18

Applications of differentiation of polynomials

Objectives

- To be able to find the equation of the **tangent** and the **normal** at a given point on a polynomial curve.
- To use the derivative of a polynomial in **rate of change** problems.
- To be able to find the **stationary points** on the graphs of certain polynomial functions and to state the nature of such points.
- To apply differentiation to **sketching graphs** of polynomial functions.
- To apply differentiation to the solution of **maximum and minimum** problems.
- To apply differentiation to problems involving **motion in a straight line**.

In this chapter we continue our study of calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The new techniques for sketching graphs of polynomial functions are a useful addition to the skills that were introduced in Chapter 6. At that stage, rather frustratingly, we were only able to determine the coordinates of turning points of cubic and quartic functions using technology. The new techniques are also used for determining maximum or minimum values for problems set in a 'real world' context.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. This was the motivation for Newton's work on calculus.

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18A Tangents and normals

The derivative of a function is a new function which gives the measure of the gradient of the tangent at each point on the curve. If the gradient is known, it is possible to find the equation of the tangent for a given point on the curve.

Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point where $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $(1, \frac{3}{2})$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent to the curve at $x = 1$ is 4.

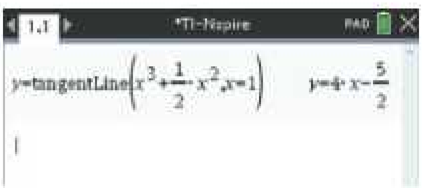
Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

which becomes $y = 4x - \frac{5}{2}$.

Using the TI-Nspire

Use (menu) > **Calculus** > **Tangent Line** to calculate the tangent to the curve at $x = 1$.

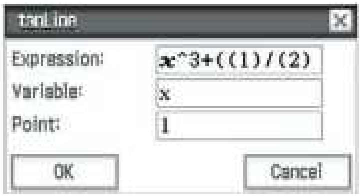


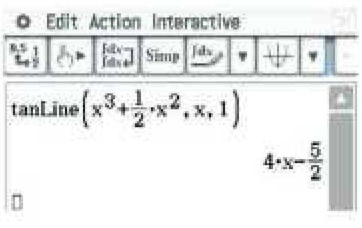
Using the Casio ClassPad

The tangent to the graph of $y = x^3 + \frac{1}{2}x^2$ at $x = 1$ can be found in two ways.

Method 1

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^3 + \frac{1}{2}x^2$.
- Select **Interactive** > **Calculation** > **line** > **tanLine**.
- The pop-up window shown will appear. Enter the value 1 for the point and tap OK.





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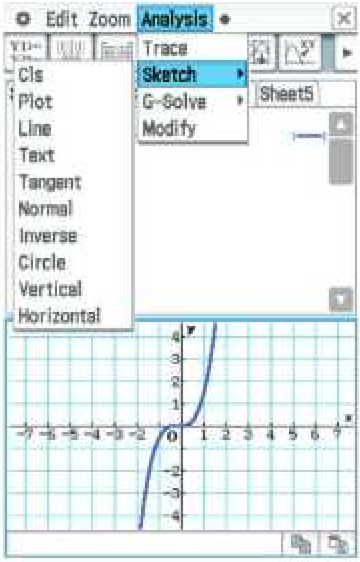
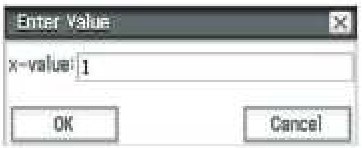
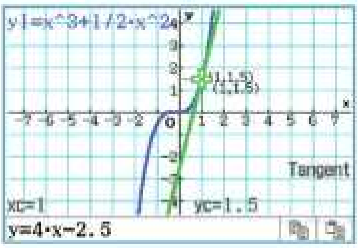
Method 2



- In Graph&Table, enter the expression $x^3 + \frac{1}{2}x^2$ in $y1$.
- Tick the box for $y1$ and select the graph icon.
- Select **Analysis** > **Sketch** > **Tangent**.
- When the graph appears, press the x-value of interest, in this case $x = 1$, and the window shown below will appear. Tap OK.



- The tangent at $x = 1$ is drawn on the graph.
- To view the equation of the tangent, tap EXE.
- The tangent equation is shown at the bottom of the screen.



The **normal** to a curve at a point on the curve is the line which passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ lies on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the tangent at $x = 1$ is -1.

Thus the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. the equation of the normal is $y = x - 2$.

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Using the TI-Nspire

Use (menu) > **Calculus** > **Normal Line** to calculate the normal to the curve at the point $(1, -1)$, i.e. when $x = 1$.

Using the Casio ClassPad

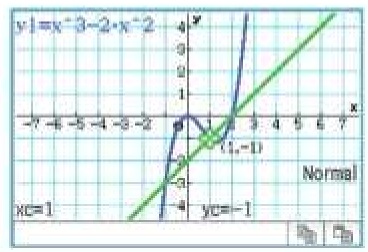
The normal to the graph of $y = x^3 - 2x^2$ at the point $(1, -1)$ can be found in two ways.

Method 1

- In Main $\sqrt{\alpha}$, enter and highlight $x^3 - 2x^2$.
- Select **Interactive** > **Calculation** > **line** > **normal**.
- In the pop-up window that appears, enter the value 1 for the point and tap OK.

Method 2

- In Graph & Table, enter the expression $x^3 - 2x^2$ in $y1$.
- Tick the box for $y1$ and select the graph icon.
- Select **Analysis** > **Sketch** > **Normal**.
- When the graph appears, press the x-value of interest, in this case $x = 1$. Tap OK.
- The normal at $x = 1$ is drawn on the graph.
- To view the equation of the normal, tap (EXE).



Summary 18A

- **Equation of a tangent line**
  Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- **Gradient of a normal line**
  If a tangent has gradient m, the normal has gradient $-\dfrac{1}{m}$.

Skill-sheet

Exercise 18A

Example 1
Example 2

1 Find the equation of the tangent and the normal at the given point for:

a $f(x) = x^2$, $(2, 4)$

b $f(x) = (2x - 1)^2$, $(2, 9)$

c $f(x) = 3x - x^2$, $(2, 2)$

d $f(x) = 9x - x^3$, $(1, 8)$

2 Find the equation of the tangent to the curve with equation $y = 3x^3 - 4x^2 + 2x - 10$ at the point of intersection with the y-axis.

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3 Find the equation of the tangent to $y = x^2$ at the point $(1, 1)$ and the equation of the tangent to $y = \frac{1}{6}x^3$ at the point $\left(2, \frac{4}{3}\right)$.
Show that these tangents are parallel and find the perpendicular distance between them.

4 Find the equations of the tangents to the curve $y = x^3 - 6x^2 + 12x + 2$ which are parallel to the line $y = 3x$.

5 The curve with the equation $y = (x-2)(x-3)(x-4)$ cuts the x-axis at the points $P = (2, 0)$, $Q = (3, 0)$ and $R = (4, 0)$.

a Prove that the tangents at P and R are parallel.

b At what point does the normal to the curve at Q cut the y-axis?

6 For the curve with equation $y = x^2 + 3$, show that $y = 2ax - a^2 + 3$ is the equation of the tangent at the point $(a, a^2 + 3)$.
Hence find the coordinates of the two points on the curve, the tangents of which pass through the point $(2, 6)$.

7 **a** Find the equation of the tangent at the point $(2, 4)$ to the curve $y = x^3 - 2x$.

b Find the coordinates of the point where the tangent meets the curve again.

8 **a** Find the equation of the tangent to the curve $y = x^3 - 9x^2 + 20x - 8$ at the point $(1, 4)$.

b At what points on the curve is the tangent parallel to the line $4x + y - 3 = 0$?

18B Rates of change

The derivative of a function was defined geometrically in the previous chapter. But, as seen in Chapter 16, the process of differentiation may be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is defined to be $f'(a)$.

Average rate of change has been discussed in Chapter 16.

The instantaneous rate of change of y with respect to x is given by $\dfrac{dy}{dx}$, that is, by the derivative of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, the change is an increase in the value of y corresponding to an increase in x.
- If $\dfrac{dy}{dx} < 0$, the change is a decrease in the value of y corresponding to an increase in x.

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Example 3

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b
$$\begin{aligned}
\text{Average rate of change} &= \frac{f(2+h) - f(2)}{2 + h - 2} \\
&= \frac{(2+h)^2 + 2(2+h) - 8}{h} \\
&= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\
&= \frac{6h + h^2}{h} \\
&= 6 + h
\end{aligned}$$

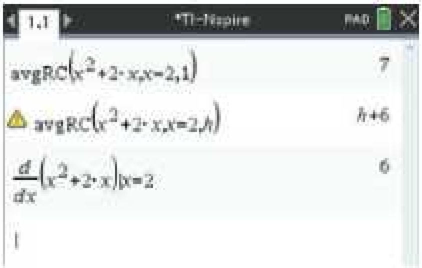
c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part **b**.

Using the TI-Nspire

- For parts **a** and **b**, use the catalog to access the **Average Rate of Change** command ([catalog] [1] [A]) and enter as:

  avgRC(*expression*, *x* = *initial value*, *step size*)

- For part **c**, use [menu] > **Calculus** > **Derivative at a Point** and complete as shown.



Example 4

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V (cm^3), at time t (seconds) is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t > 0$.

a Find the rate of change of volume after:

  i 10 seconds

  ii 20 seconds

b For how long could the model be valid?

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Solution

a $V = 600 - 10t - \frac{1}{100}t^2$

$\frac{dV}{dt} = -10 - \frac{t}{50}$

i When $t = 10$, $\frac{dV}{dt} = -10 - \frac{1}{5}$

$= -10\frac{1}{5}$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\frac{dV}{dt} = -10 - \frac{2}{5}$

$= -10\frac{2}{5}$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$.

Consider $V = 0$:

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = \frac{10 \pm \sqrt{100 + 4 \times 0.01 \times 600}}{-0.02}$$

$$\therefore \quad t = -1056.78 \quad \text{or} \quad t = 56.78 \quad \text{(to two decimal places)}$$

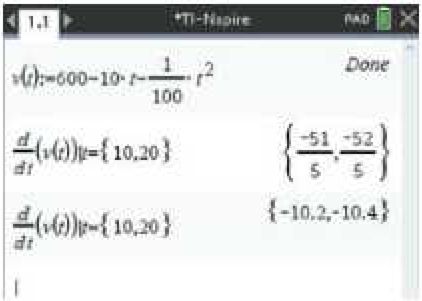
Hence the model may be suitable for $0 < t < 56.78$.

Using the TI-Nspire

Assign the function $v(t)$ as shown.

a Use (menu) > **Calculus** > **Derivative** and enter the required t-values using the | symbol ((ctrl)(=)) to evaluate the derivative of $v(t)$ at $t = 10$ and $t = 20$.

Press (ctrl)(enter) to obtain the answer as a decimal number.

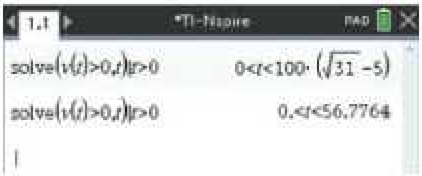


Note: If you used (menu) > **Calculus** > **Derivative at a Point** instead, then each t-value would need to be evaluated separately.

b To find the domain, use:

$$\text{solve}(v(t) > 0, t) \mid t > 0$$

Press (ctrl)(enter) to obtain the answer as a decimal number.



