7

Further functions

Objectives

- To graph **power functions** with rational non-integer index.
- To review and extend our study of all the functions of Mathematical Methods by revisiting:
  - sums, differences and products of functions
  - addition of ordinates
  - one-to-one functions, strictly increasing functions, strictly decreasing functions, odd functions and even functions
  - compositions of functions
  - inverse functions
  - transformations of functions.
- To use **functional equations** to describe properties of functions.
- To use parameters to describe **families of functions**.

In Chapter 1, we introduced:

- operations on functions, including addition, multiplication, composition and inverse
- properties of functions, including one-to-one, strictly increasing, strictly decreasing, odd and even.

In this chapter, we revisit these concepts with all the functions of Mathematical Methods at our disposal: power functions, polynomial functions, exponential and logarithmic functions, and circular functions.

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7A More power functions

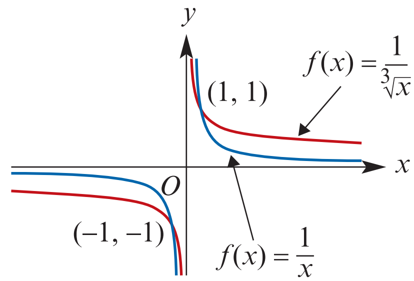
In Chapter 1 we looked at power functions of the form $f(x) = x^n$, $f(x) = x^{-n}$ and $f(x) = x^{\frac{1}{n}}$, where n is a positive integer. In Chapter 3 we looked at transformations of these functions. Here we briefly consider some other power functions to complete our collection.

The function $f(x) = x^{-\frac{1}{n}}$ where n is a positive integer

We can write $x^{-\frac{1}{n}} = \dfrac{1}{x^{\frac{1}{n}}} = \dfrac{1}{\sqrt[n]{x}}$.

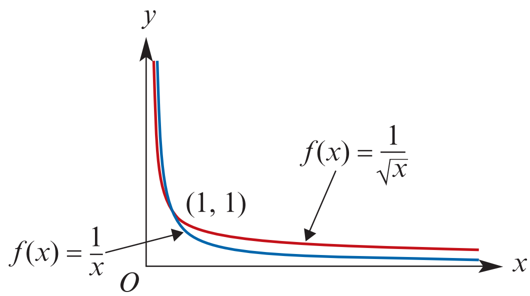
For the function f with rule $f(x) = \dfrac{1}{\sqrt[n]{x}}$:

- the maximal domain is $\mathbb{R} \setminus \{0\}$ if n is odd
- the maximal domain is $\mathbb{R}^+$ if n is even.



The first diagram shows the graphs of $y = \dfrac{1}{\sqrt[3]{x}}$ and $y = \dfrac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$.

The second diagram shows the graphs of $y = \dfrac{1}{\sqrt{x}}$ and $y = \dfrac{1}{x}$ for $x \in \mathbb{R}^+$.



Each graph has a horizontal asymptote with equation $y = 0$ and a vertical asymptote with equation $x = 0$.

If n is an odd positive integer, then $f(x) = x^{-\frac{1}{n}}$ is an odd function, since $f(-x) = -f(x)$.

Example 1

For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the range corresponding to this domain. Also state whether the function is odd, even or neither.

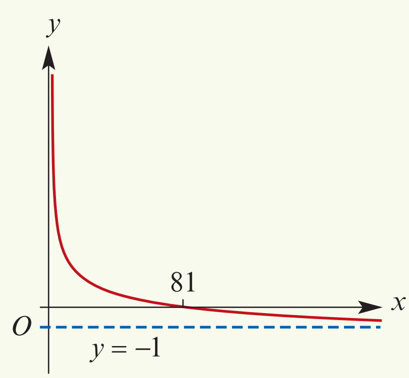
a $f(x) = 3x^{-\frac{1}{4}} - 1$ **b** $f(x) = 6x^{-\frac{1}{5}} + 1$

Solution

a The maximal domain is $\mathbb{R}^+$.
The range is $(-1, \infty)$.
The function is neither odd nor even.

To find the x-axis intercept:

$$\begin{aligned} 3x^{-\frac{1}{4}} - 1 &= 0 \\ x^{-\frac{1}{4}} &= \frac{1}{3} \\ x^{\frac{1}{4}} &= 3 \\ \therefore \quad x &= 3^4 = 81 \end{aligned}$$



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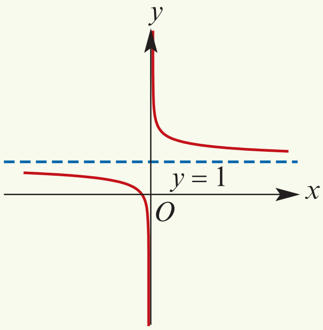
b $f(x) = 6x^{-\frac{1}{5}} + 1$

The maximal domain is $\mathbb{R} \setminus \{0\}$.
The range is $\mathbb{R} \setminus \{1\}$.
The function is neither odd nor even.

The line $x = 0$ is a vertical asymptote.
The line $y = 1$ is a horizontal asymptote.



The function $f(x) = x^{\frac{p}{q}}$ where p and q are positive integers

The special case where $p = 1$ has been considered in Chapter 1.

The expression $x^{\frac{p}{q}}$, where p and q are positive integers, can be defined as

$$x^{\frac{p}{q}} = \left(x^{\frac{1}{q}}\right)^p = (\sqrt[q]{x})^p$$

To employ this definition we will always first write the fractional power in simplest form.

For the function f with rule $f(x) = x^{\frac{p}{q}}$:

- the maximal domain is $\mathbb{R}$ if q is odd
- the maximal domain is $\mathbb{R}^+ \cup \{0\}$ if q is even.

Here are some examples of evaluating such functions:

$$8^{\frac{2}{3}} = \left(8^{\frac{1}{3}}\right)^2 = 2^2 = 4 \qquad (-8)^{\frac{2}{3}} = \left((-8)^{\frac{1}{3}}\right)^2 = (-2)^2 = 4$$

$$10\,000^{\frac{3}{4}} = \left(10\,000^{\frac{1}{4}}\right)^3 = 10^3 = 1000 \qquad 0.0001^{\frac{3}{4}} = \left(0.0001^{\frac{1}{4}}\right)^3 = 0.1^3 = 0.001$$

An investigation of these graphs with your calculator is worthwhile. Not every case will be illustrated here.

