17

Differentiation and antidifferentiation of polynomials

Objectives

- To understand the concept of **limit**.
- To understand the definition of the **derivative** of a function.
- To understand and use the notation for the **derivative** of a polynomial function.
- To find the **gradient of a tangent** to a polynomial function by calculating its derivative.
- To apply the rules for differentiating polynomials to solving problems.
- To be able to differentiate expressions of the form x^n where n is a negative integer.
- To understand and use the notation for the **antiderivative** of a polynomial function.

It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

In the previous chapter, we investigated the rate of change of one quantity with respect to another quantity. In this chapter, we will develop a technique for calculating the rate of change for polynomial functions.

To illustrate the idea, we start with an introductory example:

> On planet X, an object falls a distance of y metres in t seconds, where $y = 0.8t^2$.
> Can we find a general expression for the speed of such an object after t seconds?

(Note that, on Earth, the commonly used model is $y = 4.9t^2$.)

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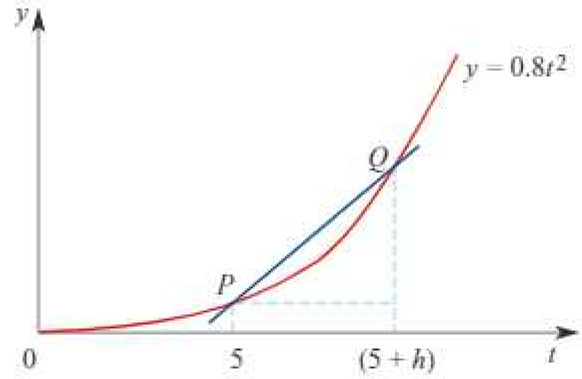
In the previous chapter, we found that we could approximate the gradient of a curve at a given point P by finding the gradient of a secant PQ, where Q is a point on the curve as close as possible to P.

The gradient of PQ approximates the speed of the object at P. The closer we make the point Q to the point P, the better the approximation.



Let P be the point on the curve where $t = 5$. Let Q be the point on the curve corresponding to h seconds after $t = 5$. That is, Q is the point on the curve where $t = 5 + h$.

$$\begin{aligned}\text{Gradient of } PQ &= \frac{0.8(5+h)^2 - 0.8 \times 5^2}{(5+h) - 5} \\ &= \frac{0.8((5+h)^2 - 5^2)}{h} \\ &= 0.8(10 + h)\end{aligned}$$

The table gives the gradient of PQ for different values of h. Use your calculator to check these.

h	Gradient of PQ
0.7	8.56
0.6	8.48
0.5	8.40
0.4	8.32
0.3	8.24
0.2	8.16
0.1	8.08

If we take values of h with smaller and smaller magnitude, then the gradient of PQ gets closer and closer to 8. So, at the point where $t = 5$, the gradient of the curve is 8.

Thus the speed of the object at the moment $t = 5$ is 8 m/s.

The speed of the object at the moment $t = 5$ is the limiting value of the gradients of PQ, as Q approaches P.

We want to find a general formula for the speed of the object at any time t.

Let P be the point with coordinates $(t, 0.8t^2)$ on the curve and let Q be the point with coordinates $(t + h, 0.8(t + h)^2)$.

$$\begin{aligned}\text{Gradient of } PQ &= \frac{0.8(t+h)^2 - 0.8t^2}{(t+h) - t} \\ &= 0.8(2t + h)\end{aligned}$$

Now consider the limit as h approaches 0, that is, the value of $0.8(2t + h)$ as h becomes arbitrarily small. This limit is $1.6t$.

So the gradient of the tangent to the curve at the point corresponding to time t is $1.6t$. Hence the speed at time t is $1.6t$ m/s.

This technique can be used to investigate the gradient of the tangent at a given point for similar functions.

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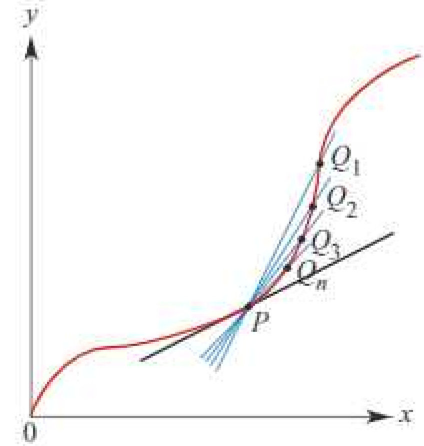
17A The derivative

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

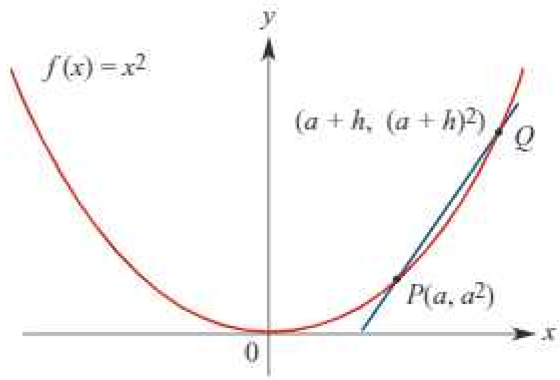
In the previous chapter we considered what happened when we looked at a sequence of secants $PQ_1, PQ_2, \dots, PQ_n, \dots$, where the points Q_i get closer and closer to P. The idea of instantaneous rate of change at P was introduced.

In this section we focus our attention on the gradient of the tangent at P.



The tangent to a curve at a point

Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.



The gradient of the secant PQ shown on the graph is

$$\begin{aligned}\text{gradient of } PQ &= \frac{(a+h)^2 - a^2}{a+h-a}\\ &= \frac{a^2 + 2ah + h^2 - a^2}{h}\\ &= 2a + h\end{aligned}$$

The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

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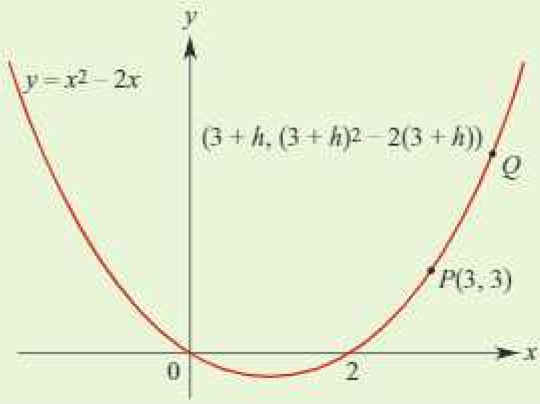
Example 1

By first considering the gradient of the secant PQ, find the gradient of the tangent line to $y = x^2 - 2x$ at the point P with coordinates $(3, 3)$.



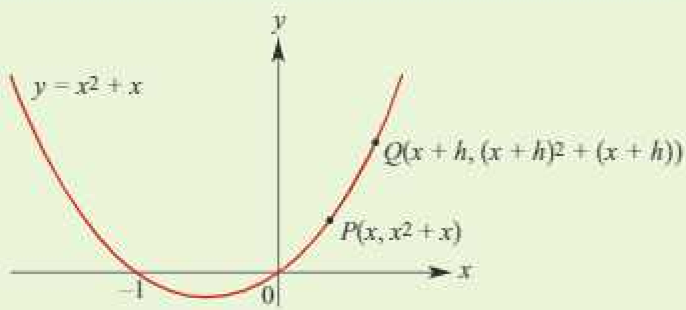
Solution

$$\text{Gradient of } PQ = \frac{(3+h)^2 - 2(3+h) - 3}{3 + h - 3}$$
$$= \frac{9 + 6h + h^2 - 6 - 2h - 3}{h}$$
$$= \frac{4h + h^2}{h}$$
$$= 4 + h$$

Now consider the gradient of PQ as h approaches 0. The gradient of the tangent line at the point $P(3, 3)$ is 4.

Example 2

Find the gradient of the secant PQ and hence find the derivative of $x^2 + x$.



Solution

$$\text{Gradient of } PQ = \frac{(x+h)^2 + (x+h) - (x^2 + x)}{x + h - x}$$
$$= \frac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$$
$$= \frac{2xh + h^2 + h}{h}$$
$$= 2x + h + 1$$

From this it is seen that the derivative of $x^2 + x$ is $2x + 1$.

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The expansion of $(a + b)^n$

You are already familiar with the identity

$$(a+b)^2 = a^2 + 2ab + b^2$$

By multiplying both sides by $a + b$, we can obtain new identities:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$
$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

You may recognise the pattern of coefficients from Pascal's triangle (see Section 10C). We will give the general expansion of $(a + b)^n$ in the next section.

Limit notation

The notation for the limit of $2x + h + 1$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h + 1)$$

The derivative of a function with rule $f(x)$ may be found by:

1 finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2 finding the limit of this expression as h approaches 0.

Example 3

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\begin{aligned}\text{Gradient of } PQ &= \frac{(2+h)^3 - 8}{2 + h - 2} \\ &= \frac{8 + 12h + 6h^2 + h^3 - 8}{h} \\ &= \frac{12h + 6h^2 + h^3}{h} \\ &= 12 + 6h + h^2\end{aligned}$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

The following example provides practice in determining limits.

Example 4

Find:

a $\lim_{h \to 0} (22x^2 + 20xh)$ **b** $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$ **c** $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} (22x^2 + 20xh) = 22x^2$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h} = \lim_{h \to 0} (3x^2 + 2h) = 3x^2$

c $\lim_{h \to 0} 4 = 4$

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Using the TI-Nspire

To calculate a limit, use [menu] > **Calculus** > **Limit** and complete as shown.

Note: The limit template can also be accessed from the 2D-template palette [template icon]. When you insert the limit template, you will notice a superscript field (small box) on the template – generally this will be left empty.

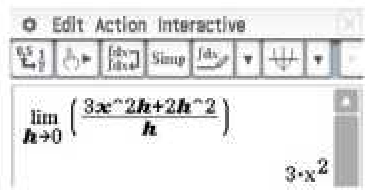
Using the Casio ClassPad

- In [Main $\sqrt{\alpha}$], enter and highlight the expression

$$\frac{3x^2h + 2h^2}{h}$$

Note: Use h from the [Var] keyboard.

- Select [lim] from the [Math2] keyboard.
- Enter h and 0 in the spaces provided as shown. Tap [EXE].



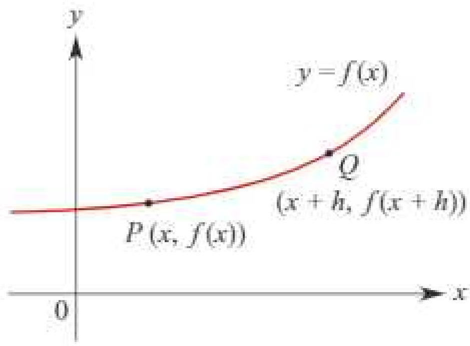
Definition of the derivative

In general, consider the graph $y = f(x)$ of a function $f: \mathbb{R} \to \mathbb{R}$.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x + h - x}$$

$$= \frac{f(x+h) - f(x)}{h}$$

The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.



Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Sections 17F and 17G.

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Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 5

For $f(x) = x^2 + 2x$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 + 2(x+h) - (x^2+2x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 + 2x + 2h - x^2 - 2x}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 + 2h}{h} \\ &= \lim_{h\to 0} (2x + h + 2) \\ &= 2x + 2 \end{aligned}$$

$$\therefore \quad f'(x) = 2x + 2$$

Example 6

For $f(x) = 2 - x^3$, find $f'(x)$ by first principles.

Solution

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2 - (x+h)^3 - (2 - x^3)}{h} \\ &= \lim_{h\to 0} \frac{2 - (x^3 + 3x^2h + 3xh^2 + h^3) - (2 - x^3)}{h} \\ &= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h} \\ &= \lim_{h\to 0} (-3x^2 - 3xh - h^2) \\ &= -3x^2 \end{aligned}$$

$$\therefore \quad f'(x) = -3x^2$$

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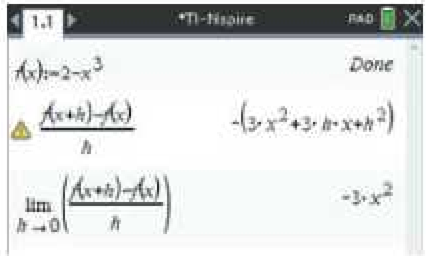
Using the TI-Nspire

- Assign the function $f(x)$ as shown.

Note: The assign symbol := is accessed using ctrl ⊞.

- Use menu > **Calculus** > **Limit** or the 2D-template palette ⊞, and complete as shown.