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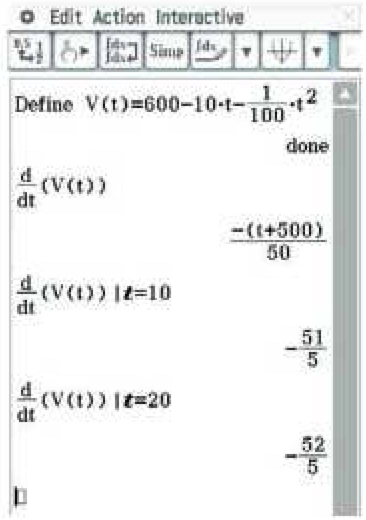
Using the Casio ClassPad

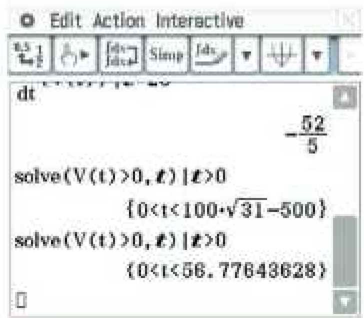
- In $\sqrt{\alpha}$ (Main), enter and highlight $600 - 10t - \frac{1}{100}t^2$.
- Select **Interactive** > **Define**. Enter the function name V using the abc keyboard, set the variable to t using the Var keyboard, and then tap OK.
- In the next entry line, enter and highlight $V(t)$.
- Go to **Interactive** > **Calculation** > **diff**. Set the variable to t and tap OK.
- To substitute values of t, insert | from the Math3 keyboard and type $t = 10$ or $t = 20$ after the derivative, as shown.



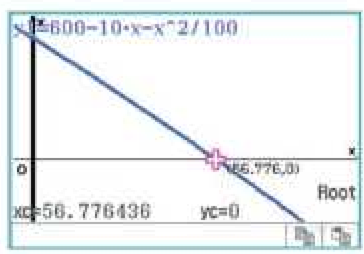
- To find the domain, select solve(from the Math3 keyboard and enter:

$$V(t) > 0, t) \mid t > 0$$

Note: You can copy and paste $V(t)$ from the entry line above.



- The domain can also be obtained graphically by finding where $V = 0$.



Summary 18B

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\frac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is $f'(a)$.

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Exercise 18B

Example 3 **1** Let $y = 35 + 12x^2$.

a Find the change in y as x changes from 1 to 2. What is the average rate of change of y with respect to x in this interval?

b Find the change in y as x changes from $2 - h$ to 2. What is the average rate of change of y with respect to x in this interval?

c Find the rate of change of y with respect to x when $x = 2$.

Example 4 **2** According to a business magazine, the expected assets, $\$M$, of a proposed new company will be given by $M = 200\,000 + 600t^2 - \frac{200}{3}t^3$, where t is the number of months after the business is set up.

a Find the rate of growth of assets at time t months.

b Find the rate of growth of assets at time $t = 3$ months.

c When will the rate of growth of assets be zero?

3 As a result of a survey, the marketing director of a company found that the revenue, $\$R$, from pricing 100 produced items at $\$P$ each is given by the rule $R = 30P - 2P^2$.

a Find $\frac{dR}{dP}$ and explain what it means.

b Calculate $\frac{dR}{dP}$ when $P = 5$ and $P = 10$.

c For what selling prices is revenue rising?

4 The population, P, of a new housing estate t years after 30 January 2017 is given by the rule $P = 100(5 + t - 0.25t^2)$. Find the rate of change of the population after:

a 1 year

b 2 years

c 3 years

5 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by $V(t) = \frac{5}{8}\left(10t^2 - \frac{t^3}{3}\right)$, $0 \le t \le 20$.

a Find the volume of water in the flask at time:

i $t = 0$ **ii** $t = 20$

b Find the rate of flow of water into the flask at time t.

c Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

6 The area, A km^2, of an oil slick is growing according to the rule $A = \frac{t}{2} + \frac{t^2}{10}$, where t is the time in hours since the leak started.

a Find the area covered at the end of 1 hour.

b Find the rate of increase of the area after 1 hour.

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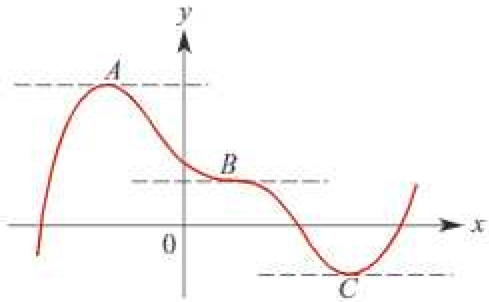
18C Stationary points

In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, for $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

In the graph shown, there are stationary points at A, B and C. At such points the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary points* becomes clear later in this chapter when we look at the application to the motion of a particle.

Example 5

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $p = 2t^3 - 5t^2 - 4t + 13$ for $t > 0$ **c** $y = 4 + 3x - x^3$

Solution

a $y = 9 + 12x - 2x^2$

$\frac{dy}{dx} = 12 - 4x$

The stationary points occur when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$, i.e. at $x = 3$.

When $x = 3$, $y = 9 + 12 \times 3 - 2 \times 3^2 = 27$.

Thus the stationary point is at $(3, 27)$.

b $p = 2t^3 - 5t^2 - 4t + 13 \quad (t > 0)$

$\frac{dp}{dt} = 6t^2 - 10t - 4 \quad (t > 0)$

Thus, $\frac{dp}{dt} = 0$ implies $2(3t^2 - 5t - 2) = 0$

$(3t + 1)(t - 2) = 0$

$\therefore \quad t = -\frac{1}{3}$ or $t = 2$

But $t > 0$, therefore the only acceptable solution is $t = 2$.

When $t = 2$, $p = 16 - 20 - 8 + 13 = 1$.

So the corresponding stationary point is at $(2, 1)$.

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c $y = 4 + 3x - x^3$

$\frac{dy}{dx} = 3 - 3x^2$

Thus, $\frac{dy}{dx} = 0$ implies $3(1 - x^2) = 0$

$\therefore \quad x = \pm 1$

The stationary points occur at $(1, 6)$ and $(-1, 2)$.

Example 6

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through $(0, 5)$ and has a stationary point at $(2, 7)$. Find the values of a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $c = 5$.

We have $\frac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\frac{dy}{dx} = 0$.

Therefore

$$0 = 12 + 4a + b \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$7 = 2^3 + 2^2a + 2b + 5$$

$$2 = 8 + 4a + 2b$$

$$4a + 2b + 6 = 0 \qquad (2)$$

Subtract (2) from (1):

$$-b + 6 = 0$$

$$\therefore \quad b = 6$$

Substitute in (1):

$$0 = 12 + 4a + 6$$

$$-18 = 4a$$

$$\therefore \quad -\frac{9}{2} = a$$

Hence $a = -\frac{9}{2}$, $b = 6$ and $c = 5$.

Summary 18C

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, for $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = a$, then $(a, f(a))$ is a stationary point.

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Skill-sheet

Exercise 18C

Example 5

1 Find the coordinates of the stationary points of each of the following functions:

a $f(x) = x^2 - 6x + 3$

b $y = x^3 - 4x^2 - 3x + 20$ for $x > 0$

c $z = x^4 - 32x + 50$

d $q = 8t + 5t^2 - t^3$ for $t > 0$

e $y = 2x^2(x - 3)$

f $y = 3x^4 - 16x^3 + 24x^2 - 10$

Example 6

2 The curve with equation $y = ax^2 + bx + c$ passes through $(0, -1)$ and has a stationary point at $(2, -9)$. Find the values of a, b and c.

3 The curve with equation $y = ax^2 + bx + c$ has a stationary point at $(1, 2)$. When $x = 0$, the slope of the curve is $45°$. Find the values of a, b and c.

4 The curve with equation $y = ax^2 + bx$ has a gradient of 3 at the point $(2, -2)$.

a Find the values of a and b.

b Find the coordinates of the turning point.

5 The curve with equation $y = x^2 + ax + 3$ has a stationary point when $x = 4$. Find the value of a.

6 The curve with equation $y = x^2 - ax + 4$ has a stationary point when $x = 3$. Find the value of a.

7 Find the coordinates of the stationary points of each of the following:

a $y = x^2 - 5x - 6$

b $y = (3x - 2)(8x + 3)$

c $y = 2x^3 - 9x^2 + 27$

d $y = x^3 - 3x^2 - 24x + 20$

e $y = (x + 1)^2(x + 4)$

f $y = (x + 1)^2 + (x + 2)^2$

8 The curve with equation $y = ax^2 + bx + 12$ has a stationary point at $(1, 13)$. Find the values of a and b.

9 The curve with equation $y = ax^3 + bx^2 + cx + d$ has a gradient of -3 at $(0, 7\frac{1}{2})$ and a turning point at $(3, 6)$. Find the values of a, b, c and d.

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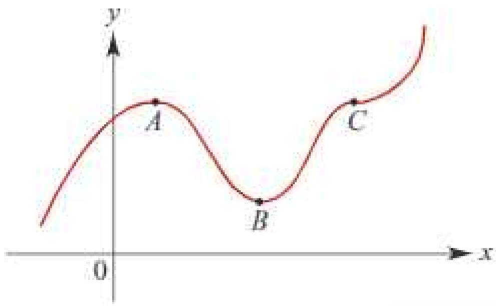
18D Types of stationary points

The graph of $y = f(x)$ below has three stationary points $A(a, f(a))$, $B(b, f(b))$, $C(c, f(c))$.



A Point A is called a **local maximum** point.
Note that $f'(x) > 0$ immediately to the left of A, and that $f'(x) < 0$ immediately to the right of A.
This means that f is strictly increasing immediately to the left of A, and that f is strictly decreasing immediately to the right of A.

gradient	+	0	−
shape of f	╱	—	╲

B Point B is called a **local minimum** point.
Note that $f'(x) < 0$ immediately to the left of B, and that $f'(x) > 0$ immediately to the right of B.
This means that f is strictly decreasing immediately to the left of B, and that f is strictly increasing immediately to the right of B.

gradient	−	0	+
shape of f	╲	—	╱

C Point C is called a **stationary point of inflection**.
Note that $f'(x) > 0$ immediately to the left and the right of C.

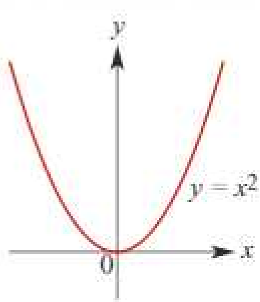
gradient	+	0	+
shape of f	╱	—	╱

Clearly it is also possible to have stationary points of inflection with $f'(x) < 0$ immediately to the left and right.

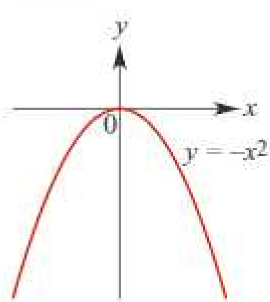
gradient	−	0	−
shape of f	╲	—	╲

Stationary points of types A and B are referred to as **turning points**.

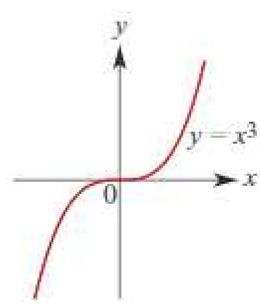
Before proceeding with some more complicated functions, it is worth referring back to some of the functions we met earlier in this book.



$f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$
Local minimum at $(0, 0)$.



$f: \mathbb{R} \to \mathbb{R}, f(x) = -x^2$
Local maximum at $(0, 0)$.



$f: \mathbb{R} \to \mathbb{R}, f(x) = x^3$
Stationary point of inflection at $(0, 0)$.

