Revision

12 Revision of Chapters 9–11

12A Technology-free questions

1 Let $y = \dfrac{x^2 - 1}{x^4 - 1}$.

a Find $\dfrac{dy}{dx}$.

b Find $\left\{x : \dfrac{dy}{dx} = 0\right\}$.

2 Let $y = (3x^2 - 4x)^4$. Find $\dfrac{dy}{dx}$.

3 Let $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^2 \log_e(2x)$. Find $f'(x)$.

4 **a** Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x+1}$. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find b.

b Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x+1} + k$ where k is a real number. The tangent to the graph of f at the point where $x = b$ passes through the point $(0, 0)$. Find k in terms of b.

5 The line $y = mx - 8$ is tangent to the curve $y = x^{\frac{1}{3}} + c$ at the point $(8, a)$. Find the values of a, c and m.

6 Find the average value of the function with rule $f(x) = \dfrac{1}{3x + 1}$ over the interval $[0, 2]$.

7 Find an antiderivative of:

a $\dfrac{3}{5x - 2}$, $x > \dfrac{2}{5}$

b $\dfrac{3}{(5x - 2)^2}$, $x \neq \dfrac{2}{5}$

8 If $f(3) = -2$ and $f'(3) = 5$, find $g'(3)$ where:

a $g(x) = 3x^2 - 5f(x)$

b $g(x) = \dfrac{3x + 1}{f(x)}$

c $g(x) = [f(x)]^2$

ISBN 978-1-009-11049-5 Cambridge University Press

Revision

9 If $f(4) = 6$ and $f'(4) = 2$, find $g'(4)$ where:

a $g(x) = \sqrt{x}\, f(x)$ **b** $g(x) = \dfrac{f(x)}{x}$

10 Given that $f'(x) = \sqrt{3x+4}$ and $g(x) = x^2 - 1$, find $F'(x)$ if $F(x) = f(g(x))$.

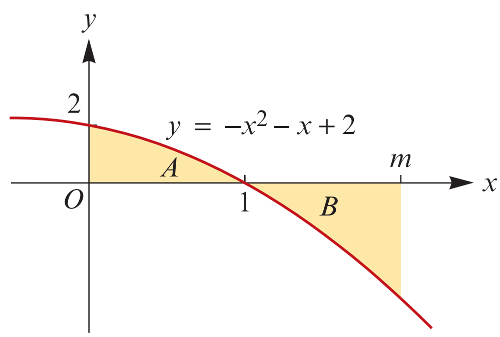
11 If $f(x) = 2x^2 - 3x + 5$, find:

a $f'(x)$ **b** $f'(0)$ **c** $\{x : f'(x) = 1\}$

12 Find the derivative of $\log_e(3f(x))$ with respect to x.

13 The tangent to the graph of $y = \sqrt{a - x}$ at $x = 1$ has a gradient of -6. Find the value of a.

14 The graph of $y = -x^2 - x + 2$ is shown. Find the value of m such that regions A and B have the same area.



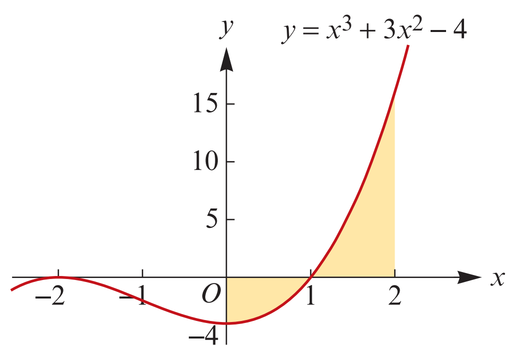
15 Let $f(x) = x^3 + 3x^2 - 4$. The graph of $y = f(x)$ is as shown. Find:

a the coordinates of the stationary points

b $\int_{-2}^{2} f(x)\,dx$

c $\int_{0}^{2} f(x)\,dx$

d the area of the shaded region.



16 If $f(x) = \dfrac{1}{3x-1}$, find $f'(2)$.

17 If $y = 1 - x^2$, prove that $x\dfrac{dy}{dx} + 2 = 2y$ for all values of x.

18 If $A = 4\pi r^2$, calculate $\dfrac{dA}{dr}$ when $r = 3$.

19 At what point on the graph of $y = 1.8x^2$ is the gradient 1?

20 If $y = 3x^2 - 4x + 7$, find the value of x such that $\dfrac{dy}{dx} = 0$.

21 If $y = \dfrac{x^2+2}{x^2-2}$, find $\dfrac{dy}{dx}$.

22 If $z = 3y + 4$ and $y = 2x - 1$, find $\dfrac{dz}{dx}$.

ISBN 978-1-009-11049-5

23 If $y = (5 - 7x)^9$, calculate $\frac{dy}{dx}$.

24 If $y = 3x^{\frac{1}{3}}$, find $\frac{dy}{dx}$ when $x = 27$.

25 If $y = \sqrt{5 + x^2}$, find $\frac{dy}{dx}$ when $x = 2$.

26 Find $\frac{dy}{dx}$ when $x = 1$, given that $y = (x^2 + 3)(2 - 4x - 5x^2)$.

27 If $y = \frac{x}{1 + x^2}$, find $\frac{dy}{dx}$ when $x = 1$.

28 If $y = \frac{2 + x}{x^2 + x + 1}$, find $\frac{dy}{dx}$ when $x = 0$.

29 Let $f(x) = \frac{1}{2x + 1}$.

a Use the definition of derivative to find $f'(x)$.

b Find the gradient of the tangent to the graph of f at the point $(0, 1)$.

30 Let $f(x) = x^3 + 3x^2 - 1$. Find:

a $\{x : f'(x) = 0\}$ **b** $\{x : f'(x) > 0\}$ **c** $\{x : f'(x) < 0\}$

31 Let $y = \frac{x}{1 - x}$.

a Find $\frac{dy}{dx}$. **b** Write $\frac{dy}{dx}$ in terms of y.

32 If $y = (x^2 + 1)^{-\frac{3}{2}}$, find $\frac{dy}{dx}$.

33 If $y = x^4$, prove that $x\frac{dy}{dx} = 4y$.

34 Show that $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x^5$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$, and if $0 > b$, then $f(0) > f(b)$.

35 Evaluate each of the following integrals:

a $\int_0^{\frac{\pi}{2}} 2\sin\left(\frac{x}{2}\right) dx$ **b** $\int_0^{\frac{3}{2}} e^{\frac{x}{2}} dx$ **c** $\int_{\frac{1}{2}}^{1} \frac{1}{2x} dx$

d $\int_{-1}^{-\frac{1}{2}} \frac{1}{2x} dx$ **e** $\int_3^4 \frac{1}{2(x-2)^2} dx$ **f** $\int_2^4 \frac{1}{(3x-2)^2} dx$

36 Let $f : (0, \infty) \to \mathbb{R}$, $f(x) = a\sqrt{x + 1} - x - 1$ where a is a constant, $a \geq 4$.

a Find the coordinates of the local maximum of the graph of $y = f(x)$ in terms of a

b **i** If $f(3) = 16$ find the value of a.

ii Find the equation of the tangent to the graph at the point $(35, 24)$

iii Find the coordinates of the intercepts of the tangent with each of the axes.