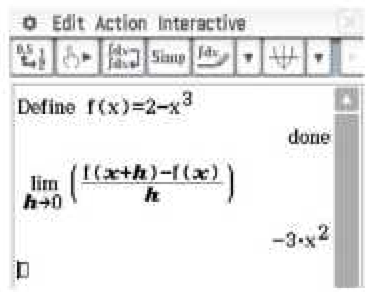
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $2 - x^3$. Select **Interactive** > **Define** and tap OK.
- Now enter and highlight the expression

$$\frac{f(x+h) - f(x)}{h}$$

Note: Select f from the Math3 keyboard and select x, h from the Var keyboard.

- Select lim from the Math2 keyboard.
- Enter h and 0 in the spaces provided as shown. Tap EXE.

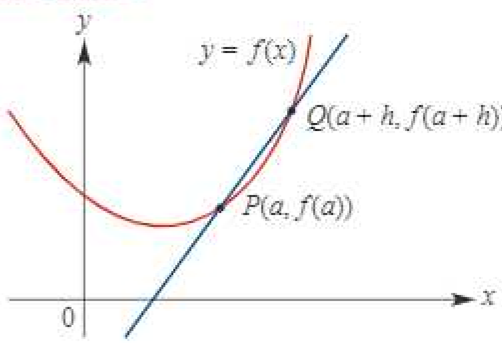


Approximating the value of the derivative

From the definition of the derivative, we can see that

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

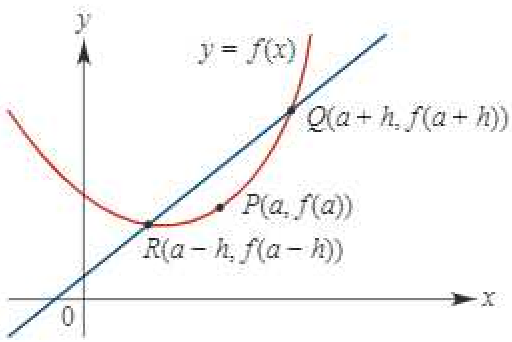
for a small value of h. This is the gradient of the secant through points $P(a, f(a))$ and $Q(a+h, f(a+h))$.



We can often obtain a better approximation by using

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

for a small value of h. This is the gradient of the secant through points $R(a-h, f(a-h))$ and $Q(a+h, f(a+h))$.



In Exercise 17A, you are asked to compare these two approximations for several functions.

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Summary 17A

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.
- The value of the derivative of f at $x = a$ can be approximated by

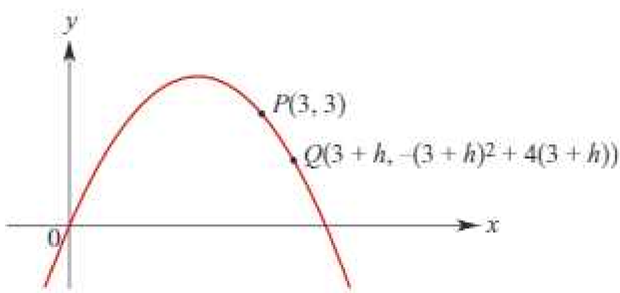
$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

for a small value of h.

Exercise 17A

Example 1

1 Let $f(x) = -x^2 + 4x$.
The graph of $y = f(x)$ is shown opposite.



a Find the gradient of PQ.

b Find the gradient of the tangent to the curve at P by considering what happens as h approaches 0.

2 Let $f(x) = x^2 - 3x$. Then the points $P(4, 4)$ and $Q(4 + h, (4 + h)^2 - 3(4 + h))$ are on the curve $y = f(x)$.

a Find the gradient of the secant PQ.

b Find the gradient of the tangent line to the curve at the point P by considering what happens as h approaches 0.

Example 2

3 The points $P(x, x^2 - 2x)$ and $Q(x + h, (x + h)^2 - 2(x + h))$ are on the curve $y = x^2 - 2x$. Find the gradient of PQ and hence find the derivative of $x^2 - 2x$.

Example 3

4 By first considering the gradient of the secant through $P(2, 16)$ and $Q(2 + h, (2 + h)^4)$ for the curve $y = x^4$, find the gradient of the tangent to the curve at the point $(2, 16)$.
Hint: $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

5 A space vehicle moves so that the distance travelled over its first minute of motion is given by $y = 4t^4$, where y is the distance travelled in metres and t the time in seconds. By finding the gradient of the secant through the points where $t = 5$ and $t = 5 + h$, calculate the speed of the space vehicle when $t = 5$.

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6 A population of insects grows so that the size of the population, P, at time t (days) is given by $P = 1000 + t^2 + t$. By finding the gradient of the secant through the points where $t = 3$ and $t = 3 + h$, calculate the rate of growth of the insect population at time $t = 3$.

Example 4

7 Find:

a $\lim_{h \to 0} (10x^2 - 5xh)$ **b** $\lim_{h \to 0} (20 - 10h)$ **c** $\lim_{h \to 0} \frac{2x^2h^3 + xh^2 + h}{h}$

d $\lim_{h \to 0} \frac{3x^2h - 2xh^2 + h}{h}$ **e** $\lim_{h \to 0} \frac{30hx^2 + 2h^2 + h}{h}$ **f** $\lim_{h \to 0} 5$

8 Find:

a $\lim_{h \to 0} \frac{(x+h)^2 + 2(x+h) - (x^2 + 2x)}{h}$ i.e. the derivative of $y = x^2 + 2x$

b $\lim_{h \to 0} \frac{(5+h)^2 + 3(5+h) - 40}{h}$ i.e. the gradient of $y = x^2 + 3x$ at $x = 5$

c $\lim_{h \to 0} \frac{(x+h)^3 + 2(x+h)^2 - (x^3 + 2x^2)}{h}$ i.e. the derivative of $y = x^3 + 2x^2$

9 For the curve with equation $y = 3x^2 - x$:

a Find the gradient of the secant PQ, where P is the point $(1, 2)$ and Q is the point $(1 + h,\ 3(1 + h)^2 - (1 + h))$.

b Find the gradient of PQ when $h = 0.1$.

c Find the gradient of the tangent to the curve at P.

10 For the curve with equation $y = \frac{2}{x}$:

a Find the gradient of the chord AB, where $A = (2, 1)$ and $B = \left(2 + h,\ \frac{2}{2+h}\right)$.

b Find the gradient of AB when $h = 0.1$.

c Find the gradient of the tangent to the curve at A.

11 For the curve with equation $y = x^2 + 2x - 3$:

a Find the gradient of the secant PQ, where P is the point $(2, 5)$ and Q is the point $(2 + h,\ (2 + h)^2 + 2(2 + h) - 3)$.

b Find the gradient of PQ when $h = 0.1$.

c Find the gradient of the tangent to the curve at P.

CAS

Example 5
Example 6

12 For each of the following, find $f'(x)$ by finding $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$:

a $f(x) = 3x^2$ **b** $f(x) = 4x$ **c** $f(x) = 3$

d $f(x) = 3x^2 + 4x + 3$ **e** $f(x) = 2x^3 - 4$ **f** $f(x) = 4x^2 - 5x$

g $f(x) = 3 - 2x + x^2$ **h** $f(x) = 2x - x^3$ **i** $f(x) = 2x - 3x^2$

13 By first considering the gradient of the secant through $P(x, f(x))$ and $Q(x + h, f(x + h))$ for the curve $f(x) = x^4$, find the derivative of x^4.
Hint: $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

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14 Consider the following two approximations for $f'(a)$, where h is small:

i $f'(a) \approx \dfrac{f(a+h) - f(a)}{h}$ **ii** $f'(a) \approx \dfrac{f(a+h) - f(a-h)}{2h}$

Compare these two approximations for each of the following:

a $f(x) = x^2,\ a = 2$ **b** $f(x) = x^3,\ a = 2$ **c** $f(x) = x^3 + 2x - 4,\ a = 2$

17B Rules for differentiation

The derivative of x^n where n is a positive integer

From your work in the first section of this chapter, you may have noticed that differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result.

> For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \dots$

Proof In order to prove this result, we use the binomial theorem, which gives the general expansion of $(a+b)^n$. The notation ${}^nC_r$ for the number of combinations of n objects in groups of size r is introduced in Chapter 10.

Binomial theorem For each $n \in \mathbb{N}$, we have

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_r a^{n-r}b^r + \cdots + {}^nC_{n-1} ab^{n-1} + b^n$$

The binomial theorem is proved in the polynomials appendix in the Interactive Textbook.

Now we let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} xh^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1} \end{aligned}$$

$\therefore$
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} xh^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

Hence $f'(x) = nx^{n-1}$.

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The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- **Constant function:** If $f(x) = c$, then $f'(x) = 0$.
- **Linear function:** If $f(x) = mx + c$, then $f'(x) = m$.
- **Multiple:** If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- **Sum:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- **Difference:** If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivative of products and quotients in Mathematical Methods Units 3 & 4.

The process of finding the derivative function is called **differentiation**.

Example 7

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $f(x) = x^5 - 2x^3 + 2$

Then $f'(x) = 5x^4 - 2(3x^2) + 0$
$= 5x^4 - 6x^2$

Explanation

We use the following results:
- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 8

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $f(x) = 3x^3 - 6x^2 + 1$

Then $f'(x) = 3(3x^2) - 6(2x) + 0$
$= 9x^2 - 12x$

$\therefore\ f'(1) = 9 - 12$
$= -3$

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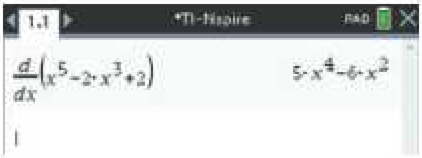
Using the TI-Nspire

For Example 7:

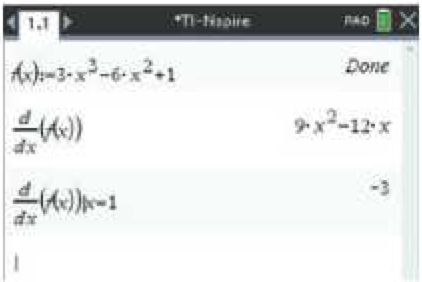
- Use (menu) > **Calculus** > **Derivative** and complete as shown.



Note: The derivative template can also be accessed from the 2D-template palette (template key). Alternatively, using (shift)(–) will paste the derivative template to the screen.

For Example 8:

- Assign the function $f(x)$ as shown.
- To find the derivative, use (menu) > **Calculus** > **Derivative**.
- To find the value of the derivative at $x = 1$, use (menu) > **Calculus** > **Derivative at a Point**.

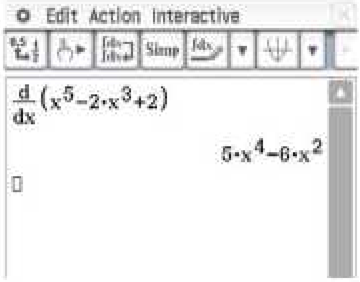


Using the Casio ClassPad

For Example 7:

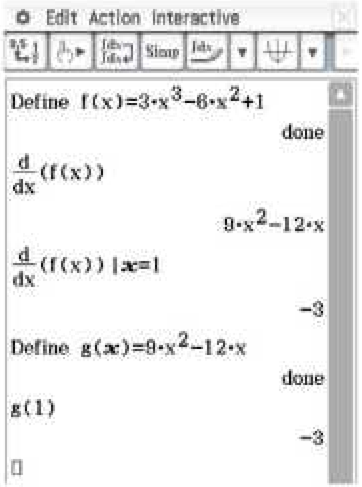
- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^5 - 2x^3 + 2$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.

Note: Alternatively, select the derivative template $\frac{d}{dx}\square$ from the (Math2) keyboard. Enter $x^5 - 2x^3 + 2$ in the main box and x in the lower box. Tap (EXE).



For Example 8:

- In Main $\sqrt{\alpha}$, enter and highlight the expression $3x^3 - 6x^2 + 1$.
- Go to **Interactive** > **Define** and tap OK.
- Go to **Interactive** > **Calculation** > **diff**. Enter $f(x)$ and tap OK.
- To find the value of the derivative at $x = 1$, tap the stylus at the end of the entry line. Select | from the (Math3) keyboard and type $x = 1$. Then tap (EXE).
- Alternatively, define the derivative as $g(x)$ and find $g(1)$.



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Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 17A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 9

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\frac{dy}{dx}$, and so we write $\frac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\frac{dy}{dx}$.

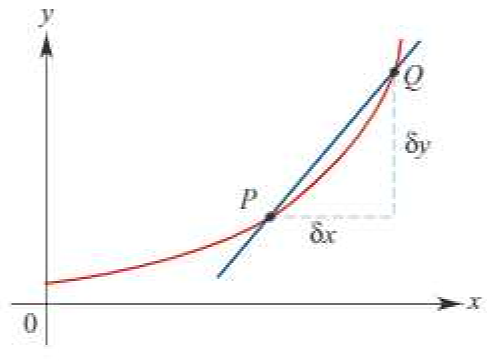
Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\frac{dz}{dt}$.

