13

Exponential functions and logarithms

Objectives

- To understand the rules for manipulating **exponential expressions**.
- To define **exponential functions** and to sketch their graphs.
- To solve **exponential equations**.
- To evaluate **logarithmic expressions**.
- To use the **logarithm laws** to simplify expressions.
- To solve equations using **logarithmic methods**.
- To sketch graphs of functions of the form $y = \log_a x$ and simple transformations of this.
- To understand and use a range of **exponential models**.

The function $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1, is called an **exponential function**.

Consider the following example of an exponential function. Assume that a particular biological organism reproduces by dividing every minute. The following table shows the population, P, after n one-minute intervals (assuming that all organisms are still alive).

n	0	1	2	3	4	5	6	n
P	1	2	4	8	16	32	64	2^n

Thus P defines a function which has the rule $P = 2^n$, an exponential function.

In this chapter, the work on functions in this course is continued. Many of the concepts introduced in Chapter 5 – such as domain, range, transformations and inverse functions – are used in the context of exponential and logarithmic functions.

In the final section, we investigate some of the many applications of exponential functions.

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13A The index laws

The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**. The plural of index is **indices**. In this section, we concentrate on indices that are integers.

We note that, if n is positive, then $0^n = 0$. But if n is negative or zero, then 0^n is undefined.

Index law 1: Multiplying powers

Index law 1

To **multiply** two powers with the same base, **add** the indices.

$$a^m \times a^n = a^{m+n}$$

If m and n are positive integers,

then $a^m = \underbrace{a \times a \times \cdots \times a}_{m \text{ terms}}$

and $a^n = \underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}$

$$\therefore \quad a^m \times a^n = (\underbrace{a \times a \times \cdots \times a}_{m \text{ terms}}) \times (\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}})$$

$$= \underbrace{a \times a \times \cdots \times a}_{(m+n) \text{ terms}}$$

$$= a^{m+n}$$

Example 1

Simplify each of the following:

a $2^3 \times 2^{12}$ **b** $x^2y^3 \times x^4y$ **c** $2^x \times 2^{x+2}$ **d** $3a^2b^3 \times 4a^3b^3$

Solution

a $2^3 \times 2^{12} = 2^{3+12}$
$= 2^{15}$

b $x^2y^3 \times x^4y = x^2 \times x^4 \times y^3 \times y$
$= x^6y^4$

c $2^x \times 2^{x+2} = 2^{x+x+2}$
$= 2^{2x+2}$

d $3a^2b^3 \times 4a^3b^3$
$= 3 \times 4 \times a^2 \times a^3 \times b^3 \times b^3$
$= 12a^5b^6$

Explanation

When multiplying powers with the same base, add the indices.

In part **b**, the indices of the base-x powers are added, and the indices of the base-y powers are added. Note that $y = y^1$.

In part **c**, we use $x + x + 2 = 2x + 2$.

In part **d**, the indices of the base-a powers are added, the indices of the base-b powers are added, and the factors that are numerals are multiplied together.

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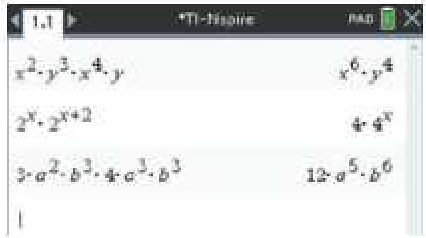
Using the TI-Nspire

Parts **b**, **c** and **d** can be simplified as shown: the TI-Nspire will simplify automatically.

(For part **c**, note that $4 \cdot 4^x = 2^2 \cdot 2^{2x} = 2^{2x+2}$.)

Note: When using [∧] to enter indices, you need to use either ► or ▼ to return to the baseline.

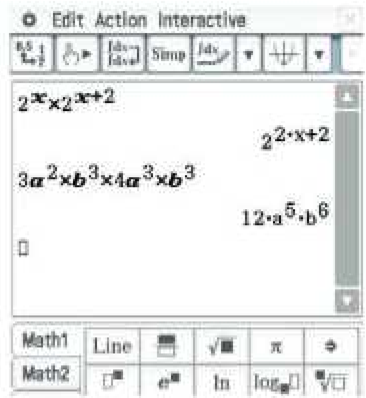


Using the Casio ClassPad

To enter a power:

- First enter the base of the power.
- Tap on the power template [□■] found in the [Math1] keyboard and then enter the index.
- Tap the stylus to the right of the expression to return the cursor to the baseline.

Note: Alternatively, you can enter indices using the button [∧] on the hard keyboard. You must enclose the index in brackets if it consists of more than a single digit or variable.



Index law 2: Dividing powers

Index law 2

To **divide** two powers with the same base, **subtract** the indices.

$$a^m \div a^n = a^{m-n}$$

If m and n are positive integers with $m > n$, then

$$a^m \div a^n = \frac{\overbrace{a \times a \times \cdots \times a}^{m \text{ terms}}}{\underbrace{a \times a \times \cdots \times a}_{n \text{ terms}}}$$

$$= \underbrace{a \times a \times \cdots \times a}_{(m-n) \text{ terms}} \qquad \text{(by cancelling)}$$

$$= a^{m-n}$$

Example 2

Simplify each of the following:

a $\dfrac{x^4y^3}{x^2y^2}$ **b** $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}}$ **c** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$

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Solution	Explanation
a $\dfrac{x^4y^3}{x^2y^2} = x^{4-2}y^{3-2} = x^2y$	When dividing powers with the same base, subtract the indices.
b $\dfrac{b^{4x} \times b^{x+1}}{b^{2x}} = b^{4x+x+1-2x} = b^{3x+1}$	In part **a**, the indices of the base-x powers are subtracted, and the indices of the base-y powers are subtracted.
c $\dfrac{16a^5b \times 4a^4b^3}{8ab} = \dfrac{16 \times 4}{8} \times a^{5+4-1} \times b^{1+3-1} = 8a^8b^3$	In parts **b** and **c**, both index law 1 and index law 2 are used. In part **c**, the factors that are numerals are grouped together and simplified.

The zero index and negative integer indices

Define $a^0 = 1$ for $a \neq 0$, and define $a^{-n} = \dfrac{1}{a^n}$ for $a \neq 0$.

Index laws 1 and 2 also hold for negative indices m, n. For example:

$$2^4 \times 2^{-2} = \frac{2^4}{2^2} = \frac{2 \times 2 \times 2 \times 2}{2 \times 2} = 2^2 \qquad (\text{i.e. } 2^{4+(-2)})$$

$$2^{-4} \div 2^2 = \frac{1}{2^4} \times \frac{1}{2^2} = \frac{1}{2^4 \times 2^2} = 2^{-6} \qquad (\text{i.e. } 2^{-4-2})$$

$$2^3 \div 2^3 = 2^3 \times \frac{1}{2^3} = 1 = 2^0 \qquad (\text{i.e. } 2^{3-3})$$

The reciprocal of a fraction such as $\frac{2}{3}$ is $\frac{3}{2}$. For fractions, the index -1 means 'the reciprocal of'. For example:

$$\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$$

When raising a fraction to other negative indices, take the reciprocal first. For example:

$$\left(\frac{5}{6}\right)^{-2} = \left(\frac{6}{5}\right)^2 = \frac{36}{25}$$

Example 3

Evaluate each of the following:

a 8^{-2} **b** $\left(\frac{1}{2}\right)^{-4}$ **c** $\left(\frac{3}{4}\right)^{-3}$

Solution

a $8^{-2} = \left(\frac{1}{8}\right)^2 = \frac{1}{64}$

b $\left(\frac{1}{2}\right)^{-4} = 2^4 = 16$

c $\left(\frac{3}{4}\right)^{-3} = \left(\frac{4}{3}\right)^3 = \frac{64}{27}$

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Index law 3: Raising the power

Consider the following:

$$(2^3)^2 = 2^3 \times 2^3 = 2^{3+3} = 2^6 = 2^{3\times2}$$

$$(4^3)^4 = 4^3 \times 4^3 \times 4^3 \times 4^3 = 4^{3+3+3+3} = 4^{12} = 4^{3\times4}$$

$$(a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{2+2+2+2+2} = a^{10} = a^{2\times5}$$

In general, $(a^m)^n = a^{m\times n}$.

Index law 3

To raise a power to another power, **multiply** the indices.

$$(a^m)^n = a^{m\times n}$$

This rule holds for all integers m and n.

Example 4

Simplify each of the following:

a $(a^5)^3$ **b** $\left(\left(\frac{1}{2}\right)^{-3}\right)^2$ **c** $(b^3)^2 \times (b^2)^{-1}$

Solution	Explanation
a $(a^5)^3 = a^{15}$	Index law 3 is used.
b $\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = \left(\frac{1}{2}\right)^{-6} = 2^6 = 64$	For part **b**, the following calculation is probably preferable: $\left(\left(\frac{1}{2}\right)^{-3}\right)^2 = (2^3)^2 = 8^2 = 64$
c $(b^3)^2 \times (b^2)^{-1} = b^6 \times b^{-2} = b^4$	In part **c**, index law 1 is also used.

Index laws 4 and 5: Products and quotients

Index law 4

$$(ab)^n = a^n b^n$$

If n is a positive integer, then

$$\begin{aligned}(ab)^n &= \underbrace{(ab) \times (ab) \times \cdots \times (ab)}_{n \text{ terms}} \\ &= \underbrace{(a \times a \times \cdots \times a)}_{n \text{ terms}} \times \underbrace{(b \times b \times \cdots \times b)}_{n \text{ terms}} \\ &= a^n b^n\end{aligned}$$

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Index law 5

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

If n is a positive integer, then

$$\left(\frac{a}{b}\right)^n = \frac{a}{b} \times \frac{a}{b} \times \cdots \times \frac{a}{b}$$
$$= \frac{a^n}{b^n}$$

Example 5

Simplify each of the following:

a $(2a^2b^3)^3 \times (3ab^4)^{-2}$ **b** $\left(\frac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3$

Solution

a $(2a^2b^3)^3 \times (3ab^4)^{-2} = 8a^6b^9 \times 3^{-2}a^{-2}b^{-8}$
$= \frac{8a^4b}{9}$

b $\left(\frac{2a^3b^2}{abc^2}\right)^3 \div (ab^{-1}c)^3 = \frac{8a^9b^6}{a^3b^3c^6} \times \frac{1}{a^3b^{-3}c^3}$
$= \frac{8a^3b^6}{c^9}$

Explanation

In part **a**, index law 4 is used, and then laws 3 and 1 and the fact that $3^{-2} = \frac{1}{9}$.

In part **b**, index law 5 is used. Dividing by a fraction always means multiply by the reciprocal of that fraction.

Working with a negative base

The power $(-a)^n$ can be written as $(-1 \times a)^n = (-1)^n(a)^n$. We note that:

- If n is even, then $(-1)^n = 1$.
- If n is odd, then $(-1)^n = -1$.

Hence, if a is a positive number, then the number $(-a)^n$ is positive when n is even and negative when n is odd.

Example 6

Simplify each of the following:

a $(-3)^4$ **b** $(-5a)^3$ **c** $(-2a)^3 \times 3a^2$

Solution

a $(-3)^4 = 81$

b $(-5a)^3 = -125a^3$

c $(-2a)^3 \times 3a^2 = -8a^3 \times 3a^2$
$= -24a^5$

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Using prime decomposition

Bases that are composite numbers are often best factorised into primes before further calculations are undertaken.

Example 7

Simplify the following, expressing the answers in positive-index form:

a $12^n \times 18^{-2n}$ **b** $\dfrac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}}$ **c** $\dfrac{3^{2n} \times 6^n}{8^n \times 3^n}$

Solution

a
$$\begin{aligned} 12^n \times 18^{-2n} &= (3 \times 2^2)^n \times (3^2 \times 2)^{-2n} \\ &= 3^n \times 2^{2n} \times 3^{-4n} \times 2^{-2n} \\ &= 3^{-3n} \times 2^0 \\ &= \frac{1}{3^{3n}} \end{aligned}$$

b
$$\begin{aligned} \frac{3^{-3} \times 6^4 \times 12^{-3}}{9^{-4} \times 2^{-2}} &= \frac{3^{-3} \times 2^4 \times 3^4 \times 2^{-6} \times 3^{-3}}{3^{-4} \times 3^{-4} \times 2^{-2}} \\ &= \frac{3^{-2} \times 2^{-2}}{3^{-8} \times 2^{-2}} \\ &= 3^6 \end{aligned}$$

c
$$\begin{aligned} \frac{3^{2n} \times 6^n}{8^n \times 3^n} &= \frac{(3^n \times 3^n) \times (3^n \times 2^n)}{2^{3n} \times 3^n} \\ &= \frac{3^n \times 3^n}{2^{2n}} \\ &= \left(\frac{3}{2}\right)^{2n} \end{aligned}$$

Explanation

The prime decomposition of 12 is

$$12 = 2^2 \times 3$$

The prime decomposition of 18 is

$$18 = 2 \times 3^2$$

Each number in this question can be expressed using powers of 2 and 3.

Index law 4 is used in each of the parts. For example:

$$\begin{aligned} 12^{-3} &= (2^2 \times 3)^{-3} \\ &= (2^2)^{-3} \times 3^{-3} \\ &= 2^{-6} \times 3^{-3} \end{aligned}$$

Summary 13A

- The expression a^n is called a **power**, where a is a non-zero number called the **base** and n is a number called the **exponent** or **index**.
- Index laws The following results hold for all non-zero numbers a and b and all integers m and n:

  1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{mn}$

  4 $(ab)^n = a^n b^n$ **5** $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$
- For every non-zero number a and positive integer n:
  - $a^0 = 1$
  - $a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$
- $0^n = 0$, if n is a positive integer, and 0^0 is undefined.