Example 2

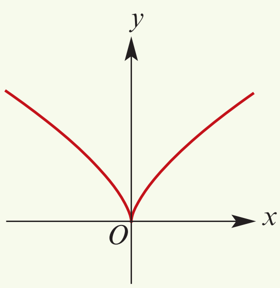
For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the range corresponding to this domain. Also state whether the function is odd, even or neither.

a $f(x) = x^{\frac{2}{3}}$ **b** $f(x) = x^{\frac{3}{2}}$

Solution

a $f(x) = x^{\frac{2}{3}}$

The maximal domain is $\mathbb{R}$.
The range is $[0, \infty)$.
The function $f(x) = x^{\frac{2}{3}}$ is even, since $f(-x) = f(x)$.



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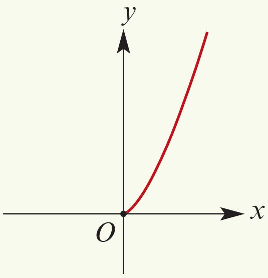
b $f(x) = x^{\frac{3}{2}}$

The maximal domain is $[0, \infty)$.

The range is $[0, \infty)$.

The function $f(x) = x^{\frac{3}{2}}$ is neither odd nor even.



Summary 7A

The power function $f(x) = x^{-\frac{1}{n}}$ where n is a positive integer

- The rule can also be written as $f(x) = \dfrac{1}{\sqrt[n]{x}}$.
- The maximal domain of f is:
  - $\mathbb{R} \setminus \{0\}$ if n is odd
  - $\mathbb{R}^+$ if n is even.
- If n is odd, then f is an odd function, since $f(-x) = -f(x)$.

The power function $f(x) = x^{\frac{p}{q}}$ where p and q are positive integers

- The expression $x^{\frac{p}{q}}$ is defined as $(\sqrt[q]{x})^p$, provided the fraction $\frac{p}{q}$ is in simplest form.
- The maximal domain of f is:
  - $\mathbb{R}$ if q is odd
  - $\mathbb{R}^+ \cup \{0\}$ if q is even.

Exercise 7A

Example 1

1 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = 2x^{-\frac{1}{4}} - 1$ **b** $f(x) = 3x^{-\frac{1}{5}} - 1$

2 Evaluate each of the following:

a $32^{\frac{2}{5}}$ **b** $(-32)^{\frac{2}{5}}$ **c** $32^{\frac{3}{5}}$

d $(-32)^{\frac{3}{5}}$ **e** $(-8)^{\frac{5}{3}}$ **f** $(-27)^{\frac{4}{3}}$

Example 2

3 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = x^{\frac{3}{5}}$ **b** $f(x) = x^{\frac{3}{4}}$

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4 For each of the following rules of functions:

i state the maximal domain of the function, the corresponding range and the equations of any asymptotes

ii sketch the graph using your calculator for assistance.

a $f(x) = \dfrac{1}{\sqrt{x}}$ **b** $f(x) = x^{\frac{5}{3}}$ **c** $f(x) = -x^{\frac{3}{5}}$

d $f(x) = \dfrac{8}{\sqrt[3]{x}}$ **e** $f(x) = \dfrac{4}{\sqrt[5]{x}}$ **f** $f(x) = x^{\frac{7}{5}}$

5 **a** Find $\left\{x : x^{\frac{3}{2}} > x^2\right\}$. **b** Find $\left\{x : x^{\frac{3}{2}} < x^{-2}\right\}$.

6 For each of the following, state whether the function is odd, even or neither:

a $f(x) = \dfrac{1}{x}$ **b** $f(x) = \dfrac{1}{x^2}$ **c** $f(x) = \sqrt[3]{x}$

d $f(x) = \dfrac{1}{\sqrt[3]{x}}$ **e** $f(x) = x^{\frac{2}{3}}$ **f** $f(x) = x^{\frac{5}{7}}$

7B Composite and inverse functions

In the previous chapters, we have considered compositions and inverses for different families of functions. In this section, we revisit these two concepts using all the functions of Mathematical Methods.

We recall the following from Chapter 1.

Composition of functions The composition of g with f is written $g \circ f$ (read 'composition of f followed by g') and the rule for the composite function is $g \circ f(x) = g(f(x))$.

If $\operatorname{ran} f \subseteq \operatorname{dom} g$, then the composition $g \circ f$ is defined and $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Inverse functions If f is a one-to-one function, then a new function f^{-1}, called the inverse of f, may be defined by

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f \text{ and } y \in \operatorname{dom} f$$

Example 3

Express each of the following as the composition of two functions:

a $h(x) = e^{x^2}$ **b** $h(x) = \sin(x^2)$ **c** $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$

Solution

a $h(x) = e^{x^2}$
Choose $f(x) = x^2$
and $g(x) = e^x$.
Then $h(x) = g \circ f(x)$.

b $h(x) = \sin(x^2)$
Choose $f(x) = x^2$
and $g(x) = \sin x$.
Then $h(x) = g \circ f(x)$.

c $h(x) = (x^2 - 2)^n, \quad n \in \mathbb{N}$
Choose $f(x) = x^2 - 2$
and $g(x) = x^n$.
Then $h(x) = g \circ f(x)$.

Note: These are not the only possible answers, but the 'natural' choices have been made.

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Example 4

Let $f(x) = e^{2x}$ and let $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in \mathbb{R}^+$. Find:

a f^{-1} **b** g^{-1} **c** $f \circ g$
d $g \circ f$ **e** $(f \circ g)^{-1}$ **f** $(g \circ f)^{-1}$

Solution

a $f^{-1}(x) = \dfrac{1}{2}\log_e x, \ \ x \in \mathbb{R}^+$

b $g^{-1}(x) = \dfrac{1}{x^2}, \ \ x \in \mathbb{R}^+$

c $f \circ g(x) = f\left(\dfrac{1}{\sqrt{x}}\right) = e^{\frac{2}{\sqrt{x}}}, \ \ x \in \mathbb{R}^+$

d $g \circ f(x) = g(e^{2x}) = \dfrac{1}{e^x}, \ \ x \in \mathbb{R}$

e For $(f \circ g)^{-1}$, let $x = e^{\frac{2}{\sqrt{y}}}$. Then

$$\log_e x = \frac{2}{\sqrt{y}}$$

$$\therefore \quad y = \left(\frac{2}{\log_e x}\right)^2$$

$$(f \circ g)^{-1}(x) = \left(\frac{2}{\log_e x}\right)^2, \ \ x \in (1, \infty)$$

f For $(g \circ f)^{-1}$, let $x = \dfrac{1}{e^y}$. Then

$$e^y = \frac{1}{x}$$

$$\therefore \quad y = \log_e\left(\frac{1}{x}\right) = -\log_e x$$

$$(g \circ f)^{-1}(x) = -\log_e x, \ \ x \in \mathbb{R}^+$$

Strictly increasing and strictly decreasing functions

We introduced strictly increasing and strictly decreasing functions in Chapter 1.