Warning: In this notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



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Example 10

a If $y = t^2$, find $\frac{dy}{dt}$.

b If $x = t^3 + t$, find $\frac{dx}{dt}$.

c If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a $y = t^2$

$\frac{dy}{dt} = 2t$

b $x = t^3 + t$

$\frac{dx}{dt} = 3t^2 + 1$

c $z = \frac{1}{3}x^3 + x^2$

$\frac{dz}{dx} = x^2 + 2x$

Example 11

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a It is first necessary to write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$
$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t+2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$
$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$
$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d

$$y = 2x^3 - 1$$
$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

Example 12

Find:

a $\frac{d}{dx}(5x - 4x^3)$

b $\frac{d}{dz}(5z^2 - 4z)$

c $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

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Example 13

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8

b $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = −6

Solution

a $y = x^3$ implies $\dfrac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\dfrac{8}{3}} = \pm\dfrac{2\sqrt{6}}{3}$

The points are $\left(\dfrac{2\sqrt{6}}{3}, \dfrac{16\sqrt{6}}{9}\right)$ and $\left(-\dfrac{2\sqrt{6}}{3}, -\dfrac{16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\dfrac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\dfrac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

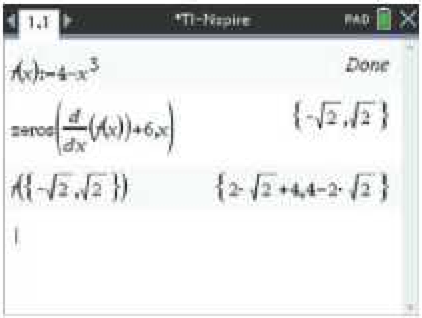
$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

Using the TI-Nspire

- Assign the function $f(x)$ as shown.
- Use (menu) > **Algebra** > **Zeros** and (menu) > **Calculus** > **Derivative** to solve the equation $\dfrac{d}{dx}(f(x)) = -6$.
- To find the y-coordinates, calculate the values of $f(-\sqrt{2})$ and $f(\sqrt{2})$ as shown.



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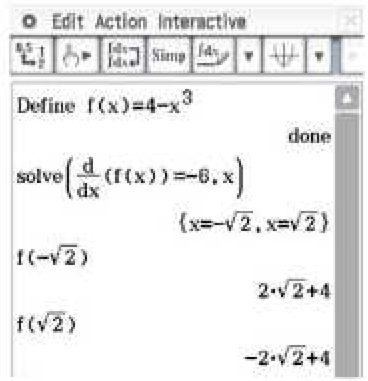
Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight the expression $4 - x^3$.
- Go to **Interactive** > **Define** and tap OK.
- In the next entry line, type and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.
- Type $= -6$ after $\frac{d}{dx}(f(x))$. Highlight the equation and use **Interactive** > **Equation/Inequality** > **solve**.
- Enter $f(-\sqrt{2})$ and $f(\sqrt{2})$ to find the y-coordinates.



Note: Alternatively, you can select commands and templates from Math2 and Math3.

Summary 17B

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Linear function: If $f(x) = mx + c$, then $f'(x) = m$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.

For example, if $f(x) = 5x^3 - 10x^2 + 7$, then $f'(x) = 5(3x^2) - 10(2x) + 0 = 15x^2 - 20x$.

Exercise 17B

Example 7

1 Find the derivative of each of the following with respect to x:

a $x^2 + 4x$ **b** $2x + 1$ **c** $x^3 - x$

d $\frac{1}{2}x^2 - 3x + 4$ **e** $5x^3 + 3x^2$ **f** $-x^3 + 2x^2$

2 For each of the following, find $f'(x)$:

a $f(x) = x^{12}$ **b** $f(x) = 3x^7$ **c** $f(x) = 5x$

d $f(x) = 5x + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^5 + 3x^4$ **h** $f(x) = 2x^4 - \frac{1}{3}x^3 - \frac{1}{4}x^2 + 2$

Example 8

3 For each of the following, find $f'(1)$:

a $f(x) = x^6$ **b** $f(x) = 4x^5$ **c** $f(x) = 5x$

d $f(x) = 5x^2 + 3$ **e** $f(x) = 3$ **f** $f(x) = 5x^2 - 3x$

g $f(x) = 10x^4 - 3x^3$ **h** $f(x) = 2x^4 - \frac{1}{3}x^3$ **i** $f(x) = -10x^3 - 2x^2 + 2$

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4 For each of the following, find $f'(-2)$:

a $f(x) = 5x^3$ **b** $f(x) = 4x^2$ **c** $f(x) = 5x^3 - 3x$ **d** $f(x) = -5x^4 - 2x^2$

Example 9

5 Find the gradient of the tangent line to the graph of f at the given point:

a $f(x) = x^2 + 3x, \quad (2, 10)$ **b** $f(x) = 3x^2 - 4x, \quad (1, -1)$

c $f(x) = -2x^2 - 4x, \quad (3, -30)$ **d** $f(x) = x^3 - x, \quad (2, 6)$

Example 10

6 **a** If $y = t^2 - 7$, find $\frac{dy}{dt}$. **b** If $x = -5t^3 + t$, find $\frac{dx}{dt}$. **c** If $z = \frac{1}{2}x^4 - x^2$, find $\frac{dz}{dx}$.

Example 11

7 For each of the following, find $\frac{dy}{dx}$:

a $y = -x$ **b** $y = 10$ **c** $y = 4x^3 - 3x + 2$

d $y = \frac{1}{3}(x^3 - 3x + 6)$ **e** $y = (x + 1)(x + 2)$ **f** $y = 2x(3x^2 - 4)$

g $y = (2x - 1)^2$ **h** $y = \frac{5x - x^2}{x}, x \neq 0$ **i** $y = \frac{10x^5 + 3x^4}{2x^2}, x \neq 0$

8 **a** For $y = (x + 4)^2$, find $\frac{dy}{dx}$. **b** For $z = (4t - 1)^2(t + 1)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^3 + 3x}{x}$, find $\frac{dy}{dx}$.

9 **a** For the curve with equation $y = x^3 + 1$, find the gradient of the tangent line at points:

i $(1, 2)$ **ii** $(a, a^3 + 1)$

b Find the derivative of $x^3 + 1$ with respect to x.

10 **a** Given that $y = x^3 - 3x^2 + 3x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x, and interpret this in terms of the graph of $y = x^3 - 3x^2 + 3x$.

b Given that $y = \frac{x^2 + 2x}{x}$, for $x \neq 0$, find $\frac{dy}{dx}$.

c Differentiate $y = (3x + 1)^2$ with respect to x.

11 For each of the following curves, find the y-coordinate of the point on the curve with the given x-coordinate, and find the gradient of the tangent line at that point: CAS

a $y = x^2 - 2x + 1, \quad x = 2$ **b** $y = x^2 + x + 1, \quad x = 0$

c $y = x^2 - 2x, \quad x = -1$ **d** $y = (x + 2)(x - 4), \quad x = 3$

e $y = 3x^2 - 2x^3, \quad x = -2$ **f** $y = (4x - 5)^2, \quad x = \frac{1}{2}$

12 **a** For each of the following, first find $f'(x)$ and $f'(1)$. Then, for $y = f(x)$, find the set $\{ (x, y) : f'(x) = 1 \}$. That is, find the coordinates of the points where the gradient of the tangent line is 1.

i $f(x) = 2x^2 - x$ **ii** $f(x) = 1 + \frac{1}{2}x + \frac{1}{3}x^2$

iii $f(x) = x^3 + x$ **iv** $f(x) = x^4 - 31x$

b What is the interpretation of $\{ (x, y) : f'(x) = 1 \}$ in terms of the graphs?

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Example 12 **13** Find:

a $\frac{d}{dt}(3t^2 - 4t)$ **b** $\frac{d}{dx}(4 - x^2 + x^3)$ **c** $\frac{d}{dz}(5 - 2z^2 - z^4)$

d $\frac{d}{dy}(3y^2 - y^3)$ **e** $\frac{d}{dx}(2x^3 - 4x^2)$ **f** $\frac{d}{dt}(9.8t^2 - 2t)$

Example 13 **14** For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line has the given value:

a $y = x^2$, gradient = 8 **b** $y = x^3$, gradient = 12

c $y = x(2 - x)$, gradient = 2 **d** $y = x^2 - 3x + 1$, gradient = 0

e $y = x^3 - 6x^2 + 4$, gradient = −12 **f** $y = x^2 - x^3$, gradient = −1

17C Differentiating x^n where n is a negative integer

In the previous section we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x^{-1}$

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = 2x + x^{-1}$

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x + 3 + x^{-2}$

Note: We have reintroduced function notation to emphasise the need to consider domains.

Example 14

Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{x}$. Find $f'(x)$ by first principles.

Solution

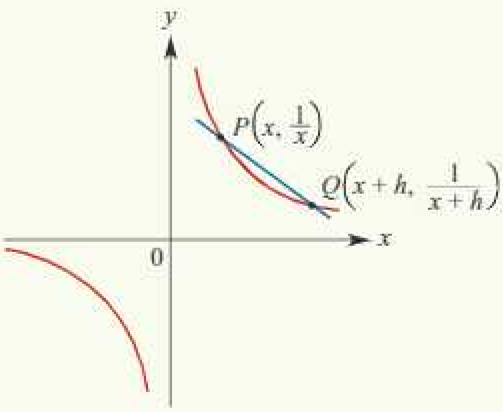
The gradient of secant PQ is given by

$$\frac{f(x+h) - f(x)}{x+h-x} = \left(\frac{1}{x+h} - \frac{1}{x}\right) \times \frac{1}{h}$$

$$= \frac{x - (x+h)}{(x+h)x} \times \frac{1}{h}$$

$$= \frac{-h}{(x+h)x} \times \frac{1}{h}$$

$$= \frac{-1}{(x+h)x}$$



So the gradient of the curve at P is

$$\lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2} = -x^{-2}$$

Hence $f'(x) = -x^{-2}$.

Example 15

Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

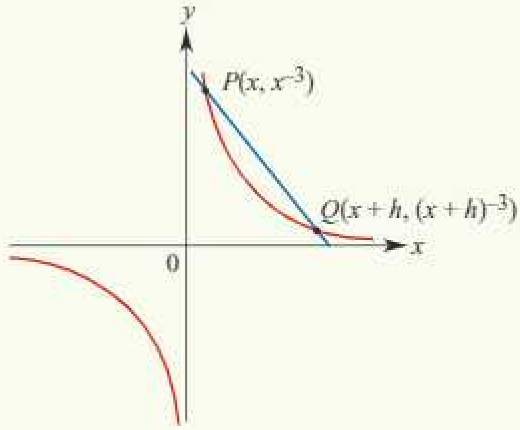
$$\frac{(x+h)^{-3} - x^{-3}}{h}$$

$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

We are now in a position to state the generalisation of the result we found in Section 17B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Note: We will consider rational indices in Chapter 20.

Example 16

Find the derivative of:

a $2x^{-3} - x^{-1} + 2, \ x \neq 0$

b $\dfrac{5x^3 + x}{x^2}, \ x \neq 0$

Solution

a Let $f(x) = 2x^{-3} - x^{-1} + 2$

Then $f'(x) = 2(-3x^{-4}) - (-x^{-2}) + 0$

$= -6x^{-4} + x^{-2}$ (for $x \neq 0$)

b Let $f(x) = \dfrac{5x^3 + x}{x^2}$

$= 5x + x^{-1}$

Then $f'(x) = 5 + (-x^{-2})$

$= 5 - \dfrac{1}{x^2}$ (for $x \neq 0$)

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Example 17

Find the derivative f' of $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^2 - 6x^{-2} + 1$.

Solution

$$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \begin{aligned} f'(x) &= 3(2x) - 6(-2x^{-3}) + 0 \\ &= 6x + 12x^{-3} \end{aligned}$$

Example 18

Find the gradient of the tangent to the curve determined by the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 + \frac{1}{x}$ at the point $(1, 2)$.

Solution

$$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \begin{aligned} f'(x) &= 2x + (-x^{-2}) \\ &= 2x - x^{-2} \end{aligned}$$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 19

Show that the derivative of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$ is always negative.

Solution

$$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad \begin{aligned} f'(x) &= -3x^{-4} \\ &= -\frac{3}{x^4} \end{aligned}$$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

Summary 17C

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Skill-sheet

Exercise 17C

Example 14

1 **a** Let $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}$, $f(x) = \frac{1}{x - 3}$. Find $f'(x)$ by first principles.

b Let $f: \mathbb{R} \setminus \{-2\} \to \mathbb{R}$, $f(x) = \frac{1}{x + 2}$. Find $f'(x)$ by first principles.

Example 15

2 **a** Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-2}$. Find $f'(x)$ by first principles.

b Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-4}$. Find $f'(x)$ by first principles.