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Skill-sheet

Exercise 13A

1 For each of the following, use the stated rule to give an equivalent expression in simplest form:

a $x^2 \times x^3$	**b** $2 \times x^3 \times x^4 \times 4$	Index law 1
c $\dfrac{x^5}{x^3}$	**d** $\dfrac{4x^6}{2x^3}$	Index law 2
e $(a^3)^2$	**f** $(2^3)^2$	Index law 3
g $(xy)^2$	**h** $(x^2y^3)^2$	Index law 4 (also use law 3 for **h**)
i $\left(\dfrac{x}{y}\right)^3$	**j** $\left(\dfrac{x^3}{y^2}\right)^2$	Index law 5 (also use law 3 for **j**)

Example 1

2 Simplify each of the following:

a $3^5 \times 3^{12}$ **b** $x^3y^2 \times x^4y^3$ **c** $3^{x+1} \times 3^{3x+2}$ **d** $5a^3b^2 \times 6a^2b^4$

Example 2

3 Simplify each of the following:

a $\dfrac{x^5y^2}{x^3y}$ **b** $\dfrac{b^{5x} \times b^{2x+1}}{b^{3x}}$ **c** $\dfrac{8a^2b \times 3a^5b^2}{6a^2b^2}$

Example 3

4 Evaluate each of the following:

a 7^{-2} **b** $\left(\dfrac{1}{4}\right)^{-3}$ **c** $\left(\dfrac{5}{2}\right)^{-3}$

Example 4

5 Simplify each of the following:

a $(b^5)^2$ **b** $\left(\left(\dfrac{1}{3}\right)^{-2}\right)^3$ **c** $(b^5)^2 \times (b^2)^{-3}$

Example 5

6 Simplify each of the following:

a $(3a^4b^3)^3 \times (4a^2b^4)^{-2}$ **b** $\left(\dfrac{5a^3b^3}{ab^2c^2}\right)^3 \div (a^2b^{-1}c)^3$

Example 6

7 Simplify each of the following:

a $(-2)^6$ **b** $(-3a)^3$ **c** $(-2a)^5 \times 3a^{-2}$

Example 7

8 Simplify the following:

a $36^n \times 12^{-2n}$ **b** $\dfrac{2^{-3} \times 8^4 \times 32^{-3}}{4^{-4} \times 2^{-2}}$ **c** $\dfrac{5^{2n} \times 10^n}{8^n \times 5^n}$

9 Simplify the following:

a $x^3 \times x^4 \times x^2$
b $2^4 \times 4^3 \times 8^2$
c $3^4 \times 9^2 \times 27^3$
d $(q^2p)^3 \times (qp^3)^2$
e $a^2b^{-3} \times (a^3b^2)^3$
f $(2x^3)^2 \times (4x^4)^3$
g $m^3p^2 \times (m^2n^3)^4 \times (p^{-2})^2$
h $2^3a^3b^2 \times (2a^{-1}b^2)^{-2}$

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10 Simplify the following:

a $\dfrac{x^3y^5}{xy^2}$ **b** $\dfrac{16a^5b \times 4a^4b^3}{8ab}$ **c** $\dfrac{(-2xy)^2 \times 2(x^2y)^3}{8(xy)^3}$ **d** $\dfrac{(-3x^2y^3)^2}{(2xy)^3} \times \dfrac{4x^4y^3}{(xy)^3}$

11 Simplify each of the following, expressing your answer in positive-index form:

a $m^3n^2p^{-2} \times (mn^2p)^{-3}$ **b** $\dfrac{x^3yz^{-2} \times 2(x^3y^{-2}z)^2}{xyz^{-1}}$ **c** $\dfrac{a^2b \times (ab^{-2})^{-3}}{(a^{-2}b^{-1})^{-2}}$

d $\dfrac{a^2b^3c^{-4}}{a^{-1}b^2c^{-3}}$ **e** $\dfrac{a^{2n-1} \times b^3 \times c^{1-n}}{a^{n-3} \times b^{2-n} \times c^{2-2n}}$

12 Simplify each of the following:

a $3^{4n} \times 9^{2n} \times 27^{3n}$ **b** $\dfrac{2^n \times 8^{n+1}}{32^n}$ **c** $\dfrac{3^{n-1} \times 9^{2n-3}}{6^2 \times 3^{n+2}}$

d $\dfrac{2^{2n} \times 9^{2n-1}}{6^{n-1}}$ **e** $\dfrac{25^{2n} \times 5^{n-1}}{5^{2n+1}}$ **f** $\dfrac{6^{x-3} \times 4^x}{3^{x+1}}$

g $\dfrac{6^{2n} \times 9^3}{27^n \times 8^n \times 16^n}$ **h** $\dfrac{3^{n-2} \times 9^{n+1}}{27^{n-1}}$ **i** $\dfrac{8 \times 2^5 \times 3^7}{9 \times 2^7 \times 81}$

13 Simplify and evaluate:

a $\dfrac{(8^3)^4}{(2^{12})^2}$ **b** $\dfrac{125^3}{25^2}$ **c** $\dfrac{81^4 \div 27^3}{9^2}$

13B Rational indices

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

Using this notation for square roots:

$$\sqrt{a} = \sqrt[2]{a} = a^{\frac{1}{2}}$$

Further, the expression a^x can be defined for rational indices, i.e. when $x = \dfrac{m}{n}$, where m and n are integers, by defining

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

To employ this definition we will always first write the fractional power in simplest form.

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Example 8

Evaluate:

a $(-64)^{\frac{1}{3}}$ **b** $9^{-\frac{1}{2}}$ **c** $16^{\frac{5}{2}}$ **d** $64^{-\frac{2}{3}}$

Solution

a $(-64)^{\frac{1}{3}} = -4$

b $9^{-\frac{1}{2}} = \dfrac{1}{9^{\frac{1}{2}}} = \dfrac{1}{\sqrt{9}} = \dfrac{1}{3}$

c $16^{\frac{5}{2}} = \left(16^{\frac{1}{2}}\right)^5 = \left(\sqrt[2]{16}\right)^5 = 4^5 = 1024$

d $64^{-\frac{2}{3}} = \dfrac{1}{64^{\frac{2}{3}}} = \dfrac{1}{\left(64^{\frac{1}{3}}\right)^2} = \dfrac{1}{\left(\sqrt[3]{64}\right)^2} = \dfrac{1}{4^2} = \dfrac{1}{16}$

Note: In the previous section, we stated the index laws for m and n integers:

1 $a^m \times a^n = a^{m+n}$ **2** $a^m \div a^n = a^{m-n}$ **3** $(a^m)^n = a^{m\times n}$

These laws are applicable for all rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Example 9

Simplify:

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}}$ **b** $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4$

Solution

a $\dfrac{3^{\frac{1}{4}} \times \sqrt{6} \times \sqrt[4]{2}}{16^{\frac{3}{4}}} = \dfrac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{\left(16^{\frac{1}{4}}\right)^3}$

$= \dfrac{3^{\frac{1}{4}} \times 3^{\frac{1}{2}} \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}}}{2^3}$

$= \dfrac{3^{\frac{3}{4}} \times 2^{\frac{3}{4}}}{2^3}$

$= \dfrac{3^{\frac{3}{4}}}{2^{\frac{12}{4}-\frac{3}{4}}} = \dfrac{3^{\frac{3}{4}}}{2^{\frac{9}{4}}}$

b $(x^{-2}y)^{\frac{1}{2}} \times \left(\dfrac{x}{y^{-3}}\right)^4 = x^{-1}y^{\frac{1}{2}} \times \dfrac{x^4}{y^{-12}}$

$= x^3 \times y^{\frac{25}{2}}$

Explanation

$\sqrt{6} = \sqrt{3} \times \sqrt{2} = 3^{\frac{1}{2}} \times 2^{\frac{1}{2}}$ and $\sqrt[4]{2} = 2^{\frac{1}{4}}$

$2^3 = 2^{\frac{12}{4}}$

$\dfrac{x^4}{y^{-12}} = x^4 \times y^{12}$

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Summary 13B

- Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a.
- Define $0^{\frac{1}{n}} = 0$, for each $n \in \mathbb{N}$.
- If n is odd, then we can define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a.
- In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

- The index laws can be extended to rational indices:

1 $a^{\frac{m}{q}} \times a^{\frac{n}{p}} = a^{\frac{m}{q}+\frac{n}{p}}$ **2** $a^{\frac{m}{q}} \div a^{\frac{n}{p}} = a^{\frac{m}{q}-\frac{n}{p}}$ **3** $\left(a^{\frac{m}{q}}\right)^{\frac{n}{p}} = a^{\frac{m}{q}\times\frac{n}{p}}$

Skill-sheet

Exercise 13B

Example 8

1 Evaluate each of the following:

a $125^{\frac{2}{3}}$ **b** $243^{\frac{3}{5}}$ **c** $81^{-\frac{1}{2}}$ **d** $64^{\frac{2}{3}}$

e $\left(\frac{1}{8}\right)^{\frac{1}{3}}$ **f** $32^{-\frac{2}{5}}$ **g** $125^{-\frac{2}{3}}$ **h** $32^{\frac{4}{5}}$

i $1000^{-\frac{4}{3}}$ **j** $10\,000^{\frac{3}{4}}$ **k** $81^{\frac{3}{4}}$ **l** $\left(\frac{27}{125}\right)^{\frac{1}{3}}$

m $(-8)^{\frac{1}{3}}$ **n** $125^{-\frac{4}{3}}$ **o** $(-32)^{\frac{4}{5}}$ **p** $\left(\frac{1}{49}\right)^{-\frac{3}{2}}$

Example 9

2 Simplify:

a $\sqrt[3]{a^2b} \div \sqrt{ab^3}$ **b** $(a^{-2}b)^3 \times \left(\frac{1}{b^{-3}}\right)^{\frac{1}{2}}$

c $\dfrac{45^{\frac{1}{3}}}{9^{\frac{3}{4}} \times 15^{\frac{3}{2}}}$ **d** $2^{\frac{3}{2}} \times 4^{-\frac{1}{4}} \times 16^{-\frac{3}{4}}$

e $\left(\dfrac{x^3y^{-2}}{3^{-3}y^{-3}}\right)^{-2} \div \left(\dfrac{3^{-3}x^{-2}y}{x^4y^{-2}}\right)^2$ **f** $\left(\sqrt[5]{a^2}\right)^{\frac{3}{2}} \times \left(\sqrt[3]{a^5}\right)^{\frac{1}{5}}$

3 Simplify each of the following:

a $(2x-1)\sqrt{2x-1}$ **b** $(x-1)^2\sqrt{x-1}$

c $(x^2+1)\sqrt{x^2+1}$ **d** $(x-1)^3\sqrt{x-1}$

e $\dfrac{1}{\sqrt{x-1}} + \sqrt{x-1}$ **f** $(5x^2+1)\sqrt[3]{5x^2+1}$

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13C Graphs of exponential functions

Two types of graphs of exponential functions will be examined.

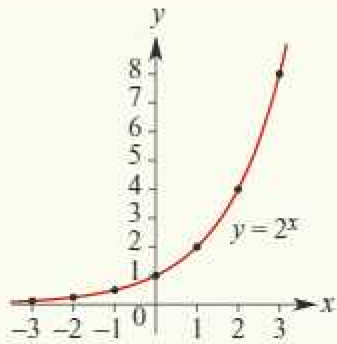
Graph of $y = a^x$ when $a > 1$

Example 10

Plot the graph of $y = 2^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

x	-3	-2	-1	0	1	2	3
$y = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



We can make the following observations about graphs of the form $y = a^x$ where $a > 1$:

- As the magnitude of the negative x-values becomes larger and larger, the y-values get closer and closer to zero, but never reach zero. That is, the graph of $y = a^x$ approaches the x-axis from above. The x-axis is said to be an **asymptote**.

  We can write: As $x \to -\infty$, $y \to 0^+$.

  This is read: As x approaches negative infinity, y approaches 0 from the positive side.
- As the x-values increase, the y-values increase.
- The y-axis intercept is at $(0, 1)$.
- The domain of the function is $\mathbb{R}$ and the range of the function is $\mathbb{R}^+$.

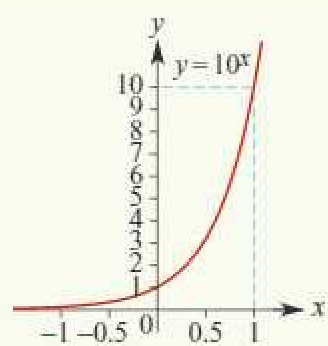
Example 11

Plot the graph of $y = 10^x$ and examine the table of values for $-1 \le x \le 1$. A calculator can be used to obtain approximate values.

Solution

x	-1	-0.5	0	0.5	1
$y = 10^x$	0.1	≈ 0.316	1	≈ 3.16	10

- The x-axis is an asymptote.
- The y-axis intercept is at $(0, 1)$.
- As the x-values increase, the y-values increase.
- For a given value of x, this rate of increase for $y = 10^x$ is greater than that for $y = 2^x$.



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Note that, for any numbers a and b greater than 1, there is a positive number k with $a^k = b$. This can be seen from the graphs of $y = 2^x$ and $y = 10^x$. Using a calculator to solve $2^k = 10$ gives $k = 3.3219\ldots$. Hence $10^x = (2^{3.3219\ldots})^x$ and therefore the graph of $y = 10^x$ can be obtained from the graph of $y = 2^x$ by a dilation of factor $\frac{1}{k} = \frac{1}{3.3219\ldots}$ from the y-axis.

All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis. (This will be discussed again later in the chapter.)