- A function f is **strictly increasing** if $a > b$ implies $f(a) > f(b)$, for all $a, b \in \text{dom} f$.
- A function f is **strictly decreasing** if $a > b$ implies $f(a) < f(b)$, for all $a, b \in \text{dom} f$.

In Section 1G we noted that, if f is a strictly increasing function, then it is one-to-one and so it has an inverse function.

Recall that a function f is **one-to-one** if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \text{dom} f$.

If f is strictly increasing, then it is a one-to-one function.

If f is strictly decreasing, then it is a one-to-one function.

Proof We prove only the first of the two statements.

Assume f is strictly increasing and let $a, b \in \text{dom} f$ with $a \neq b$. Then $a > b$ or $b > a$. Therefore $f(a) > f(b)$ or $f(b) > f(a)$, since f is strictly increasing. In both cases, we have $f(a) \neq f(b)$. Hence f is a one-to-one function.

Note: If a function f is continuous and one-to-one on an interval, then f is either strictly increasing or strictly decreasing on this interval. The proof of this result is beyond the requirements of this course.

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If f is strictly increasing, then f^{-1} is also strictly increasing.

If f is strictly decreasing, then f^{-1} is also strictly decreasing.

Proof We prove only the first of the two statements.

Assume f is strictly increasing and let $a, b \in \text{dom } f^{-1}$ with $a > b$.

If $f^{-1}(a) = f^{-1}(b)$, then $f(f^{-1}(a)) = f(f^{-1}(b))$ and so $a = b$, which is not the case.

If $f^{-1}(a) < f^{-1}(b)$, then $f(f^{-1}(a)) < f(f^{-1}(b))$ and so $a < b$, which is not the case.

Thus, we must have $f^{-1}(a) > f^{-1}(b)$, and hence f^{-1} is strictly increasing.

These results help us to understand the graphs of strictly increasing and strictly decreasing functions and their inverses.

Example 5

For each function f, find the inverse function f^{-1}, and state whether f and f^{-1} are strictly increasing, strictly decreasing or neither:

a $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{3}}$

b $f\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{3}}$

Solution

a Write $x = y^{\frac{2}{3}}$ and solve for y:

$$x = \left(y^{\frac{1}{3}}\right)^2$$

$$y^{\frac{1}{3}} = \pm\sqrt{x}$$

$$y = (\pm\sqrt{x})^3 = \pm x^{\frac{3}{2}}$$

The domain of f is $\mathbb{R}^+ \cup \{0\}$;
the range of f is $\mathbb{R}^+ \cup \{0\}$.

Hence $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = x^{\frac{3}{2}}$

Both f and f^{-1} are strictly increasing.

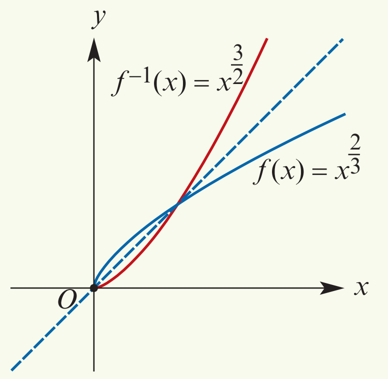
b Write $x = y^{\frac{2}{3}}$. Then $y = \pm x^{\frac{3}{2}}$.

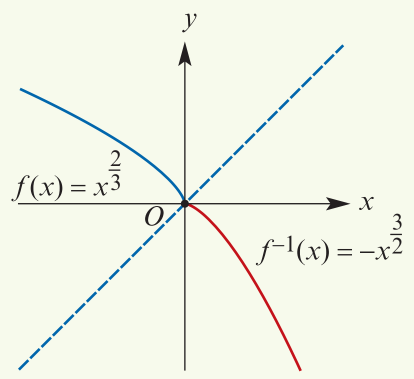
The domain of f is $\mathbb{R}^- \cup \{0\}$;
the range of f is $\mathbb{R}^+ \cup \{0\}$.

Hence $f^{-1}\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f^{-1}(x) = -x^{\frac{3}{2}}$

Both f and f^{-1} are strictly decreasing.

Explanation





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Skill-sheet

Exercise 7B

1 Express each of the following as the composition of two functions:

Example 3

a $h(x) = e^{x^3}$ **b** $h(x) = \sin(2x^2)$ **c** $h(x) = (x^2 - 2x)^n$ where $n \in \mathbb{N}$

d $h(x) = \cos(x^2)$ **e** $h(x) = \cos^2 x$ **f** $h(x) = (x^2 - 1)^4$

g $h(x) = \cos^2(2x)$ **h** $h(x) = (x^2 - 2x)^3 - 2(x^2 - 2x)$

Example 4

2 Let $f(x) = 4e^{3x}$ and let $g(x) = \dfrac{2}{\sqrt[3]{x}}$ for $x \in \mathbb{R} \setminus \{0\}$. Find:

a f^{-1} **b** g^{-1} **c** $f \circ g$ **d** $g \circ f$ **e** $(f \circ g)^{-1}$ **f** $(g \circ f)^{-1}$

Example 5

3 For each function f, find the inverse function f^{-1}, and state whether f and f^{-1} are strictly increasing, strictly decreasing or neither:

a $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{5}}$ **b** $f\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R},\ f(x) = x^{\frac{2}{5}}$

c $f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = x^{\frac{5}{2}}$

4 Let $g(x) = x^2$. For each of the following functions f:

i Find the rules $f \circ g(x)$ and $g \circ f(x)$.

ii Find the range of $y = f \circ g(x)$ and $y = g \circ f(x)$ (and state the maximal domain for each of the composite functions to exist).

a $f(x) = 3\sin(2x)$ **b** $f(x) = -2\cos(2x)$ **c** $f(x) = e^x$

d $f(x) = e^{2x} - 1$ **e** $f(x) = -2e^x - 1$ **f** $f(x) = \log_e(2x)$

g $f(x) = \log_e(x - 1)$ **h** $f(x) = -\log_e x$

5 Let $f(x) = 2x - \dfrac{\pi}{3}$ and $g(x) = \sin x$.

a Find $g \circ f$.

b Describe a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g \circ f(x)$.

6 Consider $f\colon (\frac{1}{3}, \infty) \to \mathbb{R},\ f(x) = 3x - 2$ and $g\colon (-1, \infty) \to \mathbb{R},\ g(x) = \log_e(x + 1)$.

a Find $g \circ f$.

b Describe a sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g \circ f(x)$.