Hint: $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

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Example 16

3 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\frac{3}{x^2} + 5x^2$ **c** $\frac{5}{x^3} + \frac{4}{x^2} + 1$

d $3x^2 + \frac{5}{3}x^{-4} + 2$ **e** $6x^{-2} + 3x$ **f** $\frac{3x^2 + 2}{x}$

4 Find the derivative of each of the following:

a $\frac{3z^2 + 2z + 4}{z^2}$, $z \neq 0$ **b** $\frac{3 + z}{z^3}$, $z \neq 0$ **c** $\frac{2z^2 + 3z}{4z}$, $z \neq 0$

d $9z^2 + 4z + 6z^{-3}$, $z \neq 0$ **e** $9 - z^{-2}$, $z \neq 0$ **f** $\frac{5z - 3z^2}{5z}$, $z \neq 0$

Example 17

5 **a** Find the derivative f' of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^4 - 6x^{-3} + x^{-1}$.

b Find the derivative f' of $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 5x^4 + 4x^{-2} + x^{-1}$.

6 Carefully sketch the graph of $f(x) = \frac{1}{x^2}$, $x \neq 0$.

a For points $P(1, f(1))$ and $Q(1 + h, f(1 + h))$, find the gradient of the secant PQ.

b Hence find the gradient of the tangent line to the curve $y = \frac{1}{x^2}$ at $x = 1$.

Example 18

7 For each of the following curves, find the gradient of the tangent line to the curve at the given point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = \frac{x - 2}{x}$, $x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \frac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

8 For the curve with equation $f(x) = x^{-2}$, find the x-coordinate of the point on the curve at which the gradient of the tangent line is:

a 16 **b** −16

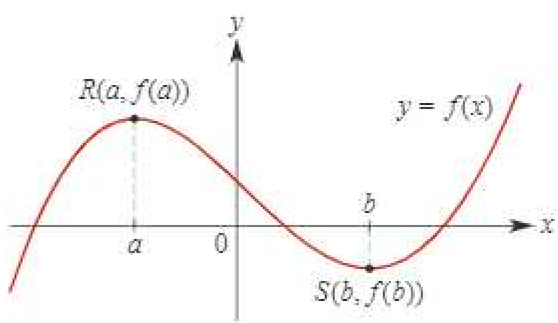
Example 19

9 Show that the derivative of the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-1}$ is always negative.

17D Graphs of the derivative function

Sign of the derivative

Consider the graph of $y = f(x)$ shown here. At a point $(x, f(x))$ on the graph, the gradient is $f'(x)$.



By noting whether the curve is sloping upwards or downwards at a particular point, we can tell the sign of the derivative at that point:

Values of x	$x < a$	$x = a$	$a < x < b$	$x = b$	$x > b$
Sign of $f'(x)$	$f'(x) > 0$	$f'(a) = 0$	$f'(x) < 0$	$f'(b) = 0$	$f'(x) > 0$

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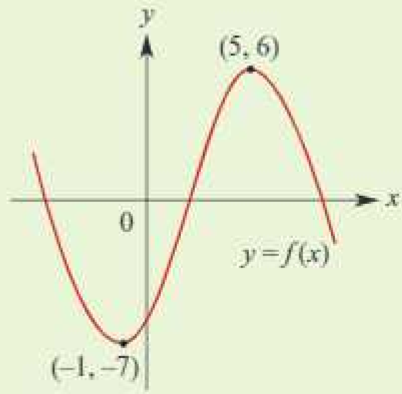
Example 20

For the graph of $f : \mathbb{R} \to \mathbb{R}$, find:

a $\{x : f'(x) > 0\}$

b $\{x : f'(x) < 0\}$

c $\{x : f'(x) = 0\}$



Solution

a $\{x : f'(x) > 0\} = \{x : -1 < x < 5\} = (-1, 5)$

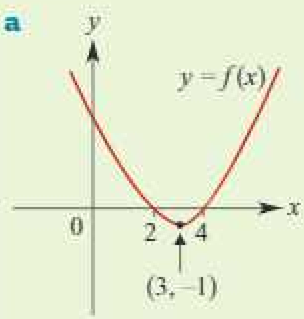
b $\{x : f'(x) < 0\} = \{x : x < -1\} \cup \{x : x > 5\} = (-\infty, -1) \cup (5, \infty)$

c $\{x : f'(x) = 0\} = \{-1, 5\}$

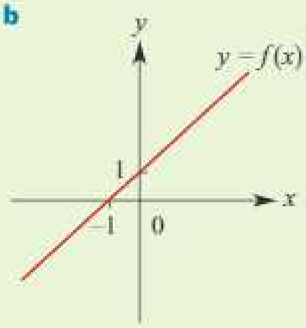
Example 21

Sketch the graph of $y = f'(x)$ for each of the following. (It is impossible to determine all features.)

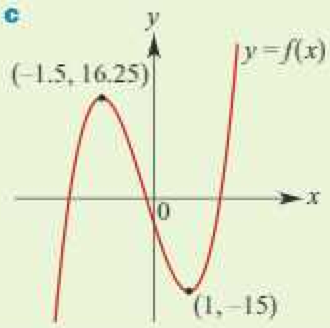
a



b



c

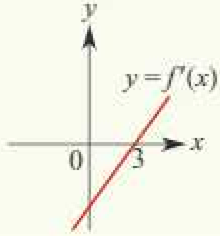


Solution

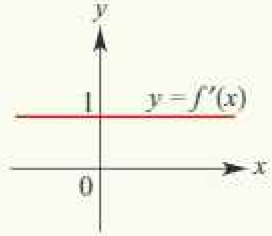
a $f'(x) > 0$ for $x > 3$

$f'(x) < 0$ for $x < 3$

$f'(x) = 0$ for $x = 3$



b $f'(x) = 1$ for all x

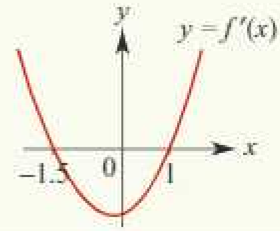


c $f'(x) > 0$ for $x > 1$

$f'(x) < 0$ for $-1.5 < x < 1$

$f'(x) > 0$ for $x < -1.5$

$f'(-1.5) = 0$ and $f'(1) = 0$



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If the rule for the function is given, then a CAS calculator can be used to plot the graph of its derivative function.

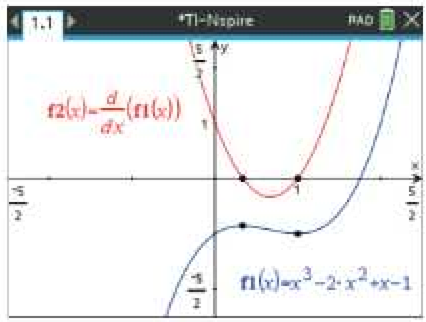
Using the TI-Nspire

Plot the graphs of

$$f1(x) = x^3 - 2x^2 + x - 1$$

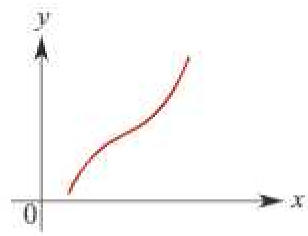
$$f2(x) = \frac{d}{dx}(f1(x))$$

Note: Access the derivative template from ⌸.



Increasing and decreasing functions

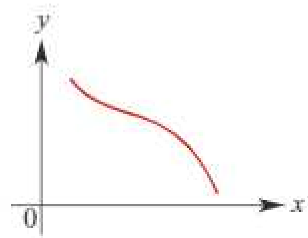
We say a function f is **strictly increasing** on an interval if $a < b$ implies $f(a) < f(b)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f: [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $a < b$ implies $f(a) > f(b)$.



For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f: (-\infty, 0] \to \mathbb{R}$, $f(x) = x^2$ is strictly decreasing.

Note: The word *strictly* refers to the use of the strict inequality signs $<, >$ rather than $\leq, \geq$.

Derivative tests

We can sometimes establish that a function is strictly increasing or strictly decreasing on an interval by checking the sign of its derivative on that interval.

- If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing. (Think of the tangents at each point – they each have positive gradient.)
- If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing. (Think of the tangents at each point – they each have negative gradient.)

Warning: These two tests are not equivalent to the definitions of strictly increasing and strictly decreasing.

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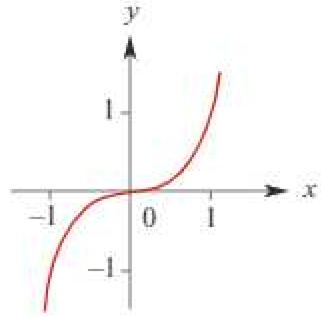
For example, the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

We can see that $f(x) = x^3$ is strictly increasing from its graph. Alternatively, consider

$$\begin{aligned} a^3 - b^3 &= (a-b)(a^2 + ab + b^2) \\ &= (a-b)\left(a^2 + ab + (\tfrac{1}{2}b)^2 + b^2 - (\tfrac{1}{2}b)^2\right) \\ &= (a-b)\left((a + \tfrac{1}{2}b)^2 + \tfrac{3}{4}b^2\right) \end{aligned}$$

We can see that if $a < b$, then $a^3 < b^3$.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If θ is the angle that a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\theta$. That is:

$$m = \tan\theta$$

For example:

- If $\theta = 45^\circ$, then $\tan\theta = 1$ and the gradient is 1.
- If $\theta = 20^\circ$, then the gradient of the straight line is $\tan 20^\circ$.
- If $\theta = 135^\circ$, then $\tan\theta = -1$ and the gradient is -1.

Example 22

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\dfrac{dy}{dx} = 2x - 7$

$2x - 7 = 1$ (as $\tan 45^\circ = 1$)

$2x = 8$

$\therefore\ x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore\ x = \dfrac{5}{2}$

The coordinates are $\left(\dfrac{5}{2}, -\dfrac{13}{4}\right)$.

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Example 23

The planned path for a flying saucer leaving a planet is defined by the equation

$$y = \frac{1}{4}x^4 + \frac{2}{3}x^3 \quad \text{for } x > 0$$

The units are kilometres. (The x-axis is horizontal and the y-axis vertical.)

a Find the direction of motion when the x-value is:

i 2 **ii** 3

b Find a point on the flying saucer's path where the path is inclined at 45° to the positive x-axis (i.e. where the gradient of the path is 1).

c Are there any other points on the path which satisfy the situation described in part **b**?

Solution

a $\frac{dy}{dx} = x^3 + 2x^2$

i When $x = 2$, $\frac{dy}{dx} = 8 + 8 = 16$

$\tan^{-1} 16 = 86.42°$ (to the x-axis)

ii When $x = 3$, $\frac{dy}{dx} = 27 + 18 = 45$

$\tan^{-1} 45 = 88.73°$ (to the x-axis)

b, c When the flying saucer is flying at 45° to the positive direction of the x-axis, the gradient of the curve of its path is given by $\tan 45°$. Thus to find the point at which this happens we consider the equation

$$\frac{dy}{dx} = \tan 45°$$

$$x^3 + 2x^2 = 1$$

$$x^3 + 2x^2 - 1 = 0$$

$$(x + 1)(x^2 + x - 1) = 0$$

$$\therefore \quad x = -1 \text{ or } x = \frac{-1 \pm \sqrt{5}}{2}$$

The only acceptable solution is $x = \frac{-1 + \sqrt{5}}{2} \approx 0.62$, as the other two possibilities give negative values for x and we are only considering positive values for x.