Graph of $y = a^x$ when $0 < a < 1$

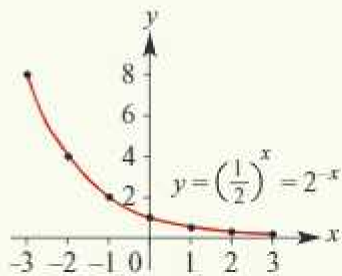
Example 12

Plot the graph of $y = (\frac{1}{2})^x$ and examine the table of values for $-3 \le x \le 3$. A calculator can be used.

Solution

Note that $y = (\frac{1}{2})^x = 2^{-x}$.

x	-3	-2	-1	0	1	2	3
$y = (\frac{1}{2})^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

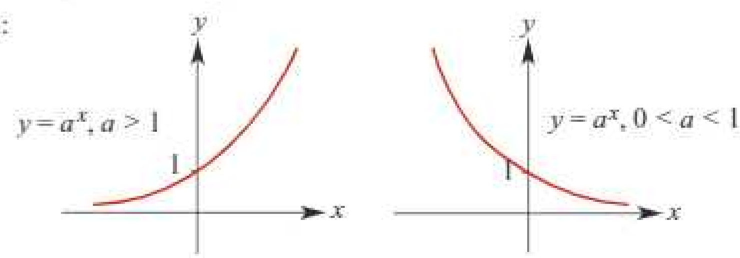


We can make the following observations about graphs of the form $y = a^x$ where $0 < a < 1$:

- The x-axis is an asymptote. As the x-values increase, the graph approaches the x-axis from above. This is written: As $x \to \infty$, $y \to 0^+$.
- The y-axis intercept is at $(0, 1)$.
- The domain of the function is $\mathbb{R}$ and the range of the function is $\mathbb{R}^+$.

Graphs of $y = a^x$ in general

In general:



In both cases $a > 1$ and $0 < a < 1$, we can write $y = a^x$ as $y = b^{-x}$, where $b = \frac{1}{a}$. The graph of $y = b^{-x}$ is obtained from the graph of $y = b^x$ by a reflection in the y-axis.

Thus, for example, the graph of $y = (\frac{1}{2})^x$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis, and vice versa. Using function notation: Let $f(x) = 2^x$ and $g(x) = (\frac{1}{2})^x$. Then

$$g(x) = (\tfrac{1}{2})^x = (2^{-1})^x = 2^{-x} = f(-x)$$

Example 13

Plot the graph of $y = 2^x$ on a CAS calculator and hence find (correct to three decimal places):

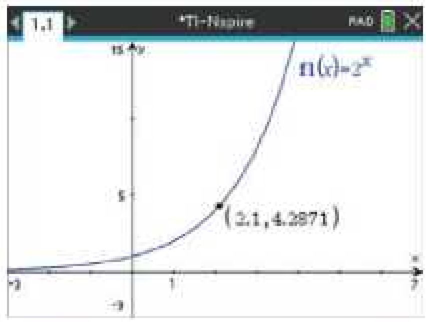
a the value of y when $x = 2.1$ **b** the value of x when $y = 9$.

Using the TI-Nspire

Plot the graph of $y = 2^x$.

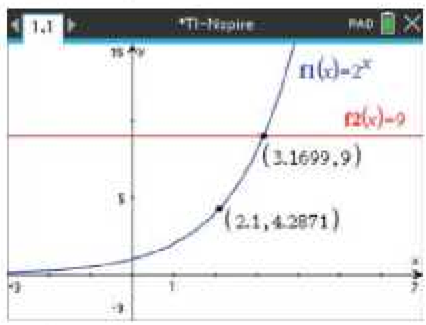
a
- To go to the point with x-coordinate 2.1, use (menu) > **Trace** > **Graph Trace** and type 2.1 (enter).
- Press (enter) to paste the coordinates to the point.
- Press (esc) to exit the **Graph Trace** tool.

When $x = 2.1$, $y = 4.287$ (correct to three decimal places).



b
- To find the value of x for which $y = 9$, plot the graph of $y = 9$ on the same screen and use (menu) > **Geometry** > **Points & Lines** > **Intersection Point(s)**.
- Press (esc) to exit the **Intersection Point(s)** tool.

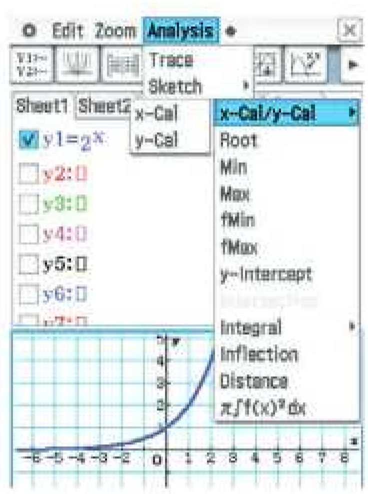
When $y = 9$, $x = 3.170$ (correct to three decimal places).



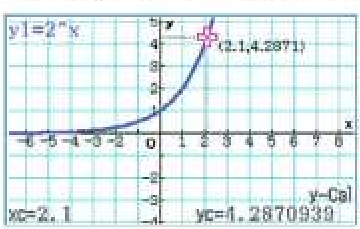
Note: Alternatively, find the intersection point using (menu) > **Analyze Graph** > **Intersection**.

Using the Casio ClassPad

In Graph&Table, enter 2^x in y1. Tick the box and select the graph icon.



a Tap in the graph window, select **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **y-Cal** and enter $x = 2.1$.



When $x = 2.1$, $y = 4.287$ (correct to three decimal places).

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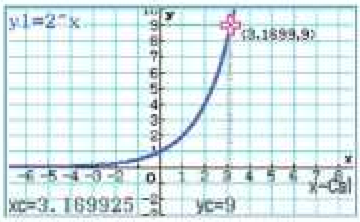
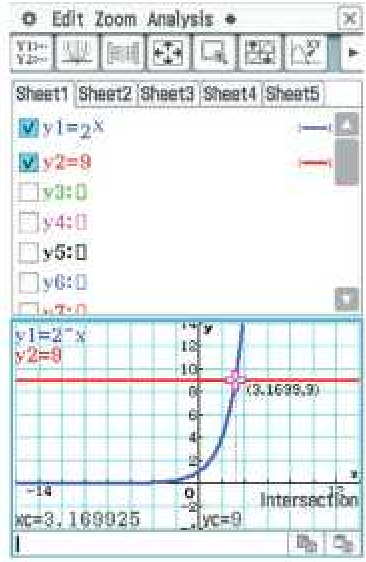
b Select **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **x-Cal** and enter $y = 9$.



When $y = 9$, $x = 3.170$ (correct to three decimal places).

Note: An alternative method for part **b** is to enter $y2 = 9$ and use **Analysis** > **G-Solve** > **Intersection**, as shown on the right.



Transformations of exponential graphs

The techniques for transformations that were introduced in earlier chapters are now applied to the graphs of exponential functions.

Example 14

Sketch the graphs of each of the following pairs of functions. For the second function in each pair, state the equation of the asymptote, the y-axis intercept and the range. (The x-axis intercepts need not be given.)

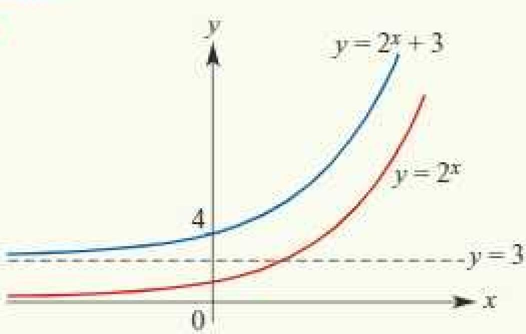
a $f: \mathbb{R} \to \mathbb{R}, f(x) = 2^x$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = 2^x + 3$

b $f: \mathbb{R} \to \mathbb{R}, f(x) = 3^x$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = 2 \times 3^x + 1$

c $f: \mathbb{R} \to \mathbb{R}, f(x) = 3^x$ and $g: \mathbb{R} \to \mathbb{R}, g(x) = -3^x + 2$

Solution

a



Graph of $g(x) = 2^x + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $g(0) = 2^0 + 3 = 4$.
- The range of the function g is $(3, \infty)$.

Explanation

The graph of $y = 2^x + 3$ is obtained by transforming the graph of $y = 2^x$ by a translation of 3 units in the positive direction of the y-axis.

The asymptote of $y = 2^x$ is the line with equation $y = 0$, which is transformed to the line with equation $y = 3$.

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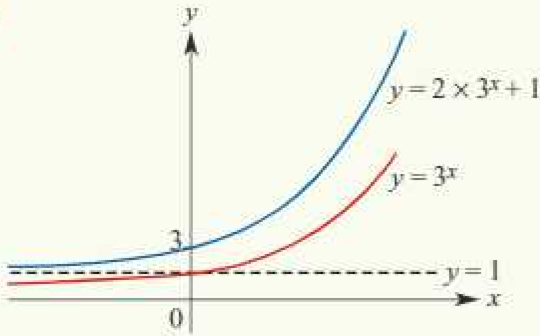
b



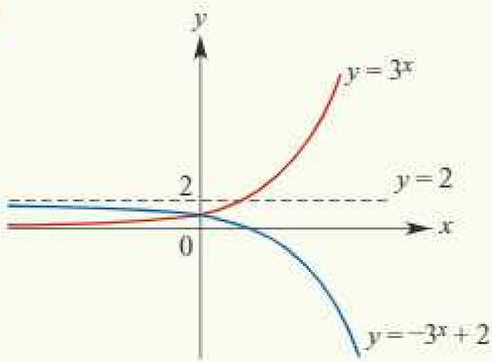
Graph of $g(x) = 2 \times 3^x + 1$:

- The asymptote has equation $y = 1$.
- The y-axis intercept is $g(0) = 2 \times 3^0 + 1 = 3$.
- The range of the function g is $(1, \infty)$.

The graph of $y = 2 \times 3^x + 1$ is obtained by transforming the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis, followed by a translation of 1 unit in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 1$.

c



Graph of $g(x) = -3^x + 2$:

- The asymptote has equation $y = 2$.
- The y-axis intercept is $g(0) = -3^0 + 2 = 1$.
- The range of the function g is $(-\infty, 2)$.

The graph of $y = -3^x + 2$ is obtained by transforming the graph of $y = 3^x$ by a reflection in the x-axis, followed by a translation of 2 units in the positive direction of the y-axis.

The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 2$.

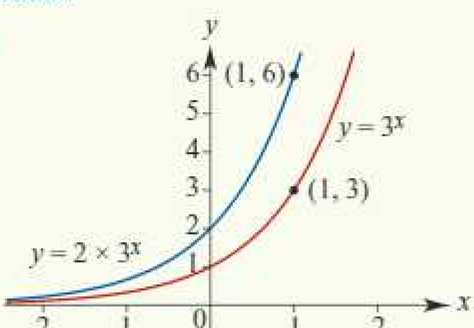
Example 15

Sketch the graph of each of the following:

a $y = 2 \times 3^x$ **b** $y = 3^{2x}$ **c** $y = -3^{-2x} + 4$

Solution

a



Explanation

The graph of $y = 2 \times 3^x$ is obtained from the graph of $y = 3^x$ by a dilation of factor 2 from the x-axis.

Both graphs have a horizontal asymptote with equation $y = 0$.

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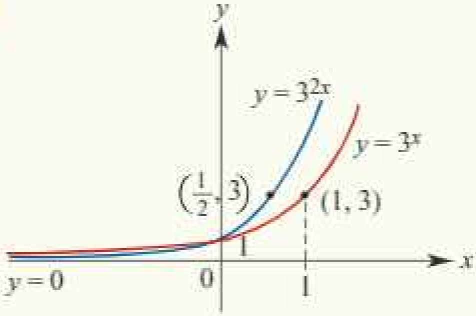
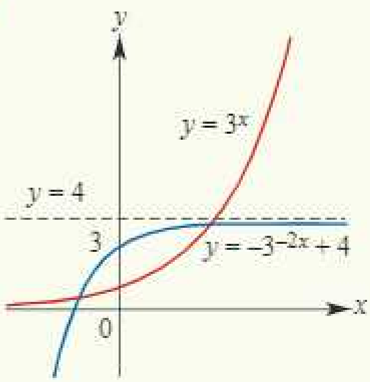
b



The graph of $y = 3^{2x}$ is obtained from the graph of $y = 3^x$ by a dilation of factor $\frac{1}{2}$ from the y-axis. (See the note below.)

Both graphs have a horizontal asymptote with equation $y = 0$.

c



The graph of $y = -3^{-2x} + 4$ is obtained from the graph of $y = 3^x$ by the sequence of transformations:

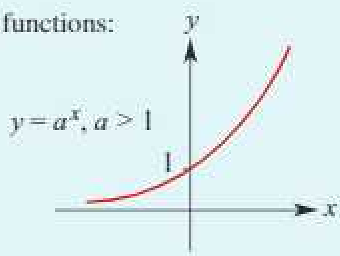
- dilation of factor $\frac{1}{2}$ from the y-axis
- reflection in the x-axis
- reflection in the y-axis
- translation of 4 units in the positive direction of the y-axis.

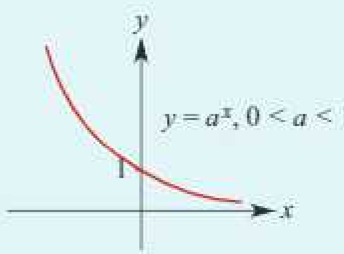
The asymptote of $y = 3^x$ is the line $y = 0$, which is transformed to the line $y = 4$.

Note: In the notation introduced in Chapter 7, write the transformation for part **b** as $(x, y) \to (\frac{1}{2}x, y)$. Then describe the transformation as $x' = \frac{1}{2}x$ and $y' = y$, and hence $x = 2x'$ and $y = y'$. The graph of $y = 3^x$ is mapped to the graph of $y' = 3^{2x'}$.