ISBN 978-1-009-11049-5 Cambridge University Press

37 Show that $f : \mathbb{R} \to \mathbb{R}, f(x) = -2x^3 + 1$ is a strictly decreasing function for $\mathbb{R}$.

38 Let $f : \mathbb{R} \to \mathbb{R}, f(x) = e^{-mx+2} + 4x$ where m is a positive rational number.

a Find the x-coordinate of the stationary point of the graph of $y = f(x)$ in terms of m.

b Find the values of m for which the x-coordinate of this stationary point is negative.

39 For each of the following functions, find the coordinates of the points on the graph at which the tangent passes through the origin:

a $y = x \sin x, \quad -\pi \le x \le \pi$

b $y = x \cos(2x), \quad -\pi \le x \le \pi$

40 Let $f(x) = 3\sin(\pi x)$ for $-2 \le x \le 2$.

a Sketch the graph of $y = f(x)$ for $-2 \le x \le 2$.

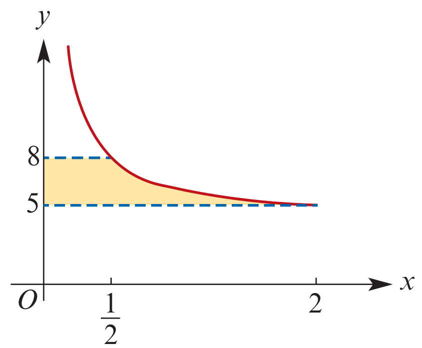
b Find the equation of the tangent to the graph where $x = \frac{1}{2}$.

c Evaluate $\int_0^{\frac{1}{2}} f(x) - \frac{x}{4}\, dx$.

41 The diagram shows the graph of the function

$$f(x) = 4 + \frac{2}{x}, \quad 0 < x \le 2$$

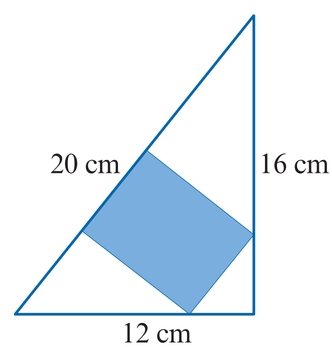
and the lines $y = 5$ and $y = 8$. Find the area of the shaded region.



42 A function h has a rule of the form $h(x) = (ax^2 + b)e^{cx}$. Find the values of the constants a, b and c, given that the function has the following three properties:

- $h(0) = -4$
- $h'(0) = 8$
- the graph of h has a local minimum at $x = -1$.

43 A right-angled triangle has sides 12 cm, 16 cm and 20 cm as shown. A rectangle is inscribed in the triangle with one side along the hypotenuse and a vertex on each of the other two sides of the triangle. What are the dimensions of the largest such rectangle?



ISBN 978-1-009-11049-5

12B Multiple-choice questions

1 The derivative of the function f is $f'(x) = x^4(x-4)(x+7)$. At how many points of f will the graph have a local maximum?

A 0 **B** 1 **C** 2 **D** 3 **E** 4

2 The absolute maximum value of $f(x) = 2x^3 - x^2 - 2x + 1$ on the closed interval $[-2, 2]$ occurs at

A -2 **B** $\frac{1-\sqrt{13}}{6}$ **C** 2 **D** $\frac{1+\sqrt{13}}{6}$ **E** 0

3 The gradient of the curve with equation $y = \sin(2x) + 1$ at $(0, 1)$ is

A 1 **B** -1 **C** 0 **D** 2 **E** -2

4 Let $f : (0, 3\pi] \to \mathbb{R},\ f(x) = e^{\frac{x}{\sqrt{3}}} \cos x$. There are values of x for which $f'(x) = 0$. The sum of these values is

A $\frac{10\pi}{3}$ **B** $\frac{17\pi}{6}$ **C** $\frac{7\pi}{2}$ **D** $\frac{4\pi}{3}$ **E** $\frac{7\pi}{3}$

5 A polynomial with rule $y = P(x)$ has a local maximum at $(-3, 7)$, a local minimum at $(2, 2)$ and a local maximum at $(6, 7)$. There are no other points on the graph of $y = P(x)$ with zero gradient. How many solutions does the equation $P(x) = 0$ have?

A 1 **B** 2 **C** 3 **D** 4 **E** 5

6 Points P and Q lie on the curve $y = x^3$. The x-coordinates of P and Q are 2 and $2 + h$ respectively. The gradient of the secant PQ is

A $\frac{h^3 - 8}{h - 2}$ **B** $12 + 6h$ **C** 12 **D** $\frac{(2+h)^3 - h^3}{h}$ **E** $12 + 6h + h^2$

7 If $f(x) = \frac{3}{x}$, then $\frac{f(x+h) - f(x)}{h}$ is equal to

A $\frac{-3}{x(x+h)}$ **B** $\frac{3}{x^2}$ **C** $\frac{-3}{x^2}$ **D** $\frac{-3}{h(x+h)}$ **E** $f'(x)$

8 The gradient of $y = ce^{2x}$ is equal to 11 when $x = 0$. The value of c is

A 0 **B** 1 **C** 5 **D** 5.5 **E** $5e^{-2}$

9 The graph of $y = bx^2 - cx$ crosses the x-axis at the point $(4, 0)$. The gradient at this point is 1. The value of c is

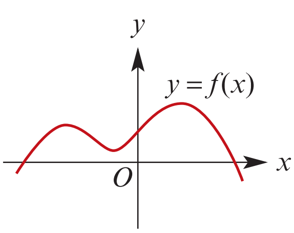
A 8 **B** 1 **C** 4 **D** -8 **E** 2

10 For the graph of $y = f(x)$ shown, $f'(x) = 0$ at

A 3 points **B** 2 points

C 5 points **D** 0 points

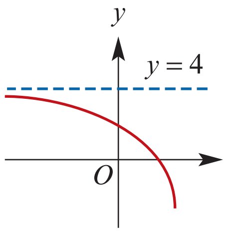
E none of these



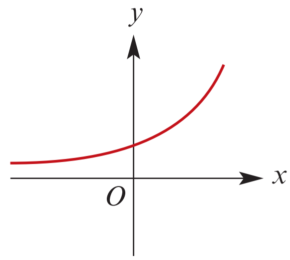
ISBN 978-1-009-11049-5

11 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 4 - e^{-2x}$. The graph of $f'(x)$ is best represented by

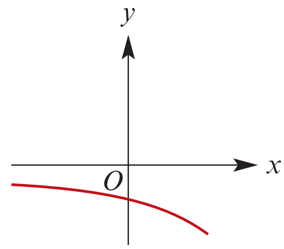
A



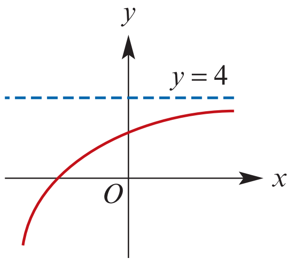
B



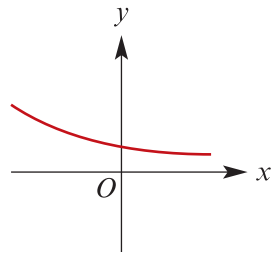
C



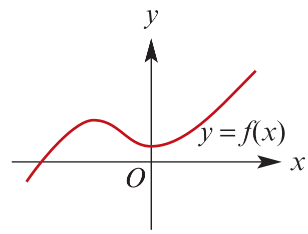
D



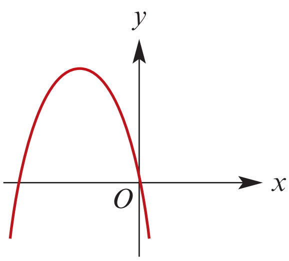
E



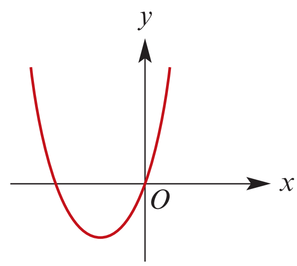
12 The graph of $y = f(x)$ is shown on the right. The graph that best represents the graph of $y = f'(x)$ is