Summary 17D

- A function f is **strictly increasing** on an interval if $a < b$ implies $f(a) < f(b)$.
- A function f is **strictly decreasing** on an interval if $a < b$ implies $f(a) > f(b)$.
- If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
- If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

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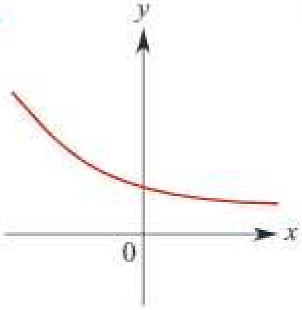
Skill-sheet

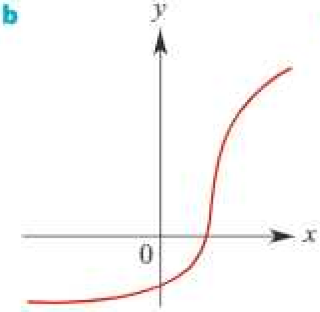
Exercise 17D

1 For which of the following curves is $\frac{dy}{dx}$ positive for all values of x?

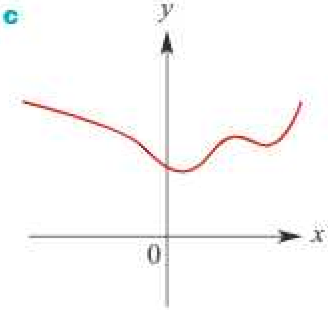
a



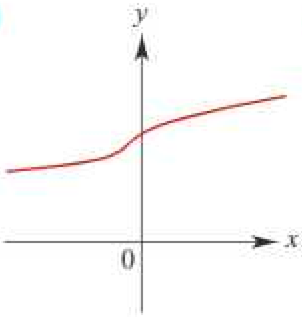
b



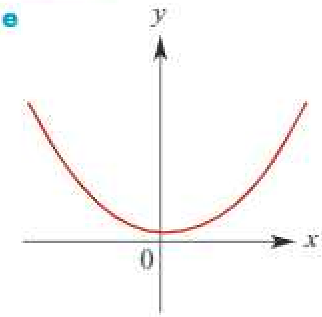
c



d

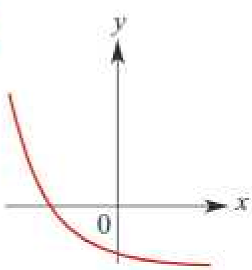


e

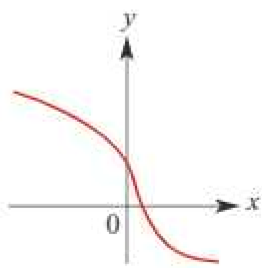


2 For which of the following curves is $\frac{dy}{dx}$ negative for all values of x?

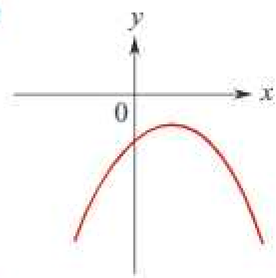
a



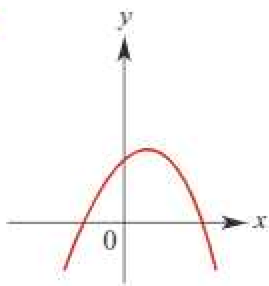
b



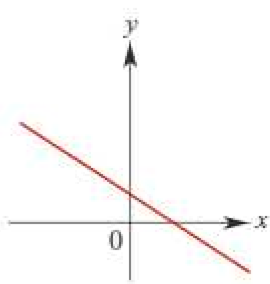
c



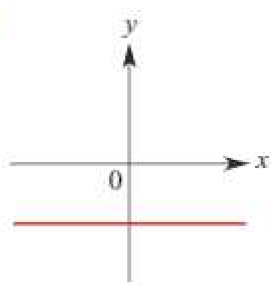
d



e



f



3 For the function $f(x) = 2(x-1)^2$, find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$ **d** $f'(x) < 0$ **e** $f'(x) = -2$

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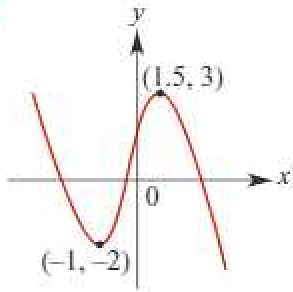
Example 20

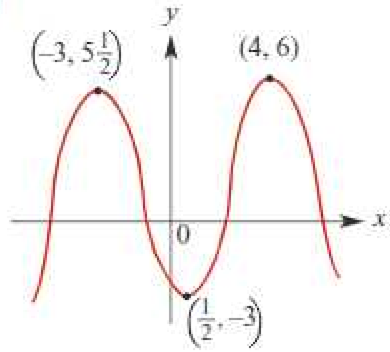
4 For each of the following, use the given graph of $y = f(x)$ to find the sets:

i $\{x : f'(x) > 0\}$ **ii** $\{x : f'(x) < 0\}$ **iii** $\{x : f'(x) = 0\}$

a

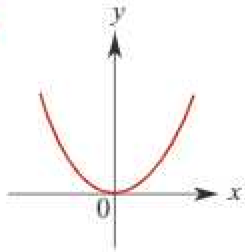


b

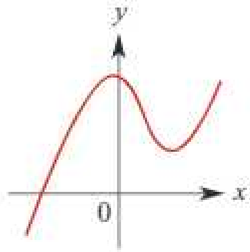


5 Which of the graphs labelled **A–F** correspond to each of the graphs labelled **a–f**?

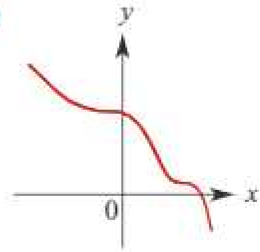
a



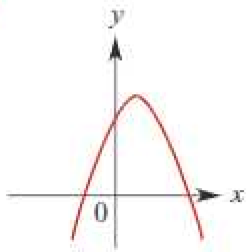
b



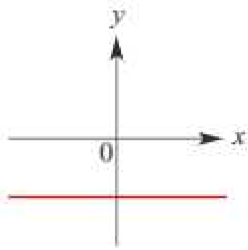
c



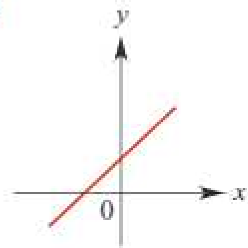
d



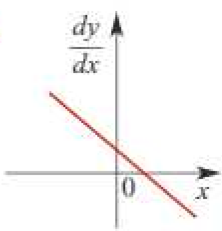
e



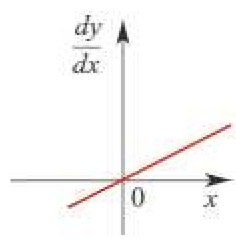
f



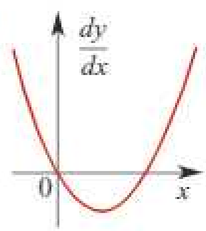
A



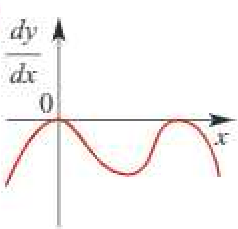
B



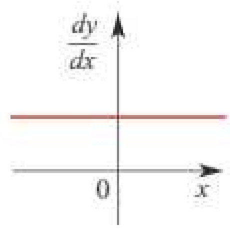
C



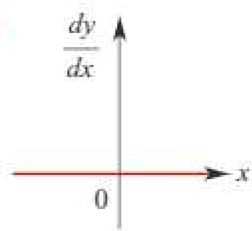
D



E



F



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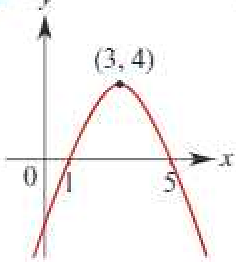
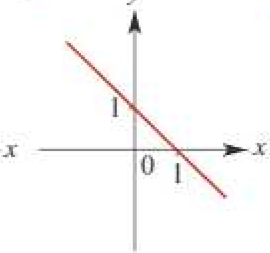
Example 21

6 Sketch the graph of $y = f'(x)$ for each of the following:

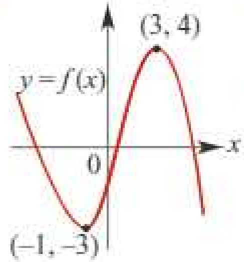
a



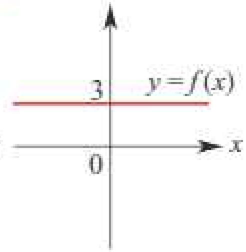
b



c



d



Example 22

7 Find the coordinates of the points on the curve $y = x^2 - 5x + 6$ at which the tangent:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 3x + 4$.

8 Find the coordinates of the points on the parabola $y = x^2 - x - 6$ at which:

a the gradient of the tangent is zero

b the tangent is parallel to the line $x + y = 6$.

9 Use a calculator to plot the graph of $y = f'(x)$ where:

a $f(x) = \sin x$ **b** $f(x) = \cos x$ **c** $f(x) = 2^x$

Example 23

10 The path of a particle is defined by the equation $y = \frac{1}{3}x^3 + \frac{2}{3}x^2$, for $x > 0$. The units are metres. (The x-axis is horizontal and the y-axis vertical.)

a Find the direction of motion when the x-value is:

i 1 **ii** 0.5

b Find a point on the particle's path where the path is inclined at 45° to the positive direction of the x-axis.

c Are there any other points on the path which satisfy the situation described in part **b**?

11 A car moves away from a set of traffic lights so that the distance, $S(t)$ metres, covered after t seconds is modelled by $S(t) = 0.2 \times t^3$.

a Find its speed after t seconds. **b** What will its speed be when $t = 1, 3, 5$?

12 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b.

b Find the coordinates of the point where the gradient is 0.

13 A rocket is launched from Cape York Peninsula so that after t seconds its height, $h(t)$ metres, is given by $h(t) = 20t^2$, $0 \le t \le 150$. After $2\frac{1}{2}$ minutes this model is no longer appropriate.

a Find the height and the speed of the rocket when $t = 150$.

b After how long will its speed be 1000 m/s?

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17E Antidifferentiation of polynomial functions

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

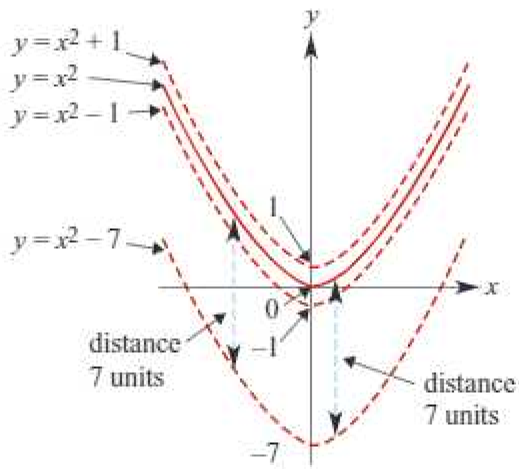
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation for antiderivatives

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\begin{aligned}\int 2x\,dx &= \{\, f(x) : f'(x) = 2x \,\}\\ &= \{\, x^2 + c : c \in \mathbb{R} \,\}\end{aligned}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.

In general:

> If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.

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Rules for antidifferentiation

We know that:

$$f(x) = x^3 \quad \text{implies} \quad f'(x) = 3x^2$$
$$f(x) = x^8 \quad \text{implies} \quad f'(x) = 8x^7$$
$$f(x) = x \quad \text{implies} \quad f'(x) = 1$$
$$f(x) = x^n \quad \text{implies} \quad f'(x) = nx^{n-1}$$

Reversing this process we have:

$$\int 3x^2 \, dx = x^3 + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int 8x^7 \, dx = x^8 + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int 1 \, dx = x + c \qquad \text{where } c \text{ is an arbitrary constant}$$
$$\int nx^{n-1} \, dx = x^n + c \qquad \text{where } c \text{ is an arbitrary constant}$$

We also have:

$$\int x^2 \, dx = \frac{1}{3}x^3 + c$$
$$\int x^7 \, dx = \frac{1}{8}x^8 + c$$
$$\int 1 \, dx = x + c$$
$$\int x^{n-1} \, dx = \frac{1}{n}x^n + c$$

From this we see that:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \quad n \in \mathbb{N} \cup \{0\}$$

Note: This result can be extended to include x^n where n is a negative integer other than -1. The extension is covered in Chapter 20. You will see the antiderivative of x^{-1} in Mathematical Methods Units 3 & 4.

We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\quad \int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$

Difference $\quad \int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$

Multiple $\quad \int k f(x) \, dx = k \int f(x) \, dx$, where k is a real number

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Example 24

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$ **b** $3x^2 + 4x^3 + 3$

Solution

a
$$\int 3x^5\,dx = 3\int x^5\,dx = 3 \times \frac{x^6}{6} + c = \frac{x^6}{2} + c$$

b
$$\int 3x^2 + 4x^3 + 3\,dx = 3\int x^2\,dx + 4\int x^3\,dx + 3\int 1\,dx = \frac{3x^3}{3} + \frac{4x^4}{4} + \frac{3x}{1} + c = x^3 + x^4 + 3x + c$$

Given extra information, we can find a unique antiderivative.

Example 25

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2\,dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

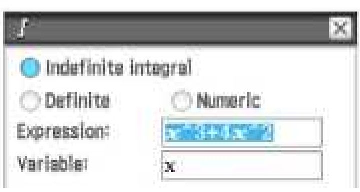
Thus $f(x) = \dfrac{x^4}{4} + \dfrac{4x^3}{3} + c$ for some real number c.

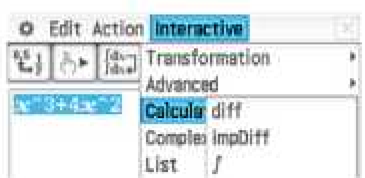
Since $f(0) = 0$, we have $c = 0$.