Summary 13C

- Graphs of exponential functions:





- In both cases $a > 1$ and $0 < a < 1$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- All graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis. Similarly, all graphs of the form $y = a^x$, where $0 < a < 1$, are related to each other by dilations from the y-axis.
- Let $a > 1$. If $f(x) = a^x$ and $g(x) = (\frac{1}{a})^x$, then $g(x) = f(-x)$ and so the graph of $y = g(x)$ is the reflection in the y-axis of the graph of $y = f(x)$.

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Exercise 13C

CAS

1 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 1.8^x$ b $y = 2.4^x$ c $y = 0.9^x$ d $y = 0.5^x$

2 Using a calculator, plot the graphs of the following and comment on the similarities and differences between them:

a $y = 2 \times 3^x$ b $y = 5 \times 3^x$ c $y = -2 \times 3^x$ d $y = -5 \times 3^x$

Example 13 3 Plot the graph of $y = 2^x$ on a CAS calculator and hence find (correct to three decimal places):

a the value of y when $x = 3.1$ b the value of x when $y = 14$.

4 Plot the graph of $y = 10^x$ on a CAS calculator and hence find the solution of the equation $10^x = 6$.

Example 14 5 Sketch the graphs of the following functions. For each function, give the equation of the asymptote, the y-axis intercept and the range. (The x-axis intercepts need not be given.)

a $f: \mathbb{R} \to \mathbb{R}, f(x) = 3 \times 2^x + 2$ b $f: \mathbb{R} \to \mathbb{R}, f(x) = 3 \times 2^x - 3$

c $f: \mathbb{R} \to \mathbb{R}, f(x) = -3^x - 2$ d $f: \mathbb{R} \to \mathbb{R}, f(x) = -2 \times 3^x + 2$

Example 15 6 Sketch the graph of each of the following:

a $y = 3^{3x}$ b $y = 5^{\frac{x}{2}}$ c $y = (\frac{1}{2})^x + 2$ d $y = -2^{-3x} + 2$

13D Solving exponential equations and inequalities

Solution of equations

One method without using a calculator is to express both sides of the equation as powers with the same base and then equate the indices (since $a^x = a^y$ implies $x = y$, for any $a \in \mathbb{R}^+ \setminus \{1\}$).

Example 16

Find the value of x for which:

a $4^x = 256$ b $3^{x-1} = 81$ c $5^{2x-4} = 25^{-x+2}$

Solution

a
$$4^x = 256$$
$$4^x = 4^4$$
$$\therefore x = 4$$

b
$$3^{x-1} = 81$$
$$3^{x-1} = 3^4$$
$$\therefore x - 1 = 4$$
$$x = 5$$

c
$$5^{2x-4} = 25^{-x+2}$$
$$= (5^2)^{-x+2}$$
$$= 5^{-2x+4}$$
$$\therefore 2x - 4 = -2x + 4$$
$$4x = 8$$
$$x = 2$$

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In the next example, an exponential equation is solved by first solving a quadratic equation.

Example 17

Solve:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We have $(3^x)^2 = 12 \times 3^x - 27$.

Let $a = 3^x$. The equation becomes

$$a^2 = 12a - 27$$
$$a^2 - 12a + 27 = 0$$
$$(a - 3)(a - 9) = 0$$

Therefore

$$a - 3 = 0 \quad \text{or} \quad a - 9 = 0$$
$$a = 3 \quad \text{or} \quad a = 9$$

Hence $3^x = 3^1$ or $3^x = 3^2$

and so $x = 1$ or $x = 2$

b We have $(3^x)^2 = 27 - 6 \times 3^x$.

Let $a = 3^x$. The equation becomes

$$a^2 = 27 - 6a$$
$$a^2 + 6a - 27 = 0$$
$$(a + 9)(a - 3) = 0$$

Therefore

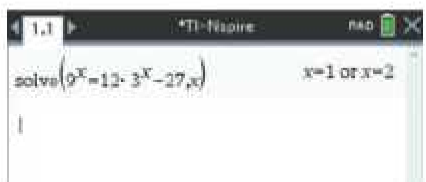
$$a = -9 \quad \text{or} \quad a = 3$$

Hence $3^x = -9$ or $3^x = 3^1$

There is only one solution, $x = 1$, since $3^x > 0$ for all values of x.

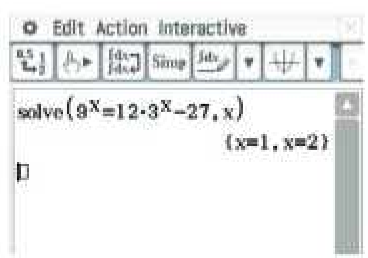
Using the TI-Nspire

Use menu > **Algebra** > **Solve** to solve the equation.



Using the Casio ClassPad

- Go to the $\sqrt{\alpha}$ Main screen and turn on the keyboard.
- Enter the equation $9^x = 12 \times 3^x - 27$.
- Highlight the equation using the stylus and select **Interactive** > **Equation/Inequality** > **solve**.
- Tap on OK to obtain the solution. (Note that the default variable is x.)



The calculator can be used to obtain approximate answers as shown in the following example. For the equation $5^x = 10$ we can find an exact solution, but logarithms are involved in the final answer. Logarithms are discussed in the following section.

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Example 18

Solve $5^x = 10$ correct to two decimal places.

Using the TI-Nspire

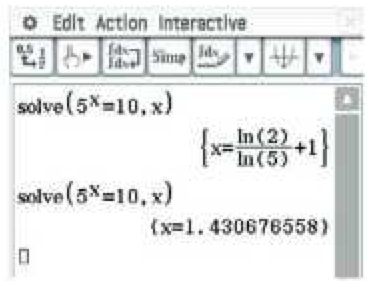
Press ctrl enter to obtain the answer as a decimal number.

The solution is $x = 1.43$ (correct to two decimal places).

Using the Casio ClassPad

To answer the question as required, you may need to highlight the answer and tap the conversion button to convert from the exact solution to a decimal approximation.

The solution is $x = 1.43$ (correct to two decimal places).



Solution of inequalities

The following two properties are useful when solving inequalities:

- $a^x > a^y \Leftrightarrow x > y$, when $a \in (1, \infty)$
- $a^x > a^y \Leftrightarrow x < y$, when $a \in (0, 1)$.

Example 19

Solve for x in each of the following:

a $16^x > 2$

b $2^{-3x+1} < \frac{1}{16}$

Solution

a
$$2^{4x} > 2^1$$
$$\Leftrightarrow \quad 4x > 1$$
$$\Leftrightarrow \quad x > \frac{1}{4}$$

b
$$2^{-3x+1} < 2^{-4}$$
$$\Leftrightarrow \quad -3x + 1 < -4$$
$$\Leftrightarrow \quad -3x < -5$$
$$\Leftrightarrow \quad x > \frac{5}{3}$$

Note: A CAS calculator can be used to help 'visualise' the inequality. For Example 19 **a**, plot the graphs of $y = 16^x$ and $y = 2$, and then find the point of intersection.

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Summary 13D

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in \mathbb{R}^+ \setminus \{1\}$).

  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$
- To solve an exponential inequality, first proceed as for an equation and then use the appropriate property:
  - $a^x > a^y \Leftrightarrow x > y$, when $a \in (1, \infty)$
  - $a^x > a^y \Leftrightarrow x < y$, when $a \in (0, 1)$.

  For example: $2^{x+1} > 8 \Leftrightarrow 2^{x+1} > 2^3 \Leftrightarrow x + 1 > 3 \Leftrightarrow x > 2$

Skill-sheet

Exercise 13D

Example 16

1 Solve for x in each of the following:

a $3^x = 27$ **b** $4^x = 64$ **c** $49^x = 7$
d $16^x = 8$ **e** $125^x = 5$ **f** $5^x = 625$
g $16^x = 256$ **h** $4^{-x} = \frac{1}{64}$ **i** $5^{-x} = \frac{1}{125}$

2 Solve for n in each of the following:

a $5^n \times 25^{2n-1} = 125$ **b** $3^{2n-4} = 1$ **c** $3^{2n-1} = \frac{1}{81}$
d $\frac{3^{n-2}}{9^{1-n}} = 1$ **e** $3^{3n} \times 9^{-2n+1} = 27$ **f** $2^{-3n} \times 4^{2n-2} = 16$
g $2^{n-6} = 8^{2-n}$ **h** $9^{3n+3} = 27^{n-2}$ **i** $4^{n+1} = 8^{n-2}$
j $32^{2n+1} = 8^{4n-1}$ **k** $25^{n+1} = 5 \times 390\,625$ **l** $125^{4-n} = 5^{6-2n}$
m $4^{2-n} = \frac{1}{2048}$

3 Solve the following exponential equations:

a $2^{x-1} \times 4^{2x+1} = 32$ **b** $3^{2x-1} \times 9^x = 243$ **c** $(27 \times 3^x)^2 = 27^x \times 3^{\frac{1}{2}}$

Example 17

4 Solve for x:

a $4(2^{2x}) = 8(2^x) - 4$ **b** $8(2^{2x}) - 10(2^x) + 2 = 0$
c $3 \times 2^{2x} - 18(2^x) + 24 = 0$ **d** $9^x - 4(3^x) + 3 = 0$

Example 18

5 Use a calculator to solve each of the following, correct to two decimal places:

a $2^x = 5$ **b** $4^x = 6$ **c** $10^x = 18$ **d** $10^x = 56$

CAS

Example 19

6 Solve for x in each of the following:

a $7^x > 49$ **b** $8^x > 2$ **c** $25^x \le 5$ **d** $3^{x+1} < 81$
e $9^{2x+1} < 243$ **f** $4^{2x+1} > 64$ **g** $3^{2x-2} \le 81$

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13E Logarithms

Consider the statement

$$2^3 = 8$$

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as 'the logarithm of 8 to the base 2 is equal to 3'.

For $a \in \mathbb{R}^+ \setminus \{1\}$, the **logarithm function** with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

The logarithm function with base a is the inverse of the exponential function with base a. We will discuss this in Section 13G.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 20

Without the aid of a calculator, evaluate the following:

a $\log_2 32$

b $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

Note: To find $\log_2 32$, we ask 'What power of 2 gives 32?'
To find $\log_3 81$, we ask 'What power of 3 gives 81?'

Laws of logarithms

The index laws can be used to establish rules for computations with logarithms. For each index law, there is a corresponding logarithm law.

Law 1: Logarithm of a product

The logarithm of a product is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

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Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using index law 1)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000\\ &= 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using index law 2)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4\\ &= 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p \log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using index law 3)}$$

Hence $\log_a(m^p) = xp = p \log_a m$.

For example: $\log_2 32 = \log_2(2^5) = 5$.

Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example: $\log_a(\frac{1}{2}) = \log_a(2^{-1}) = -\log_a 2$.

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Law 5

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$. Since $a^1 = a$, we have $\log_a a = 1$.

We can use the logarithm laws to simplify expressions involving logarithms of the same base.

Example 21

Without using a calculator, simplify the following:

$$2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned}2\log_{10} 3 + \log_{10} 16 - 2\log_{10}\left(\frac{6}{5}\right) &= \log_{10}(3^2) + \log_{10} 16 - \log_{10}\left(\frac{6}{5}\right)^2 \\ &= \log_{10} 9 + \log_{10} 16 - \log_{10}\left(\frac{36}{25}\right) \\ &= \log_{10}\left(9 \times 16 \times \frac{25}{36}\right) \\ &= \log_{10} 100 \\ &= 2\end{aligned}$$

Example 22

Solve each of the following equations for x:

a $\log_5 x = 3$

b $\log_5(2x + 1) = 2$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

Solution

a $\log_5 x = 3 \;\Leftrightarrow\; x = 5^3 = 125$

b $\log_5(2x + 1) = 2 \;\Leftrightarrow\; 2x + 1 = 5^2$

$$\begin{aligned}\therefore\; 2x + 1 &= 25 \\ 2x &= 24 \\ x &= 12\end{aligned}$$

c $\log_2(2x + 1) - \log_2(x - 1) = 4$

$$\begin{aligned}\log_2\left(\frac{2x+1}{x-1}\right) &= 4 \\ \therefore\; \frac{2x+1}{x-1} &= 2^4 \\ 2x + 1 &= 16(x - 1) \\ 17 &= 14x \\ x &= \frac{17}{14}\end{aligned}$$

d $\log_3(x - 1) + \log_3(x + 1) = 1$

$$\begin{aligned}\log_3[(x - 1)(x + 1)] &= 1 \\ \log_3(x^2 - 1) &= 1 \\ x^2 - 1 &= 3 \\ x &= \pm 2\end{aligned}$$

But the original expression is not defined for $x = -2$, and therefore $x = 2$.