7 **a** Given that $[g(x)]^2 - 7g(x) + 12 = 0$, find possible rules for $g(x)$.

b Given that $[g(x)]^2 - 7xg(x) + 12x^2 = 0$, find possible rules for $g(x)$.

8 Given that $e^{g(x)} = 2x - 1$, find the rule for $g(x)$.

9 The functions f and g are defined by $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = e^{4x}$ and $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(x) = 2\sqrt{x}$. Find each of the following:

a $g \circ f(x)$ **b** $(g \circ f)^{-1}(x)$ **c** $f \circ g^{-1}(x)$

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10 The functions f and g are defined by $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-2x}$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = x^3 + 1$.

a Find the inverse function of each of these functions.

b Find the rules $f \circ g(x)$ and $g \circ f(x)$ and state the range of each of these composite functions.

11 The function f is defined by $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \dfrac{1}{x+1}$.

a Find f^{-1}.

b Solve the equation $f(x) = f^{-1}(x)$ for x.

12 The functions f and g are defined by $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \log_e(x+1)$ and $g\colon (-1, \infty) \to \mathbb{R}$, $g(x) = x^2 + 2x$.

a Define f^{-1} and g^{-1}, giving their rules and domains.

b Find the rule for $f \circ g(x)$.

13 The functions f and g are defined by $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = \log_e x$ and $g\colon (0, \infty) \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. Find $f \circ g(x)$ and simplify $f(x) + f \circ g(x)$.

14 The functions g and h are defined by $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = 5x^2 + 3$ and $h\colon [3, \infty) \to \mathbb{R}$, $h(x) = \sqrt{\dfrac{x-3}{5}}$. Find $h(g(x))$.

15 For $f(x) = (x-4)(x-6)$ and $g(x) = x^2 - 4$:

a Find $f(g(x))$ and $g(f(x))$.

b Solve the equation $g(f(x)) - f(g(x)) = 158$ for x.

16 For $f(x) = 4 - x^2$, solve the equation $f(f(x)) = 0$ for x.

17 For $f(x) = e^x - e^{-x}$, show that:

a $f(-x) = -f(x)$

b $[f(x)]^3 = f(3x) - 3f(x)$

18 The inverse function of the linear function $f(x) = ax + b$ is $f^{-1}(x) = 6x + 3$. Find the values of a and b.

19 Show that $f = f^{-1}$ for $f(x) = \dfrac{x+2}{x-1}$.

20 Let $g\colon \mathbb{R} \to \mathbb{R}$ such that $\log_e(g(x)) = ax + b$. Given that $g(0) = 1$ and $g(1) = e^6$, find a and b and hence find $g(x)$.

21 **a** Let $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = \dfrac{e^x + e^{-x}}{2}$. Find f^{-1}.

b Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = \dfrac{e^x - e^{-x}}{2}$. Find g^{-1}.

c Is f a strictly increasing function on the stated interval?

d Is g a strictly increasing function on the stated interval?

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22 Let f and g be functions such that the composite $g \circ f$ is defined.

a Prove that, if both f and g are strictly increasing, then $g \circ f$ is strictly increasing.

b Prove that, if both f and g are strictly decreasing, then $g \circ f$ is strictly increasing.

c What can be said about the composite $g \circ f$ if one of the two functions f and g is strictly increasing and the other is strictly decreasing?

7C Sums and products of functions and addition of ordinates

In Chapter 1 we saw that, for functions f and g, the new functions $f + g$ and fg can be defined by

$$(f+g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)\,g(x)$$

$$\operatorname{dom}(f+g) = \operatorname{dom} f \cap \operatorname{dom} g \qquad \operatorname{dom}(fg) = \operatorname{dom} f \cap \operatorname{dom} g$$

We also considered graphing by addition of ordinates. The new functions that have been defined in Chapters 4 to 6 may now be included.

Example 6

For $f(x) = \cos x$ and $g(x) = e^{-x}$:

a Find the rules for $(f+g)(x)$ and $(fg)(x)$.

b Evaluate $(f+g)(0)$ and $(fg)(0)$.

Solution

a $(f+g)(x) = \cos x + e^{-x}$ and $(fg)(x) = e^{-x}\cos x$

b $(f+g)(0) = 1 + 1 = 2$ and $(fg)(0) = 1 \times 1 = 1$

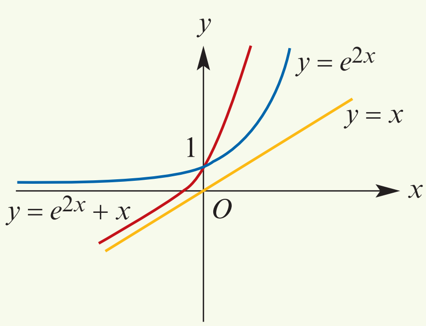
Example 7

For $f(x) = x$ and $g(x) = e^{2x}$, sketch the graph of $y = (f+g)(x)$.

Solution

Note that $(f+g)(0) = 0 + e^0 = 1$ and that $(f+g)(x) = 0$ implies $x + e^{2x} = 0$. This equation cannot be solved analytically, but a calculator can be used to find the approximate solution $x = -0.43$, correct to two decimal places.

Also note that, as $x \to -\infty$, $(f+g)(x) \to x$ from 'above'.



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Summary 7C

- Sum of functions $(f+g)(x) = f(x) + g(x)$, where $\mathrm{dom}(f+g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$
- Difference of functions $(f-g)(x) = f(x) - g(x)$, where $\mathrm{dom}(f-g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$
- Product of functions $(f \cdot g)(x) = f(x) \cdot g(x)$, where $\mathrm{dom}(f \cdot g) = \mathrm{dom}\, f \cap \mathrm{dom}\, g$
- Addition of ordinates This technique can be used to help sketch the graph of the sum of two functions. Key points to consider when sketching $y = (f+g)(x)$:
  - When $f(x) = 0$, $(f+g)(x) = g(x)$.
  - When $g(x) = 0$, $(f+g)(x) = f(x)$.
  - If $f(x)$ and $g(x)$ are positive, then $(f+g)(x) > g(x)$ and $(f+g)(x) > f(x)$.
  - If $f(x)$ and $g(x)$ are negative, then $(f+g)(x) < g(x)$ and $(f+g)(x) < f(x)$.
  - If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f+g)(x) < f(x)$.
  - Look for values of x for which $f(x) + g(x) = 0$.

Exercise 7C

Example 6

1 Let $f(x) = e^{2x}$ and $g(x) = -2x$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(-\frac{1}{2})$. **ii** Evaluate $(fg)(-\frac{1}{2})$.