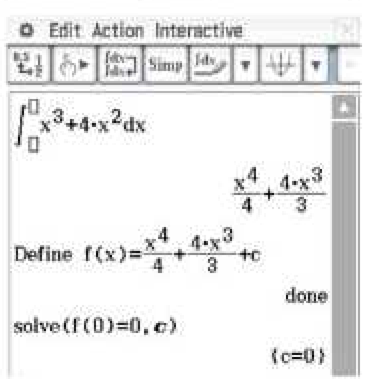
$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$$

Using the Casio ClassPad

- To find the general antiderivative of $x^3 + 4x^2$, enter and highlight the expression in $\sqrt{\alpha}$ Main.
- Select **Interactive** > **Calculation** > $\int$.
- Ensure that 'Indefinite integral' is selected, as shown below. Tap OK.







- Remember to add a constant c to the answer.
- To find the specific antiderivative, define the family of functions $f(x)$.
- Solve $f(0) = 0$ for c.

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Example 26

If the gradient of the tangent at a point (x, y) on a curve is given by $5x$ and the curve passes through $(0, 6)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5x$.

$$\int 5x \, dx = \frac{5x^2}{2} + c$$

$$\therefore \quad f(x) = \frac{5x^2}{2} + c$$

This describes the family of curves for which $f'(x) = 5x$. Here we are given the additional information that the curve passes through $(0, 6)$, i.e. $f(0) = 6$.

Hence $c = 6$ and so $f(x) = \dfrac{5x^2}{2} + 6$.

Example 27

Find y in terms of x if:

a $\dfrac{dy}{dx} = x^2 + 2x$, and $y = 1$ when $x = 1$

b $\dfrac{dy}{dx} = 3 - x$, and $y = 2$ when $x = 4$

Solution

a $\displaystyle\int x^2 + 2x \, dx = \frac{x^3}{3} + x^2 + c$

$$\therefore \quad y = \frac{x^3}{3} + x^2 + c$$

As $y = 1$ when $x = 1$,

$$1 = \frac{1}{3} + 1 + c$$

$$c = -\frac{1}{3}$$

Hence $y = \dfrac{x^3}{3} + x^2 - \dfrac{1}{3}$

b $\displaystyle\int 3 - x \, dx = 3x - \frac{x^2}{2} + c$

$$\therefore \quad y = 3x - \frac{x^2}{2} + c$$

As $y = 2$ when $x = 4$,

$$2 = 3 \times 4 - \frac{4^2}{2} + c$$

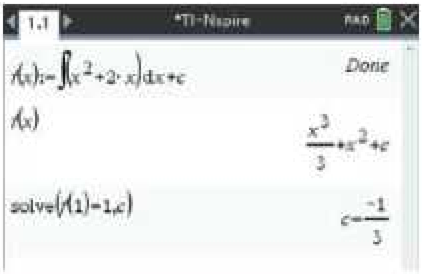
$$c = -2$$

Hence $y = 3x - \dfrac{x^2}{2} - 2$

Using the TI-Nspire

For Example 27 **a**:

- To find the general antiderivative, assign the function $f(x)$ using menu > **Calculus** > **Integral** as shown.
- Check that c has not been assigned a value.
- For the specific antiderivative, find the value of c by solving $f(1) = 1$.



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Summary 17E

- Antiderivative of x^n, for $n \in \mathbb{N} \cup \{0\}$:

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$$

- Rules of antidifferentiation:
  - $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$
  - $\int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$
  - $\int k f(x) \, dx = k \int f(x) \, dx$, where k is a real number

Exercise 17E

Example 24

1 Find:

a $\int \frac{1}{2}x^3 \, dx$ **b** $\int 3x^2 - 2 \, dx$ **c** $\int 5x^3 - 2x \, dx$

d $\int \frac{4}{5}x^3 - 2x^2 \, dx$ **e** $\int (x-1)^2 \, dx$ **f** $\int x(x + \frac{1}{x}) \, dx, \ x \neq 0$

g $\int 2z^2(z-1) \, dz$ **h** $\int (2t-3)^2 \, dt$ **i** $\int (t-1)^3 \, dt$

Example 25

2 It is known that $f'(x) = 4x^3 + 6x^2 + 2$ and $f(0) = 0$. Find $f(x)$.

Example 26

3 If the gradient at a point (x, y) on a curve is given by $6x^2$ and the curve passes through $(0, 12)$, find the equation of the curve.

Example 27

4 Find y in terms of x in each of the following:

a $\frac{dy}{dx} = 2x - 1$, and $y = 0$ when $x = 1$

b $\frac{dy}{dx} = 3 - x$, and $y = 1$ when $x = 0$

c $\frac{dy}{dx} = x^2 + 2x$, and $y = 2$ when $x = 0$

d $\frac{dy}{dx} = 3 - x^2$, and $y = 2$ when $x = 3$

e $\frac{dy}{dx} = 2x^4 + x$, and $y = 0$ when $x = 0$

5 Assume that $\frac{dV}{dt} = t^2 - t$ for $t > 1$, and that $V = 9$ when $t = 3$.

a Find V in terms of t.

b Calculate the value of V when $t = 10$.

6 The gradient of the tangent at any point $(x, f(x))$ on the curve with equation $y = f(x)$ is given by $3x^2 - 1$. Find $f(x)$ if the curve passes through the point $(1, 2)$, i.e. $f(1) = 2$.

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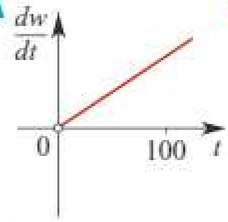
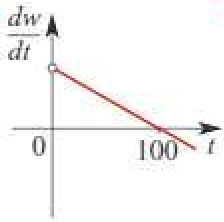
7 **a** Which one of the following graphs represents $\frac{dw}{dt} = 2000 - 20t,\ t > 0$?

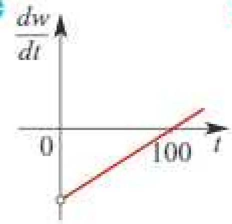
A



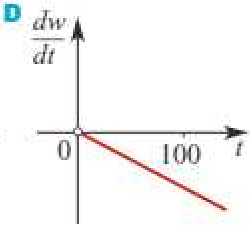
B



C



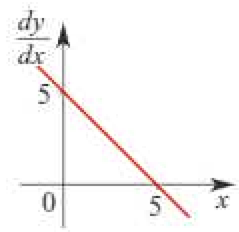
D



b Find w in terms of t if when $t = 0$, $w = 100\,000$.

8 The graph shows $\frac{dy}{dx}$ against x for a certain curve with equation $y = f(x)$.

Find $f(x)$, given that the point $(0, 4)$ lies on the curve.



9 Find the equation of the curve $y = f(x)$ which passes through the point $(2, -6)$ and for which $f'(x) = x^2(x - 3)$.

10 The curve $y = f(x)$ for which $f'(x) = 4x + k$, where k is a constant, has a turning point at $(-2, -1)$.

a Find the value of k.

b Find the coordinates of the point at which the curve meets the y-axis.

11 Given that $\frac{dy}{dx} = ax^2 + 1$ and that when $x = 1$, $\frac{dy}{dx} = 3$ and $y = 3$, find the value of y when $x = 2$.

12 The curve for which $\frac{dy}{dx} = 2x + k$, where k is a constant, is such that the tangent at $(3, 6)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k

b the equation of the curve.

13 The curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a turning point at $(2, 1)$.

a Find the value of k.

b Find the value of $f(x)$ when $x = 7$.

14 Suppose that a point moves along some unknown curve $y = f(x)$ in such a way that, at each point (x, y) on the curve, the tangent line has slope x^2. Find an equation for the curve, given that it passes through $(2, 1)$.

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17F Limits and continuity

Limits

It is not the intention of this course to provide a formal introduction to limits. We require only an intuitive understanding of limits and some fairly obvious rules for how to handle them.

The notation $\lim_{x \to a} f(x) = \ell$ says that the limit of $f(x)$, as x approaches a, is ℓ.

We can also say: 'As x approaches a, $f(x)$ approaches ℓ.'

This means that we can make the value of $f(x)$ as close as we like to ℓ, provided we choose x-values close enough to a.

We have met a similar idea earlier in the course. For example, we have seen that $\lim_{x \to \infty} f(x) = 4$ for the function with rule $f(x) = \frac{1}{x} + 4$. The graph of $y = f(x)$ can get as close as we like to the line $y = 4$, just by taking larger and larger values of x.

As we will see, for many functions (in particular, for polynomial functions), the limit at a particular point is simply the value of the function at that point.

Example 28

Find $\lim_{x \to 2} 3x^2$.

Solution	Explanation
$\lim_{x \to 2} 3x^2 = 3(2)^2 = 12$	As x gets closer and closer to 2, the value of $3x^2$ gets closer and closer to 12.

If the function is not defined at the value for which the limit is to be found, a different procedure is used.

Example 29

For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, find $\lim_{x \to 2} f(x)$.

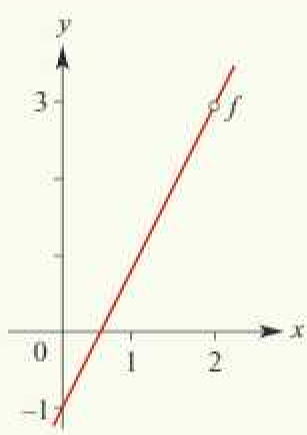
Solution

Observe that

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 2}{x - 2} \\ &= \frac{(2x - 1)(x - 2)}{x - 2} \\ &= 2x - 1 \qquad \text{(for } x \neq 2\text{)} \end{aligned}$$

Hence $\lim_{x \to 2} f(x) = 3$.

The graph of $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = 2x - 1$ is shown.



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We can investigate Example 29 further by looking at the values of the function as we take x-values closer and closer to 2.

Observe that $f(x)$ is defined for $x \in \mathbb{R} \setminus \{2\}$. Examine the behaviour of $f(x)$ for values of x close to 2.

$x < 2$	$x > 2$
$f(1.7) = 2.4$	$f(2.3) = 3.6$
$f(1.8) = 2.6$	$f(2.2) = 3.4$
$f(1.9) = 2.8$	$f(2.1) = 3.2$
$f(1.99) = 2.98$	$f(2.01) = 3.02$
$f(1.999) = 2.998$	$f(2.001) = 3.002$

From the table, it is apparent that, as x takes values closer and closer to 2 (regardless of whether x approaches 2 from the left or from the right), the values of $f(x)$ become closer and closer to 3. That is, $\lim_{x \to 2} f(x) = 3$.

Note that the limit exists, but the function is not defined at $x = 2$.

Algebra of limits

The following important results are useful for the evaluation of limits.

Suppose that f and g are functions and that a is a real number. Assume that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

- **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- **Multiple:** $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$, where k is a given real number.
- **Product:** $\lim_{x \to a} (f(x)\, g(x)) = \lim_{x \to a} f(x) \, \lim_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- **Quotient:** $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

Example 30

Find:

a $\lim_{h \to 0} (3h + 4)$ **b** $\lim_{x \to 2} 4x(x + 2)$ **c** $\lim_{x \to 3} \dfrac{5x + 2}{x - 2}$ **d** $\lim_{x \to 5} \dfrac{x^2 - 7x + 10}{x^2 - 25}$

Solution

a $$\begin{aligned}\lim_{h \to 0} (3h + 4) &= \lim_{h \to 0} (3h) + \lim_{h \to 0} (4)\\ &= 0 + 4\\ &= 4\end{aligned}$$

b $$\begin{aligned}\lim_{x \to 2} 4x(x + 2) &= \lim_{x \to 2} (4x) \, \lim_{x \to 2} (x + 2)\\ &= 8 \times 4\\ &= 32\end{aligned}$$

c $$\begin{aligned}\lim_{x \to 3} \frac{5x + 2}{x - 2} &= \lim_{x \to 3} (5x + 2) \div \lim_{x \to 3} (x - 2)\\ &= 17 \div 1\\ &= 17\end{aligned}$$

d $$\begin{aligned}\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 25} &= \lim_{x \to 5} \frac{(x - 2)(x - 5)}{(x + 5)(x - 5)}\\ &= \frac{\lim_{x \to 5} (x - 2)}{\lim_{x \to 5} (x + 5)} = \frac{3}{10}\end{aligned}$$

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Left and right limits

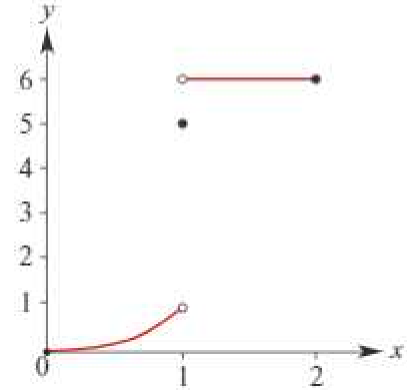
An idea which is useful in the following discussion is the existence of limits from the left and from the right. This is particularly useful when talking about piecewise-defined functions.