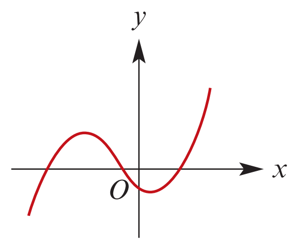
A



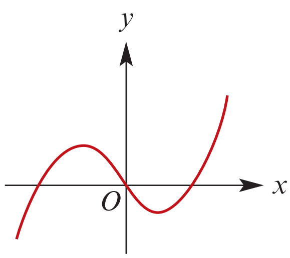
B



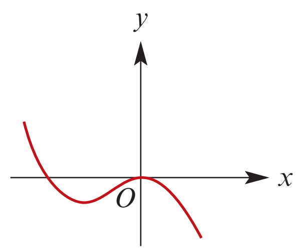
C



D



E



13 Consider all right cylinders for which the sum of the height and the circumference is 30 cm. What is the radius of the cylinder with maximum volume?

A 3 cm **B** $\frac{1}{\pi^2}$ cm **C** $\frac{10}{\pi}$ cm **D** $\frac{30}{\pi^2}$ cm **E** $\frac{1}{\pi}$ cm

ISBN 978-1-009-11049-5

14 Let $f(x) = 3x^2 + 2$. If $g'(x) = f'(x)$ and $g(2) = 29$, then $g(x) =$

A $3x^3 + 5$ **B** $3x^2 - 3$ **C** $\frac{x^3}{3} + 2x$ **D** $3x^2 + 17$ **E** $6x + 17$

15 If $f(x) = e^{kx} + e^{-kx}$, then $f'(x) > 0$ for

A $x \in \mathbb{R}$ **B** $x \geq 0$ **C** $x < 0$ **D** $x \leq 0$ **E** $x > 0$

16 If g is a differentiable function and $g(x) < 0$ for all real numbers x and if $f'(x) = (x^2 - 9)g(x)$, which of the following is true?

A f has a local maximum at $x = -3$ and a local minimum at $x = 3$

B f has a local minimum at $x = -3$ and a local maximum at $x = 3$

C f has a local minimums at $x = -3$ and $x = 3$

D f has a local maximums at $x = -3$ and $x = 3$

E f has stationary points of inflexion at $x = -3$ and $x = 3$

17 Rainwater is being collected in a water tank. The volume, V m^3, of water in the tank after time t minutes is given by $V = 2t^2 + 3t + 1$. The average rate of change of volume of water between times $t = 2$ and $t = 4$, in m^3/min, is

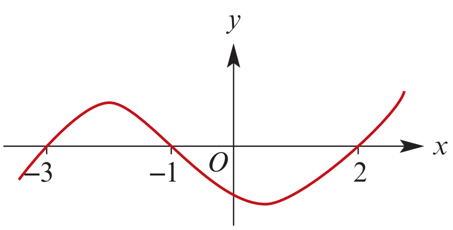
A 11 **B** 13 **C** 15 **D** 17 **E** 19

18 $P(x, f(x))$ and $Q(x + h, f(x + h))$ are two points on the graph of the function $f(x) = x^2 - 2x + 1$. The gradient of the line joining P and Q is given by

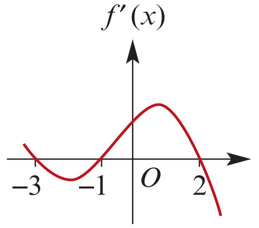
A $2x - 2$ **B** $2xh - 4x - 2h + 2$ **C** $2xh - 2h - h^2$

D $2xh - 2h + h^2$ **E** $2x - 2 + h$

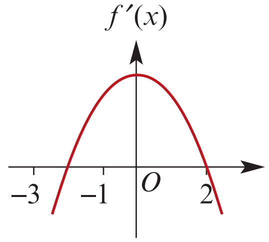
19 The graph of $y = f(x)$ is shown.
A possible graph of the gradient function f' with rule given by $f'(x)$ is



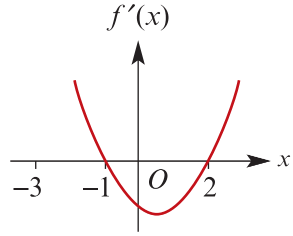
A



B

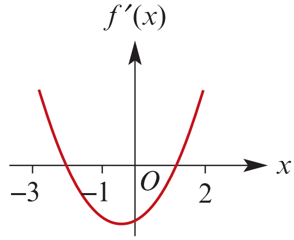


C

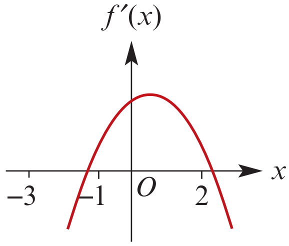


ISBN 978-1-009-11049-5 Cambridge University Press

D



E



20 Which one of the following gives the gradient of the tangent to a curve with the equation $y = f(x)$ at the point $x = 2$?

A $\dfrac{f(x+h) - f(x)}{h}$ **B** $f(2+h) - f(2)$ **C** $\dfrac{f(2+h) - f(2)}{h}$

D $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$ **E** $\lim\limits_{h\to 0} \dfrac{f(2+h) - f(2)}{h}$

21 Let $f(x) = \begin{cases} -6 & \text{if } x \le -3 \\ 2x & \text{if } -3 < x < 1 \\ -2(x-2)^2 + 10 & \text{if } x \ge 1 \end{cases}$

The maximal set of values of x for which f is strictly increasing is

A $[-3, 2]$ **B** $(-3, 2)$ **C** $(-3, 1) \cup (1, 2)$

D $(-3, 1) \cup (1, 2]$ **E** $[-3, 1) \cup (1, 2]$

22 The maximum value of $-x^2 + 4x + 3$ is

A 2 **B** 3 **C** $2 + 2\sqrt{7}$ **D** 7 **E** 15

23 The functions f and g are differentiable and $g(x) \neq 0$ for all x. Let $h(x) = f(x) \times g(x)$. If $f(2) = 4$, $g(2) = -3$, $f'(2) = -6$ and $g'(2) = 7$ then $h'(2)$ is equal to.

A 0 **B** −40 **C** −42 **D** −46 **E** 46

24 The graph of the curve with equation $y = x^2 - x^3$ has stationary points where x is equal to

A 0 and $\dfrac{2}{3}$ **B** 0 and 1 **C** −1 and 0 **D** 0 and $\dfrac{3}{2}$ **E** 2 and −3

25 Consider the tangent to the graph of $y = x^2 + 3x$ at the point $(2, 10)$. Which of the following points lies on this tangent?