Example 7

2 Sketch the graphs of $f(x) = e^{-2x}$ and $g(x) = -2x$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = e^{-2x} - 2x$.

3 Sketch the graphs of $f(x) = 2e^{2x}$ and $g(x) = x + 2$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = 2e^{2x} + x + 2$.

4 Let $f(x) = \sin\left(\frac{\pi x}{2}\right)$ and $g(x) = -2x$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(1)$. **ii** Evaluate $(fg)(1)$.

5 Let $f(x) = \cos\left(\frac{\pi x}{2}\right)$ and $g(x) = e^x$.

a **i** Find the rule for $f+g$. **ii** Find the rule for fg.

b **i** Evaluate $(f+g)(0)$. **ii** Evaluate $(fg)(0)$.

6 Prove that any function f with domain $\mathbb{R}$ can be expressed as the sum of an even function and an odd function.

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7D Function notation and identities

This section contains material not in the study design for MM Units 3 and 4, but it can certainly add to the understanding of function notation and its implementation.

Many of the properties which have been investigated for the functions introduced in the previous chapters may be expressed using function notation.

For example, the rules for logarithms

$$\log_e(x) + \log_e(y) = \log_e(xy) \qquad \log_e(x) - \log_e(y) = \log_e\left(\frac{x}{y}\right)$$

can be written in the following way if $f(x) = \log_e x$:

$$f(x) + f(y) = f(xy) \qquad f(x) - f(y) = f\left(\frac{x}{y}\right)$$

The rules for exponential functions

$$e^{x+y} = e^x \times e^y \qquad e^{x-y} = \frac{e^x}{e^y}$$

can be written in the following way if $f(x) = e^x$:

$$f(x+y) = f(x)f(y) \qquad f(x-y) = \frac{f(x)}{f(y)}$$

Example 8

a For the function with rule $f(x) = 2x$, show that $f(x+y) = f(x) + f(y)$ for all x and y.

b For the function with rule $f(x) = x + 2$, show that $f(x+y) \neq f(x) + f(y)$ for all x and y.

Solution

a $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

b $f(x+y) = (x+y) + 2 = (x+2) + (y+2) - 2 = f(x) + f(y) - 2$

If $f(x) + f(y) - 2 = f(x) + f(y)$, then $-2 = 0$, which is a contradiction.

Example 9

If $f(x) = \dfrac{1}{x}$, verify that $f(x) + f(y) = (x+y)\,f(xy)$ for all non-zero real numbers x and y.

Solution

$$f(x) + f(y) = \frac{1}{x} + \frac{1}{y} = \frac{y+x}{xy} = (x+y) \times \frac{1}{xy} = (x+y)\,f(xy)$$

Example 10

For the function $f(x) = \cos x$, give an example to show that $f(x+y) \neq f(x) + f(y)$ for some x and y.

Solution

For $x = 0$ and $y = \pi$:

$$f(0+\pi) = f(\pi) = -1$$

$$f(0) + f(\pi) = 1 + (-1) = 0$$

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Summary 7D

Functional equations can be useful for describing properties of functions. For example, the exponential function $f(x) = e^x$ satisfies $f(x + y) = f(x)\, f(y)$.

Note that, in general,

$$f(x + y) \neq f(x) + f(y)$$
$$f(xy) \neq f(x)\, f(y)$$

Exercise 7D

Example 8

1 **a** For the function with rule $f(x) = 2x$, show that $f(x - y) = f(x) - f(y)$ for all x and y.

b For the function with rule $f(x) = x - 3$, verify that $f(x - y) \neq f(x) - f(y)$ for all x and y.

2 For $f(x) = kx$, find an equivalent expression for $f(x - y)$ in terms of $f(x)$ and $f(y)$.

3 For $f(x) = 2x + 3$, show that $f(x + y)$ can be written in the form $f(x) + f(y) + a$ and give the value of a.

Example 9

4 If $f(x) = \dfrac{3}{x}$, show that $f(x) + f(y) = (x + y)\, f(xy)$ for all non-zero real numbers x and y.

5 A function g satisfies the property that $[g(x)]^2 = g(x)$. Find the possible values of $g(x)$.

6 A function g satisfies the property that $\dfrac{1}{g(x)} = g(x)$. Find the possible values of $g(x)$.

Example 10

7 For the function with rule $f(x) = x^3$, give an example to show that $f(x + y) \neq f(x) + f(y)$ for some x and y.

8 For the function $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = \sin x$, give an example to show that $f(x + y) \neq f(x) + f(y)$ for some x and y.

9 For the function with rule $f(x) = \dfrac{1}{x^2}$, show that $f(x) + f(y) = (x^2 + y^2)\, f(xy)$.

10 **a** For $h(x) = x^2$, give an example to show that $h(x + y) \neq h(x) + h(y)$ for some x and y.

b Show that $h(x + y) = h(x) + h(y)$ implies $x = 0$ or $y = 0$.

11 For $g(x) = 2^{3x}$, show that $g(x + y) = g(x)\, g(y)$.

12 Show that the functions with rules of the form $f(x) = x^n$, where n is a natural number, satisfy the identities $f(xy) = f(x)\, f(y)$ and $f\left(\dfrac{x}{y}\right) = \dfrac{f(x)}{f(y)}$.

13 For the function with rule $f(x) = ax$ where $a \in \mathbb{R} \setminus \{0, 1\}$, give an example to show that $f(xy) \neq f(x)\, f(y)$ for some x and y.

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14 For the function $f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x+1}$, show that $f(f(x)) + f(x+1) = 1$ for all $x \in \mathbb{R} \setminus \{-1, -2\}$.

15 Give an example to show that in general $f \circ (g + h) \neq (f \circ g) + (f \circ h)$.

16 Prove that $(g + h) \circ f = (g \circ f) + (h \circ f)$.

17 Let $g : \mathbb{R}^+ \to \mathbb{R}$, $g(x) = xe^x$ and $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e x$. Show that $f(g(x)) - f(x) = x$ for all $x > 0$ and show that $\dfrac{g(f(x))}{f(x)} = x$ for all $x > 1$.

7E Families of functions and solving literal equations

In Chapter 2 we used parameters to describe the solutions of simultaneous equations. In this section we use parameters to describe families of functions.

Here are some families of functions:

$f : \mathbb{R} \to \mathbb{R},\ f(x) = mx$ where $m \in \mathbb{R}$

$f : \mathbb{R} \to \mathbb{R},\ f(x) = ax^3$ where $a \in \mathbb{R} \setminus \{0\}$

$f : \mathbb{R} \to \mathbb{R},\ f(x) = mx + 2$ where $m \in \mathbb{R}^+$

$f : \mathbb{R} \to \mathbb{R},\ f(x) = ke^{mx}$ where $m \in \mathbb{R} \setminus \{0\}$ and $k \in \mathbb{R} \setminus \{0\}$

The use of parameters makes it possible to describe general properties.