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Summary 13E

- For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function base a is defined as follows:

  $$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- The expression $\log_a y$ is defined for all positive real numbers y.
- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- Laws of logarithms
  1. $\log_a(mn) = \log_a m + \log_a n$
  2. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
  3. $\log_a(m^p) = p \log_a m$
  4. $\log_a(m^{-1}) = -\log_a m$
  5. $\log_a 1 = 0$ and $\log_a a = 1$

Skill-sheet

Exercise 13E

Example 20

1 Without using a calculator, evaluate the following:

a $\log_2 128$ **b** $\log_3 81$ **c** $\log_5 125$ **d** $\log_{10} 0.1$

2 Use the stated rule for each of the following to give an equivalent expression in simplest form:

a $\log_2 10 + \log_2 a$ **b** $\log_{10} 5 + \log_{10} 2$ Law 1
c $\log_2 9 - \log_2 4$ **d** $\log_2 10 - \log_2 5$ Law 2
e $\log_2(a^3)$ **f** $\log_2(8^3)$ Law 3
g $\log_5\left(\frac{1}{6}\right)$ **h** $\log_5\left(\frac{1}{25}\right)$ Law 4

3 Without using a calculator, evaluate each of the following:

a $\log_3 27$ **b** $\log_5 625$
c $\log_2\left(\frac{1}{128}\right)$ **d** $\log_4\left(\frac{1}{64}\right)$
e $\log_x(x^4)$ **f** $\log_2 0.125$
g $\log_{10} 10\,000$ **h** $\log_{10} 0.000\,001$
i $-3 \log_5 125$ **j** $-4 \log_{16} 2$
k $2 \log_3 9$ **l** $-4 \log_{16} 4$

Example 21

4 Without using a calculator, simplify each of the following:

a $\frac{1}{2} \log_{10} 16 + 2 \log_{10} 5$ **b** $\log_2 16 + \log_2 8$
c $\log_2 128 + \log_3 45 - \log_3 5$ **d** $\log_4 32 - \log_9 27$
e $\log_b(b^3) - \log_b \sqrt{b}$ **f** $2 \log_x a + \log_x(a^3)$
g $x \log_2 8 + \log_2(8^{1-x})$ **h** $\frac{3}{2} \log_a a - \log_a \sqrt{a}$

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Example 22

5 Solve for x:

a $\log_3 9 = x$
b $\log_3 x = 3$
c $\log_5 x = -3$
d $\log_{10} x = \log_{10} 4 + \log_{10} 2$
e $\log_{10} 2 + \log_{10} 5 + \log_{10} x - \log_{10} 3 = 2$
f $\log_{10} x = \frac{1}{2}\log_{10} 36 - 2\log_{10} 3$
g $\log_x 64 = 2$
h $\log_5(2x - 3) = 3$
i $\log_3(x + 2) - \log_3 2 = 1$
j $\log_x 0.01 = -2$

6 Solve each of the following for x:

a $\log_x\left(\dfrac{1}{25}\right) = -2$
b $\log_4(2x - 1) = 3$
c $\log_4(x + 2) - \log_4 6 = 1$
d $\log_4(3x + 4) + \log_4 16 = 5$
e $\log_3(x^2 - 3x - 1) = 0$
f $\log_3(x^2 - 3x + 1) = 0$

7 If $\log_{10} x = a$ and $\log_{10} y = c$, express $\log_{10}\left(\dfrac{100x^3y^{-\frac{1}{2}}}{y^2}\right)$ in terms of a and c.

8 Prove that $\log_{10}\left(\dfrac{ab^2}{c}\right) + \log_{10}\left(\dfrac{c^2}{ab}\right) - \log_{10}(bc) = 0$.

9 If $\log_a\left(\dfrac{11}{3}\right) + \log_a\left(\dfrac{490}{297}\right) - 2\log_a\left(\dfrac{7}{9}\right) = \log_a k$, find k.

10 Solve each of the following equations for x:

a $\log_{10}(x^2 - 2x + 8) = 2\log_{10} x$
b $\log_{10}(5x) - \log_{10}(3 - 2x) = 1$
c $3\log_{10}(x - 1) = \log_{10} 8$
d $\log_{10}(20x) - \log_{10}(x - 8) = 2$
e $2\log_{10} 5 + \log_{10}(x + 1) = 1 + \log_{10}(2x + 7)$
f $1 + 2\log_{10}(x + 1) = \log_{10}(2x + 1) + \log_{10}(5x + 8)$

13F Using logarithms to solve exponential equations and inequalities

If $a \in \mathbb{R}^+ \setminus \{1\}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations.

Example 23

Solve for x if $2^x = 11$.

Solution

We immediately see that $x = \log_2 11$.

This can be evaluated with a calculator:

$$x = \log_2 11 \approx 3.45943$$

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Note: We will show an alternative method for solving Example 23 because it illustrates the relationship between logarithms of different bases. Take $\log_{10}$ of both sides of the equation $2^x = 11$. Then

$$\log_{10}(2^x) = \log_{10} 11$$
$$x \log_{10} 2 = \log_{10} 11$$
$$x = \frac{\log_{10} 11}{\log_{10} 2}$$

We have shown that

$$\log_2 11 = \frac{\log_{10} 11}{\log_{10} 2}$$

In general, if a, b and c are positive numbers with $a \neq 1$ and $b \neq 1$, then

$$\log_a c = \frac{\log_b c}{\log_b a}$$

Example 24

Solve $3^{2x-1} = 28$.

Solution

$$2x - 1 = \log_3 28$$
$$2x = 1 + \log_3 28$$
$$x = \frac{1 + \log_3 28}{2}$$
$$\approx 2.017 \quad \text{(to three decimal places)}$$

Example 25

Solve the inequality $0.7^x \geq 0.3$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10}(0.7^x) \geq \log_{10} 0.3$$
$$x \log_{10} 0.7 \geq \log_{10} 0.3$$
$$x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \quad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0\text{)}$$
$$x \leq 3.376 \quad \text{(to three decimal places)}$$

Alternatively, we can solve the inequality $0.7^x \geq 0.3$ directly as follows:

Note that $0 < 0.7 < 1$ and thus, as x decreases, $y = 0.7^x$ increases. Therefore the inequality holds for $x \leq \log_{0.7} 0.3$.

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Exponential graphs revisited

In Section 13C we graphed exponential functions, but often we could not find the x-axis intercept. Now that we have defined logarithms this can be done.

Example 26

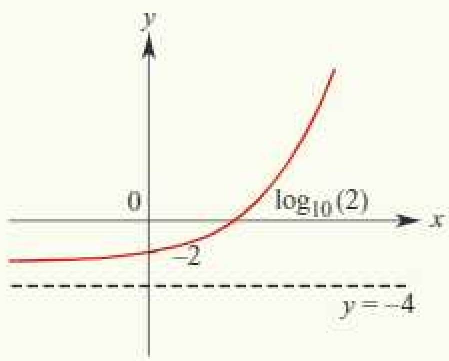
Sketch the graph of $f(x) = 2 \times 10^x - 4$, giving the equation of the asymptote and the axis intercepts.

Solution

- As $x \to -\infty$, $y \to -4^+$. The equation of the horizontal asymptote is $y = -4$.
- The graph crosses the y-axis when $x = 0$, and so the y-axis intercept is given by $f(0) = 2 \times 10^0 - 4 = 2 - 4 = -2$.
- The graph crosses the x-axis when $f(x) = 0$:

$$2 \times 10^x - 4 = 0$$
$$2 \times 10^x = 4$$
$$10^x = 2$$
$$x = \log_{10} 2 \approx 0.3010$$

(correct to four decimal places)



Summary 13F

- If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities.

  For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \geq 5 \Leftrightarrow x \geq \log_2 5$
  - $(0.3)^x = 5 \Leftrightarrow x = \log_{0.3} 5$
  - $(0.3)^x \geq 5 \Leftrightarrow x \leq \log_{0.3} 5$
- The x-axis intercepts of exponential graphs can now be found.

Skill-sheet

Exercise 13F

Example 23

1 Solve each of the following equations correct to two decimal places:

a $2^x = 7$ **b** $2^x = 0.4$ **c** $3^x = 14$

d $4^x = 3$ **e** $2^{-x} = 6$ **f** $0.3^x = 2$

Example 24

2 Solve each of the following equations correct to two decimal places:

a $5^{2x-1} = 90$ **b** $3^{x-1} = 10$ **c** $0.2^{x+1} = 0.6$

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Example 25 **3** Solve for x. Give values correct to two decimal places if necessary.

a $2^x > 8$ **b** $3^x < 5$ **c** $0.3^x > 4$ **d** $3^{x-1} \le 7$ **e** $0.4^x \le 0.3$

Example 26 **4** For each of the following, sketch the graph of $y = f(x)$, giving the equation of the asymptote and the axis intercepts:

a $f(x) = 2^x - 4$ **b** $f(x) = 2 \times 3^x - 6$ **c** $f(x) = 3 \times 10^x - 5$

d $f(x) = -2 \times 10^x + 4$ **e** $f(x) = -3 \times 2^x + 6$ **f** $f(x) = 5 \times 2^x - 6$

13G Graphs of logarithm functions

Graph of $y = \log_a x$ when $a > 1$

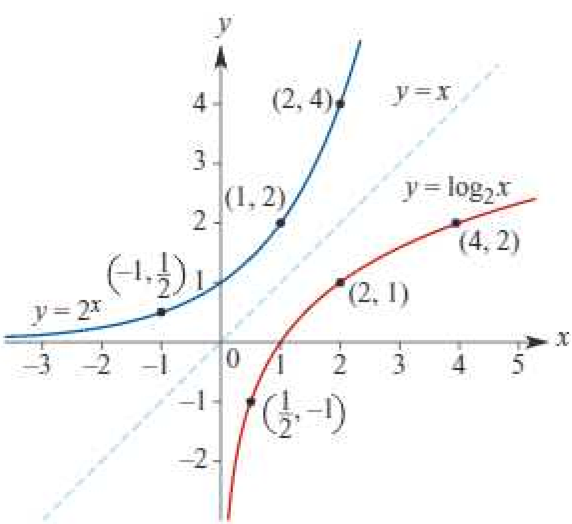
We look at the graphs of $y = \log_2 x$ and $y = \log_{10} x$. We note that $\log_{10} x = \log_{10} 2 \times \log_2 x$, and hence the graph of $y = \log_{10} x$ is the image of the graph of $y = \log_2 x$ under a dilation from the x-axis of factor $\log_{10} 2 \approx 0.3010$.

More generally, all graphs of the form $y = \log_a x$, where $a > 1$, are related to each other by dilations from the x-axis.

$y = \log_2 x$

A table of values for $y = \log_2 x$ is given below, and the graphs of $y = 2^x$ and $y = \log_2 x$ are drawn on the one set of axes.

x	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$y = \log_2 x$	−2	−1	0	1	2	3



Notes:

- The point $(1, 2)$ is on the graph of $y = 2^x$ and the point $(2, 1)$ is on the graph of $y = \log_2 x$.
- The point $(2, 4)$ is on the graph of $y = 2^x$ and the point $(4, 2)$ is on the graph of $y = \log_2 x$.
- The graph of $y = \log_2 x$ is the reflection of the graph of $y = 2^x$ in the line $y = x$.

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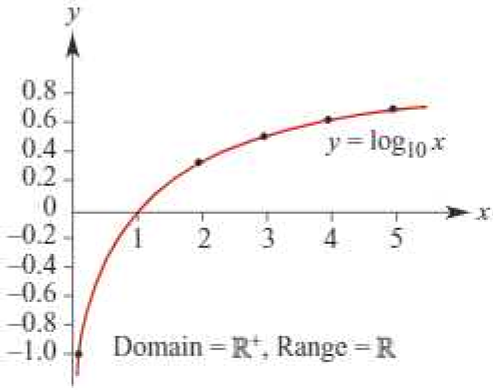
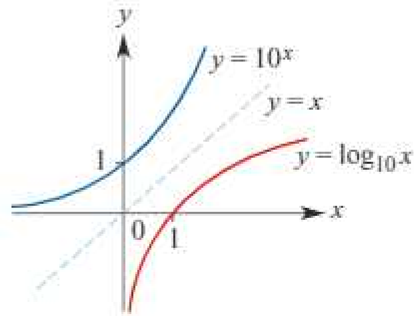
$y = \log_{10} x$

A table of values for $y = \log_{10} x$ is given below (the values are correct to two decimal places). Use your calculator to check these values.

x	0.1	1	2	3	4	5
$y = \log_{10} x$	−1	0	≈ 0.30	≈ 0.48	≈ 0.60	≈ 0.70



Domain = $\mathbb{R}^+$, Range = $\mathbb{R}$



The graph of $y = \log_{10} x$ is the reflection in the line $y = x$ of the graph of $y = 10^x$.

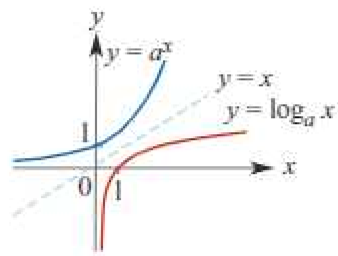
In general

Properties of $y = a^x$, $a > 1$

- domain = $\mathbb{R}$
- range = $\mathbb{R}^+$
- $a^0 = 1$
- as $x \to -\infty$, $y \to 0^+$
- $y = 0$ is an asymptote

Properties of $y = \log_a x$, $a > 1$

- domain = $\mathbb{R}^+$
- range = $\mathbb{R}$
- $\log_a 1 = 0$
- as $x \to 0^+$, $y \to -\infty$
- $x = 0$ is an asymptote

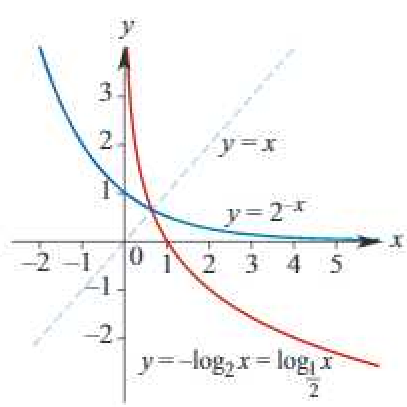


Graph of $y = \log_a x$ when $0 < a < 1$

Consider $y = \log_{\frac{1}{2}} x$. We observe the following:

$$\begin{aligned}\log_{\frac{1}{2}} x = y &\Leftrightarrow (\tfrac{1}{2})^y = x\\ &\Leftrightarrow 2^{-y} = x\\ &\Leftrightarrow \log_2 x = -y\\ &\Leftrightarrow y = -\log_2 x\end{aligned}$$

So we have $\log_{\frac{1}{2}} x = -\log_2 x$.



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Transformations of logarithm graphs

Transformations can be applied to the graphs of logarithm functions. This is shown in the following example.