A $(2, 3)$ **B** $(1, 4)$ **C** $(-1, -2)$ **D** $(-2, -18)$ **E** $(10, 7)$

26 If $f(x) = \int_0^x \sqrt{t^3 + 4t}\, dt$ then $f'(1)$ is equal to

A $\dfrac{3}{2}$ **B** $\dfrac{9}{4}$ **C** 7 **D** $\sqrt{5}$ **E** $\sqrt{7}$

27 If $f'(x) = x^2 + \dfrac{1}{x}$ and $f(1) = \dfrac{1}{3}$, then $f(x)$ is equal to

A $\dfrac{x^3}{3} + \log_e x$ **B** $\dfrac{x^3}{3} + \log_e x + \dfrac{2}{3}$ **C** $\dfrac{x^3}{3} - \log_e x - \dfrac{1}{3}$

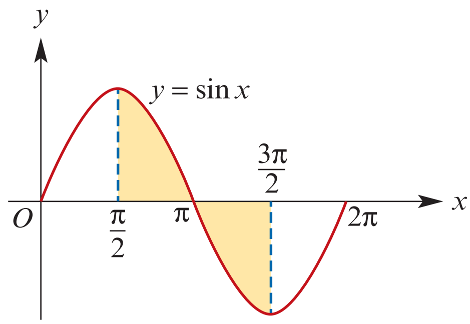
ISBN 978-1-009-11049-5 Cambridge University Press

D $\dfrac{-x^3}{3} + \log_e x + \dfrac{2}{3}$ **E** $\dfrac{x^3}{3} - \log_e x + \dfrac{1}{3}$

28 If $y = F(x)$ and $\dfrac{dy}{dx} = f(x)$, then $\int_2^3 f(x)\, dx$ is equal to

A $f(3) - f(2)$ **B** $F'(3) - F'(2)$ **C** $F(3) - F(2)$ **D** $f(x) + c$ **E** $F(3) - f(2)$

29 The area of the shaded region of the graph is given by



A $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\, dx$

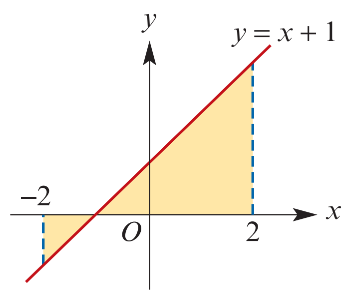
B $\int_{\pi}^{\frac{3\pi}{2}} \sin x\, dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx$

C $\int_{\frac{3\pi}{2}}^{\pi} \sin x\, dx + \int_{\frac{\pi}{2}}^{\pi} \sin x\, dx$

D $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin x\, dx + \int_{\pi}^{\frac{\pi}{2}} \sin x\, dx$

E $\pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 x\, dx$

30 The area of the shaded region of the graph is given by



A $\int_0^2 (x+1)\, dx - \int_2^0 (x+1)\, dx$

B $\int_{-2}^2 (x+1)\, dx$

C $\int_0^2 (x+1)\, dx + \int_{-2}^0 (x+1)\, dx$

D $\int_{-1}^2 (x+1)\, dx - \int_{-2}^{-1} (x+1)\, dx$

E $\int_{-1}^2 (x+1)\, dx + \int_{-2}^{-1} (x+1)\, dx$

31 If $\dfrac{dy}{dx} = \dfrac{1}{x^2}$ and $y = 2$ when $x = 1$, then

A $y = \dfrac{-1}{x}$ **B** $y = \dfrac{-1}{x} + 3$ **C** $y = \dfrac{-2}{x^3}$ **D** $y = \dfrac{2}{x^3}$ **E** $y = \dfrac{1}{x} + 1$

32 If $\int_0^{16} \dfrac{1}{2x+1}\, dx = \log_e k$, then k is

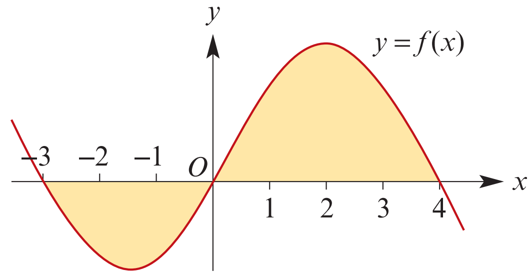
A 33 **B** $\sqrt{33}$ **C** $\dfrac{17}{2}$ **D** $\dfrac{2}{17}$ **E** -33

33 Let $f : (-1, \infty) \to \mathbb{R},\ f(x) = 8\sqrt{x+1} - x - 1$. The tangent to $y = f(x)$ at the point $(a, f(a))$ is parallel to the line connecting the positive x-axis intercept and the y-axis intercept. The value of a is

A $\dfrac{1}{9}$ **B** $\dfrac{77}{4}$ **C** 7 **D** 19 **E** 20

ISBN 978-1-009-11049-5

34 The area of the shaded region of the graph is given by



A $\int_{-3}^{4} f(x)\,dx$

B $\int_{-3}^{0} f(x)\,dx + \int_{0}^{4} f(x)\,dx$

C $\int_{-3}^{1} f(x)\,dx + \int_{1}^{4} f(x)\,dx$

D $\int_{4}^{0} f(x)\,dx + \int_{-3}^{0} f(x)\,dx$

E none of these

35 The area bounded by the curve $y = \dfrac{1}{3-x}$, the x-axis, the y-axis and the line $x = 2$ is

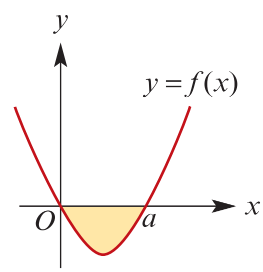
A $\log_e 3$ **B** $\log_e\left(\dfrac{1}{3}\right)$ **C** $-\log_e(3-x)$ **D** $\log_e 2$ **E** $\log_e\left(\dfrac{1}{2}\right)$

36 If $\int_{a}^{b} \sin(2x)\,dx = 0$, then

A $b = \dfrac{3\pi}{4},\ a = \dfrac{\pi}{4}$ **B** $b = \dfrac{\pi}{2},\ a = 0$ **C** $b = \pi,\ a = \dfrac{\pi}{2}$

D $b = \dfrac{\pi}{6},\ a = \dfrac{\pi}{3}$ **E** $b = \pi,\ a = \dfrac{\pi}{4}$

37 The area of the shaded region of the graph is given by



A $\int_{0}^{a} -f(x)\,dx$

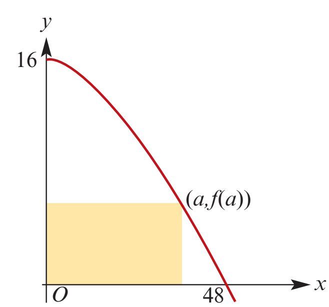
B $\int_{0}^{a} f(x)\,dx$

C $\int_{0}^{a} x - f(x)\,dx$

D $\int_{0}^{a} f(x) - x\,dx$

E none of these

38 Let $f : [0, \infty) \to \mathbb{R},\ f(x) = 8\sqrt{x+16} - x - 16$. A rectangle is formed by using a point $(a, f(a))$ on the curve $y = f(x)$ and the coordinare axes as shown in the diagram. The maximum area is



A 256 **B** $128 + \sqrt{5}$ **C** $128 - \sqrt{5}$

D $12 - 12\sqrt{5}$ **E** 56

39 The function $f(x) = x^3 + 3x^2 - 9x + 7$ is strictly increasing only when

A $x > 0$ **B** $-3 < x < 1$ **C** $x \le -1$ or $x \ge 3$

D $x \le -3$ or $x \ge 1$ **E** $-1 \le x \le 3$

ISBN 978-1-009-11049-5 Cambridge University Press

40 The average value of the function with with rule $f(x) = x^3 - 3x^2$ over the interval $[1, a]$, where $a > 1$, is 43. The value of a is