What can be said in general about each of these families? The following example explores the family of functions of the form $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + 2$ where $m \in \mathbb{R}^+$.

Example 11

For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = mx + 2$ where $m \in \mathbb{R}^+$:

a Find the x-axis intercept of the graph of $y = f(x)$.

b For which values of m is the x-axis intercept less than -2?

c Find the inverse function of f.

d Find the equation of the line perpendicular to the graph of $y = f(x)$ at the point $(0, 2)$.

Solution

a When $y = 0$, we have $mx + 2 = 0$ and so $x = -\dfrac{2}{m}$. The x-axis intercept is $-\dfrac{2}{m}$.

b Consider $-\dfrac{2}{m} < -2$ and solve for m:

$$\frac{2}{m} > 2$$

$$2 > 2m \qquad \text{Multiply both sides of the inequality by } m\ (m > 0).$$

$$\therefore \quad m < 1$$

Therefore the x-axis intercept is less than -2 for $0 < m < 1$.

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c Consider $x = my + 2$ and solve for y:

$$my = x - 2$$

$$\therefore \quad y = \frac{x-2}{m}$$

Therefore $f^{-1}(x) = \frac{1}{m}x - \frac{2}{m}$. The domain of f^{-1} is $\mathbb{R}$.

d The perpendicular line has gradient $-\frac{1}{m}$.

The equation is $y - 2 = -\frac{1}{m}(x - 0)$, which rearranges to $y = -\frac{1}{m}x + 2$.

Example 12

The graph of a quadratic function passes through the points $(1, 6)$ and $(2, 4)$. Find the coefficients of the quadratic rule in terms of c, the y-axis intercept of the graph.

Solution

Let $f(x) = ax^2 + bx + c$ be a function in this family. Then $f(1) = 6$ and $f(2) = 4$.

The following equations are obtained:

$$a + b + c = 6 \quad \text{and} \quad 4a + 2b + c = 4$$

Solving these gives

$$a = \frac{c-8}{2} \quad \text{and} \quad b = \frac{20-3c}{2}$$

The equation of the quadratic in terms of c is

$$y = \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c$$

The following example demonstrates how to solve literal equations involving exponential and logarithmic functions.

Example 13

Solve each of the following for x. All constants are positive reals.

a $ae^{bx} - c = 0$ **b** $\log_e(x - a) = b$ **c** $\log_e(cx - a) = 1$

Solution

a
$$\begin{aligned} ae^{bx} - c &= 0 \\ e^{bx} &= \frac{c}{a} \\ bx &= \log_e\left(\frac{c}{a}\right) \\ x &= \frac{1}{b}\log_e\left(\frac{c}{a}\right) \end{aligned}$$

b
$$\begin{aligned} \log_e(x - a) &= b \\ x - a &= e^b \\ x &= e^b + a \end{aligned}$$

c
$$\begin{aligned} \log_e(cx - a) &= 1 \\ cx - a &= e \\ cx &= a + e \\ x &= \frac{a+e}{c} \end{aligned}$$

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Example 14

A transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the rule

$$(x, y) \to (5x + 5, ky + 2)$$

where k is a non-zero real number.

a Let $(x', y') = (5x + 5, ky + 2)$. Find x and y in terms of x' and y'.

b Find the image of the curve with equation $y = \dfrac{1}{x}$ under this transformation.

c Find the value of k if the image passes through the origin.

Solution

a From the matrix equation:

$$5x + 5 = x'$$
$$ky + 2 = y'$$

Therefore $x = \dfrac{x' - 5}{5}$ and $y = \dfrac{y' - 2}{k}$.

b The image has equation $\dfrac{y' - 2}{k} = \dfrac{5}{x' - 5}$, which can be written as $y = \dfrac{5k}{x - 5} + 2$.

c If the graph passes through the origin, then $0 = -k + 2$ and so $k = 2$.

Skill-sheet

Exercise 7E

Example 11

1 Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = mx - 4$ where $m \in \mathbb{R} \setminus \{0\}$.

a Find the x-axis intercept of the graph of $y = f(x)$.

b For which values of m is the x-axis intercept less than or equal to 1?

c Find the inverse function of f.

d Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

e Find the equation of the line perpendicular to the line $y = f(x)$ at the point $(0, -4)$.

2 Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -2x + c$ where $c \in \mathbb{R}$.

a Find the x-axis intercept of the graph of $y = f(x)$.

b For which values of c is the x-axis intercept less than or equal to 1?

c Find the inverse function of f.

d Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

e Find the equation of the line perpendicular to the line $y = f(x)$ at the point $(0, c)$.

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3 Consider the family of quadratics with rules of the form $y = x^2 - bx$, where b is a non-zero real number.

a Find the x-axis intercepts.

b Find the coordinates of the vertex of the parabola.

c **i** Find the coordinates of the points of intersection of the graph of $y = x^2 - bx$ with the line $y = -x$ in terms of b.

ii For what value(s) of b is there one intersection point?

iii For what value(s) of b are there two intersection points?

Example 12

4 The graph of a quadratic function passes through the points $(-1, 6)$ and $(1, 4)$. Find the coefficients of the quadratic rule in terms of c, the y-axis intercept of the graph.

5 **a** The graph of $f(x) = x^2$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 8$.

b The graph of $f(x) = x^2$ is transformed to the graph of $y = f(ax)$. Find the possible values of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quadratic with equation $y = ax^2 + bx$ has vertex with coordinates $(1, 8)$. Find the values of a and b.

6 Consider the family of functions with rules of the form $f(x) = \sqrt{2a - x}$, where a is a positive real number.

a State the maximal domain of f.

b Find the coordinates of the point of intersection of the graph of $y = f(x)$ with the graph of $y = x$.

c For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(1, 1)$?

d For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates $(2, 2)$?

e For what value of a does the line with equation $y = x$ intersect the graph of $y = f(x)$ at the point with coordinates (c, c), where c is a positive real number?

7 Consider the function with rule $f(x) = (x^2 - ax)^2$.

a State the coordinates of the x-axis intercepts.

b State the coordinates of the y-axis intercept.

c For $a > 0$, find the maximum value of the function in the interval $[0, a]$.

d Find the possible values of a for which the point $(-1, 16)$ lies on the graph of $y = f(x)$.