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Example 7

For the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$
$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $(-\frac{2}{3}, f(-\frac{2}{3}))$ and $(\frac{2}{3}, f(\frac{2}{3}))$, that is, at $(-\frac{2}{3}, 2\frac{7}{9})$ and $(\frac{2}{3}, -\frac{7}{9})$.
So $f'(x)$ is of constant sign for each of

$$\{x : x < -\tfrac{2}{3}\}, \quad \{x : -\tfrac{2}{3} < x < \tfrac{2}{3}\} \quad \text{and} \quad \{x : x > \tfrac{2}{3}\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$

$f'(0) = 0 - 4 = -4 < 0$

$f'(1) = 9 - 4 = 5 > 0$

We can now put together the chart shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	/	—	\	—	/

There is a local maximum at $(-\frac{2}{3}, 2\frac{7}{9})$ and a local minimum at $(\frac{2}{3}, -\frac{7}{9})$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.
Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$.
By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

Now $(x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$.
We have

$$3x^2 + 3x - 1 = 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right]$$
$$= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right]$$
$$= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right)$$

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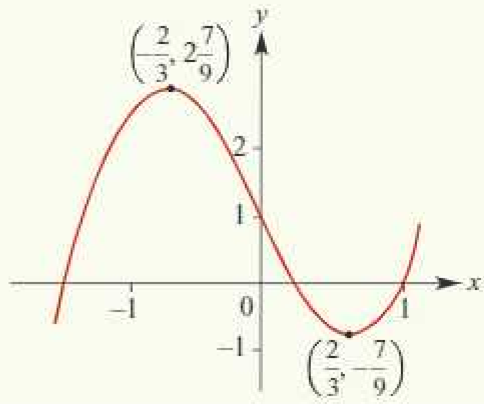
Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \quad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \quad x = 1$$

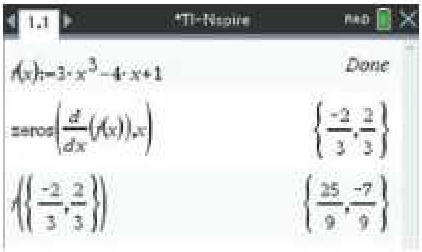
For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.

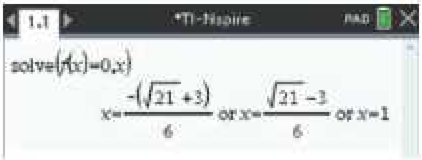


Using the TI-Nspire

- Assign the function $f(x)$ as shown.
- Use (menu) > **Algebra** > **Zeros** and (menu) > **Calculus** > **Derivative** to solve the equation $\frac{d}{dx}(f(x)) = 0$ and determine the coordinates of the stationary points.



- Find the x-axis intercepts by solving the equation $f(x) = 0$.

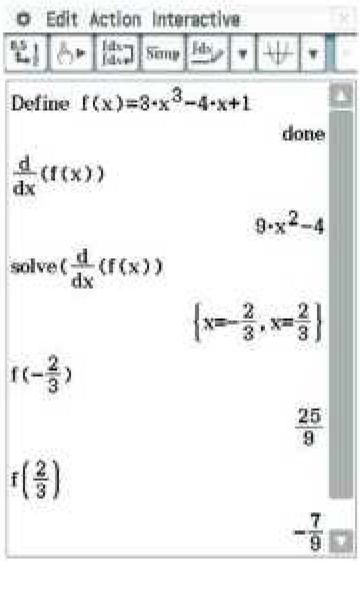


Using the Casio ClassPad

To determine the exact coordinates of the stationary points:

- In Main $\sqrt{\alpha}$, define the function $f(x) = 3x^3 - 4x + 1$.
- Solve the equation $\frac{d}{dx}(f(x)) = 0$ for x.
- Evaluate the function f at each x-value to find the corresponding y-value.

Note: Recall that it is not necessary to enter the right-hand side of an equation if it is zero.



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To find the x-axis intercepts:

- Solve the equation $f(x) = 0$.

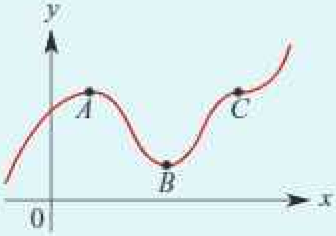
solve(f(x)

$\left\{x=1, x=\frac{-\sqrt{21}}{6}-\frac{1}{2}, x=\frac{\sqrt{21}}{6}-\frac{1}{2}\right\}$

Summary 18D

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points



A Point A is a **local maximum**:

- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:

- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are called **turning points**.

Exercise 18D

1 Each of the following is a gradient chart for a curve $y = f(x)$ with two stationary points. For each chart, complete the last row and state the nature of the stationary points.

a

x		1		3	
$f'(x)$	−	0	+	0	−
shape of f					

b

x		2		5	
$f'(x)$	+	0	−	0	−
shape of f					

Example 7

2 For each of the following, find all stationary points and state their nature. Sketch the graph of each function.

a $y = 9x^2 - x^3$ **b** $y = x^3 - 3x^2 - 9x$ **c** $y = x^4 - 4x^3$

3 Find the stationary points (and state their type) for each of the following functions:

a $y = x^2(x - 4)$ **b** $y = x^2(3 - x)$ **c** $y = x^4$

d $y = x^5(x - 4)$ **e** $y = x^3 - 5x^2 + 3x + 2$ **f** $y = x(x - 8)(x - 3)$

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4 Sketch the graph of each of the following functions:

a $y = 2 + 3x - x^3$ **b** $y = 2x^2(x - 3)$ **c** $y = x^3 - 3x^2 - 9x + 11$

5 The graph corresponding to each of the following equations has a stationary point at $(-2, 10)$. For each graph, find the nature of the stationary point at $(-2, 10)$.

a $y = 2x^3 + 3x^2 - 12x - 10$

b $y = 3x^4 + 16x^3 + 24x^2 - 6$

6 For the function $y = x^3 - 6x^2 + 9x + 10$:

a Find the values of x for which $\frac{dy}{dx} > 0$, i.e. find $\left\{ x : \frac{dy}{dx} > 0 \right\}$.

b Find the stationary points on the curve corresponding to $y = x^3 - 6x^2 + 9x + 10$.

c Sketch the curve carefully between $x = 0$ and $x = 4$.

7 For the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 1 + 12x - x^3$, determine the values of x for which $f'(x) > 0$.

8 Let $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = 3 + 6x - 2x^3$.

a Find the values of x such that $f'(x) > 0$.

b Find the values of x such that $f'(x) < 0$.

9 Let $f(x) = x(x + 3)(x - 5)$.

a Find the values of x for which $f'(x) = 0$.

b Sketch the graph of $y = f(x)$ for $-5 \le x \le 6$, giving the coordinates of the intersections with the axes and the coordinates of the turning points.

10 Sketch the graph of $y = x^3 - 6x^2 + 9x - 4$. State the coordinates of the axis intercepts and the turning points.

11 Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 45x + 2$ where the tangent is parallel to the x-axis.

12 Let $f(x) = x^3 - 3x^2$.

a Find:

i $\{ x : f'(x) < 0 \}$ **ii** $\{ x : f'(x) > 0 \}$ **iii** $\{ x : f'(x) = 0 \}$

b Sketch the graph of $y = f(x)$.

13 Sketch the graph of $y = x^3 - 9x^2 + 27x - 19$ and state the coordinates of the stationary points. CAS

14 Sketch the graph of $y = x^4 - 8x^2 + 7$. All axis intercepts and all turning points should be identified and their coordinates given.

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18E Applications to maximum and minimum problems

Many practical problems involve finding a maximum or minimum value of a function. We have solved some of these in Chapters 3 and 6. In the case of quadratic functions, we wrote the quadratic in turning point form and hence determined the maximum or minimum value. In the case of cubic functions, we used a CAS calculator to find the maximum or minimum values.

In this section we use calculus to solve problems which involve finding a local maximum or local minimum.

Example 8

A loop of string of length 100 cm is to be formed into a rectangle. Find the maximum area of this rectangle.

Solution

Let the length of the rectangle be x cm and the width y cm.

Then $2x + 2y = 100$. Thus $x + y = 50$ and hence

$$y = 50 - x \qquad (1)$$

It is clear that, for this problem, we must have $0 \leq x \leq 50$.

The area, A cm^2, is given by the formula $A = xy$.

Substituting from (1) gives

$$\begin{aligned} A &= x(50 - x) \\ &= 50x - x^2 \end{aligned}$$

Differentiating with respect to x:

$$\frac{dA}{dx} = 50 - 2x$$

Thus $\frac{dA}{dx} = 0$ implies $x = 25$.

Since the coefficient of x^2 is negative, this stationary point is a local maximum. (Alternatively, we could check the sign of $A'(x)$ immediately to the left and the right of $x = 25$.)

The maximum area is formed when the rectangle is a square with side length 25 cm, and so the maximum area is 625 cm^2.

Note: It is clear that we could have completed this question without calculus by using our knowledge of quadratic functions.

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Example 9

Given that $x + 2y = 4$, calculate the minimum value of $x^2 + xy - y^2$.

Solution

Rearranging $x + 2y = 4$, we have $x = 4 - 2y$.

Let $P = x^2 + xy - y^2$. Substituting for x gives

$$P = (4 - 2y)^2 + (4 - 2y)y - y^2$$
$$= 16 - 16y + 4y^2 + 4y - 2y^2 - y^2$$
$$= 16 - 12y + y^2$$
$$\therefore \quad \frac{dP}{dy} = -12 + 2y$$

Stationary values occur when $\frac{dP}{dy} = 0$:

$$-12 + 2y = 0$$
$$y = 6$$

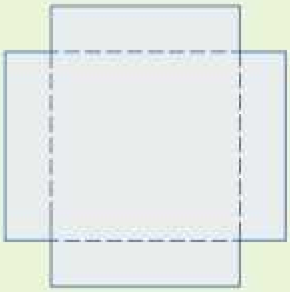
y		6	
$\frac{dP}{dy}$	$-$	0	$+$
shape of P	\	—	/

From the chart, there is a minimum when $y = 6$.

When $y = 6$, $x = -8$. Thus the minimum value of $x^2 + xy - y^2$ is -20.

Example 10

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d Find the value(s) of x for which $V = 0.5$ m^3.

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$V = (2 - 2x)^2 x$$
$$= (4 - 8x + 4x^2)x$$
$$= 4x^3 - 8x^2 + 4x$$

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b Let $V = 4x^3 - 8x^2 + 4x$. The maximum volume will occur when $\frac{dV}{dx} = 0$.

We have $\frac{dV}{dx} = 12x^2 - 16x + 4$, and so $\frac{dV}{dx} = 0$ implies that

$$12x^2 - 16x + 4 = 0$$
$$3x^2 - 4x + 1 = 0$$
$$(3x - 1)(x - 1) = 0$$
$$\therefore \quad x = \frac{1}{3} \text{ or } x = 1$$

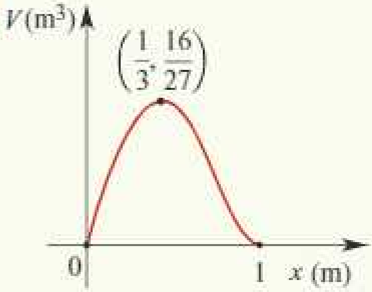
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

The maximum occurs when $x = \frac{1}{3}$.

$$\therefore \text{Maximum volume} = \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} = \frac{16}{27} \text{ m}^3$$

x		$\frac{1}{3}$		1
$\frac{dV}{dx}$	+	0	−	0
shape of V	/	—	\	—

c



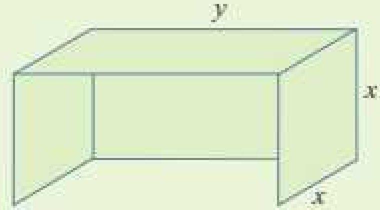
d To find the value(s) of x for which $V = 0.5$ m^3, we need to solve the equation $V = 0.5$, i.e. $4x^3 - 8x^2 + 4x = 0.5$.