A 6 **B** 7 **C** 8 **D** 9 **E** 10

41 The polynomial $f(x) = 2x^3 + ax^2 + bx$ has a local maximum when $x = -1$ and a local minimum when $x = 4$. The values of a and b are

A -1 and 4 **B** 9 and 24 **C** -4 and 1 **D** -9 and -24 **E** 2 and -9

42 If $f'(x) = \sin(2x)$ and $f(0) = 3$, then

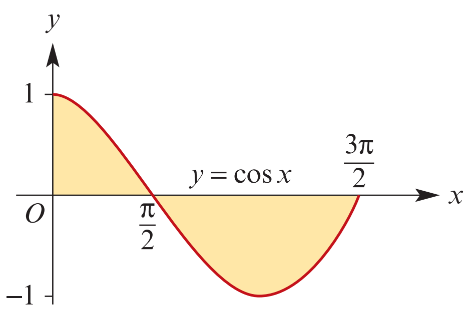
A $f(x) = -\frac{1}{2}\cos(2x) + 3$ **B** $f(x) = \frac{1}{2}\cos(2x) + 3$ **C** $f(x) = -\frac{1}{2}\cos(2x) + 3\frac{1}{2}$
D $f(x) = -\frac{1}{2}\cos(2x) + 2\frac{1}{2}$ **E** $f(x) = \frac{1}{2}\cos(2x) + 2\frac{1}{2}$

43 The function $f(x) = x^3 - x^2 - x + 2$ has a local minimum at the point

A $(-1, 0)$ **B** $(1, 1)$ **C** $(2, 0)$ **D** $(-1, 1)$ **E** $(1, 0)$

44 The total area, in square units, of the shaded regions is

A 3 **B** -1 **C** 1 **D** 2 **E** -2



45 Given that $x + y = 1$, the maximum value of $P = x^2 + xy - y^2$ occurs for x equal to

A 2 **B** -1 **C** $\frac{3}{2}$ **D** 1 **E** $\frac{2}{3}$

46 The gradient of the normal to the curve of $y = e^{-\cos x}$ at the point where $x = \frac{\pi}{3}$ is

A $\frac{\sqrt{3}}{2e^{\frac{1}{2}}}$ **B** $\frac{-2e^{\frac{1}{2}}}{\sqrt{3}}$ **C** $\frac{1}{2e^{\frac{\sqrt{3}}{2}}}$ **D** $\frac{2e^{\frac{1}{2}}}{\sqrt{3}}$ **E** $-e^{-\frac{1}{2}}$

47 Let f be differentiable for all values of x in $[0, 2]$. The graph with equation $y = f(x)$ has a maximum point at $(1, 3)$. The equation of the tangent at $(1, 3)$ is

A $x + 3y = 0$ **B** $x = 1$ **C** $y = 3$ **D** $x - 3y = 0$ **E** $3x + y = 0$

48 The maximum value of $P = -x^2 + 6x + 4$ is

A 3 **B** $-6 + 2\sqrt{5}$ **C** 4 **D** 13 **E** 24

49 The graph of the function whose rule is $f(x) = x^3 - x^2 - 1$ has stationary points when x equals

A $\frac{2}{3}$ only **B** 0 and $\frac{2}{3}$ **C** 0 and $-\frac{2}{3}$ **D** $-\frac{1}{3}$ and 1 **E** $\frac{1}{3}$ and -1

ISBN 978-1-009-11049-5 Cambridge University Press

50 If $\dfrac{f(2+h)-f(2)}{h} = h^2 + 6h + 12$, then $f'(2)$ equals

A 4 **B** 6 **C** 10 **D** 12 **E** 28

51 If $y = x^2e^x$, then the minimum value of y is

A -2 **B** 0 **C** $4e^{-2}$ **D** $-4e^{-2}$ **E** e

52 If $f(x) = a\sin(3x)$ where a is constant and $f'(\pi) = 2$, then a is equal to

A -3 **B** $-\dfrac{3}{2}$ **C** $\dfrac{3}{2}$ **D** $\dfrac{2}{3}$ **E** $-\dfrac{2}{3}$

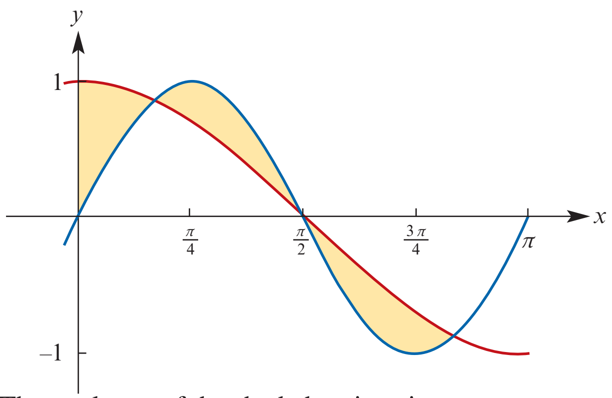
53 The maximal domain for which $y = -x^2 + 2ax + 3$ is strictly decreasing is $[2, \infty)$. The value of a is

A -2 **B** -1 **C** 0 **D** 2 **E** 1

54 A is the point $(4, 0)$ and B is point (x, y) on the graph of $y = \sqrt{x}$ such that AB is the minimum distance from the point A to the graph. This minimum distance is

A $\dfrac{\sqrt{15}}{2}$ **B** 4 **C** 2 **D** $\dfrac{7}{2}$ **E** $\dfrac{\sqrt{7}}{2}$

55 Part of the graphs of $f : \mathbb{R} \to \mathbb{R}, f(x) = \sin 2x$ and $g : \mathbb{R} \to \mathbb{R}, g(x) = \cos x$ are shown in the diagram below.



The total area of the shaded regions is

A $\int_0^{\frac{5\pi}{6}} f(x) - g(x)\,dx$

B $2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) - g(x)\,dx + \int_0^{\frac{\pi}{6}} g(x) - f(x)\,dx$

C $\int_0^{\frac{\pi}{6}} g(x) - f(x)\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) - g(x)\,dx + \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} f(x) - g(x)\,dx$

D $2\int_0^{\frac{\pi}{2}} f(x) - g(x)\,dx$

E $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} f(x) - g(x)\,dx + 2\int_0^{\frac{\pi}{6}} f(x) - g(x)\,dx$

56 A tangent to the graph of $y = 2e^{3x} - 1$ has gradient equal to 6. This tangent will cross the x-axis at

A 6 **B** -6 **C** $-\dfrac{1}{6}$ **D** 1 **E** 5

57 If $\int_a^0 f(x)\,dx = n$, then $\int_0^a 2f(x) - 1\,dx$ is equal to

A $2n - a$ **B** $2n + a$ **C** $-a - 2n$ **D** $2a - n$ **E** $\dfrac{n}{2} + a$

ISBN 978-1-009-11049-5 Cambridge University Press

58 Let $f(x) = \dfrac{a}{x^2} + x - 2,\ x \neq 0$ and a a real constant. There is a stationary point on the graph of f where $x = 1$ The value of a is