Example 13

8 Solve each of the following for x. All constants are positive reals.

a $-ae^{bx} + c = 0$

b $c \log_e(x + a) = b$

c $\log_e(cx - a) = 0$

d $e^{ax+b} = c$

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9 Consider the family of functions with rules of the form $f(x) = c\log_e(x - a)$, where a and c are positive constants.

a State the equation of the vertical asymptote.

b State the coordinates of the x-axis intercept.

c State the coordinates of the point where the graph crosses the line $y = 1$.

d If the graph of the function crosses the line $y = 1$ when $x = 2$, find the value of c in terms of a.

10 Consider the family of functions with rules of the form $f(x) = e^{x-1} - b$, where $b > 0$.

a State the equation of the horizontal asymptote.

b State the coordinates of the x-axis intercept.

c Give the values of b for which the x-axis intercept is:

i at the origin **ii** a negative number.

11 The graph of a cubic function passes through the points $(-1, 6)$, $(1, -2)$ and $(2, 4)$. Find the coefficients of the cubic rule in terms of d, the y-axis intercept of the graph.

12 A quadratic function has rule $f(x) = \left(\frac{c-8}{2}\right)x^2 + \left(\frac{20-3c}{2}\right)x + c$. Find the values of c for which:

a the graph of $y = f(x)$ touches the x-axis

b the graph of $y = f(x)$ has two distinct x-axis intercepts.

13 The graph of a cubic function passes through the points $(-2, 8)$, $(1, 1)$ and $(3, 4)$. Find the coefficients of the quadratic rule in terms of d, the y-axis intercept of the graph.

Example 14

14 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by is defined by the rule

$$(x, y) \to (-4x + 3, ky + 2)$$

where k is a non-zero real number.

a Let $(x', y') = (-4x + 3, ky + 2)$. Find x and y in terms of x' and y'.

b Find the image of the curve with equation $y = \frac{1}{x}$ under this transformation.

c Find the value of k if the image passes through the origin.

15 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by is defined by the rule

$$(x, y) \to (-4x + a, 2y - 2)$$

where a is a non-zero real number.

a $(x', y') = (-4x + a, 2y - 2)$. Find x and y in terms of x' and y'.

b Find the image of the curve with equation $y = 2^x$ under this transformation.

c Find the value of a if the image passes through the origin.

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Technology-free questions

Assign-ment

Nrich

1 For each of the following, use your calculator to help sketch the graph of $y = f(x)$ for the maximal domain. State this maximal domain and the corresponding range. Also state whether the function is odd, even or neither.

a $f(x) = 3x^{-\frac{1}{4}} + 1$ **b** $f(x) = 2x^{-\frac{1}{5}} - 2$

2 Evaluate each of the following:

a $243^{\frac{2}{5}}$ **b** $(-243)^{\frac{2}{5}}$ **c** $243^{\frac{3}{5}}$ **d** $(-243)^{\frac{3}{5}}$ **e** $(-27)^{\frac{5}{3}}$ **f** $(-125)^{\frac{4}{3}}$

3 Let $g(x) = x^2$. For each of the following functions f:

i Find the rules $f \circ g(x)$ and $g \circ f(x)$.

ii Find the range of $y = f \circ g(x)$ and $y = g \circ f(x)$ (and state the maximal domain for each of the composite functions to exist).

a $f(x) = 3\cos(2x)$ **b** $f(x) = \log_e(3x)$

c $f(x) = \log_e(2 - x)$ **d** $f(x) = -\log_e(2x)$

4 Express each of the following as the composition of two functions:

a $h(x) = \cos(x^2)$ **b** $h(x) = (x^2 - x)^n$ where $n \in \mathbb{N}$

c $h(x) = \log_e(\sin x)$ **d** $h(x) = -2\sin^2(2x)$

e $h(x) = (x^2 - 3x)^4 - 2(x^2 - 3x)^2$

5 Let $f(x) = 2\cos\left(\dfrac{\pi x}{2}\right)$ and $g(x) = e^{-x}$.

a **i** Find the rule for $f + g$. **ii** Find the rule for fg.

b **i** Evaluate $(f + g)(0)$. **ii** Evaluate $(fg)(0)$.

6 Let $f: [a, \infty) \to \mathbb{R}$ where $f(x) = -(3x - 2)^2 + 3$.

a Find the smallest value of a such that f is one-to-one.

b With this value of a, state the range of f.

c Sketch the graph of f.

d Find f^{-1} and state the domain and range of f^{-1}.

e Sketch the graphs of f and f^{-1} on the one set of axes.

7 Consider the family of functions with rules of the form $f(x) = c\log_e(x - a)$, where a and c are positive constants.

a State the equation of the vertical asymptote.

b State the coordinates of the x-axis intercept.

c State the coordinates of the point where the graph crosses the line $y = c$.

d Find the inverse function f^{-1} of f.

e State the range of f^{-1}.

f If $f^{-1}(1) = 2$ and $f^{-1}(2) = 4$, find the exact values of a and c.

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8 The inverse function of the linear function $f(x) = ax + b$ is $f^{-1}(x) = 4x - 6$. Find the values of a and b.

9 Find the inverse function of each of the following functions:

a $f(x) = 3x^{\frac{1}{3}} + 1$
b $f(x) = 4x^{\frac{1}{3}} - 2$
c $f(x) = (3x - 2)^3 + 4$
d $f(x) = -2x^3 + 3$

10 Let a be a positive constant. Let $f : [-a, a] \to \mathbb{R},\ f(x) = \sqrt{a^2 - x^2}$ and let $g : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R},\ g(x) = a \sin x$
Find $f \circ g$, stating its rule, domain and range.

Multiple-choice questions

1 The graph of the function with rule $h(x) = \dfrac{x^4 + 2}{x^2}$ can be drawn by adding the ordinates of the graphs of two functions f and g. The rules for f and g could be

A $f(x) = x^4,\quad g(x) = \dfrac{2}{x^2}$
B $f(x) = x^2,\quad g(x) = \dfrac{2}{x^2}$
C $f(x) = x^4 + 2,\quad g(x) = x^2$
D $f(x) = x^4 + 2,\quad g(x) = \dfrac{2}{x^2}$
E $f(x) = x^2,\quad g(x) = 2$

2 Which one of the following functions is not a one-to-one function?

A $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \dfrac{1}{x^2}$
B $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
C $f : \mathbb{R} \to \mathbb{R},\ f(x) = 10^x$
D $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_{10} x$
E $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$

3 For the function with rule $f(x) = e^x$, which one of the following is *not* correct for all positive real x and y?

A $f(x + y) = f(x)f(y)$
B $f^{-1}(xy) = f^{-1}(x) + f^{-1}(y)$
C $f^{-1}(x^y) = yf^{-1}(x)$
D $f^{-1}(1) = 0$
E $f^{-1}(x) = \dfrac{1}{f(x)}$

