13

Discrete random variables and their probability distributions

Objectives

- To review the basic concepts of **probability**.
- To define **discrete random variables**.
- To define the **probability distribution** of a discrete random variable.
- To calculate and interpret **expected value** (**mean**) for a discrete random variable.
- To calculate and interpret **variance** and **standard deviation** for a discrete random variable.

Uncertainty is involved in much of the reasoning we undertake every day of our lives. We are often required to make decisions based on the chance of a particular occurrence. Some events can be predicted from our present store of knowledge, such as the time of the next high tide. Others, such as whether a head or tail will show when a coin is tossed, are not predictable.

Ideas of uncertainty are pervasive in everyday life, and the use of chance and risk models makes an important impact on many human activities and concerns. Probability is the study of chance and uncertainty.

In this chapter we will extend our knowledge of probability by introducing the concept of the probability distribution (also known as the probability mass function) for a discrete random variable. Using this distribution we can determine the theoretical values of two important parameters which describe the random variable: the mean and the standard deviation. We will see that together the mean and the standard deviation tell us a lot about the distribution of the variable under consideration.

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13A Sample spaces and probability

In this section we will review the fundamental concepts of probability, the numerical value which we assign to give a measure of the likelihood of an outcome of an experiment. Probability takes a value between 0 and 1, where a probability of 0 means that the outcome is impossible, and a probability of 1 means that it is certain. Generally, the probability of an outcome will be somewhere in between, with a higher value meaning that the outcome is more likely.

Sample spaces and events

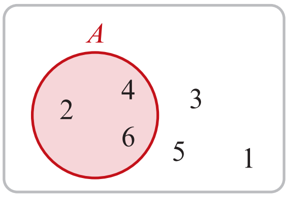
When a six-sided die is rolled, the possible outcomes are the numbers 1, 2, 3, 4, 5, 6. Rolling a six-sided die is an example of a **random experiment**, since while we can list all the possible outcomes, we do not know which one will be observed.

The possible outcomes are generally listed as the elements of a set, and the set of all possible outcomes is called the **sample space** and denoted by the Greek letter ε (epsilon). Thus, for this example:

$$\varepsilon = \{1, 2, 3, 4, 5, 6\}$$

An **event** is a subset of the sample space, usually denoted by a capital letter. If the event A is defined as 'an even number when a six-sided die is rolled', we write

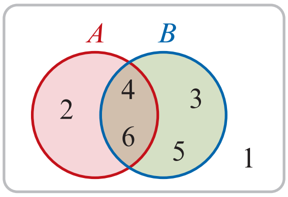
$$A = \{2, 4, 6\}$$



If A and B are two events, then the **union** of A and B, denoted by $A \cup B$, is equivalent to either event A or event B or both occurring.

Thus, if event A is 'an even number when a six-sided die is rolled' and event B is 'a number greater than 2 when a six-sided die is rolled', then $A = \{2, 4, 6\}$, $B = \{3, 4, 5, 6\}$ and

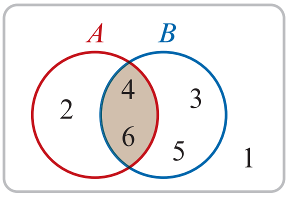
$$A \cup B = \{2, 3, 4, 5, 6\}$$



The **intersection** of A and B, denoted by $A \cap B$, is equivalent to both event A and event B occurring.

Thus, using the events A and B already described:

$$A \cap B = \{4, 6\}$$



In some experiments, it is helpful to list the elements of the sample space systematically by means of a tree diagram.

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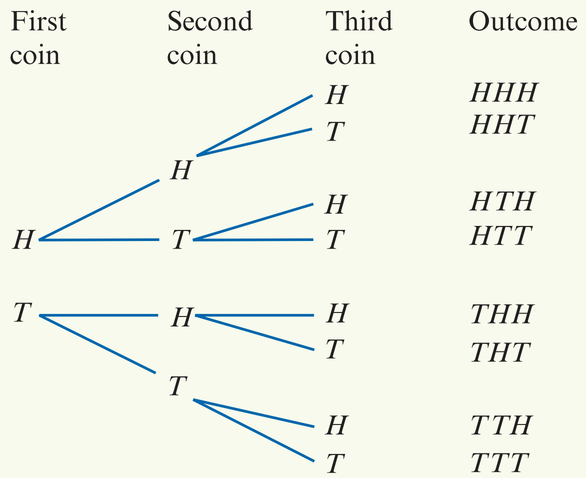
Example 1

Find the sample space when three coins are tossed and the results noted.

Solution

To list the elements of the sample space, construct a tree diagram:



Each path along the branches of the tree identifies an outcome, giving the sample space as

$$\varepsilon = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Determining probabilities for equally likely outcomes

Probability is a numerical measure of the chance of a particular event occurring. There are many approaches to determining probability, but often we assume that all of the possible outcomes are equally likely.

We require that the probabilities of all the outcomes in the sample space sum to 1, and that the probability of each outcome is a non-negative number. This means that the probability of each outcome must lie in the interval $[0, 1]$. Since six equally likely outcomes are possible when rolling a die, we can assign the probability of each outcome to be $\frac{1}{6}$. That is,

$$\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = \frac{1}{6}$$

When the sample space is finite, the **probability of an event** is equal to the sum of the probabilities of the outcomes in that event.