Using a CAS calculator gives $x = \frac{1}{2}$ or $x = \frac{3 \pm \sqrt{5}}{4}$.

But the domain of V is $[0, 1]$. Hence $x = \frac{1}{2}$ or $x = \frac{3 - \sqrt{5}}{4}$.

Example 11

A canvas shelter is made up with a back, two square sides and a top. The area of canvas available is 24 m^2. Let V m^3 be the volume enclosed by the shelter.



a Find the dimensions of the shelter that will create the largest possible enclosed volume.

b Sketch the graph of V against x for a suitable domain.

c Find the values of x and y for which $V = 10$ m^3.

Solution

a The volume $V = x^2y$. One of the variables must be eliminated.
We know that the area is 24 m^2.

$\therefore \quad 2x^2 + 2xy = 24$

Rearranging gives $y = \dfrac{24 - 2x^2}{2x}$, i.e. $y = \dfrac{12}{x} - x$.
Substituting in the formula for volume gives

$$V = 12x - x^3$$

Differentiation now gives

$$\frac{dV}{dx} = 12 - 3x^2$$

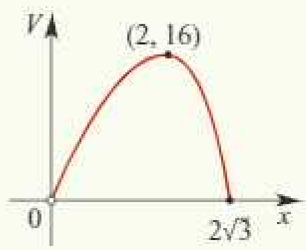
Stationary points occur when $\dfrac{dV}{dx} = 0$, which implies $12 - 3x^2 = 0$.

So stationary points occur when $x^2 = 4$, i.e. when $x = \pm 2$. But we must have $x > 0$, and so the only solution is $x = 2$.

x		2	
$\dfrac{dV}{dx}$	+	0	−
shape of V	/	—	\

Hence the maximum is at $x = 2$.
The dimensions are 2 m, 2 m, 4 m.

b Note that $x > 0$ and $y \geq 0$.
This implies $x > 0$ and $12 - x^2 \geq 0$, i.e. $0 < x \leq 2\sqrt{3}$.



c Using a CAS calculator, solve the equation $12x - x^3 = 10$ for $0 < x \leq 2\sqrt{3}$.
The solutions are $x = 0.8926$ and $x = 2.9305$, correct to four decimal places.

Possible dimensions to the nearest centimetre are 0.89 m, 0.89 m, 12.55 m and 2.93 m, 2.93 m, 1.16 m.

Maximum or minimum at an endpoint

Calculus can be used to find a local maximum or local minimum, but these are often not the actual maximum or minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

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Example 12

Let $f\colon [-2, 4] \to \mathbb{R}$, $f(x) = x^2 + 2$. Find the absolute maximum value and the absolute minimum value of the function.

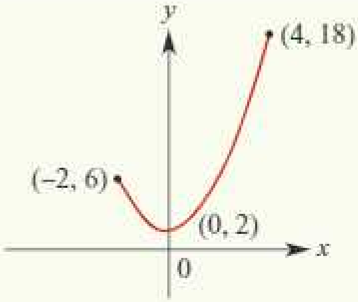
Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

The minimum value occurs at a stationary point of the graph, but the endpoint $(4, 18)$ is not a stationary point.

The absolute maximum value is 18 and the absolute minimum value is 2.



Example 13

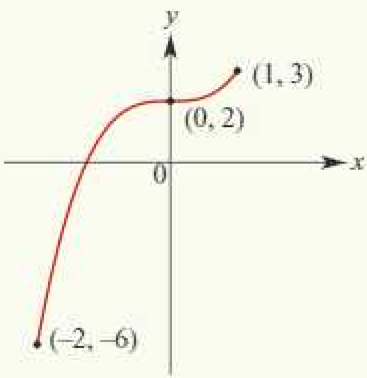
Let $f\colon [-2, 1] \to \mathbb{R}$, $f(x) = x^3 + 2$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

The absolute maximum and the absolute minimum do not occur at stationary points.



Example 14

In Example 10, the maximum volume of a box was found. The maximum value corresponded to a local maximum of the graph of $V = 4x^3 - 8x^2 + 4x$. This was also the absolute maximum value.

If the height of the box must be at most 0.3 m (i.e. $x \le 0.3$), what will be the maximum volume of the box?

Solution

The local maximum of $V(x)$ for $x \in [0, 1]$ was at $x = \frac{1}{3}$. But $\frac{1}{3}$ is greater than 0.3.

For the new problem, we have $V'(x) > 0$ for all $x \in [0, 0.3]$, and so $V(x)$ is strictly increasing on the interval $[0, 0.3]$.

Therefore the maximum volume occurs when $x = 0.3$ and is $0.588\ \text{m}^3$.

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Summary 18E

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Skill-sheet

Exercise 18E

Example 8

1 A loop of string of length 200 cm is to be formed into a rectangle. Find the maximum area of this rectangle.

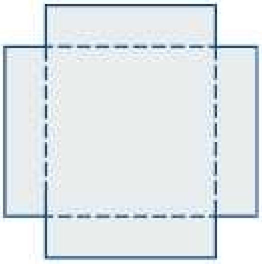
2 Find the maximum value of the product of two numbers x and $10 - x$.

Example 9

3 Given that $x + y = 2$, calculate the minimum value of $x^2 + y^2$.

Example 10

4 From a square piece of metal of side length 6 m, four squares are removed as shown in the diagram. The metal is folded along the dashed lines to form an open box with height x m.

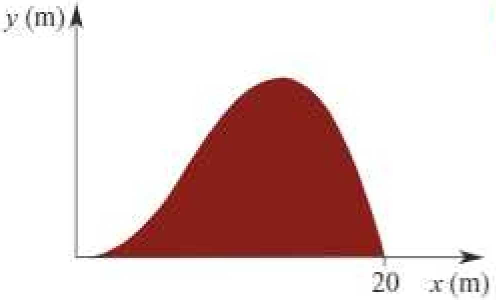


a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 24x^2 + 36x$.

b Find the value of x that gives the box its maximum volume and find the maximum volume.

5 A bank of earth has cross-section as shown in the diagram. The curve defining the bank has equation

$$y = \frac{x^2}{400}(20 - x) \quad \text{for } x \in [0, 20]$$



a Find the height of the bank where:

i $x = 5$ **ii** $x = 10$ **iii** $x = 15$

b Find the value of x for which the height is a maximum and state the maximum height of the bank.

c Find the values of x for which:

i $\frac{dy}{dx} = \frac{1}{8}$ **ii** $\frac{dy}{dx} = -\frac{1}{8}$

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Example 11 **6** A cuboid has a total surface area of 150 cm^2 and a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \frac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

d If the maximum side length of the square base of the cuboid is 4 cm, what is the maximum volume possible?

7 The volume of a cylinder is given by the formula $V = \pi r^2 h$. Find the maximum value of V if $r + h = 12$.

8 A rectangular sheet of metal measures 50 cm by 40 cm. Congruent squares of side length x cm are cut from each of the corners and not used further. The sheet is then folded up to make a tray of depth x cm. Find the value of x for which the volume of the tray is a maximum.

Example 12 **9** Let $f : [-2, 2] \to \mathbb{R}$, $f(x) = 2 - 8x^2$. Find the absolute maximum value and the absolute minimum value of the function.

Example 13 **10** Let $f : [-2, 1] \to \mathbb{R}$, $f(x) = x^3 + 2x + 3$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

11 Let $f : [0, 4] \to \mathbb{R}$, $f(x) = 2x^3 - 6x^2$. Find the absolute maximum and the absolute minimum values of the function.

12 Let $f : [-2, 5] \to \mathbb{R}$, $f(x) = 2x^4 - 8x^2$. Find the absolute maximum and the absolute minimum values of the function.

Example 14 **13** A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, is given by $V = 15x^2 - 12x^3$.

b Find the derivative $\frac{dV}{dx}$.

c Find the local maximum for the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

14 For the variables x, y and z, it is known that $x + y = 20$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the maximum and minimum values of z.

15 For the variables x, y and z, it is known that $z = x^2 y$ and $2x + y = 50$. Find the maximum value of z if:

a $x \in [0, 25]$ **b** $x \in [0, 10]$ **c** $x \in [5, 20]$

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16 A piece of string 10 metres long is cut into two pieces to form two squares.

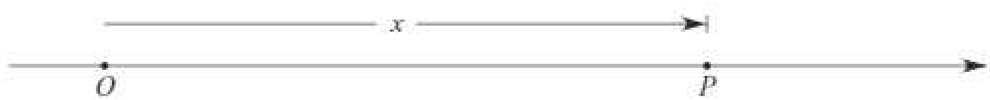
- **a** If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.
- **b** Find $\frac{dA}{dx}$.
- **c** Find the value of x that makes A a minimum.
- **d** What is the minimum total area of the two squares?

18F Applications to motion in a straight line

In this section we continue our study of motion in a straight line from Section 16E.

Position

The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is considered to be positive.



Consider a particle which starts at O and begins to move. The position of the particle at any instant can be specified by a real number x. For example, if the unit is metres and if $x = -3$, the position is 3 m to the left of O; while if $x = 3$, the position is 3 m to the right of O.

Sometimes there is a rule that enables the position at any instant to be calculated. In this case, we can view x as being a function of t. Hence $x(t)$ is the position at time t.

For example, imagine that a stone is dropped from the top of a vertical cliff 45 metres high. Assume that the stone is a particle travelling in a straight line. Let $x(t)$ metres be the downwards position of the particle from O, the top of the cliff, t seconds after the particle is dropped. If air resistance is neglected, then an approximate model for the position is

$$x(t) = 5t^2 \quad \text{for } 0 \le t \le 3$$

Example 15

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \ge 0$.

- **a** Find its initial position.
- **b** Find its position at $t = 4$.

Solution

- **a** At $t = 0$, $x = +6$, i.e. the particle is 6 cm to the right of O.
- **b** At $t = 4$, $x = (4)^2 - 7(4) + 6 = -6$, i.e. the particle is 6 cm to the left of O.

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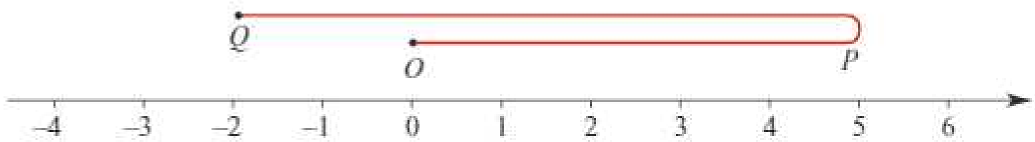
Distance and displacement

The **displacement** of a particle is defined as the change in position of the particle.

It is important to distinguish between the scalar quantity **distance** and the vector quantity displacement (which has a direction).

For example, consider a particle that starts at O and moves first 5 units to the right to point P, and then 7 units to the left to point Q.



The difference between its final position and its initial position is -2. So the displacement of the particle is -2 units. However, the distance it has travelled is 12 units.

Velocity

In this section we focus on the instantaneous rates of change which arise when studying the motion of a particle travelling in a straight line. In particular, we define the velocity and acceleration of a particle.

Average velocity

The average rate of change of position with respect to time is **average velocity**.

A particle's average velocity for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

where x_1 is the position at time t_1 and x_2 is the position at time t_2.

Instantaneous velocity

The instantaneous rate of change of position with respect to time is **instantaneous velocity**. We will refer to the instantaneous velocity as simply the **velocity**.

If a particle's position, x, at time t is given as a function of t, then the velocity of the particle at time t is determined by differentiating the rule for position with respect to time.

If x is the position of a particle at time t, then

$$\text{velocity } v = \frac{dx}{dt}$$

Velocity may be positive, negative or zero. If the velocity is positive, the particle is moving to the right, and if it is negative, the particle is moving to the left. A velocity of zero means the particle is instantaneously at rest.