Example 27

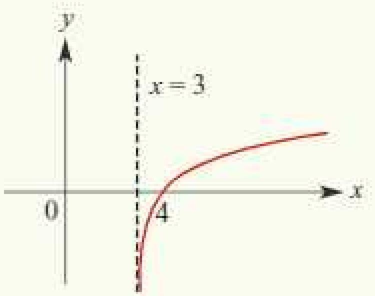
Sketch the graph of each of the following. Give the maximal domain, the equation of the asymptote and the axis intercepts.

a $f(x) = \log_2(x-3)$ **b** $f(x) = \log_2(x+2)$ **c** $f(x) = \log_2(3x)$

d $f(x) = -\log_2(x)$ **e** $f(x) = \log_2(-x)$

Solution

a $f(x) = \log_2(x-3)$



- The maximal domain is $(3, \infty)$.
- Equation of asymptote is $x = 3$.
- x-axis intercept: $\log_2(x-3) = 0$ implies $x - 3 = 2^0$, i.e. $x = 4$.

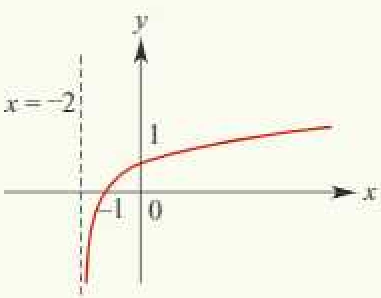
Explanation

The graph of $y = \log_2 x$ is translated 3 units in the positive direction of the x-axis.

Maximal domain: For $f(x) = \log_2(x-3)$ to be defined, we need $x - 3 > 0$, i.e. $x > 3$.

Asymptote: As $x \to 3^+$, $y \to -\infty$.

b $f(x) = \log_2(x+2)$



- The maximal domain is $(-2, \infty)$.
- Equation of asymptote is $x = -2$.
- y-axis intercept: $f(0) = \log_2(2) = 1$.
- x-axis intercept: $\log_2(x+2) = 0$ implies $x + 2 = 2^0$, i.e. $x = -1$.

The graph of $y = \log_2 x$ is translated 2 units in the negative direction of the x-axis.

Maximal domain: For $f(x) = \log_2(x+2)$ to be defined, we need $x + 2 > 0$, i.e. $x > -2$.

Asymptote: As $x \to -2^+$, $y \to -\infty$.

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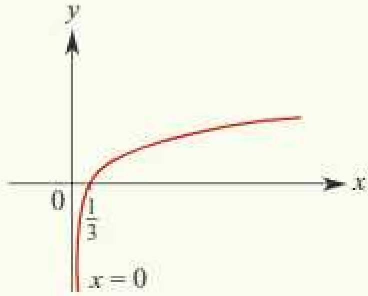
c $f(x) = \log_2(3x)$



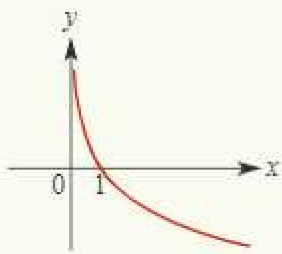
- The maximal domain is $(0, \infty)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $\log_2(3x) = 0$ implies $3x = 2^0$, i.e. $x = \frac{1}{3}$.

The graph of $y = \log_2 x$ is dilated by a factor of $\frac{1}{3}$ from the y-axis.

Maximal domain: For $f(x) = \log_2(3x)$ to be defined, we need $3x > 0$, i.e. $x > 0$.

Asymptote: As $x \to 0^+$, $y \to -\infty$.

d $f(x) = -\log_2(x)$



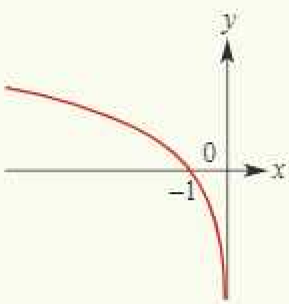
- The maximal domain is $(0, \infty)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $-\log_2(x) = 0$ implies $x = 2^0 = 1$.

The graph of $y = \log_2 x$ is reflected in the x-axis.

Maximal domain: For $f(x) = -\log_2(x)$ to be defined, we need $x > 0$.

Asymptote: As $x \to 0^+$, $y \to \infty$.

e $f(x) = \log_2(-x)$



- The maximal domain is $(-\infty, 0)$.
- Equation of asymptote is $x = 0$.
- x-axis intercept: $\log_2(-x) = 0$ implies $-x = 2^0$, i.e. $x = -1$.

The graph of $y = \log_2 x$ is reflected in the y-axis.

Maximal domain: For $f(x) = \log_2(-x)$ to be defined, we need $-x > 0$, i.e. $x < 0$.

Asymptote: As $x \to 0^-$, $y \to -\infty$.

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Inverses

The inverse of a one-to-one function was introduced in Section 5G.

- The inverse of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x$ is the function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_2 x$.
- The inverse of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 10^x$ is the function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_{10} x$.

Let $a \in \mathbb{R}^+ \setminus \{1\}$. The function with rule $y = \log_a x$, for $x > 0$, is the inverse of the function with rule $y = a^x$, for $x \in \mathbb{R}$.

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

Example 28

Find the inverse of:

a $f(x) = 10^{2x}$

b $g(x) = \log_{10}(2x)$

Solution

a Let $y = 10^{2x}$.

Interchanging x and y gives:

$$x = 10^{2y}$$
$$2y = \log_{10} x$$
$$y = \tfrac{1}{2}\log_{10} x$$

Thus $f^{-1}(x) = \frac{1}{2}\log_{10} x$.

b Let $y = \log_{10}(2x)$.

Interchanging x and y gives:

$$x = \log_{10}(2y)$$
$$10^x = 2y$$
$$y = \tfrac{1}{2} \times 10^x$$

Thus $g^{-1}(x) = \frac{1}{2} \times 10^x$.

Example 29

Find the rule for the inverse function of each of the following and state the domain of f^{-1}:

a $f(x) = \log_2(x - 2)$

b $f(x) = 5 \times 2^x + 3$

Solution

a Let $y = \log_2(x - 2)$.

Interchanging x and y gives:

$$x = \log_2(y - 2)$$
$$2^x = y - 2$$
$$y = 2^x + 2$$
$$\therefore \quad f^{-1}(x) = 2^x + 2$$

The domain of f^{-1} is $\mathbb{R}$.

b Let $y = 5 \times 2^x + 3$.

Interchanging x and y gives:

$$x = 5 \times 2^y + 3$$
$$\frac{x-3}{5} = 2^y$$
$$y = \log_2\left(\frac{x-3}{5}\right)$$
$$\therefore \quad f^{-1}(x) = \log_2\left(\frac{x-3}{5}\right)$$

The domain of f^{-1} is $(3, \infty)$.

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We observed in Section 13C that all graphs of the form $y = a^x$, where $a > 1$, are related to each other by dilations from the y-axis. This can be established by using the fact that $a^{\log_a x} = x$ for all $x > 0$.

For example, if $y = 2^x$, then

$$y = 2^x = (10^{\log_{10} 2})^x = 10^{(\log_{10} 2)x}$$

Therefore the graph of $y = 2^x$ is the image of the graph of $y = 10^x$ under a dilation of factor $\dfrac{1}{\log_{10} 2}$ from the y-axis.

Summary 13G

Let $a \in \mathbb{R}^+ \setminus \{1\}$. The functions $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ and $g: \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_a x$ are inverse functions. That is, $g = f^{-1}$.

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

Exercise 13G

Example 27 **1** Sketch the graph of each of the following. Give the maximal domain, the equation of the asymptote and the axis intercepts.

- **a** $y = \log_2(x - 4)$
- **b** $y = \log_2(x + 3)$
- **c** $y = \log_2(2x)$
- **d** $y = \log_2(x + 2)$
- **e** $y = \log_2\left(\dfrac{x}{3}\right)$
- **f** $y = \log_2(-2x)$
- **g** $y = -2\log_{10} x$
- **h** $y = -\log_{10}(-2x)$
- **i** $y = \log_{10}(\frac{1}{2}x)$

Example 28 **2** Determine the inverse of each of the following:

- **a** $y = 10^{0.5x}$
- **b** $y = 3\log_{10} x$
- **c** $y = 10^{3x}$
- **d** $y = 2\log_{10}(3x)$

Example 29 **3** Find the rule for the inverse function of each of the following:

- **a** $f(x) = 3^x + 2$
- **b** $f(x) = \log_2(x - 3)$
- **c** $f(x) = 4 \times 3^x + 2$
- **d** $f(x) = 5^x - 2$
- **e** $f(x) = \log_2(3x)$
- **f** $f(x) = \log_2\left(\dfrac{x}{3}\right)$
- **g** $f(x) = \log_2(x + 3)$
- **h** $f(x) = 5 \times 3^x - 2$

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4 Use a calculator to solve each of the following equations correct to two decimal places:

a $2^{-x} = x$

b $\log_{10}(x) + x = 0$

5 Use a calculator to plot the graphs of $y = \log_{10}(x^2)$ and $y = 2\log_{10} x$ for $x \in [-10, 10]$, $x \neq 0$.

6 On the same set of axes, plot the graphs of $y = \log_{10}(\sqrt{x})$ and $y = \frac{1}{2}\log_{10} x$ for $x \in (0, 10]$.

7 Use a calculator to plot the graphs of $y = \log_{10}(2x) + \log_{10}(3x)$ and $y = \log_{10}(6x^2)$.

13H Exponential models and applications

In this section we begin by looking at several situations that can be modelled by exponential functions, and then discuss an example where data is fitted by an exponential function.

Exponential growth and decay

In the following, we consider a variable A that is subject to exponential change.

Let A be the quantity at time t. Then

$$A = A_0 b^t$$

where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. Then after t minutes the number of cells, N, is given by the formula

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**. Here we are only dealing with the type of reproduction where the cell divides in two. For most known bacteria that can be cultured, generation times range from about 15 minutes to 1 hour.

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Example 30

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D = \dfrac{240}{\log_2 20} \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay so that the amount of radioactive material, A, present at time t (in years) is given by the formula

$$A = A_0 2^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

Example 31

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0 2^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$0.647A_0 = A_0 2^{-1000k}$$

$$0.647 = 2^{-1000k}$$

$$-1000k = \log_2 0.647$$

$$k = \frac{-\log_2 0.647}{1000} \approx 0.000628$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$A_0 2^{-kt} = \tfrac{1}{2}A_0$$

$$2^{-kt} = \tfrac{1}{2}$$

$$-kt = \log_2(\tfrac{1}{2})$$

$$-kt = -1$$

$$t = \frac{1}{k} \approx 1591.95$$

The half-life of radium-226 is approximately 1592 years.

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Population growth

It is sometimes possible to model population growth through exponential models.

Example 32

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$\begin{aligned} G &= 10R \\ G_0(1.11)^n &= 10 \times 10G_0(0.95)^n \\ (1.11)^n &= 100(0.95)^n \end{aligned}$$

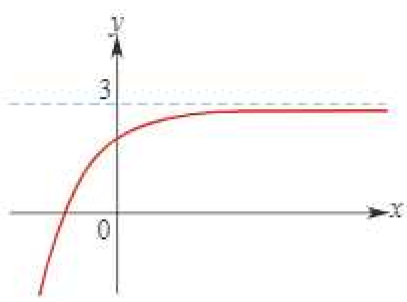
Taking $\log_{10}$ of both sides:

$$\begin{aligned} \log_{10}((1.11)^n) &= \log_{10}(100(0.95)^n) \\ n\log_{10} 1.11 &= \log_{10} 100 + n\log_{10} 0.95 \\ n\log_{10} 1.11 &= 2 + n\log_{10} 0.95 \\ n &= \frac{2}{\log_{10} 1.11 - \log_{10} 0.95} \approx 29.6 \end{aligned}$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

Limiting values of exponential functions

Consider the exponential function $f(x) = 3 - 2^{-x}$. We can see from the graph that, as x increases, the value of $f(x)$ gets closer and closer to 3.



In general, for an exponential function f, if there is a real number ℓ such that, as $x \to \infty$, $f(x) \to \ell$, then we say that ℓ is the **limiting value** of f.

There are many applications of exponential functions where the limiting value is significant. For example, if an object is dropped in a liquid, then a possible formula for its speed, v m/s, at time t seconds could be

$$v = 0.5(1 - 3^{-0.4t})$$

It is easily seen that, as $t \to \infty$, $v \to 0.5$. The object's speed gradually approaches 0.5 m/s, but does not ever reach 0.5 m/s.

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Example 33

An ice pack is taken out of a freezer, which is kept at $-20°C$. The temperature, $T°C$, of the ice pack at time t minutes is given by

$$T = -50 \times (0.98)^t + 30$$

a Determine the limiting value of T as $t \to \infty$. What does this value represent?

b Determine the value of t for which $T = 0$.

c Sketch the graph of T against t.

d Comment on this model.

Solution

a As $t \to \infty$, we have

$$0.98^t \to 0$$
$$\therefore \quad T \to 30$$

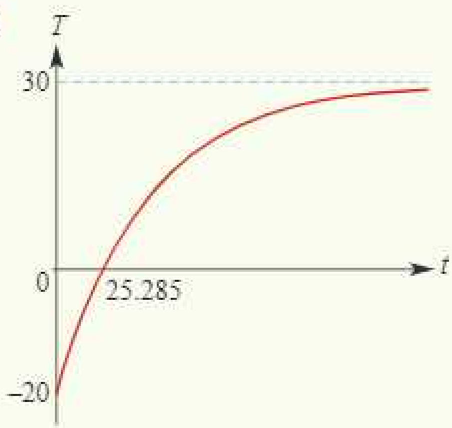
The limiting value of T is 30.
The outside temperature is 30°C.

b $-50 \times (0.98)^t + 30 = 0$

$$0.98^t = 0.6$$
$$t \approx 25.285$$

The temperature of the ice pack is 0°C after approximately 25 minutes.

c



d According to the model, the ice pack approaches but never reaches the outside temperature. The temperature of the ice pack passes 29.9°C after about 5 hours. You would probably consider that it has reached outside temperature by this stage.

