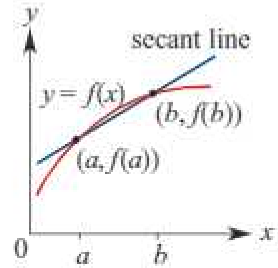
Chapter summary

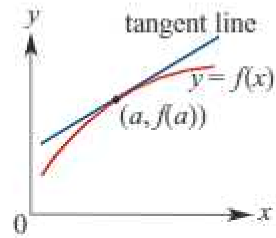
Assign-ment

Nrich

- Where a real-life situation is modelled by a straight-line graph, the gradient represents the rate of change of one quantity with respect to another.
- Average speed $= \dfrac{\text{total distance travelled}}{\text{total time taken}}$
- For a function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the secant line through $(a, f(a))$ and $(b, f(b))$.



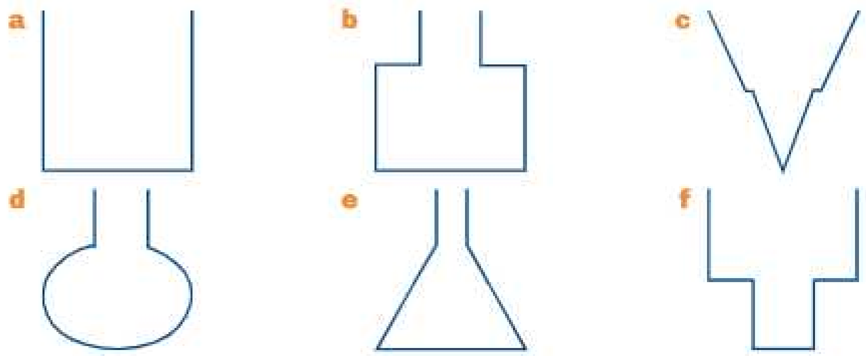
- For a function $y = f(x)$, the **instantaneous rate of change** of y with respect to x at the point $(a, f(a))$ is the gradient of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.



- The **position** of a particle moving along a straight line specifies its location relative to a reference point O.
- The **average velocity** of the particle is the average rate of change in position with respect to time over a given time interval.
- The **instantaneous velocity** is the instantaneous rate of change in position with respect to time at a given moment.

Technology-free questions

1 A liquid is poured at a constant rate into each of the containers shown below. For each container, draw a graph to show how the depth of the water varies with time.



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2 a A car travels from New Delhi to Agra, a distance of 200 km, in 3 hours. Assuming the car travels at a constant speed, draw a distance–time graph and calculate this constant speed. For the distance–time graph, use minutes as the unit of time.

b A girl walks at a constant speed of 1.5 m/s for 20 seconds. Draw a distance–time graph to illustrate this.

c A car travels along a road for $6\frac{1}{2}$ hours. The trip can be described in three sections:

Section A Car travels at a constant speed of 40 km/h for 1 hour.

Section B Car travels at a constant speed of 80 km/h for $3\frac{1}{2}$ hours.

Section C Car travels at a constant speed of 90 km/h for 2 hours.

Draw a distance–time graph which illustrates this motion.

3 The surface area, s cm^2, of a cube is given by the formula $s = 6x^2$, where x cm is the length of each edge of the cube. Find the average rate at which the surface area changes with respect to x, as x increases from $x = 2$ to $x = 4$.

4 Let $y = x^3$. Find the average rate at which y changes with respect to x over each of the following intervals:

a $[0, 1]$ b $[1, 3]$

5 Let $s(t) = 4t - 6t^3$ be the position function of a particle moving in a straight line, where t is in seconds and s is in metres.

a Find the average velocity for the time interval $[0, 1]$.

b Find the average velocity for the time interval $[0.9, 1]$.

c Estimate the instantaneous velocity for $t = 1$.