- If the value of $f(x)$ approaches the number ℓ as x approaches a from the right-hand side, then it is written as $\lim_{x \to a^+} f(x) = \ell$.
- If the value of $f(x)$ approaches the number ℓ as x approaches a from the left-hand side, then it is written as $\lim_{x \to a^-} f(x) = \ell$.
- The limit as x approaches a exists only if both the limit from the left and the limit from the right exist and are equal. Then $\lim_{x \to a} f(x) = \ell$.

Piecewise-defined function

The following is an example of a piecewise-defined function where the limit does not exist for a particular value.

Let $f(x) = \begin{cases} x^3 & \text{if } 0 \leq x < 1 \\ 5 & \text{if } x = 1 \\ 6 & \text{if } 1 < x \leq 2 \end{cases}$



It is clear from the graph of f that $\lim_{x \to 1} f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand, if x is allowed to approach 1 from the right, then $f(x)$ approaches 6. Also note that $f(1) = 5$.

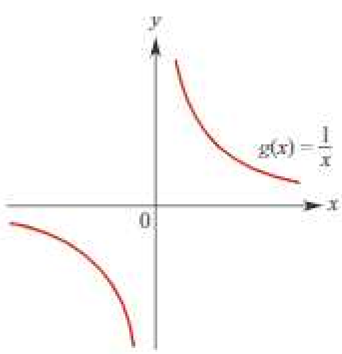
Rectangular hyperbola

As mentioned at the start of this section, the notation of limits is used to describe the asymptotic behaviour of graphs.

Consider $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. The behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x \to 0^-} g(x) = -\infty \qquad \text{and} \qquad \lim_{x \to 0^+} g(x) = \infty$$



Now examine this function as the magnitude of x becomes very large. It can be seen that, as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise, as x decreases without bound through negative values, the corresponding values of $g(x)$ also approach zero.

Symbolically this is written as:

$$\lim_{x \to \infty} g(x) = 0^+ \qquad \text{and} \qquad \lim_{x \to -\infty} g(x) = 0^-$$

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Continuity at a point

We only require an intuitive understanding of continuity.

> A function with rule $f(x)$ is said to be continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise, there is said to be a discontinuity at $x = a$.

We can give a more formal definition of continuity using limits. A function f is continuous at the point $x = a$ provided $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ all exist and are equal.

We can state this equivalently as follows:

> A function f is **continuous** at the point $x = a$ if the following conditions are met:
> - $f(x)$ is defined at $x = a$
> - $\lim_{x \to a} f(x) = f(a)$

The function is **discontinuous** at a point if it is not continuous at that point.

A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere. In contrast, the function

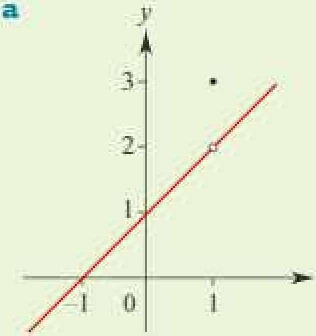
$$h(x) = \begin{cases} x^3 & \text{if } x < 1 \\ 6 & \text{if } x \geq 1 \end{cases}$$

is defined for all real numbers but is not continuous at $x = 1$.

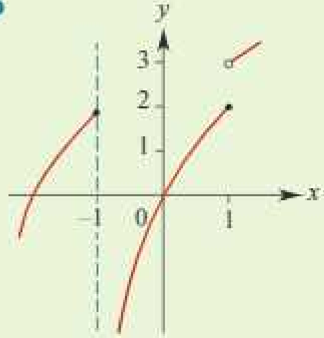
Example 31

State the values for x for which the functions shown below have a discontinuity:

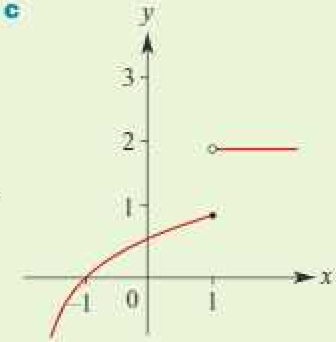
a



b



c



Solution

a There is a discontinuity at $x = 1$, since $f(1) = 3$ but $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$.

b There is a discontinuity at $x = 1$, since $f(1) = 2$ and $\lim_{x \to 1^-} f(x) = 2$ but $\lim_{x \to 1^+} f(x) = 3$. There is also a discontinuity at $x = -1$, since $f(-1) = 2$ and $\lim_{x \to -1^-} f(x) = 2$ but $\lim_{x \to -1^+} f(x) = -\infty$.

c There is a discontinuity at $x = 1$, since $f(1) = 1$ and $\lim_{x \to 1^-} f(x) = 1$ but $\lim_{x \to 1^+} f(x) = 2$.

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Example 32

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of $f(a)$, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to explain why:

a $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x+1 & \text{if } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

b $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

c $f(-1) = -1$ but $\lim_{x\to -1^+} f(x) = 1$, therefore there is a discontinuity at $x = -1$.
$f(0) = 1$ but $\lim_{x\to 0^-} f(x) = 0$, therefore there is a discontinuity at $x = 0$.

d No discontinuity. **e** No discontinuity.

Summary 17F

- A function f is **continuous** at the point $x = a$ if the following conditions are met:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x\to a} f(x) = f(a)$
- The function is **discontinuous** at a point if it is not continuous at that point.
- A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere.
- **Algebra of limits** Suppose that f and g are functions and that a is a real number. Assume that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
  - $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
    That is, the limit of the sum is the sum of the limits.
  - $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$, where k is a given real number.
  - $\lim_{x\to a} (f(x)\,g(x)) = \lim_{x\to a} f(x) \; \lim_{x\to a} g(x)$
    That is, the limit of the product is the product of the limits.
  - $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
    That is, the limit of the quotient is the quotient of the limits.

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Exercise 17F

Example 28
Example 29

1 Find the following limits:

a $\lim_{x \to 3} 15$ **b** $\lim_{x \to 6} (x - 5)$ **c** $\lim_{x \to \frac{1}{2}} (3x - 5)$

d $\lim_{t \to -3} \dfrac{t - 2}{t + 5}$ **e** $\lim_{t \to -1} \dfrac{t^2 + 2t + 1}{t + 1}$ **f** $\lim_{x \to 0} \dfrac{(x + 2)^2 - 4}{x}$

g $\lim_{t \to 1} \dfrac{t^2 - 1}{t - 1}$ **h** $\lim_{x \to 9} \sqrt{x + 3}$ **i** $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

j $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$ **k** $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}$ **l** $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

Example 30

2 Find the following limits:

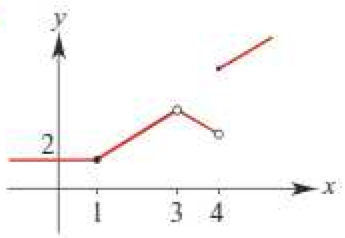
a $\lim_{h \to 0} (5h - 1)$ **b** $\lim_{h \to 0} h(2x + 3)$ **c** $\lim_{h \to 2} (5 - 2h)$

d $\lim_{x \to 1} \dfrac{3x - 1}{x + 1}$ **e** $\lim_{h \to 6} \dfrac{h^2 - 3h - 18}{h^2 - 6h}$ **f** $\lim_{t \to -1} \dfrac{t^2 - 1}{t - 1}$

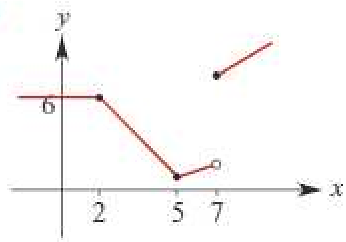
Example 31

3 For each of the following graphs, give the values of x at which a discontinuity occurs. Give reasons.

a



b



Example 32

4 For the following functions, state each value of x at which there is a discontinuity. Use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity.

a $f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \geq 0 \end{cases}$

5 The rule of a particular function is given below. For what values of x is the graph of this function discontinuous?

$$y = \begin{cases} 2 & \text{if } x < 1 \\ (x - 4)^2 - 9 & \text{if } 1 \leq x < 7 \\ x - 7 & \text{if } x \geq 7 \end{cases}$$

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17G When is a function differentiable?

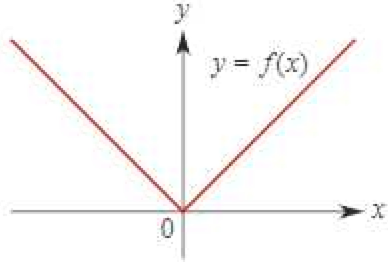
A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

The polynomial functions considered in this chapter are differentiable for all real numbers. However, this is not true for all functions.

For example, consider the function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The graph of $y = f(x)$ is shown on the right. (This is called the modulus function.)



Now consider the gradient of the secant through the points $(0, 0)$ and $(h, f(h))$ on the graph of $y = f(x)$:

$$\frac{f(0+h) - f(0)}{h} = \frac{f(h)}{h} = \begin{cases} 1 & \text{if } h > 0 \\ -1 & \text{if } h < 0 \end{cases}$$

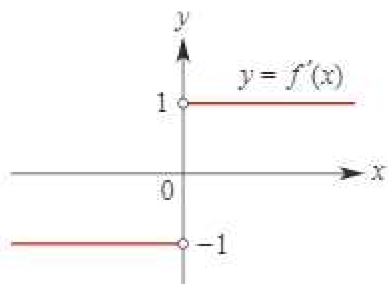
The gradient of the secant does not approach a unique value as $h \to 0$.

We say that $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

The gradient of the curve $y = f(x)$ is 1 to the right of 0, and -1 to the left of 0. Therefore the derivative function is given by

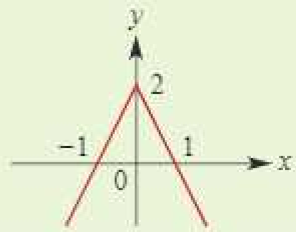
$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

The graph of $y = f'(x)$ is shown on the right.



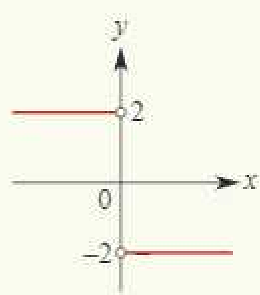
Example 33

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$,
i.e. the function is not differentiable at $x = 0$.



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It was shown in the previous section that some piecewise-defined functions are continuous everywhere. Similarly, there are piecewise-defined functions which are differentiable everywhere. The smoothness of the 'joins' determines whether this is the case.

Example 34

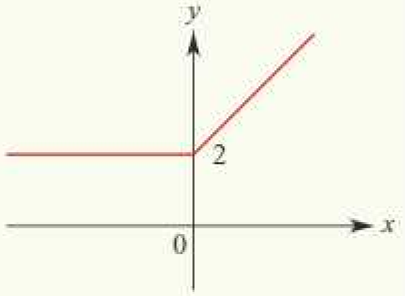
For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$

In particular, $f'(0)$ is defined and is equal to 2. The two sections of the graph of $y = f(x)$ join smoothly at the point (0, 1).



Example 35

For the function with rule

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

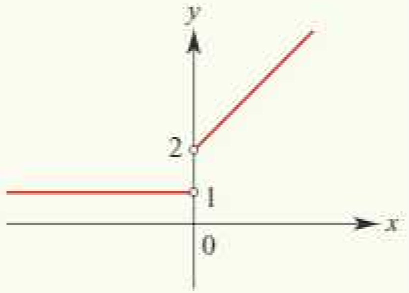
state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$.

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$

$f'(0)$ is not defined as the limits from the left and right are not equal.

The function f is differentiable for $\mathbb{R} \setminus \{0\}$.



Summary 17G

A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

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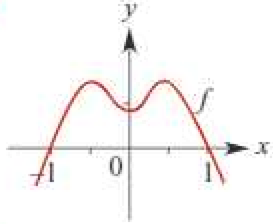
Exercise 17G

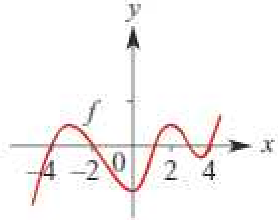
Example 33

1 In each of the figures below, the graph of a function f is given. Sketch the graph of f'. Obviously your sketch of f' cannot be exact; but $f'(x)$ should be zero at values of x for which the gradient of f is zero, and $f'(x)$ should be negative where the original graph slopes downwards, and so on.