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Speed and average speed

- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is given by $\dfrac{\text{distance travelled}}{t_2 - t_1}$

Units of measurement

Common units for velocity (and speed) are:

$$\begin{aligned}
&1 \text{ metre per second} &&= 1 \text{ m/s} &&= 1 \text{ m s}^{-1}\\
&1 \text{ centimetre per second} &&= 1 \text{ cm/s} &&= 1 \text{ cm s}^{-1}\\
&1 \text{ kilometre per hour} &&= 1 \text{ km/h} &&= 1 \text{ km h}^{-1}
\end{aligned}$$

The first and third units are connected in the following way:

$$\begin{aligned}
1 \text{ km/h} &= 1000 \text{ m/h}\\
&= \frac{1000}{60 \times 60} \text{ m/s}\\
&= \frac{5}{18} \text{ m/s}\\
\therefore \quad 1 \text{ m/s} &= \frac{18}{5} \text{ km/h}
\end{aligned}$$

Example 16

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 7t + 6$, $t \geq 0$.

a Find its initial velocity.

b When does its velocity equal zero, and what is its position at this time?

c What is its average velocity for the first 4 seconds?

d Determine its average speed for the first 4 seconds.

Solution

a $x = t^2 - 7t + 6$

$v = \dfrac{dx}{dt} = 2t - 7$

At $t = 0$, $v = -7$. The particle is initially moving to the left at 7 cm/s.

b $\dfrac{dx}{dt} = 0$ implies $2t - 7 = 0$, i.e. $t = 3.5$

When $t = 3.5$, $x = (3.5)^2 - 7(3.5) + 6$
$= -6.25$

So, at $t = 3.5$ seconds, the particle is at rest 6.25 cm to the left of O.

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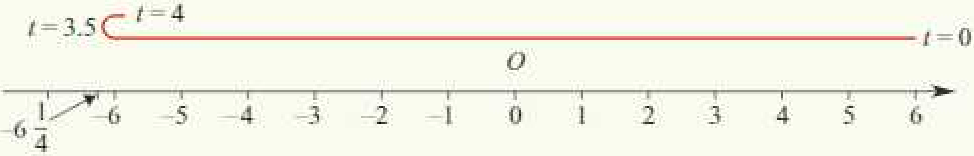
c Average velocity = $\dfrac{\text{change in position}}{\text{change in time}}$

Position is given by $x = t^2 - 7t + 6$. So at $t = 4$, $x = -6$, and at $t = 0$, $x = 6$.

$\therefore$ Average velocity = $\dfrac{-6-6}{4} = -3$ cm/s

d Average speed = $\dfrac{\text{distance travelled}}{\text{change in time}}$



The particle stopped at $t = 3.5$ and began to move in the opposite direction. So we must consider the distance travelled in the first 3.5 seconds (from $x = 6$ to $x = -6.25$) and then the distance travelled in the final 0.5 seconds (from $x = -6.25$ to $x = -6$).

Total distance travelled = $12.25 + 0.25 = 12.5$

$\therefore$ Average speed = $\dfrac{12.5}{4} = 3.125$ cm/s

Note: Remember that speed is the magnitude of the velocity. However, we can see from this example that average speed is *not* the magnitude of the average velocity.

Acceleration

The acceleration of a particle is the rate of change of its velocity with respect to time.

- **Average acceleration** for the time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Note: The second derivative $\dfrac{d^2x}{dt^2}$ is just the derivative of the derivative. It will be discussed further in Chapter 20.

Acceleration may be positive, negative or zero. Zero acceleration means the particle is moving at a constant velocity.

The direction of motion and the acceleration need not coincide. For example, a particle may have a positive velocity, indicating it is moving to the right, but a negative acceleration, indicating it is slowing down.

Also, although a particle may be instantaneously at rest, its acceleration at that instant need not be zero. If acceleration has the same sign as velocity, then the particle is 'speeding up'. If the sign is opposite, the particle is 'slowing down'.

The most commonly used units for acceleration are cm/s^2 and m/s^2.

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Example 17

A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Solution

a $x = t^3 - 6t^2 + 5$

$$v = \frac{dx}{dt} = 3t^2 - 12t$$

$$a = \frac{dv}{dt} = 6t - 12$$

So when $t = 0$, we have $x = 5$, $v = 0$ and $a = -12$.

Initially, the particle is instantaneously at rest 5 cm to the right of O, with an acceleration of -12 cm/s^2.

b $v = 0$ implies $3t^2 - 12t = 0$

$$3t(t - 4) = 0$$

$$\therefore \quad t = 0 \text{ or } t = 4$$

The particle is initially at rest and stops again after 4 seconds.

At $t = 0$, $x = 5$ and $a = -12$.

At $t = 4$, $x = (4)^3 - 6(4)^2 + 5 = -27$ and $a = 6(4) - 12 = 12$.

After 4 seconds, the particle's position is 27 cm to the left of O, and its acceleration is 12 cm/s^2.

Example 18

A car starts from rest and moves a distance s metres in t seconds, where $s = \frac{1}{6}t^3 + \frac{1}{4}t^2$.

What is the initial acceleration and the acceleration when $t = 2$?

Solution

We are given

$$s = \frac{1}{6}t^3 + \frac{1}{4}t^2$$

The car's velocity is given by

$$v = \frac{ds}{dt} = \frac{1}{2}t^2 + \frac{1}{2}t$$

The car's acceleration is given by

$$a = \frac{dv}{dt} = t + \frac{1}{2}$$

When $t = 0$, $a = \frac{1}{2}$, and when $t = 2$, $a = 2\frac{1}{2}$.

Hence the required accelerations are $\frac{1}{2}$ m/s^2 and $2\frac{1}{2}$ m/s^2.

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Summary 18F

- The **position** of a particle moving in a straight line is determined by its distance from a fixed point O on the line, called the **origin**, and whether it is to the right or left of O. By convention, the direction to the right of the origin is positive.
- **Average velocity** for a time interval $[t_1, t_2]$ is given by

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}} = \frac{x_2 - x_1}{t_2 - t_1}$$

  where x_2 is the position at time t_2 and x_1 is the position at time t_1.
- The instantaneous rate of change of position with respect to time is called the **instantaneous velocity**, or simply the **velocity**.
  If x is the position of the particle at time t, then its velocity is $v = \dfrac{dx}{dt}$
- **Speed** is the magnitude of the velocity.
- **Average speed** for a time interval $[t_1, t_2]$ is $\dfrac{\text{distance travelled}}{t_2 - t_1}$
- **Average acceleration** for a time interval $[t_1, t_2]$ is given by $\dfrac{v_2 - v_1}{t_2 - t_1}$, where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1.
- **Instantaneous acceleration** $a = \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{dx}{dt}\right) = \dfrac{d^2x}{dt^2}$

Skill-sheet

Exercise 18F

Example 15

1 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 12t + 11$, $t \geq 0$.

- **a** Find its initial position.
- **b** Find its position at $t = 3$.

Example 16

2 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = t^2 - 12t + 11$, $t \geq 0$.

- **a** Find its initial velocity.
- **b** When does its velocity equal zero, and what is its position at this time?
- **c** What is its average velocity for the first 3 seconds?
- **d** Determine its average speed for the first 3 seconds.

3 The position of a body moving in a straight line, x cm from the origin, at time t seconds ($t \geq 0$) is given by $x = \frac{1}{3}t^3 - 12t + 6$.

- **a** Find the rate of change of position with respect to time at $t = 3$.
- **b** Find the time at which the velocity is zero.

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Example 17

4 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds is given by $x = 4t^3 - 6t^2 + 5$, $t \geq 0$.

a Find its initial position, velocity and acceleration, and hence describe its motion.

b Find the times when it is instantaneously at rest and determine its position and acceleration at those times.

Example 18

5 A car starts from rest and moves a distance s metres in t seconds, where $s = t^4 + t^2$.

a What is the acceleration when $t = 0$?

b What is the acceleration when $t = 2$?

6 The position, x metres, at time t seconds ($t \geq 0$) of a particle moving in a straight line is given by $x = t^2 - 7t + 10$.

a When does its velocity equal zero?

b Find its acceleration at this time.

c Find the distance travelled in the first 5 seconds.

d When does its velocity equal -2 m/s, and what is its position at this time?

7 A particle moves along a straight line so that after t seconds its position, s m, relative to a fixed point O on the line is given by $s = t^3 - 3t^2 + 2t$.

a When is the particle at O?

b What is its velocity and acceleration at these times?

c What is the average velocity during the first second?

8 A particle moves in a straight line so that its position, x cm, relative to O at time t seconds ($t \geq 0$) is given by $x = t^2 - 7t + 12$.

a Find its initial position.

b What is its position at $t = 5$?

c Find its initial velocity.

d When does its velocity equal zero, and what is its position at this time?

e What is its average velocity in the first 5 seconds?

f What is its average speed in the first 5 seconds?

9 A particle moving in a straight line has position x cm relative to the point O at time t seconds ($t \geq 0$), where $x = t^3 - 11t^2 + 24t - 3$.

a Find its initial position and velocity.

b Find its velocity at any time t.

c At what times is the particle stationary?

d What is the position of the particle when it is stationary?

e For how long is the particle's velocity negative?

f Find its acceleration at any time t.

g When is the particle's acceleration zero? What is its velocity and its position at that time?

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10 A particle moves in a straight line so that after t seconds its position, s metres, is given by $s = t^4 + 3t^2$.

a Find the acceleration when $t = 1, t = 2, t = 3$.

b Find the average acceleration between $t = 1$ and $t = 3$.

11 A particle is moving in a straight line in such a way that its position, x cm, relative to the point O at time t seconds ($t \geq 0$) satisfies $x = t^3 - 13t^2 + 46t - 48$. When does the particle pass through O, and what is its velocity and acceleration at those times?

12 Two particles are moving along a straight path so that their positions, x_1 cm and x_2 cm, relative to a fixed point P at any time t seconds are given by $x_1 = t + 2$ and $x_2 = t^2 - 2t - 2$.

a Find the time when the particles are at the same position.

b Find the time when the particles are moving with the same velocity.

18G Families of functions and transformations

In the earlier chapters of this book we looked at families of functions. We can now use calculus to explore such families further. It is assumed that a CAS calculator will be used throughout this section.

Example 19

Consider the family of functions with rules of the form $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(a, 0)$ is always a local maximum.

d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.

b The coordinates of the stationary points are $(a, 0)$ and $\left(\frac{a + 2b}{3}, \frac{4(a - b)^3}{27}\right)$.

c If $x < a$, then $f'(x) > 0$, and if $a < x < \frac{a + 2b}{3}$, then $f'(x) < 0$.

Therefore the stationary point at $(a, 0)$ is a local maximum.

d Since $a < b$, we must have $a = 3$ and $\frac{a + 2b}{3} = 4$. Therefore $b = \frac{9}{2}$.

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Example 20

The graph of the function $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.

b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.

b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

Skill-sheet

Exercise 18G

Example 19

1 Consider the family of functions with rules of the form $f(x) = (x-2)^2(x-b)$, where b is a positive constant with $b > 2$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(2, 0)$ is always a local maximum.

d Find the value of b if the stationary points occur where $x = 2$ and $x = 4$.

Example 20

2 The graph of the function $y = x^4 - 12x^3$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the stationary points of the graph of $y = x^4 - 12x^3$.

b Find the coordinates of the stationary points of its image.