Multiple-choice questions

1 A bushwalker walks 12 km in 2 hours, stops for 45 minutes and then walks a further 8 km in another 1.25 hours. The average walking speed of the bushwalker over the entire walk is

A 10 km/h B 9 km/h C 5 km/h D 4 km/h E 7.2 km/h

2 Postal workers sort 12 000 parcels during the normal day shift of 8 hours and, with a reduced workforce during the 2 hours overtime shift, they sort a further 2500 parcels. The average rate of parcel sorting per hour is

A 1375 parcels per hour B 1450 parcels per hour C 1300 parcels per hour
D 1400 parcels per hour E 1500 parcels per hour

3 The average rate of change of the function $y = 3 \times 2^x$ over the interval $[0, 2]$ is

A 9 B 4.5 C 12 D 6 E 5

4 Given $f(x) = 2x^3 + 3x$, the average rate of change of $f(x)$ with respect to x for the interval $[-2, 2]$ is

A 0 B -22 C -11 D 22 E 11

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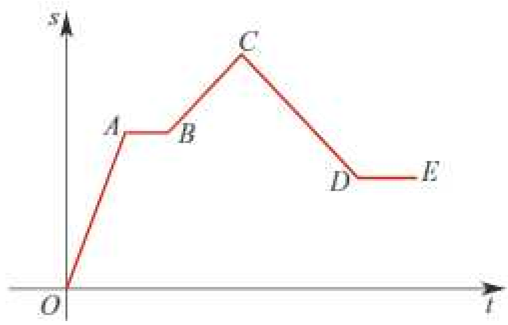
Questions 5–7 refer to the following information:

The graph shows the movement of a vehicle over a period of time. It represents the distance (s) from a fixed point at a given time (t).



5 The line segment OA represents a stage of the movement during which the vehicle is

A speeding up **B** slowing down **C** travelling north
D travelling at a constant speed **E** stationary

6 The line segment AB represents a stage of the movement during which the vehicle is

A speeding up **B** slowing down **C** travelling east
D travelling at a constant speed greater than zero **E** stationary

7 The sections of the graph which represent the vehicle when it is stationary are

A only at O **B** at A and C **C** between C and D
D between A and B and between D and E **E** at no time

8 The population of trout in a trout pond is growing. If the population, P, after t weeks is given by $P = 10 \times 1.1^t$, the average rate of growth of the population during the 5th week is closest to

A 16 trout per week **B** 15 trout per week **C** 1.5 trout per week
D 4 trout per week **E** 15.35 trout per week

Extended-response questions

1 A rock falls from the top of a high cliff. It falls y metres in t seconds, where $y = 4.9t^2$.

a Find the average speed of the rock between:

i $t = 0$ and $t = 2$ **ii** $t = 2$ and $t = 4$

b **i** How far has the rock fallen between $t = 4 - h$ and $t = 4$?

ii What is the average speed between $t = 4 - h$ and $t = 4$?

iii Find the average speed when $h = 0.2, 0.1, 0.05, 0.01, 0.001$.

2 A vending machine in a bus terminus contains cans of soft drink. On a typical day:

- the machine starts one-quarter full
- no drinks are sold between 1 a.m. and 6 a.m.
- the machine is filled at 2 p.m.

Sketch a graph to show how the number of cans in the machine may vary from 6 a.m. until midnight.

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Review

3 **a** $P(a, a^2)$ and $Q(b, b^2)$ are two points on the curve with equation $y = x^2$.
Find the gradient of the line joining the points. (Answer in terms of a and b.)

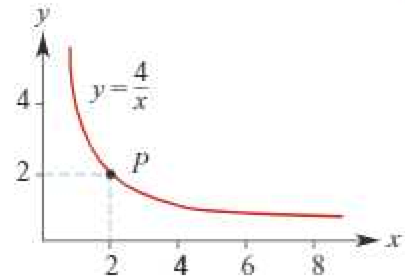
b Use this result to find the gradient of the line for points with $a = 1$ and $b = 2$.

c Use this result to find the gradient of the line for points with $a = 2$ and $b = 2.01$.

4 The figure shows part of the curve with equation $y = \frac{4}{x}$ and the point $P(2, 2)$.

CAS

a A_1 and A_2 are points on the curve whose x-coordinates are 1.5 and 2.5 respectively. Use your calculator to find their y-coordinates and hence find the gradient of A_1A_2.