4 If $f(x) = \cos x$ and $g(x) = 3x^2$, then $g\left(f\left(\frac{\pi}{3}\right)\right)$ is equal to

A $\cos\left(\dfrac{\pi^2}{9}\right)$
B $\dfrac{1}{\sqrt{2}}$
C 1
D $\dfrac{3}{4}$
E $\dfrac{4}{3}$

5 Which of the following is not an even function?

A $f : \mathbb{R} \to \mathbb{R},\ f(x) = 4x^2$
B $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos^2 x$
C $f : \mathbb{R} \to \mathbb{R},\ f(x) = \cos x$
D $f : \mathbb{R} \to \mathbb{R},\ f(x) = 4x^2 - 3$
E $f : \mathbb{R} \to \mathbb{R},\ f(x) = (x - 2)^2$

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6 It is known that the graph of the function with rule $y = 2ax + \cos(2x)$ has an x-axis intercept when $x = \pi$. The value of a is

A 2 **B** $\dfrac{1}{2\pi}$ **C** 2π **D** -2π **E** $\dfrac{-1}{2\pi}$

7 Let $g(x) = \log_e(x - 5)$ for $x > 5$. If $2[g(x)] = g(f(x))$, then $f(x)$ is equal to

A $5x - 8$ **B** $x^2 - 10x + 30$ **C** $5x^2$
D $(2x - 10)^2$ **E** $2x - 2$

8 If the equation $f(3x) = 3f(x)$ is true for all real values of x, then the rule for f could be

A x^2 **B** $3x + 3$ **C** $4x$ **D** $\log_e(x + 3)$ **E** $x - 5$

9 The function $g\colon [-a, a] \to \mathbb{R}$, $g(x) = 3\sin(2x)$ has an inverse function. The maximum possible value of a is

A 3 **B** $\dfrac{\pi}{6}$ **C** $\dfrac{\pi}{3}$ **D** $\dfrac{\pi}{4}$ **E** $\dfrac{\pi}{2}$

10 If $f\colon (-\infty, 3) \to \mathbb{R}$, $f(x) = 4\log_e(3 - x)$ and $g\colon [2, \infty) \to \mathbb{R}$, $g(x) = 4\sqrt{x - 2}$, then the maximal domain of the function $f + g$ is

A $\mathbb{R}$ **B** $[-2, 3)$ **C** $[2, 3)$ **D** $[-3, 3)$ **E** $(2, 3]$

11 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the curve with equation $y = \sin x$ onto the curve $y = -4\sin\left(3x - \dfrac{\pi}{3}\right) + 2$ is given by the rule

A $(x, y) \to \left(\dfrac{1}{9}x + \dfrac{\pi}{3}, -4y + 2\right)$ **B** $(x, y) \to \left(\dfrac{1}{3}x + \dfrac{\pi}{9}, -4y + 2\right)$
C $(x, y) \to \left(3x + \dfrac{\pi}{3}, -4y + 2\right)$ **D** $(x, y) \to \left(-\dfrac{1}{3}x + \dfrac{\pi}{3}, 4y - 2\right)$
E $(x, y) \to \left(\dfrac{1}{3}x + \dfrac{\pi}{3}, -4y + 2\right)$

12 A transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ that maps the curve with equation $y = 2\sin\left(2x - \dfrac{\pi}{4}\right) - 3$ onto the curve $y = \sin x$ is given by the rule

A $(x, y) \to \left(2x + \dfrac{\pi}{4}, 2y + 3\right)$ **B** $(x, y) \to \left(2x - \dfrac{\pi}{4}, 2y - 3\right)$
C $(x, y) \to \left(\dfrac{1}{2}x + \dfrac{\pi}{4}, \dfrac{1}{2}y + 3\right)$ **D** $(x, y) \to \left(2x - \dfrac{\pi}{4}, \dfrac{1}{2}y + \dfrac{3}{2}\right)$
E $(x, y) \to \left(2x + \dfrac{\pi}{4}, -2y + 3\right)$

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Extended-response questions

1 Consider $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = e^{-x}$ and $g\colon (-\infty, 1) \to \mathbb{R}$, $g(x) = \dfrac{1}{x-1}$.

a State the ranges of f and g.

b Find f^{-1} and g^{-1}.

c **i** Find $g \circ f$. **ii** Sketch the graph of $y = g \circ f(x)$.

d **i** Find $(g \circ f)^{-1}$. **ii** Sketch the graph of $y = (g \circ f)^{-1}(x)$.

2 **a** For $f\colon [5, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x-3}$:

i Sketch the graph of $y = f(x)$ for $x \in [5, \infty)$.

ii State the range.

iii Find f^{-1}.

b For $h\colon [4, \infty) \to \mathbb{R}$, $h(x) = \sqrt{x-p}$ with inverse function h^{-1} that has domain $[1, \infty)$:

i Find p.

ii Find the rule for h^{-1}.

iii Sketch the graphs of $y = h(x)$ and $y = h^{-1}(x)$ on the one set of axes.

3 Let $f\colon (0, \pi) \to \mathbb{R}$ with $f(x) = \sin x$ and $g\colon [1, \infty) \to \mathbb{R}$ with $g(x) = \dfrac{1}{x}$.

a Find the range of f.

b Find the range of g.

c Give a reason why $f \circ g$ is defined and find $f \circ g(x)$.

d State, with reason, whether $g \circ f$ is defined.

e Find g^{-1}, giving its domain and range.

f Give a reason why $g^{-1} \circ f$ is defined and find $g^{-1} \circ f(x)$. Also state the domain and range of this function.

4 Let $f\colon [-a, \infty) \to \mathbb{R}$, $f(x) = k\log_e(x+a) + c$ where k, a and c are positive constants.

- The graph of f has a vertical asymptote $x = -2$.
- The graph has y-axis intercept 2.
- There is a point on the graph with coordinates $(d, 12)$.

a State the value of a.

b Find the value of c in terms of k.

c Find k in terms of d.

d If $d = 2e - 2$, find the value of k.

5 Let $f : [0, \infty) \to \mathbb{R}$ where $f(x) = \sqrt{x+9}$
and let $g : (-\infty, b] \to \mathbb{R}$ where $g(x) = x^2 + 6x - 4$.

a Find the largest value of b such that $f(g(x))$ is defined.

b Using this value of b state the rule, domain and range of the function with rule $y = f(g(x))$.

c Let h be the inverse function of the function with rule $y = f(g(x))$. Give the rule, domain and range of h.

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