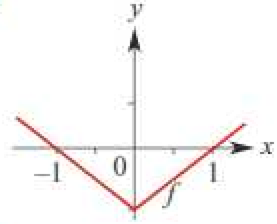
a



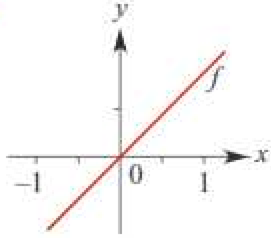
b



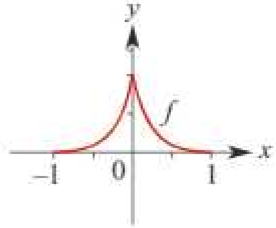
c



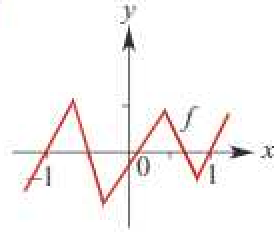
d



e



f



Example 34

2 For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$$

Example 35

3 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$$

4 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 - 3x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$$

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Review

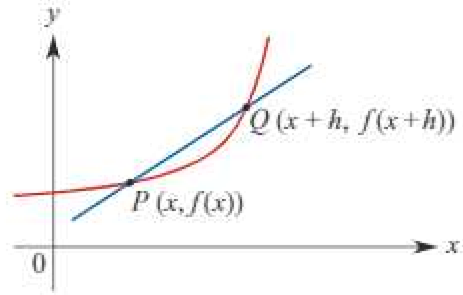
Chapter summary

Assign-ment

Nrich

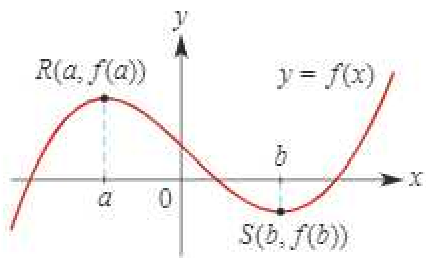
The derivative

- The notation for the limit as h approaches 0 is $\lim\limits_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
  $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
  $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- At a point $(x, f(x))$ on the curve $y = f(x)$, the gradient is $f'(x)$.

  For the graph shown:
  - $f'(x) > 0$ for $x < a$ and for $x > b$
  - $f'(x) < 0$ for $a < x < b$
  - $f'(x) = 0$ for $x = a$ and for $x = b$.



Rules for differentiation

- For $f(x) = c$, $f'(x) = 0$. That is, the derivative of a constant function is zero.

  For example:
  - $f(x) = 1$, $f'(x) = 0$
  - $f(x) = 27.3$, $f'(x) = 0$
- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

  For example:
  - $f(x) = x^2$, $f'(x) = 2x$
  - $f(x) = x^3$, $f'(x) = 3x^2$
  - $f(x) = x^{-1}$, $f'(x) = -x^{-2}$
  - $f(x) = x^{-3}$, $f'(x) = -3x^{-4}$
- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.

  For example:
  - $f(x) = 3x^2$, $f'(x) = 3(2x) = 6x$
  - $f(x) = 5x^3$, $f'(x) = 5(3x^2) = 15x^2$
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.

  For example:
  - $f(x) = x^2 + x^3$, $f'(x) = 2x + 3x^2$
  - $f(x) = 3x^2 + 5x^3$, $f'(x) = 6x + 15x^2$

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Antiderivatives

- If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + c$, where $n \in \mathbb{N} \cup \{0\}$
- $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
- $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number.

Algebra of limits

- $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

  That is, the limit of the sum is the sum of the limits.
- $\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x)$, where k is a real number.
- $\lim_{x\to a}(f(x)\,g(x)) = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$

  That is, the limit of the product is the product of the limits.
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.

  That is, the limit of the quotient is the quotient of the limits.

Continuity

- A function f is **continuous** at the point $x = a$ if:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x\to a} f(x) = f(a)$
- The function is **discontinuous** at a point if it is not continuous at that point.

Technology-free questions

1 Find the derivative of each of the following by first principles:

a $y = 3x + 1$ **b** $y = 4 - x^2$ **c** $y = x^2 + 5x$

d $y = x^3 + x$ **e** $y = x^2 + 2x + 1$ **f** $y = 3x^2 - x$

2 Find $\dfrac{dy}{dx}$ when:

a $y = 3x^2 - 2x + 6$ **b** $y = 5$ **c** $y = 2x(2 - x)$

d $y = 4(2x - 1)(5x + 2)$ **e** $y = (x + 1)(3x - 2)$ **f** $y = (x + 1)(2 - 3x)$

3 Find $\dfrac{dy}{dx}$ when:

a $y = -x$ **b** $y = 10$ **c** $y = \dfrac{(x + 3)(2x + 1)}{4}$

d $y = \dfrac{2x^3 - x^2}{3x}$ **e** $y = \dfrac{x^4 + 3x^2}{2x^2}$

4 For each of the following functions, find the y-coordinate and the gradient of the tangent at the point on the curve for the given value of x:

a $y = 4x^3 - 2x,\quad x = 2$ **b** $y = 3x^2 - 5x^3,\quad x = -1$

c $y = 5 - 2x + 3x^2 - 4x^3,\quad x = 1$ **d** $y = (x - 5)(x - 6),\quad x = 4$

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Review

5 Find the coordinates of the points on the curves given by the following equations at which the gradient of the tangent at that point has the given value:

a $y = x^2 - 3x + 1, \quad \frac{dy}{dx} = 0$ **b** $y = x^3 - 6x^2 + 4, \quad \frac{dy}{dx} = -12$

c $y = x^2 - x^3, \quad \frac{dy}{dx} = -1$ **d** $y = x^3 - 2x + 7, \quad \frac{dy}{dx} = 1$

e $y = x^4 - 2x^3 + 1, \quad \frac{dy}{dx} = 0$ **f** $y = x(x-3)^2, \quad \frac{dy}{dx} = 0$

6 For the function with rule $f(x) = 3(2x-1)^2$, find the values of x for which:

a $f(x) = 0$ **b** $f'(x) = 0$ **c** $f'(x) > 0$

d $f'(x) < 0$ **e** $f(x) > 0$ **f** $f'(x) = 3$

7 Find the derivative of each of the following with respect to x:

a x^{-4} **b** $2x^{-3}$ **c** $-\frac{1}{3x^2}$ **d** $-\frac{1}{x^4}$

e $\frac{3}{x^5}$ **f** $\frac{x^2 + x^3}{x^4}$ **g** $\frac{3x^2 + 2x}{x^2}$ **h** $5x^2 - \frac{2}{x}$

8 The curve with equation $y = ax^2 + bx$ has a tangent with gradient 3 at the point $(1, 1)$.

a Find the values of a and b.

b Find the coordinates of the points on the curve where the gradient is 0.

9 Find:

a $\int \frac{1}{2}\, dx$ **b** $\int \frac{1}{2}x^2\, dx$ **c** $\int x^2 + 3x\, dx$ **d** $\int (2x+3)^2\, dx$

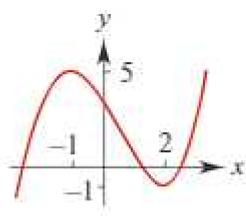
e $\int at\, dt$ **f** $\int \frac{1}{3}t^3\, dt$ **g** $\int (t+1)(t-2)\, dt$ **h** $\int (2-t)(t+1)\, dt$

10 The curve with equation $y = f(x)$ passes through the point $(3, -1)$ and $f'(x) = 2x + 5$. Find $f(x)$.

11 The curve with equation $y = f(x)$ passes through the origin and $f'(x) = 3x^2 - 8x + 3$.

a Find $f(x)$. **b** Find the intercepts of the curve with the x-axis.

12 The graph of $y = f(x)$ is shown. Sketch the graph of $y = f'(x)$. (Not all details can be determined, but the x-axis intercepts and the shape of graph can be determined.)

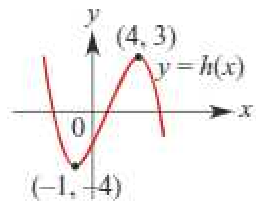


13 For the graph of $y = h(x)$, find:

a $\{x : h'(x) > 0\}$

b $\{x : h'(x) < 0\}$

c $\{x : h'(x) = 0\}$



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Multiple-choice questions

1 The gradient of the curve $y = x^3 + 4x$ at the point where $x = 2$ is

A 12 **B** 4 **C** 10 **D** 16 **E** 8

2 The gradient of the chord of the curve $y = 2x^2$ between the points where $x = 1$ and $x = 1 + h$ is given by

A $2(x+h)^2 - 2x^2$ **B** $4 + 2h$ **C** 4
D $4x$ **E** $4 + h$

3 If $y = 2x^4 - 5x^3 + 2$, then $\frac{dy}{dx}$ equals

A $8x^3 - 5x^2 + 2$ **B** $4x^4 - 15x^2 + 2$ **C** $4x^4 - 10x^2$
D $8x^3 - 15x + 2$ **E** $8x^3 - 15x^2$

4 If $f(x) = x^2(x+1)$, then $f'(-1)$ equals

A -1 **B** 1 **C** 2 **D** -2 **E** 5

5 If $f(x) = (x-3)^2$, then $f'(x)$ equals

A $x - 3$ **B** $x - 6$ **C** $2x - 6$ **D** $2x + 9$ **E** $2x$

6 If $y = \frac{2x^4 + 9x^2}{3x}$, then $\frac{dy}{dx}$ equals

A $\frac{2x^4}{3} + 6x$ **B** $2x + 3$ **C** $2x^2 + 3$ **D** $\frac{8x^3 + 18x}{3}$ **E** $8x^3 + 18x$

7 Given that $y = x^2 - 6x + 9$, the values of x for which $\frac{dy}{dx} \geq 0$ are

A $x \geq 3$ **B** $x > 3$ **C** $x \geq -3$ **D** $x \leq -3$ **E** $x < 3$

8 If $y = 2x^4 - 36x^2$, the points at which the tangent to the curve is parallel to the x-axis are

A 1, 0 and 3 **B** 0 and 3 **C** -3 and 3 **D** 0 and -3 **E** -3, 0 and 3

9 The coordinates of the point on the graph of $y = x^2 + 6x - 5$ at which the tangent is parallel to the line $y = 4x$ are

A $(-1, -10)$ **B** $(-1, -2)$ **C** $(1, 2)$ **D** $(-1, 4)$ **E** $(-1, 10)$

10 If $y = -2x^3 + 3x^2 - x + 1$, then $\frac{dy}{dx}$ equals

A $6x^2 + 6x - 1$ **B** $-6x^2 + 6x$ **C** $-6x^2 + 3x - 1$
D $-6x^2 + 6x - 1$ **E** $6x^2 - 6x - 1$

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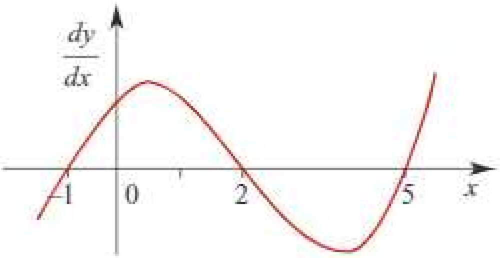
Review

Extended-response questions

1 The diagram to the right shows part of the graph of $\frac{dy}{dx}$ against x.

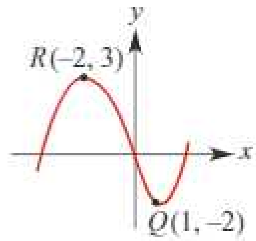


Sketch a possible shape for the graph of y against x over the same interval if:

- $y = -1$ when $x = -1$
- $y = 0$ when $x = 0$
- $y = 1$ when $x = 2$.

2 The graph shown is that of a polynomial function of the form $P(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

Note: $Q(1, -2)$ is not a turning point.



3 A body moves in a path described by the equation $y = \frac{1}{5}x^5 + \frac{1}{2}x^4$, for $x \geq 0$. Units are in kilometres, and x and y are the horizontal and vertical axes respectively.

CAS

a What will be the direction of motion (give your answer as the angle between the direction of motion and the x-axis) when the x-value is:

i 1 km **ii** 3 km?

b Find a value of x for which the gradient of the path is 32.

4 A trail over a mountain pass can be modelled by the curve $y = 2 + 0.12x - 0.01x^3$, where x and y are the horizontal and vertical distances respectively, measured in kilometres, and $0 \leq x \leq 3$.

a Find the gradients at the beginning and the end of the trail.

b Calculate the point where the gradient is zero, and calculate also the height of the pass.

5 A tadpole begins to swim vertically upwards in a pond and after t seconds it is $25 - 0.1t^3$ cm below the surface.

a How long does the tadpole take to reach the surface, and what is its speed then?

b What is the average speed over this time?

6 **a** Show that the gradients of the curve $y = x(x - 2)$ at the points $(0, 0)$ and $(2, 0)$ only differ in sign. What is the geometrical interpretation for this?

b If the gradients of the curve $y = x(x - 2)(x - 5)$ at the points $(0, 0)$, $(2, 0)$ and $(5, 0)$ are ℓ, m and n respectively, show that $\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} = 0$.

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