Determining exponential rules

We have looked at determining rules for functions in Chapters 2 to 6. We look at one very useful case for exponential functions.

Example 34

The points $(1, 6)$ and $(5, 96)$ are known to lie on the curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Find the values of a and b.

Solution

We can write

$$a \times b^1 = 6 \qquad (1)$$
$$a \times b^5 = 96 \qquad (2)$$

Dividing equation (2) by equation (1) gives $b^4 = 16$. Thus $b = 16^{\frac{1}{4}} = 2$, and substituting into equation (1) gives $a = 3$.

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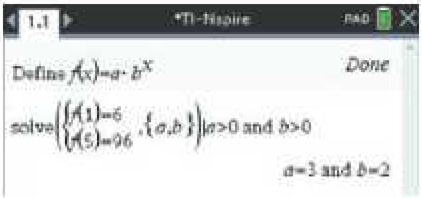
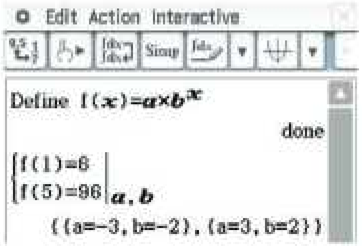
Using the TI-Nspire

- Define $f(x) = a \times b^x$.
- Solve for $a > 0$ and $b > 0$ using the simultaneous equations template with $f(1) = 6$ and $f(5) = 96$ as shown.



Using the Casio ClassPad

- Define $f(x) = a \times b^x$.
- Solve the simultaneous equations $f(1) = 6$ and $f(5) = 96$ for $a > 0$ and $b > 0$ as shown.

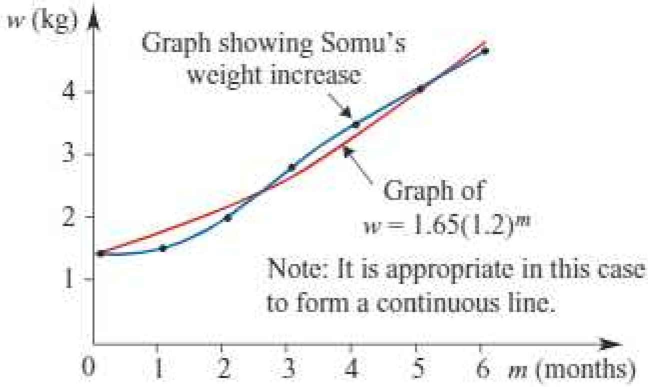


Fitting data

As an example, we consider the increase in weight of Somu, an orangutan born at the Eastern Plains Zoo. The following table shows Somu's weight for the first 6 months.

Months, m	0	1	2	3	4	5	6
Weight (kg), w	1.65	1.7	2.2	3.0	3.7	4.2	4.8

These data values are plotted on the same set of axes as the graph of the exponential function $w = 1.65(1.2)^m$, $0 \le m \le 6$.



We can see from the graph that the exponential model $w = 1.65(1.2)^m$ approximates the actual weight gain, and would be a useful model to predict weight gains for any future orangutan births at the zoo. This model describes a growth rate of 20% per month for the first 6 months.

We can use a CAS calculator to fit an exponential model to data values.

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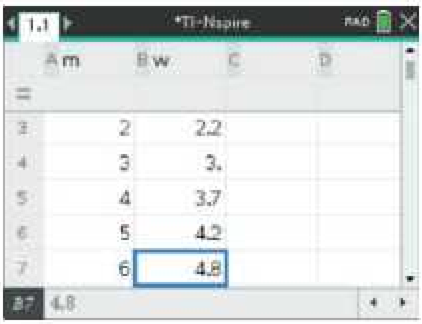
Example 35

The table below shows the increase in weight of Somu, an orangutan born at the Eastern Plains Zoo. Using a CAS calculator, find an exponential function to model Somu's weight.

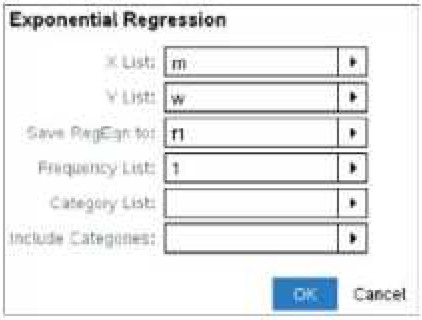
Months, m	0	1	2	3	4	5	6
Weight (kg), w	1.65	1.7	2.2	3.0	3.7	4.2	4.8

Using the TI-Nspire

- Enter the data either in a **Calculator** application as lists or in a **Lists & Spreadsheet** application as shown.



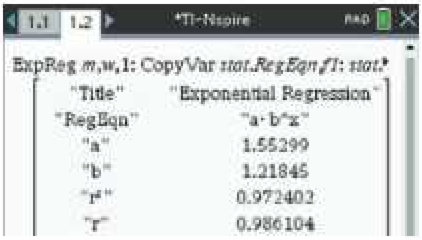
- Insert a **Calculator** page and use (menu) > **Statistics** > **Stat Calculations** > **Exponential Regression**. Complete as shown:
  - Use (tab) to move between fields.
  - Use the selection tool to open a field. Then use the arrows ▲ and ▼ to choose the correct entry, and select this entry using the selection tool.



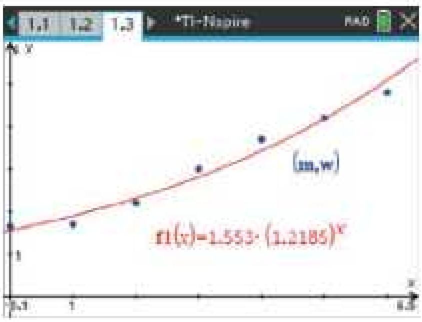
- This now gives the values of a and b, and the function has been stored as $f1$.
- Hence the exponential function

$$w = 1.55(1.22)^m$$

can be used to model Somu's weight.



- The curve can be shown in a **Graphs** application together with the scatter plot ((menu) > **Graph Type** > **Scatter Plot**) using an appropriate window ((menu) > **Window/Zoom**).



Note: Alternatively, the scatter plot and regression curve can be obtained using the **Data & Statistics** application.

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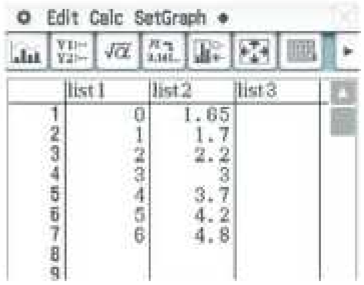
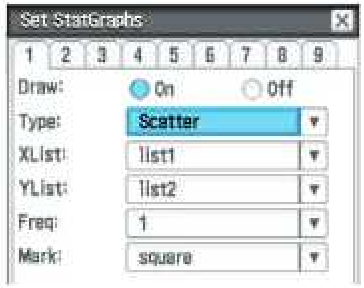
Using the Casio ClassPad

- In **Statistics**, enter the data in list1 and list2 as shown.



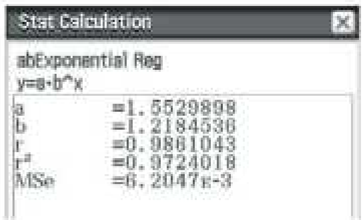
- Select the graph icon and ensure that you set the graph type to Scatter and the lists to list1 and list2. Tap Set.
- Tap to view the scatter plot.



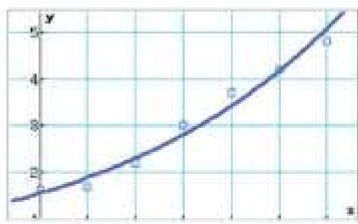
- Select **Calc** > **Regression** > **abExponential Reg**. Tap OK to confirm the settings.
- Hence the exponential function

$$w = 1.55(1.22)^m$$

can be used to model Somu's weight.



- Tap OK to view the graph of the regression curve.



Summary 13H

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then $A = A_0 b^t$, where A_0 is the initial quantity and b is a positive constant.

If $b > 1$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $b < 1$, the model represents **decay**:

- radioactive decay
- cooling of materials

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Skill-sheet

Exercise 13H

Example 30

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

Example 31

2 The half-life of plutonium-239 is 24 000 years. If 10 grams are present now, how long will it take until only 10% of the original sample remains? (Give your answer to the nearest year.)

3 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

4 The pressure, P, in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 5 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 400 millibars. (Give your answer to the nearest metre.)

5 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

6 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 10^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

7 Iodine-131 is a radioactive isotope used in the treatment of the thyroid gland. It decays so that, after t days, 1 unit of the isotope is reduced to 0.9174^t units. How many days does it take for the amount to fall to less than 0.2 units?

Example 32

8 The populations (in millions), p and q, of two neighbouring American states, P and Q, over a period of 50 years from 1950 are modelled by functions $p = 1.2 \times 2^{0.08t}$ and $q = 1.7 \times 2^{0.04t}$, where t is the number of years since 1950.

a Plot the graphs of both functions using a calculator.

b Find when the population of state P is:

i equal to the population of state Q

ii twice the population of state Q.

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9 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount S kg of undissolved sugar remaining is given by $S = 5 \times 10^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

Example 33 **10** A tub of ice-cream is taken out of a freezer, which is kept at -15°C. The temperature, T°C, of the ice-cream at time t minutes is given by

$$T = -40 \times (0.98)^t + 25$$

a Determine the limiting value of T as $t \to \infty$. What does this value represent?

b Determine the value of t for which $T = 0$.

c Sketch the graph of T against t.

d Comment on this model.

11 An object is falling such that its speed, V m/s, at time t seconds is given by

$$V = 80(1 - 3^{-0.4t})$$

a Determine the limiting speed, V_∞ m/s, as $t \to \infty$.

b When does the object have speed 40 m/s?

c Sketch the graph of V against t.

12 A company begins an advertising campaign for a new perfume. The percentage, $p(t)$%, of the target market that buys the perfume after t days of the advertising campaign is modelled by

$$p(t) = 100(1 - 3^{-0.05t}), \quad t \geq 1$$

a What is the limiting value of $p(t)$ as $t \to \infty$?

b How many days does the advertising campaign need to run to have 75% of the target market buy the perfume?

c Sketch the graph of $p(t)$ against t.

Example 34 **13** Each of the following pairs of points is known to lie on a curve $y = a \times b^x$, where $a > 0$ and $b > 0$. Find the values of a and b in each case.

a $(1, 15)$ and $(4, 1875)$ **b** $(2, 1)$ and $\left(5, \frac{1}{8}\right)$ **c** $\left(1, \frac{15}{2}\right)$ and $\left(\frac{1}{2}, \frac{5\sqrt{6}}{2}\right)$

14 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and five hours later there were 10 000 bacteria.

a Find the values of a and b.

b Find, to the nearest minute, when there were 5000 bacteria.

c Find, to the nearest minute, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

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15 Find a and k such that the graph of $y = a10^{kx}$ passes through the points $(2, 6)$ and $(5, 20)$.

Example 35 **16** Find an exponential model of the form $y = ab^x$ to fit the following data:

x	0	2	4	5	10
y	1.5	0.5	0.17	0.09	0.006

17 Find an exponential model of the form $p = ab^t$ to fit the following data:

t	0	2	4	6	8
p	2.5	4.56	8.3	15.12	27.56

18 A sheet of paper 0.2 mm thick is cut in half, and one piece is stacked on top of the other.

a If this process is repeated, complete the following table:

Cuts, n	Sheets	Total thickness, T (mm)
0	1	0.2
1	2	0.4
2	4	0.8
3	8	
⋮	⋮	⋮
10		

b Write down a formula which shows the relationship between T and n.

c Draw a graph of T against n for $n \leq 10$.

d What would be the total thickness, T, after 30 cuts?