3 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x - ax^2$, where a is a real number with $a > 0$.

a Determine the intervals for which $f'(x)$ is:

i positive **ii** negative.

b Find the equation of the tangent to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

4 Consider the cubic function with rule $f(x) = (x-a)^2(x-1)$, where $a > 1$.

a Find the coordinates of the turning points of the graph of $y = f(x)$.

b State the nature of each of the turning points.

c Find the equation of the tangent at which:

i $x = 1$ **ii** $x = a$ **iii** $x = \frac{a+1}{2}$

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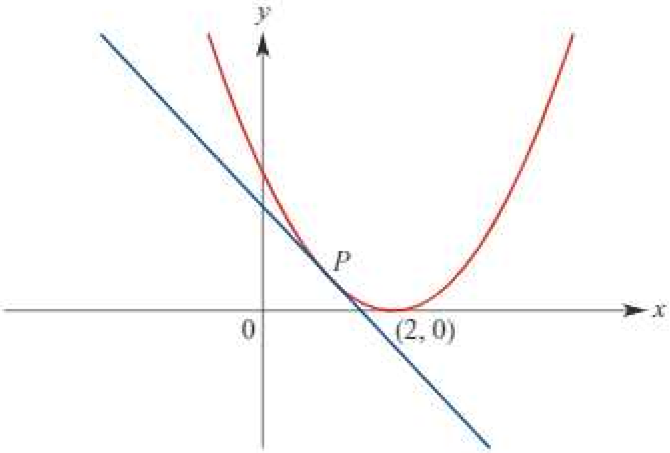
5 A line with equation $y = mx + c$ is a tangent to the curve $y = (x-2)^2$ at a point P where $x = a$ such that $0 < a < 2$.



a **i** Find the gradient of the curve where $x = a$, for $0 < a < 2$.

ii Hence express m in terms of a.

b State the coordinates of the point P, expressing your answer in terms of a.

c Find the equation of the tangent where $x = a$.

d Find the x-axis intercept of the tangent.

6 **a** The graph of $f(x) = x^3$ is translated to the graph of $y = f(x+h)$. Find the value of h if $f(1+h) = 27$.

b The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the value of a if the graph of $y = f(ax)$ passes through the point $(1, 27)$.

c The cubic with equation $y = ax^3 - bx^2$ has a turning point with coordinates $(1, 8)$. Find the values of a and b.

7 The graph of the function $y = x^4 + 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^4 + 4x^2$.

b Find the coordinates of the turning points of its image.

8 Consider the quartic function with rule $f(x) = (x-1)^2(x-b)^2$, where $b > 1$.

a Find the derivative of f.

b Find the coordinates of the turning points of f.

c Find the value of b such that the graph of $y = f(x)$ has a turning point at $(2, 1)$.

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18H Newton's method for finding solutions to equations

Newton's method is used for finding approximate solutions to equations of the form $f(x) = 0$. The method involves finding the tangent at successive points on the graph of $y = f(x)$.

Solving the equation $x^2 - 2 = 0$ for $x > 0$

Let $f(x) = x^2 - 2$. Then $f'(x) = 2x$.

We will use Newton's method to look for an approximation to the solution of the equation $f(x) = 0$, where $x > 0$. Part of the graph of $y = f(x)$ is shown below.

First step Start with $x = 4$. The equation of the tangent to the curve $y = f(x)$ at the point $(4, f(4))$ is $y - f(4) = f'(4)(x - 4)$.

The x-axis intercept of this tangent occurs when $y = 0$:

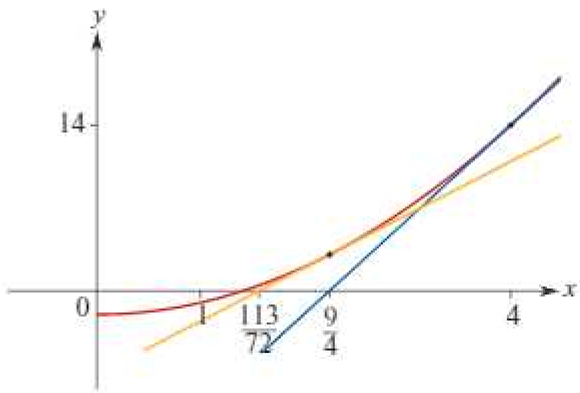
$$-f(4) = f'(4)(x-4)$$

$$-\frac{f(4)}{f'(4)} = x - 4$$

$$x = 4 - \frac{f(4)}{f'(4)}$$

$$x = 4 - \frac{14}{8}$$

$$\therefore \quad x = \frac{9}{4}$$



This is our first approximate solution (not counting $x = 4$) to the equation $f(x) = 0$.

Second step Now find the x-axis intercept of the tangent to the curve when $x = \frac{9}{4}$.

$$-f\left(\frac{9}{4}\right) = f'\left(\frac{9}{4}\right)\left(x - \frac{9}{4}\right)$$

$$-\frac{f(\frac{9}{4})}{f'(\frac{9}{4})} = x - \frac{9}{4}$$

$$x = \frac{9}{4} - \frac{f(\frac{9}{4})}{f'(\frac{9}{4})}$$

$$\therefore \quad x = \frac{113}{72} \approx 1.56944$$

Next step To go from one approximation to the next, we use the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{where } n = 0, 1, 2, \dots$$

In the special case for $f(x) = x^2 - 2$, this formula becomes

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} \qquad \text{where } n = 0, 1, 2, \dots$$

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In searching for the solution of $x^2 - 2 = 0$, we obtain the sequence of approximations $x_0 = 4$, $x_1 = 1.25$, $x_2 \approx 1.56944$, $x_3 \approx 1.42189, \ldots$

The process is continued in a spreadsheet as shown. You can see the speed of convergence to a very good approximation.

n	x_n	$f(x_n)$	$f'(x_n)$
0	4.00000000	14.00000000	8.00000000
1	2.25000000	3.06250000	4.50000000
2	1.56944444	0.46315586	3.13888889
3	1.42189036	0.02177221	2.84378073
4	1.41423429	0.00005862	2.82846857
5	1.41421356	0.00000000	2.82842713
6	1.41421356	0.00000000	2.82842712

You can produce this sequence on your calculator by defining the function

$$g(a) = a - \frac{a^2 - 2}{2a}$$

and repeatedly applying this function starting with $x = 4$. Or you can use nests such as $g(g(g(g(4))))$, which gives four iterations at once.

The general procedure

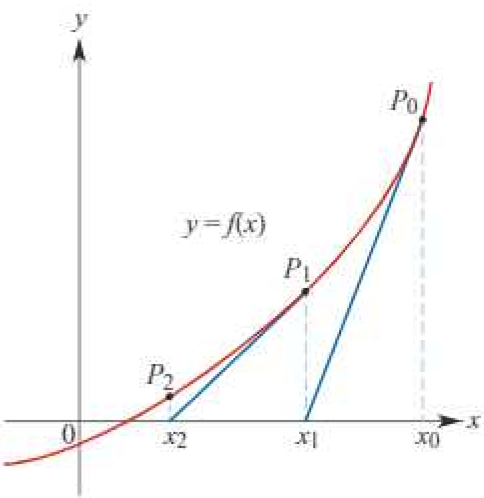
Of course, this process can be used for other functions.

Suppose that the equation $f(x) = 0$ has a solution at $x = \alpha$. Choose x_0 close to α.

Start with the point $P_0(x_0, f(x_0))$ on the curve $y = f(x)$.

Let x_1 be the x-axis intercept of the tangent to the curve at P_0. In general, x_1 will be a better approximation to the solution α.

Next consider the point $P_1(x_1, f(x_1))$.



The process is repeated to give a sequence of values $x_1, x_2, x_3, \ldots$ with each one closer to α.

We can go from x_n to x_{n+1} by using the iterative formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{where } n = 0, 1, 2, \ldots$$

The process does not always work, as we will see later.

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Example 21

Use Newton's method to solve the equation $-x^3 + 5x^2 - 3x + 4 = 0$ for $x \in [0, \infty)$.

Solution

Let $f(x) = -x^3 + 5x^2 - 3x + 4$.

The derivative is $f'(x) = -3x^2 + 10x - 3$, and so the iterative formula is

$$x_{n+1} = x_n - \frac{-x_n^3 + 5x_n^2 - 3x_n + 4}{-3x_n^2 + 10x_n - 3}$$

By starting at $x_0 = 3.8$, we obtain the spreadsheet shown.

n	x_n	$f(x_n)$	$f'(x_n)$
0	3.80000000	9.92800000	−8.32000000
1	4.99326923	−10.81199119	−27.86552053
2	4.60526316	−1.44403339	−20.57271468
3	4.53507148	−0.04308844	−19.34990517
4	4.53284468	−0.00004266	−19.31159580
5	4.53284247	0.00000000	−19.31155781
6	4.53284247	0.00000000	−19.31155781

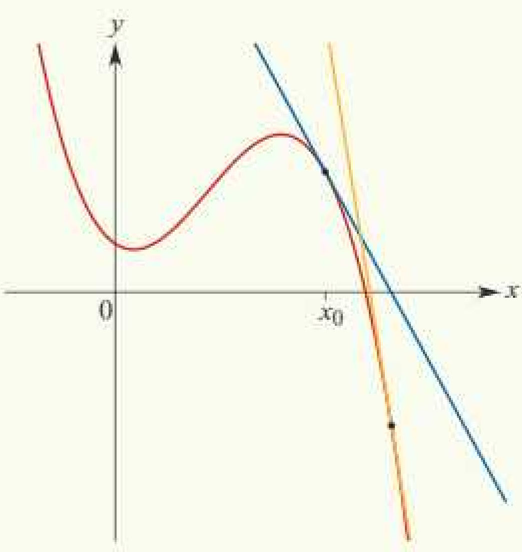
Alternatively, define the function

$$g(a) = a - \frac{-a^3 + 5a^2 - 3a + 4}{-3a^2 + 10a - 3}$$

and apply repeatedly starting with $a = 3.8$.

The solution is $x \approx 4.53284247$.

The graph on the right shows the first two tangent lines when starting at $x_0 = 3.8$. Note that the second tangent line is through a point on the curve below the x-axis.



Note: In Example 21, if you start at a point on the other side of the local maximum you can still have 'convergence' to the solution. For example, starting at $x_0 = 2.7$, it takes over 100 iterations to arrive at $x \approx 4.53284247$. Starting at $x_0 = -5$, it takes only 12 iterations.

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Newton's method does not always work

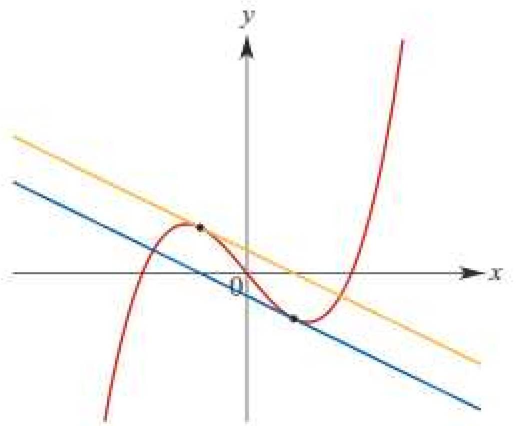
The function $f(x) = x^3 - 5x$ can be used to illustrate the problems that can occur when using Newton's method.

1 Oscillating sequence

If you start with $x_0 = 1$, the tangent is $y = -2 - 2x$. This gives $x_1 = -1$. The tangent at $x = -1$ is $y = 2 - 2x$. So you get the sequence $1, -1, 1, -1, \dots$

2 Terminating sequence

There are stationary points at $x = \pm\frac{\sqrt{15}}{3}$. The tangents at these points are parallel to the x-axis, and you do not get a solution.