A $\dfrac{1}{2}$ **B** 1 **C** -1 **D** 4 **E** 2

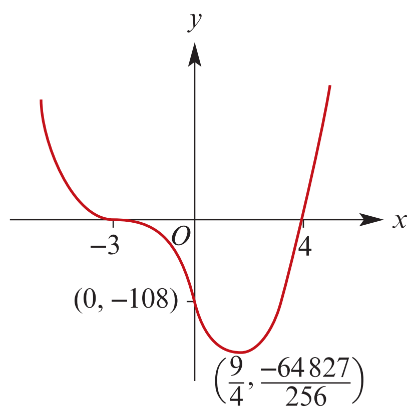
59 The tangent to the graph of $y = 2x^3 + ax^2 + 1$ at $x = -1$ passes through the origin. The value of a is

A 1 **B** $-\dfrac{7}{3}$ **C** $\dfrac{7}{3}$ **D** 5 **E** -5

60 Let f be a one-to-one differentiable function such that $f(4) = 11$, $f(6) = 4$, $f'(6) = 4$, $f(4) = 12$, $f'(4) = 12$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x.
$g'(4)$ is equal to

A $\dfrac{1}{4}$ **B** 1 **C** $\dfrac{1}{12}$ **D** $\dfrac{3}{8}$ **E** $\dfrac{5}{6}$

61 The graph shown is of a function with rule $y = (x + 3)^3(x - 4)$. Which of the following is *not* true?



A $\dfrac{dy}{dx} = 0$ when $x = \dfrac{9}{4}$ and $x = -3$ and at no other point.

B There is only one turning point on the graph.

C The x-axis is a tangent to the graph where $x = -3$.

D There is only one stationary point on the graph.

E $y \geq \dfrac{-64\,827}{256}$ for all values of x.

62 The total area of the regions enclosed by the curve $y = e^{5x} - 2\sin 4x$, the x-axis and the lines $x = -1$ and $x = 1$ correct to two decimal places is

A 0.17 **B** 29.55 **C** 29.68 **D** 29.85 **E** 30.02

12C Extended-response questions

1 The amount of salt (s grams) in 100 litres of salt solution at time t minutes is given by $s = 50 + 30e^{-\frac{1}{5}t}$.

a Find the amount of salt in the mixture after 10 minutes.

b Sketch the graph of s against t for $t \geq 0$.

c Find the rate of change of the amount of salt at time t (in terms of t).

d Find the rate of change of the amount of salt at time t (in terms of s).

e Find the concentration (grams per litre) of salt at time $t = 0$.

f Find the value of t for which the salt solution first reaches a concentration of 0.51 grams per litre.

ISBN 978-1-009-11049-5 Cambridge University Press

Revision

2 A medium is kept at a constant temperature of 20°C. An object is placed in this medium. The temperature, T°C, of the object at time t minutes is given by

$$T = 40e^{-0.36t} + 20, \quad t \geq 0$$

a Find the initial temperature of the object.

b Sketch the graph of T against t for $t \geq 0$.

c Find the rate of change of temperature with respect to time (in terms of t).

d Find the rate of change of temperature with respect to time (in terms of T).

3 A certain food is susceptible to contamination from bacterial spores of two types, F and G. In order to kill the spores, the food is heated to a temperature of 120°C. The number of live spores after t minutes can be approximated by $f(t) = 1000e^{-0.5t}$ for F-type spores and by $g(t) = 1200e^{-0.7t}$ for G-type spores.

a Find the time required to kill 50% of the F-type spores.

b Find the total number of live spores of both types when $t = 0$, and find the percentage of these that are still alive when $t = 5$.

c Find the rate at which the total number of live spores is decreasing when $t = 5$.

d Find the value of t for which the number of live F-type spores and the number of live G-type spores are equal.

e On the same set of axes, sketch the graphs of $y = f(t)$ and $y = g(t)$ for $t \geq 0$.

4 An object falls from rest in a medium and its velocity, V m/s, after t seconds is given by $V = 100(1 - e^{-0.2t})$.

a Sketch the graph of V against t for $t \geq 0$.

b Express the acceleration at any instant:

i in terms of t **ii** in terms of V.

c Find the value of t for which the velocity of the object is 80 m/s.

5 A manufacturer determines that the total cost, \$$C$ per year, of producing a product is given by $C = 0.05x^2 + 5x + 500$, where x is the number of units produced per year. At what level of output will the average cost per unit be a minimum? (Use a continuous function to model this discrete situation.)

6 An object that is at a higher temperature than its surroundings cools according to Newton's law of cooling: $T = T_0 e^{-kt}$, where T_0 is the original excess of temperature and T is the excess of temperature after time t minutes.

a Prove that $\frac{dT}{dt}$ is proportional to T.

b If the original temperature of the object is 100°C, the temperature of its surroundings is 30°C and the object cools to 70°C in 20 minutes, find the value of k correct to three decimal places.

c At what rate is the temperature decreasing after 30 minutes?

ISBN 978-1-009-11049-5 Cambridge University Press

7 Suppose that the spread of a cold virus through a population is such that the proportion, $p(t)$, of the population which has had the virus up to time t days after its introduction into the population is given by

$$p(t) = 0.2 - 0.2e^{\frac{-t}{20}} + 0.1e^{\frac{-t}{10}}, \quad \text{for } t \geq 0$$

a **i** Find, correct to four decimal places, the proportion of the population which has had the virus up to 10 days after its introduction.

ii Find the proportion of the population that eventually catches the virus.

b The number of new cases on day t is proportional to $p'(t)$. Find how long after the introduction of the virus the number of new cases per day is at a maximum.

8 A real-estate firm owns the Shantytown Apartments, consisting of 70 garden-type apartments. The firm can find a tenant for all the apartments at \$500 each per month. However, for every \$20 per month increase, there will be two vacancies with no possibility of filling them. What price per apartment will maximise monthly revenue? (Use a continuous function to model this discrete situation.)

9 The amount of liquid, V m^3, in a large pool at time t days is given by $V = \dfrac{5 \times 10^4}{(t+1)^2}$ for $t \geq 0$.

a Find the initial volume of the pool.

b Find the rate of change of volume with respect to time when $t = 1$.

c Find the average rate of change for the interval $t = 1$ to $t = 4$.

d When is the amount of water in the pool less than 1 cubic metre?

e Sketch the graph of V against t for $t \geq 0$.

10 Each week a factory produced N thousand bottle tops and the cost of production is reckoned to be \$1000C, where $C = (N^3 + 16)^{\frac{1}{4}}$.

a Sketch the graph of C against N. (Use a continuous model.)

b Calculate $\dfrac{dC}{dN}$.

c What does $\dfrac{dC}{dN}$ represent?

11 A company produces items at a cost price of \$2 per item. Market research indicates that the likely number of items sold per month will be $\dfrac{800}{p^2}$, where p dollars is the selling price of each item. Find the value of p for which the company would expect to maximise its total monthly profit, and the corresponding number of items sold.

12 A curve with equation $y = (ax + b)^{-2}$ has y-axis intercept $(0, \frac{1}{4})$ and at this point the gradient is $-\frac{3}{4}$. Find the value(s) of a and b and sketch the graph.

13 The cost of running a ship at a constant speed of V km/h is $160 + \dfrac{1}{100}V^3$ dollars per hour.

a Find the cost of a journey of 1000 km at a speed of 10 km/h.

b Find the cost, \$$C$, of a journey of 1000 km at a speed of V km/h.

ISBN 978-1-009-11049-5 Cambridge University Press

c Sketch the graph of C against V.