19 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$, where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

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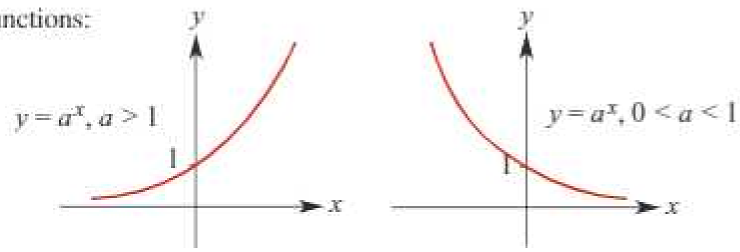
Chapter summary

- **Index laws**
  - To **multiply** two powers with the same base, **add** the indices: $a^m \times a^n = a^{m+n}$
  - To **divide** two powers with the same base, **subtract** the indices: $a^m \div a^n = a^{m-n}$
  - To raise a power to another power, **multiply** the indices: $(a^m)^n = a^{m \times n}$
- Rational indices:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{and} \quad a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m$$

- Graphs of exponential functions:



- For $a \in \mathbb{R}^+ \setminus \{1\}$:

$$\text{if } a^x = a^y, \text{ then } x = y$$

- For $a \in \mathbb{R}^+ \setminus \{1\}$:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- **Laws of logarithms**

  1 $\log_a(mn) = \log_a m + \log_a n$

  2 $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$

  3 $\log_a(m^p) = p \log_a m$

  4 $\log_a(m^{-1}) = -\log_a m$

  5 $\log_a 1 = 0$ and $\log_a a = 1$

Technology-free questions

1 Simplify each of the following, expressing your answer with positive index:

a $\frac{a^6}{a^2}$ **b** $\frac{b^8}{b^{10}}$ **c** $\frac{m^3n^4}{m^5n^6}$ **d** $\frac{a^3b^2}{(ab^2)^4}$

e $\frac{6a^8}{4a^2}$ **f** $\frac{10a^7}{6a^9}$ **g** $\frac{8(a^3)^2}{(2a)^3}$ **h** $\frac{m^{-1}n^2}{(mn^{-2})^3}$

i $(p^{-1}q^{-2})^2$ **j** $\frac{(2a^{-4})^3}{5a^{-1}}$ **k** $\frac{6a^{-1}}{3a^{-2}}$ **l** $\frac{a^4 + a^8}{a^2}$

2 Use logarithms to solve each of the following equations:

a $2^x = 7$ **b** $2^{2x} = 7$ **c** $10^x = 2$ **d** $10^x = 3.6$

e $10^x = 110$ **f** $10^x = 1010$ **g** $2^{5x} = 100$ **h** $2^x = 0.1$

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3 Evaluate each of the following:

a $\log_2 4$ **b** $\log_3 27$ **c** $\log_4 64$ **d** $\log_2\left(\frac{1}{2}\right)$

4 For each of the following, give an equivalent expression in simplest form:

a $\log_a x + \log_a y$ **b** $\log_2 x + \log_2(x+3)$ **c** $\log_b(2x) - \log_b(3y)$

d $3\log_a 4 - \log_a 8$ **e** $\log_3\left(\frac{1}{9}\right)$ **f** $\log_3(x^2) + 4\log_3(x)$

5 Solve each of the following equations for x:

a $2^{2x-1} = 16$ **b** $3^{5x-2} = 27$ **c** $2^{1-x} = 8$

d $2^{x+2} = \frac{1}{4}$ **e** $3^{3x-8} = 1$ **f** $5^{2x+1} = \frac{1}{5}$

6 Solve for x:

a $16^x = 64$ **b** $8^x = 32$ **c** $27^x = 81$ **d** $25^x = 5$

7 Solve for x:

a $\log_3 x = 2$ **b** $\log_2 x = 3$ **c** $\log_x 16 = 4$ **d** $\log_5(x-1) = 2$

8 Evaluate each of the following:

a $\log_2 64$ **b** $\log_{10}(10^7)$ **c** $\log_a(a^2)$ **d** $\log_4 1$

e $\log_3 27$ **f** $\log_2\left(\frac{1}{4}\right)$ **g** $\log_{10} 0.001$ **h** $\log_2 16$

9 Express each of the following as a single logarithm:

a $\log_{10} 2 + \log_{10} 3$ **b** $\log_{10} 4 + 2\log_{10} 3 - \log_{10} 6$

c $2\log_{10} a - \log_{10} b$ **d** $2\log_{10} a - 3 - \log_{10} 25$

e $\log_{10} x + \log_{10} y - \log_{10} x$ **f** $2\log_{10} a + 3\log_{10} b - \log_{10} c$

10 Solve each of the following for x:

a $3^x(3^x - 27) = 0$ **b** $(2^x - 8)(2^x - 1) = 0$

c $2^{2x} - 2^{x+1} = 0$ **d** $2^{2x} - 12 \times 2^x + 32 = 0$

11 Sketch the graph of:

a $y = 2 \times 2^x$ **b** $y = -3 \times 2^x$ **c** $y = 5 \times 2^{-x}$

d $y = 2^{-x} + 1$ **e** $y = 2^x - 1$ **f** $y = 2^x + 2$

12 Sketch the graph of each of the following:

a $y = \log_3(x+2)$ **b** $y = \log_3(4-x)$ **c** $y = -\log_3(2x)$

13 Solve for x:

a $\log_2(x-2) + \log_2(x-4) = 3$ **b** $\log_2(x+1) + \log_2(x-1) = 4$

c $\log_{29} x = 1 - \log_{29}(x-0.4)$

14 Solve for x:

a $2^{2x} - 6 \times 2^x + 8 = 0$ **b** $2 \times 3^{2x+1} + 3^x - 1 = 0$

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15 Solve the equation $\log_{10} x + \log_{10}(2x) - \log_{10}(x+1) = 0$.

16 Given $3^x = 4^y = 12^z$, show that $z = \dfrac{xy}{x+y}$.

17 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

18 **a** Given that $\log_p 7 + \log_p k = 0$, find k.
b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

19 Solve:
a $2 \times 4^{a+1} = 16^{2a}$ for a **b** $\log_2(y^2) = 4 + \log_2(y+5)$ for y

Multiple-choice questions

1 $8x^3 \div 4x^{-3} =$
A 2 **B** $2x^0$ **C** $2x^6$ **D** $2x^{-1}$ **E** $\dfrac{2}{x^9}$

2 The expression $\dfrac{a^2b}{(2ab^2)^3} \div \dfrac{ab}{16a^0}$ simplifies to
A $\dfrac{2}{a^2b^6}$ **B** $\dfrac{2a^2}{b^6}$ **C** $2a^2b^6$ **D** $\dfrac{2}{ab^6}$ **E** $\dfrac{1}{128ab^5}$

3 The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 3 \times 2^x - 1$ has range
A $\mathbb{R}$ **B** $\mathbb{R} \setminus \{-1\}$ **C** $(-1, \infty)$ **D** $(1, \infty)$ **E** $[1, \infty)$

4 The function $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_2(3x)$ has an inverse function f^{-1}. The rule for f^{-1} is given by
A $f^{-1}(x) = 2^x$ **B** $f^{-1}(x) = 3^x$ **C** $f^{-1}(x) = \dfrac{2^x}{3}$
D $f^{-1}(x) = 2^{\frac{x}{3}}$ **E** $f^{-1}(x) = \log_2\left(\dfrac{x}{3}\right)$

5 If $\log_{10}(x-2) - 3\log_{10}(2x) = 1 - \log_{10} y$, then y is equal to
A $\dfrac{80x^3}{x-2}$ **B** $1 + \dfrac{8x^3}{x-2}$ **C** $\dfrac{60x}{x-2}$ **D** $1 + \dfrac{6x}{x-2}$ **E** $1 - \dfrac{x-2}{8x^3}$

6 The solution of the equation $5 \times 2^{5x} = 10$ is x equals
A $\dfrac{1}{2}$ **B** $\dfrac{1}{5}$ **C** $\dfrac{1}{5}\log_2 10$ **D** $\dfrac{1}{2}\log_2 5$ **E** $\dfrac{1}{5} \times 2^5$

7 The equation of the asymptote of $y = 3\log_2(5x) + 2$ is
A $x = 0$ **B** $x = 2$ **C** $x = 3$ **D** $x = 5$ **E** $y = 2$

8 Which one of the following functions has a graph with a vertical asymptote with equation $x = b$?
A $y = \log_2(x-b)$ **B** $y = \dfrac{1}{x+b}$ **C** $y = \dfrac{1}{x+b} - b$
D $y = 2^x + b$ **E** $y = 2^{x-b}$

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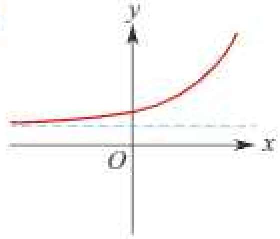
9 Which of the following graphs could be the graph of the function $f(x) = 2^{ax} + b$, where a and b are positive?

A



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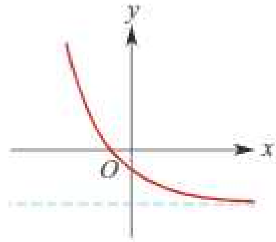

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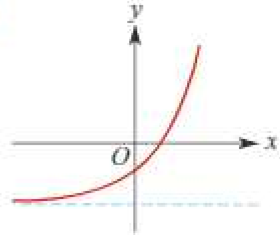

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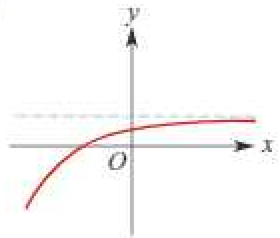
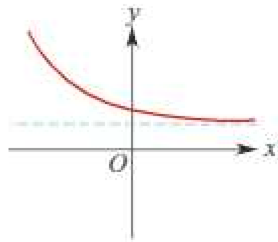


E

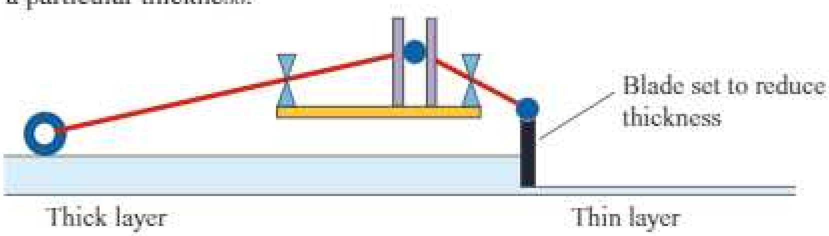


10 The expression $\dfrac{2mh}{(3mh^2)^3} \div \dfrac{mh}{81m^2}$ is equal to

A $\dfrac{6}{mh^6}$ **B** $\dfrac{6m^2}{h^6}$ **C** $6m^2h^6$ **D** $\dfrac{6}{m^2h^6}$ **E** $\dfrac{1}{128mh^5}$

Extended-response questions

1 Research is being carried out to investigate the durability of paints of different thicknesses. The automatic machine shown in the diagram is proposed for producing a coat of paint of a particular thickness.



The paint is spread over a plate and a blade sweeps over the plate reducing the thickness of the paint. The process involves the blade moving at three different speeds.

a Operating at the initial setting, the blade reduces the paint thickness to one-eighth of the original thickness. This happens n times. What fraction of the paint thickness remains? Express this as a power of $\frac{1}{2}$.

b The blade is then reset so that it removes three-quarters of the remaining paint. This happens $n - 1$ times. At the end of this second stage, express the remaining thickness as a power of $\frac{1}{2}$.

c The third phase of the process involves the blade being reset to remove half of the remaining paint. This happens $n - 3$ times. At what value of n would the machine have to be set to reduce a film of paint 8192 units thick to 1 unit thick?

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2 A hermit has little opportunity to replenish supplies of tea and so, to eke out supplies for as long as possible, he dries out the tea leaves after use and then stores the dried tea in an airtight box. He estimates that after each re-use of the leaves the amount of tannin in the used tea will be half the previous amount. He also estimates that the amount of caffeine in the used tea will be one-quarter of the previous amount.
The information on the label of the tea packet states that the tea contains 729 mg of caffeine and 128 mg of tannin.

- **a** Write down expressions for the amount of caffeine when the tea leaves are re-used for the first, second, third and nth times.
- **b** Do the same for the amount of tannin remaining.
- **c** Find the number of times he can re-use the tea leaves if a 'tea' containing more than three times as much tannin as caffeine is undrinkable.

3 A new type of red synthetic carpet was produced in two batches. The first batch had a brightness of 15 units and the second batch 20 units. After a period of time it was discovered that the first batch was losing its brightness at the rate of 5% per year while the second was losing brightness at the rate of 6% per year.

- **a** Write down expressions for the brightness of each batch after n years.
- **b** A person bought some carpet from the first batch when it was a year old and some new carpet from the second batch. How long would it be before the brightness of the two carpets was the same?

4 SWG

- The value of shares in Company X increased linearly over a two-year period according to the model $x = 0.8 + 0.17t$, where t is the number of months from the beginning of January 2019 and $\$x$ is the value of the shares at time t.
- The value of shares in Company Y increased over the same period of time according to the model $y = 100^{0.015t}$, where $\$y$ is the value of these shares at time t months.
- The value of shares in Company Z increased over the same period according to the model $z = 1.7\log_{10}(5(t+1))$, where $\$z$ is the value of the shares at time t months.

Use a calculator to sketch the graphs of the three functions on the one screen.

- **a** Find the values of the shares in each of the three companies at the end of June 2019.
- **b** Find the values of the shares in the three companies at the end of September 2020.
- **c** During which months were shares in Company X more valuable than shares in Company Y?
- **d** For how long and during which months were the shares in Company X the most valuable?

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5 In an African game park, it was estimated that there were approximately 700 wildebeest and that their population was increasing at the rate of 3% per year. At the same time, in the park there were approximately 1850 zebras and their population was decreasing at the rate of 4% per year. Use a calculator to plot the graphs of both functions.

CAS

a After how many years were there more wildebeest than zebras in the park?

b It is also estimated that there were 1000 antelope and their numbers were increasing by 50 per year. After how many years were there more antelope than zebras?

6 Students conducting a science experiment on cooling rates measure the temperature of a beaker of liquid over a period of time. The following measurements were taken.

Time (minutes)	3	6	9	12	15	18	21
Temperature (°C)	71.5	59	49	45.5	34	28	23.5

a Find an exponential model to fit the data collected.

b Use this model to estimate:

i the initial temperature of the liquid **ii** the temperature of the liquid at $t = 25$.

It is suspected that one of the temperature readings was incorrect.

c Re-calculate the model to fit the data, omitting the incorrect reading.

d Use the new model to estimate:

i the initial temperature of the liquid **ii** the temperature of the liquid at $t = 12$.

e If the room temperature is 15°C, find the approximate time at which the cooling of the liquid ceased.

7 The graph of $y = ab^x$ passes through the points $(1, 1)$ and $(2, 5)$.

a Find the values of a and b.

b Let $b^x = 10^z$.

i Take logarithms of both sides (base 10) to find an expression for z in terms of x.

ii Find the values of k and a such that the graph of $y = a10^{kx}$ passes through the points $(1, 1)$ and $(2, 5)$.

8 **a** Find an exponential model of the form $y = a \cdot b^x$ to fit the following data:

CAS

x	0	2	4	5	10
y	2	5	13	20	200

b Express the model you have found in part **a** in the form $y = a10^{kx}$.

c Hence find an expression for x in terms of y.

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