Newton's method can be used successfully with the function $f(x) = x^3 - 5x$:

- For any starting point in the interval $(-1, 1)$, you will get convergence to $x = 0$.
- For any starting point in $(1, \infty)$ except for $\frac{\sqrt{15}}{3}$, you will get convergence to $x = \sqrt{5}$.
- For any starting point in $(-\infty, -1)$ except for $-\frac{\sqrt{15}}{3}$, you will get convergence to $x = -\sqrt{5}$.

Using pseudocode for Newton's method

In Section 6K we used pseudocode to describe the bisection method for finding approximate solutions to polynomial equations. Here we use pseudocode for Newton's method.

Note: An introduction to pseudocode is given in Appendix A. The Interactive Textbook also includes online appendices on coding using the language *Python*, the TI-Nspire and the Casio ClassPad.

The following algorithm can be used to solve Example 21. The table shows the result of executing the algorithm. The first row gives the initial values of x and $f(x)$. The next rows give the values that are printed at the end of each pass of the `while` loop.

```
define f(x):
  return −x^3 + 5x^2 − 3x + 4

define Df(x):
  return −3x^2 + 10x − 3

x ← 3.8
while f(x) > 10^−6 or f(x) < −10^−6
    x ← x − f(x)/Df(x)
    print x, f(x)
end while
```

	x	$f(x)$
Initial	3.8	9.928
Pass 1	4.99326923	−10.81199119
Pass 2	4.60526316	−1.44403339
Pass 3	4.53507148	−0.04308844
Pass 4	4.53284468	−0.00004266
Pass 5	4.53284247	0.00000000

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Skill-sheet

Exercise 18H

Example 21

1 Use Newton's method to find approximate solutions for each of the following equations in the given interval. The desired accuracy is stated.

a $x^3 - x - 1 = 0$	$[1, 2]$	2 decimal places
b $x^4 + x - 3 = 0$	$[1, 3]$	3 decimal places
c $x^3 - 5x + 4.2 = 0$	$[1, 2]$	3 decimal places
d $x^3 - 2x^2 + 2x - 5 = 0$	$[2, 3]$	3 decimal places
e $2x^4 - 3x^2 + 2x - 6 = 0$	$[-2, -1]$	2 decimal places

2 For $f(x) = x^3 - 3$, show that Newton's method gives the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2}$$

Hence find an approximation for $3^{\frac{1}{3}}$ with your calculator. Start with $x_0 = 2$ and use the function $g(a) = \dfrac{2a^3 + 3}{3a^2}$ repeatedly.

3 For $f(x) = x^3 - 2x - 1$, show that Newton's method gives the iterative formula

$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 - 2}$$

Hence find an approximation to a solution near $x = 2$ for $x^3 - 2x - 1 = 0$ with your calculator. Start with $x_0 = 2$ and use the function $g(a) = \dfrac{2a^3 + 1}{3a^2 - 2}$ repeatedly.

4 For each of the following equations:

i Write a pseudocode algorithm that uses Newton's method to find an approximate solution of the given equation with the given starting value x_0.

ii Find the values of x_1, x_2 and x_3 by completing three passes of the `while` loop.

a $x^3 - x - 4 = 0$	$x_0 = 1.5$
b $x^4 - x - 13 = 0$	$x_0 = 2$
c $-x^3 - 2x^2 + 1 = 0$	$x_0 = 0.5$
d $-x^3 - 2x + 40 = 0$	$x_0 = 3.5$

Review

Chapter summary

Assign-ment

Nrich

- **Tangents and normals**

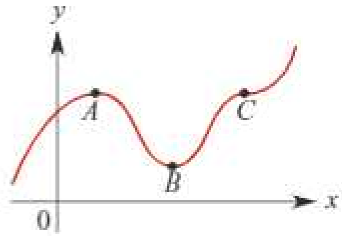
Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then

- the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
- the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

- **Stationary points**

A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.

The graph shown has three stationary points: A, B and C.



A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are referred to as **turning points**.

- **Maximum and minimum values**

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \le M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \ge N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

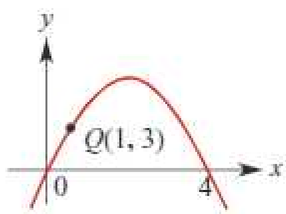
- **Motion in a straight line**

For an object moving in a straight line with position x at time t:

$$\text{velocity } v = \frac{dx}{dt} \qquad \text{acceleration } a = \frac{dv}{dt}$$

Technology-free questions

1 The graph of $y = 4x - x^2$ is shown.



a Find $\dfrac{dy}{dx}$.

b Find the gradient of the tangent to the curve at $Q(1, 3)$.

c Find the equation of the tangent at Q.

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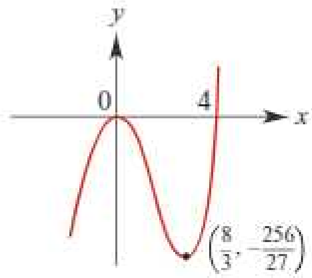
2 The graph of $y = x^3 - 4x^2$ is shown.



a Find $\frac{dy}{dx}$.

b Find the gradient of the tangent to the curve at the point $(2, -8)$.

c Find the equation of the tangent at the point $(2, -8)$.

d Find the coordinates of the point Q where the tangent crosses the curve again.

3 Let $y = x^3 - 12x + 2$.

a Find $\frac{dy}{dx}$ and the value(s) of x for which $\frac{dy}{dx} = 0$.

b State the nature of each of these stationary points.

c Find the corresponding y-value for each of these.

4 Write down the values of x for which each of the following derivative functions are zero. For each of the corresponding stationary points, determine whether it is a local maximum, local minimum or stationary point of inflection.

a $\frac{dy}{dx} = 3x^2$

b $\frac{dy}{dx} = -3x^3$

c $f'(x) = (x-2)(x-3)$

d $f'(x) = (x-2)(x+2)$

e $f'(x) = (2-x)(x+2)$

f $f'(x) = -(x-1)(x-3)$

g $\frac{dy}{dx} = -x^2 + x + 12$

h $\frac{dy}{dx} = 15 - 2x - x^2$

5 For each of the following, find all stationary points and state the nature of each:

a $y = 4x - 3x^3$

b $y = 2x^3 - 3x^2 - 12x - 7$

c $y = x(2x-3)(x-4)$

6 Sketch the graph of each of the following. Give the coordinates of the stationary points and the axis intercepts.

a $y = 3x^2 - x^3$

b $y = x^3 - 6x^2$

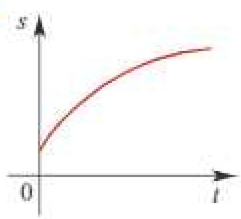
c $y = (x+1)^2(2-x)$

d $y = 4x^3 - 3x$

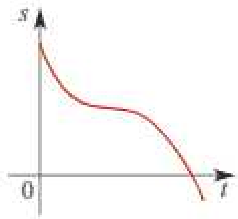
e $y = x^3 - 12x^2$

7 Match each position–time graph shown with its velocity–time graph:

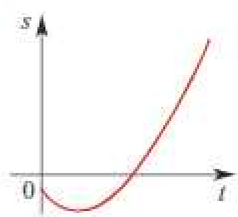
a



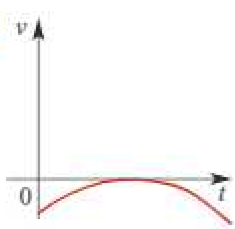
b



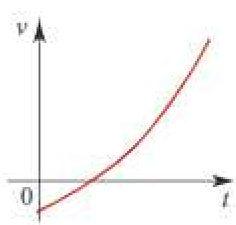
c



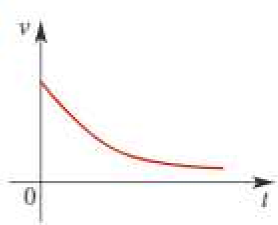
A



B



C



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Review

8 A boy stands on the edge of a cliff of height 60 m. He throws a stone vertically upwards so that its distance, h m, above the cliff top is given by $h = 20t - 5t^2$.

a Calculate the maximum height reached by the stone above the cliff top.

b Calculate the time which elapses before the stone hits the beach (vertically below).

c Calculate the speed with which the stone hits the beach.

9 Find the least possible value of $x^2 + y^2$ given that $x + y = 12$.

Multiple-choice questions

1 The equation of the tangent to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = x$ B $y = 5x$ C $y = 5x + 2$
D $y = 5x - 2$ E $y = x - 2$

2 The equation of the normal to the curve $y = x^3 + 2x$ at the point $(1, 3)$ is

A $y = -5x$ B $y = -5x + 2$ C $y = \frac{1}{5}x + \frac{12}{5}$
D $y = -\frac{1}{5}x + \frac{12}{5}$ E $y = -\frac{1}{5}x + \frac{16}{5}$

3 The equation of the tangent to the curve $y = 2x - 3x^3$ at the origin is

A $y = 2$ B $y = -2x$ C $y = x$ D $y = -x$ E $y = 2x$

4 The average rate of change of the function $f(x) = 4x - x^2$ between $x = 0$ and $x = 1$ is

A 3 B −3 C 4 D −4 E 0

5 A particle moves in a straight line so that its position, S m, relative to O at a time t seconds ($t \geq 0$) is given by $S = 4t^3 + 3t - 7$. The initial velocity of the particle is

A 0 m/s B −7 m/s C 3 m/s D −4 m/s E 15 m/s

6 The function $y = x^3 - 12x$ has stationary points at $x =$

A 0 and 12 B −4 and 4 C −2 and 4 D −2 and 2 E 2 only

7 The curve $y = 2x^3 - 6x$ has a gradient of 6 at $x =$

A 2 B $\sqrt{2}$ C −2 and 2 D $-\sqrt{2}$ and $\sqrt{2}$ E 0 and $\sqrt{2}$

8 The rate of change of the function $f(x) = 2x^3 - 5x^2 + x$ at $x = 2$ is

A 5 B −2 C 2 D −5 E 6

9 The average rate of change of the function $y = \frac{1}{2}x^4 + 2x^2 - 5$ between $x = -2$ and $x = 2$ is

A 0 B 5.5 C 11 D 22 E 2.75

10 The minimum value of the function $y = x^2 - 8x + 1$ is

A 1 B 4 C −15 D 0 E −11

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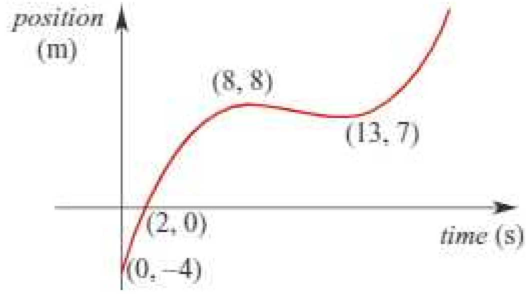
Review

Questions 11 and 12 refer to the following information:

A particle moves along a horizontal line. The graph of the particle's position relative to the origin over time is shown.



11 The particle has a velocity of zero at

A 8 s and 13 s **B** 2 s **C** 0 s **D** 8 s and 7 s **E** −4 s

12 The time interval(s) during which the particle has a negative velocity are

A $8 < t < 13$ **B** $0 < t < 2$ and $8 < t < 13$ **C** $0 < t < 2$
D $7 < t < 8$ **E** $t > 0$

Extended-response questions

Rate of change problems

1 The height, in metres, of a stone thrown vertically upwards from the surface of a planet is $2 + 10t - 4t^2$ after t seconds.

a Calculate the velocity of the stone after 3 seconds.

b Find the acceleration due to gravity.

2 A dam is being emptied. The quantity of water, V litres, remaining in the dam at any time t minutes after it starts to empty is given by $V(t) = 1000(30 - t)^3$, for $t \geq 0$.