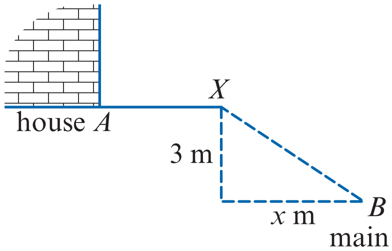
d Find the most economical speed for the journey, and the minimum cost.

e If the ship has a maximum speed of 16 km/h, find the minimum cost.

14 **a** A camper is on an island shore at point A, which is 12 km from the nearest point B on the straight shore of the mainland. He wishes to reach a town C, which is 30 km along the shore from B, in the least possible time. If he can row his boat at 5 km/h and walk at 8 km/h, how far along the shore from B towards C should he land?

b Repeat **a** if C is only 24 km from B.

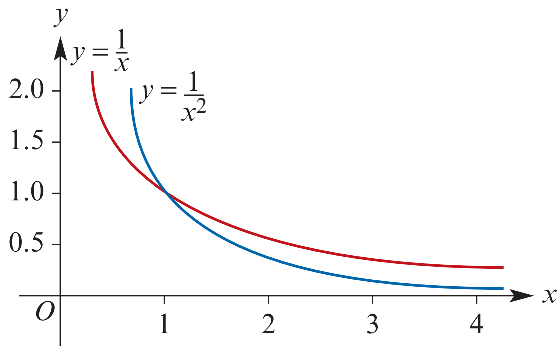
15 To connect a house to a gas supply, a pipe must be installed connecting the point A on the house to the point B on the main, where B is 3 m below ground level and at a horizontal distance of 4 m from the building. If it costs \$25 per metre to lay pipe underground and \$10 per metre on the surface, find the length of pipe which should be on the surface to minimise costs.



16 Define the functions

$$g : \mathbb{R}^+ \to \mathbb{R},\ g(x) = \frac{1}{x}$$

and $$h : \mathbb{R}^+ \to \mathbb{R},\ h(x) = \frac{1}{x^2}$$



a Find $\{ x : g(x) > h(x) \}$.

b Find $\{ x : g'(x) > h'(x) \}$, i.e. find the set of x for which the gradient of g is greater than the gradient of h.

c On one set of axes, sketch the graphs of

$$f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \frac{1}{x^3} \quad \text{and} \quad h : \mathbb{R}^+ \to \mathbb{R},\ h(x) = \frac{1}{x^2}$$

Find $\{ x : h(x) > f(x) \}$ and $\{ x : h'(x) > f'(x) \}$.

d For $f_1 : \mathbb{R}^+ \to \mathbb{R},\ f_1(x) = \frac{1}{x^n}$ and $f_2 : \mathbb{R}^+ \to \mathbb{R},\ f_2(x) = \frac{1}{x^{n+1}}$, find $\{ x : f_1(x) > f_2(x) \}$ and $\{ x : f_1'(x) > f_2'(x) \}$.

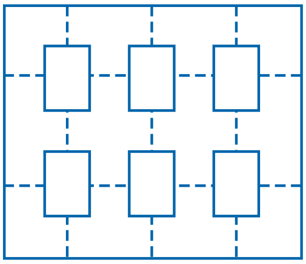
17 **a** Find the points $P\left(x, \frac{1}{x}\right)$ on the curve $y = \frac{1}{x}$ for which the distance OP is a minimum, where O is the origin $(0, 0)$.

b Find the points $P\left(x, \frac{1}{x^2}\right)$ on the curve $y = \frac{1}{x^2}$ for which the distance OP is a minimum.

c Find the points $P\left(x, \frac{1}{x^n}\right)$ on the curve $y = \frac{1}{x^n}$ for which the distance OP is a minimum, where n is a positive integer.

ISBN 978-1-009-11049-5 Cambridge University Press

18 The figure represents an intended basic design for a workshop wall which is to have six equal windows spaced so that each dashed line has length 2 m. The total area of window space is to be 36 m^2.



- **a** Express the total area, A m^2, of brickwork as a function of the window height, x m.
- **b** Sketch the graph of A against x.
- **c** Find the dimensions of each window which will give a minimum amount of brickwork.
- **d** If building regulations require that both the height and the width of a window must not be less than 1 m, find the maximum amount of brickwork that could be used.

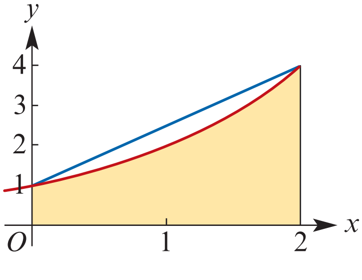
19
- **a** Sketch the graph of the equation $y = x^2 - a^2$. Label the points A, B at which it cuts the x-axis. Write down the coordinates of A and B.
- **b** Find the area of the region between the x-axis and the graph.
- **c** Draw a rectangle $ABCD$ on your sketch, lying *below* the x-axis, with area equal to the area found in part **b**. What is the length of the side BC?
- **d** If the vertex of the parabola is at point V, calculate the ratio $\dfrac{\text{length of } BC}{\text{length of } OV}$.

20
- **a** Calculate $\int_{-3}^{1} (1 - t^2)\, dt$ and illustrate the region of the Cartesian plane for which this integral gives the signed area.
- **b** Show that $\int_{a}^{1} (1 - t^2)\, dt = 0$ implies $a^3 - 3a + 2 = 0$.
- **c** Find the values of a for which $\int_{a}^{1} (1 - t^2)\, dt = 0$.

21 The rate of flow of water into a tank is given by $\dfrac{dV}{dt} = 10e^{-(t+1)}(5 - t)$ for $0 \le t \le 5$, where V litres is the amount of water in the tank at time t minutes. Initially the tank is empty.

- **a**
  - **i** Find the initial rate of flow of water into the tank.
  - **ii** Find the value of t for which $\dfrac{dV}{dt} = 0$.
  - **iii** Find the time, to the nearest second, when the rate is 1 litre per minute.
  - **iv** Find the first time, to the nearest second, when $\dfrac{dV}{dt} < 0.1$.
- **b** Find the amount of water in the tank when $t = 5$.
- **c** Find the time, to the nearest second, when there are 10 litres of water in the tank.

22 It can be shown that $\int 2^x\, dx = \dfrac{2^x}{\ln 2} + c$.



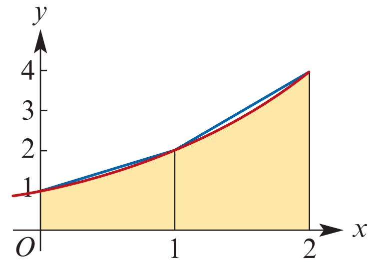
- **a** Evaluate the definite integral $\int_0^2 2^x\, dx$.
- **b**
  - **i** Find an approximation, A_1, to the definite integral using one trapezium as shown.
  - **ii** Find the error $E_1 = A_1 - \int_0^2 2^x\, dx$.

ISBN 978-1-009-11049-5 Cambridge University Press

c i Find an approximation, A_2, to the definite integral using two trapeziums as shown.

ii Find the error $E_2 = A_2 - \int_0^2 2^x \, dx$.

d Continuing in this way, find A_4 and E_4, then find A_8 and E_8. (You will notice that doubling the number of trapeziums decreases the error by about a factor of 4.)