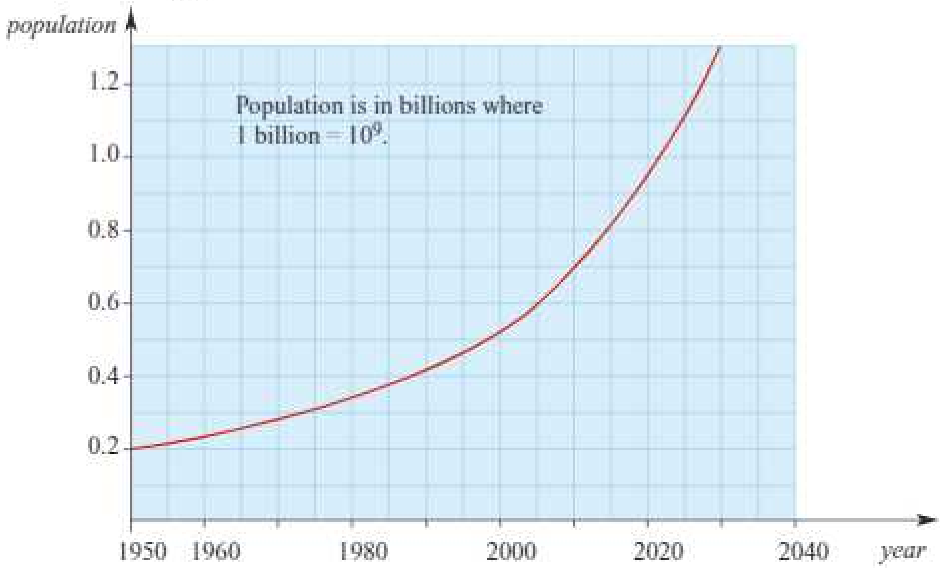


b Repeat for B_1 and B_2 whose x-coordinates are 1.9 and 2.1 respectively.

c Repeat for C_1 and C_2 whose x-coordinates are 1.99 and 2.01 respectively.

d Repeat for D_1 and D_2 whose x-coordinates are 1.999 and 2.001 respectively.

5 The graph below shows exponential growth in the size of the population of Acubaland. In exponential growth, the rate of increase of the population at any time is proportional to the size of the population at that time.



a From the graph, find the population of Acubaland in:

i 1960 **ii** 2000

b Calculate the average annual rate of population increase (in billions per year) over the years from 1960 to 2000.

c From the graph, estimate the rate of population increase in:

i 1960 **ii** 2000

d How many years do you expect that it will take to double the 2020 population? Explain your reasoning.

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6 **a** Draw the graph of $y = 10^x$ and find the gradient of the secant through the points:

i $x = 2.5$ and $x = 2.8$ **ii** $x = 2.6$ and $x = 2.8$

iii $x = 2.7$ and $x = 2.8$ **iv** $x = 2.75$ and $x = 2.8$

b Comment on your result and investigate further.

7 **a** Use the result that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ to find an expression for the gradient of the line joining points $P(a, a^3)$ and $Q(b, b^3)$ on the curve with equation $y = x^3$.

b Find the gradient of the line for $a = 1$ and $b = 2$.

c Find the gradient of the line for $a = 2$ and $b = 2.01$.

d For your expression for the gradient in terms of a and b (from part **a**), let $a = b$ and write your new expression in simplest terms. Interpret this result.

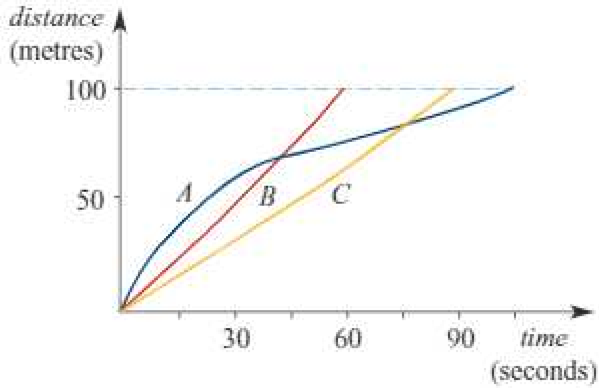
8 The rough sketch graph below shows what happens when three swimmers compete in a 100-metre race. (The vertical axis shows distance travelled by a swimmer.)

a Who wins the race?

b Who is in front at the 50 m mark?

c What is the approximate distance separating first and third place when the winner finishes?

d What is the approximate time difference between first and third place?



e What is the average speed of each swimmer?

f Describe the race as if you were a commentator.

9 In the following, $f(x)$ is the rule for a well-behaved function f.
Assume that, for $y = f(x)$, the average rate of change of y with respect to x is m, over the interval $[a, b]$. Find the average rate of change of y with respect to x over the same interval $[a, b]$ for:

a $y = f(x) + c$ **b** $y = cf(x)$ **c** $y = -f(x)$

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