CAS

a Sketch the graph of V against t.

b Find the time at which there are:

i 2 000 000 litres of water in the dam **ii** 20 000 000 litres of water in the dam.

c At what rate is the dam being emptied at any time t?

d How long does it take to empty the dam?

e At what time is the water flowing out at 8000 litres per minute?

f Sketch the graphs of $y = V(t)$ and $y = V'(t)$ on the one set of axes.

3 In a certain area of Victoria the quantity of blackberries, W tonnes, ready for picking x days after 1 September is given by

$$W = \frac{x}{4000}\left(48\,000 - 2600x + 60x^2 - \frac{x^3}{2}\right) \qquad \text{for } 0 \leq x \leq 60$$

a Sketch the graph of W against x for $0 \leq x \leq 60$.

b After how many days will there be 50 tonnes of blackberries ready for picking?

c Find the rate of increase of W, in tonnes per day, when $x = 20$, 40 and 60.

d Find the value of W when $x = 30$.

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Review

4 A newly installed central heating system has a thermometer which shows the water temperature as it leaves the boiler (y°C). It also has a thermostat which switches off the system when $y = 65$.

The relationship between y and t, the time in minutes, is given by $y = 15 + \frac{1}{80}t^2(30 - t)$.

a Find the temperature at $t = 0$.

b Find the rate of increase of y with respect to t, when $t = 0, 5, 10, 15$ and 20.

c Sketch the graph of y against t for $0 \le t \le 20$.

5 The sweetness, S, of a pineapple t days after it begins to ripen is found to be given by $S = 4000 + (t - 16)^3$ units.

a At what rate is S increasing when $t = 0$?

b Find $\frac{dS}{dt}$ when $t = 4, 8, 12$ and 16.

c The pineapple is said to be unsatisfactory when our model indicates that the rate of increase of sweetness is zero. When does this happen?

d Sketch the graph of S against t up to the moment when the pineapple is unsatisfactory.

CAS

6 A slow train which stops at every station passes a certain signal box at 12 p.m. The motion of the train between the two stations on either side of the signal box is such that it is s km past the signal box at t minutes past 12 p.m., where

$$s = \frac{1}{3}t + \frac{1}{9}t^2 - \frac{1}{27}t^3$$

(Note that, before the train reaches the signal box, both s and t will be negative.)

a Use a calculator to help sketch the graphs of s against t and $\frac{ds}{dt}$ against t on the one set of axes. Sketch for $t \in [-2, 5]$.

b Find the time of departure from the first station and the time of arrival at the second.

c Find the distance of each station from the signal box.

d Find the average velocity between the stations.

e Find the velocity with which the train passes the signal box.

7 Water is draining from a tank. The volume, V L, of water at time t (hours) is given by $V(t) = 1000 + (2 - t)^3$, for $t \ge 0$ and $V(t) \ge 0$.

a What are the possible values of t?

b Find the rate of draining when:

i $t = 5$ **ii** $t = 10$

8 The position of an object moving in a straight line is given by $s(t) = kt - t^2$ for $t \ge 0$, where s is in metres, t is in seconds and k is a constant.

a If $k = 2$, find the object's average velocity over the first 3 seconds.

b Find the value of k if the average velocity over the first 3 seconds is equal to 1 m/s.

c If $k = 2$, estimate the object's instantaneous velocity when $t = 3$ by considering the average velocity between $t = 3$ and $t = 3.1$.

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9 A mountain path can be approximately described by the following rule, where y is the elevation, in metres above sea level, and x is the horizontal distance travelled in kilometres:

$$y = \frac{1}{5}(4x^3 - 8x^2 + 192x + 144) \qquad \text{for } 0 \le x \le 7$$

a How high above sea level is the start of the track, i.e. $x = 0$?

b When $x = 6$, what is the value of y?

c Use a calculator to draw a graph of the path. Sketch this graph.

d Does this model for the path make sense for $x > 7$?

e Find the gradient of the graph for the following distances (be careful of units):

i $x = 0$ **ii** $x = 3$ **iii** $x = 7$

CAS

Review

Maximum and minimum problems

10 **a** On the one set of axes sketch the graphs of $y = x^3$ and $y = 2 + x - x^2$.

b Note that $2 + x - x^2 \ge x^3$ for $x \le 0$. Find the value of x, with $x \le 0$, for which the vertical distance between the two curves is a minimum and find the minimum distance.

Hint: Consider the function with rule $y = 2 + x - x^2 - x^3$ for $x \le 0$.

11 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos. (First plot the graph of $y = M(x)$ using a calculator.)

CAS

12 Given that $x + y = 5$ and $P = xy$, find:

a y in terms of x

b P in terms of x

c the maximum value of P and the corresponding values of x and y.

13 Given that $2x + y = 10$ and $A = x^2y$, where $0 \le x \le 5$, find:

a y in terms of x

b A in terms of x

c the maximum value of A and the corresponding values of x and y.

14 Given that $xy = 10$ and $T = 3x^2y - x^3$, find the maximum value of T for $0 < x < \sqrt{30}$.

15 The sum of two numbers x and y is 8.

a Write down an expression for y in terms of x.

b Write down an expression for s, the sum of the squares of these two numbers, in terms of x.

c Find the least value of the sum of their squares.

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Review

16 Find two positive numbers whose sum is 4, such that the sum of the cube of the first and the square of the second is as small as possible.

17 A rectangular patch of ground is to be enclosed with 100 metres of fencing wire. Find the dimensions of the rectangle so that the area enclosed will be a maximum.

18 The sum of two numbers is 24. If one number is x, find the value of x such that the product of the two numbers is a maximum.

19 A factory that produces n items per hour has overhead costs of $\$C$ per hour, where $C = 400 - 16n + \frac{1}{4}n^2$. How many items should be produced every hour to keep the overhead costs to a minimum?

20 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$, and find the maximum value of P.

21 A farmer has 4 km of fencing wire and wishes to fence in a rectangular piece of land through which a straight river flows. The river is to form one side of the enclosure. How can this be done to enclose as much land as possible?

22 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

23 A beam has a rectangular cross-section of depth x cm and width y cm. The perimeter of the cross-section of the beam is 120 cm. The strength, S, of the beam is given by $S = 5x^2y$.

a Find y in terms of x.

b Express S in terms of x.

c What are the possible values for x?

d Sketch the graph of S against x.

e Find the values of x and y which give the strongest beam.

f If the cross-sectional depth of the beam must be less than or equal to 19 cm, find the maximum strength of the beam.

24 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$, with x representing the temperature of the water in degrees (°C). (This model is valid only if $6 \le x \le 20$.) Find the water temperature that results in the maximum number of salmon swimming upstream.

25 A piece of wire 360 cm long is used to make the twelve edges of a rectangular box for which the length is twice the breadth.

a Denoting the breadth of the box by x cm, show that the volume of the box, V cm^3, is given by $V = 180x^2 - 6x^3$.

b Find the domain, S, of the function $V : S \to \mathbb{R}$, $V(x) = 180x^2 - 6x^3$ which describes the situation.

c Sketch the graph of the function with rule $y = V(x)$.

d Find the dimensions of the box that has the greatest volume.

e Find the values of x (correct to two decimal places) for which $V = 20\,000$.

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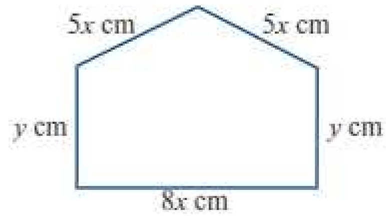
26 A piece of wire of length 90 cm is bent into the shape shown in the diagram.



a Show that the area, A cm^2, enclosed by the wire is given by $A = 360x - 60x^2$.

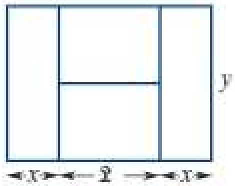
b Find the values of x and y for which A is a maximum.

27 A piece of wire 100 cm in length is to be cut into two pieces, one piece of which is to be shaped into a circle and the other into a square.

a How should the wire be cut if the sum of the enclosed areas is to be a minimum? (Give your answer to the nearest centimetre.)

b How should the wire be used to obtain a maximum area?

28 A roll of tape 36 metres long is to be used to mark out the edges and internal lines of a rectangular court of length $4x$ metres and width y metres, as shown in the diagram. Find the length and width of the court for which the area is a maximum.



29 A rectangular chicken run is to be built on flat ground. A 16-metre length of chicken wire will be used to form three of the sides; the fourth side, of length x metres, will be part of a straight wooden fence.

a Let y be the width of the rectangle. Find an expression for A, the area of the chicken run, in terms of x and y.

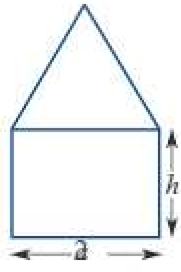
b Find an expression for A in terms of x.

c Find the possible values of x.

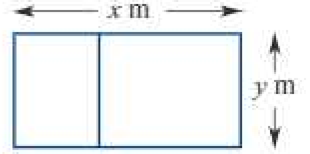
d Sketch the graph of A against x for these values of x.

e What is the largest area of ground the chicken run can cover?

30 The diagram illustrates a window that consists of an equilateral triangle and a rectangle. The amount of light that comes through the window is directly proportional to the area of the window.
If the perimeter of such a window must be 8000 mm, find the values of h and a (correct to the nearest mm) which allow the maximum amount of light to pass.



31 A rectangular frame with a cross-strut is made from a 10-metre length of steel. Find the frame's maximum possible area.



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Review

32 A cross-section of an aeroplane wing has its upper boundary described by the curve $y = ax(x-3)^2$, $0 \le x \le 3$, where a is a positive constant. The height of the wing is y m at a distance of x m across the wing (relative to a reference point on the leading edge of the wing).

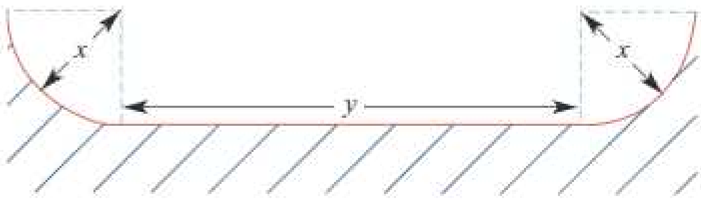
a Show that the gradient of the curve is given by $\frac{dy}{dx} = 3a(x-3)(x-1)$.

b Hence, find the value of x at which this cross-section of the wing reaches its maximum height.

c If the maximum height is 40 cm, find the value of a.

d Using the value of a found in part **c**, locate the point on the upper boundary of the wing where the gradient is -0.3.

33 The diagram shows a cross-section of an open drainage channel. The flat bottom of the channel is y metres across and the sides are quarter circles of radius x metres. The total length of the bottom plus the two curved sides is 10 metres.



a Express y in terms of x.

b State the possible values that x can take.

c Find an expression for A, the area of the cross-section, in terms of x.

d Sketch the graph of A against x, for possible values of x.

e Find the value of x which maximises A.

f Comment on the cross-sectional shape of the drain.

34 A cylinder closed at both ends has a total surface area of 1000 cm^2. The radius of the cylinder is x cm and the height h cm. Let V cm^3 be the volume of the cylinder.

a Find h in terms of x.

b Find V in terms of x.

c Find $\frac{dV}{dx}$.

d Find $\left\{ x : \frac{dV}{dx} = 0 \right\}$.

e Sketch the graph of V against x for a suitable domain.

f Find the maximum volume of the cylinder.

g Find the value(s) of x and h for which $V = 1000$, correct to two decimal places.

35 A cylindrical aluminium can able to contain half a litre of drink is to be manufactured. The volume of the can must therefore be 500 cm^3.

a Find the radius and height of the can which will use the least aluminium and therefore be the cheapest to manufacture.

b If the radius of the can must be no greater than 5 cm, find the radius and height of the can that will use the least aluminium.

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