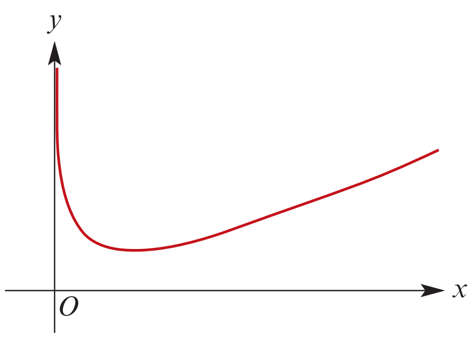


e Repeat this procedure for the definite integral $\int_0^2 x^2 \, dx$. Find the approximations and errors using one, two, four and eight trapeziums. How many trapeziums would be needed for an approximation to be within 10^{-6} of the definite integral?

23 The graph of the function

$$f(x) = x - \ln x, \quad x > 0$$

is shown on the right.



a Determine $f'(x)$ and show that:

i $f'(x) < 0$ for $0 < x < 1$

ii $f'(x) = 0$ for $x = 1$

iii $0 < f'(x) < 1$ for $x > 1$.

b Hence state the coordinates of the local minimum on the graph of $y = f(x)$.

c Let n be an integer with $n \geq 2$. Find the value of x such that $f'(x) = \frac{1}{n}$.

d Find the value of a such that the tangent to the graph of $y = f(x)$ at point $P(a, f(a))$ passes through the origin.

e Determine the equation of the tangent to the graph of $y = f(x)$ at $x = e^{-1}$.

f Determine the equation of the tangent to the graph of $y = f(x)$ at $x = e^n$, where n is a positive integer, and state the y-axis intercept of this tangent.

g Differentiate $x \ln x$ and hence find an anti-derivative of $x - \ln x$.

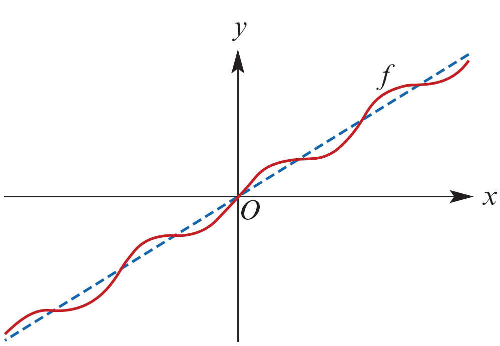
h Evaluate $\int_1^e f(x) \, dx$.

24 Consider the function f given by $f(x) = x + \sin x$ for $-4\pi \leq x \leq 4\pi$.

a Find $f'(x)$ and $f''(x)$.

b Show that $f'(x) \geq 0$ for all x.

c Find the coordinates of the stationary points of inflection on the graph of f.



Now consider $g(x) = \frac{x}{2} + \sin x$ for $-2\pi \leq x \leq 2\pi$.

d Solve the equation $g'(x) = 0$ for $-2\pi \leq x \leq 2\pi$.

e Find the coordinates of the stationary points on the graph of g.

ISBN 978-1-009-11049-5 Cambridge University Press

12D Algorithms and pseudocode

An introduction to pseudocode is given in Appendix A of this book and the reader is referred to that appendix for explanations of the terms used in this section. You may like to use a device to implement the algorithms in this section; see the coding appendices in the Interactive Textbook for instructions.

Skill-sheet

1 Consider the following two approximations for $f'(a)$, where h is small:

- $f'(a) \approx \dfrac{f(a+h) - f(a)}{h}$
- $f'(a) \approx \dfrac{f(a+h) - f(a-h)}{2h}$

For each of the following functions describe an algorithm using pseudocode to compare these approximations at the given values.

a $f(x) = \sin(x),\ x = \dfrac{\pi}{3}$ **b** $f(x) = \log_e(x),\ x = 2.5$ **c** $f(x) = x^4 - \log_e x,\ x = 1$

Note: Use a **while loop** based on the closeness to the exact value of the derivative. Start with $h = 0.5$ and decrease by a factor of 2. That is, $0.5, 0.25, 0.125, \ldots$ Record the iteration number and use this to comment on the 'speed of convergence' of each method.

2 Newton's method

The following algorithm can be used to solve $-x^3 + 5x^2 - 3x + 4 = 0$ near $x = 4$. The table shows the result of executing the algorithm. The first row gives the initial values of x and $f(x)$. The next rows give the values that are printed at the end of each pass of the `while` loop.

```
define f(x):
  return -x^3 + 5x^2 - 3x + 4

define Df(x):
  return -3x^2 + 10x - 3

x ← 3.8
while f(x) > 10^(-6) or f(x) < -10^(-6)
    x ← x - f(x)/Df(x)
    print x, f(x)
end while
```

	x	$f(x)$
Initial	3.8	9.928
Pass 1	4.99326923	−10.81199119
Pass 2	4.60526316	−1.44403339
Pass 3	4.53507148	−0.04308844
Pass 4	4.53284468	−0.00004266
Pass 5	4.53284247	0.00000000

The while instruction can be written more efficiently, using the absolute value function as `while` $|f(x)| > 10^{-6}$.

a Use the pseudocode algorithm for Newton's method to find an approximate solution of the given equation with the given starting value x_0 using a tolerance of 10^{-4}. Preferably use a device to implement your code..

i $\sin 2x = x,\ x_0 = 1$ **ii** $\cos 2x = x,\ x_0 = 1$

iii $\log_e x = 0.25x,\ x_0 = 1$ **iv** $e^x - \log_e x = 3,\ x_0 = 1$

v $\sin x - \log_e x = 3,\ x_0 = 2$ **vi** $(x-2)^2 - \log_e x = 0,\ x_0 = 2$

b Write the iterative rule for Newton's method applied to each of these functions.

ISBN 978-1-009-11049-5 © Michael Evans et al 2023 Cambridge University Press

Revision

3 Using Newton's method find the values of x_1, x_2 and x_3 by completing three passes of the `while` loop.

a $x^3 - x - 4 = 0, \quad x_0 = 1.5$

b $x^4 - x - 13 = 0, \quad x_0 = 2$

c $-x^3 - 2x^2 + 1 = 0, \quad x_0 = 0.5$

d $e^x + x + 1 = 0, \quad x_0 = 0.6$

4 **Halley's method** Let f be a nicely behaved function, (f', f'' and f''' all defined). Define the iterative rule:

$$x_{n+1} = x_n - \frac{2f(x_n)f'(x_n)}{2[f'(x_n)]^2 - f(x_n)f''(x_n)}$$

Describe an algorithm using Halley's method to solve the equation $\sin x = \frac{x}{4}$. Use $x_0 = 3$

5 **The trapezium method for approximating areas** In this question, we use pseudocode to describe algorithms including the trapezium method for estimating the area of the region between a graph and the x-axis

The algorithm given on the right finds the trapezoidal estimate for $\int_0^5 f(x)\,dx$ using 10 strips, where $f(x) = x^3 + 2x^2 + 3$.

a Find the estimate given by this algorithm. (Preferably use a device to implement the algorithm.)

b Modify the code to find:

i the left-endpoint estimate using 10 rectangles

ii the right-endpoint estimate using 10 rectangles.

c Modify the code to find an estimate of $\int_0^3 2^x\,dx$ using 100 strips.

```
define f(x):
  return x^3 + 2x^2 + 3

a ← 0
b ← 5
n ← 10
h ← (b − a)/n
left ← a
right ← a + h
sum ← 0
for i from 1 to n
    strip ← 0.5 × (f(left) + f(right)) × h
    sum ← sum + strip
    left ← left + h
    right ← right + h
end for
print sum
```

ISBN 978-1-009-11049-5 Cambridge University Press