For example, let A be the event that an even number is rolled on the die. Then $A = \{2, 4, 6\}$ and $\Pr(A) = \Pr(2) + \Pr(4) + \Pr(6) = \frac{1}{2}$. Since the outcomes are equally likely, we can calculate this more easily as

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{3}{6} = \frac{1}{2}$$

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Equally likely outcomes

In general, if the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each of these outcomes.

Then the probability of any event A which contains m of these outcomes is the ratio of the number of elements in A to the number of elements in ε. That is,

$$\Pr(A) = \frac{n(A)}{n(\varepsilon)} = \frac{m}{n}$$

where the notation $n(S)$ is used to represent the number of elements in set S.

We will see that there are other methods of determining probabilities. But whichever method is used, the following rules of probability will hold:

- $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
- $\Pr(\varepsilon) = 1$
- The sum of the probabilities of all outcomes of an experiment is 1.
- $\Pr(\varnothing) = 0$, where $\varnothing$ represents the empty set
- $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
- $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**

When two events A and B have no outcomes in common, i.e. when they cannot occur together, they are called **mutually exclusive** events. In this case, we have $\Pr(A \cap B) = 0$ and so the addition rule becomes:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B), \quad \text{the addition rule when } A \text{ and } B \text{ are mutually exclusive}$$

We illustrate some of these rules in the following example.

Example 2

If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a an ace **b** not a heart **c** an ace or a heart **d** either a king or an ace?

Solution

a Let A be the event 'the card drawn is an ace'. A standard deck of cards contains four aces, so

$$\Pr(A) = \frac{4}{52} = \frac{1}{13}$$

b Let H be the event 'the card drawn is a heart'. There are 13 cards in each suit, so

$$\Pr(H) = \frac{13}{52} = \frac{1}{4}$$

and therefore

$$\Pr(H') = 1 - \Pr(H) = 1 - \frac{1}{4} = \frac{3}{4}$$

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c Using the addition rule:

$$\Pr(A \cup H) = \Pr(A) + \Pr(H) - \Pr(A \cap H)$$

Now $\Pr(A \cap H) = \frac{1}{52}$, since the event $A \cap H$ corresponds to drawing the ace of hearts. Therefore

$$\Pr(A \cup H) = \frac{1}{13} + \frac{1}{4} - \frac{1}{52} = \frac{4}{13}$$

d Let K be the event 'the card drawn is a king'. We observe that $K \cap A = \varnothing$. That is, the events K and A are mutually exclusive. Hence

$$\Pr(K \cup A) = \Pr(K) + \Pr(A) = \frac{1}{13} + \frac{1}{13} = \frac{2}{13}$$

Example 3

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?

	Age < 25	Age ≥ 25	Total
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected from these 500. Find the probability that:

a the person regularly uses social media

b the person is less than 25 years of age

c the person is less than 25 years of age and does not regularly use social media.

Solution

a $\Pr(\text{Yes}) = \frac{300}{500} = \frac{3}{5}$

Explanation

There are 300 out of 500 people who say yes.

b $\Pr(\text{Age} < 25) = \frac{240}{500} = \frac{12}{25}$

There are 240 out of 500 people who are less than 25 years of age.

c $\Pr(\text{No} \cap \text{Age} < 25) = \frac{40}{500} = \frac{2}{25}$

There are 40 out of 500 people who are less than 25 years of age and say no.

Other methods of determining probabilities

When we are dealing with a random experiment which does not have equally likely outcomes, other methods of determining probability are required.

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Subjective probabilities

Sometimes, the probability is assigned a value on the basis of judgement. For example, a farmer may look at the weather conditions and determine that there is a 70% chance of rain that day, and take appropriate actions. Such probabilities are called **subjective probabilities**.

Probabilities from data

A better way to estimate an unknown probability is by experimentation: by performing the random experiment many times and recording the results. This information can then be used to estimate the chances of the event happening again in the future. The proportion of trials that resulted in this event is called the **relative frequency** of the event. (For most purposes we can consider proportion and relative frequency as interchangeable.) That is,

$$\text{Relative frequency of event } A = \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}}$$

This information can then be used to estimate the probability of the event.

When the number of trials is sufficiently large, the observed relative frequency of an event A becomes close to the probability $\Pr(A)$. That is,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

If the experiment was repeated, it would generally be found that the results were slightly different. One might conclude that relative frequency is not a very good way of estimating probability. In many situations, however, experiments are the only way to get at an unknown probability. One of the most valuable lessons to be learnt is that such estimates are not exact, and will in fact vary from sample to sample.

Understanding the variation between estimates is extremely important in the study of statistics, and this is the topic of Chapter 17. At this stage it is valuable to realise that the variation does exist, and that the best estimates of the probabilities will result from using as many trials as possible.

Example 4

Suppose that a die is tossed 1000 times and the following outcomes observed:

Outcome	1	2	3	4	5	6
Frequency	135	159	280	199	133	97

a Use this information to estimate the probability of observing a 6 when this die is rolled.

b What outcome would you predict to be most likely the next time the die is rolled?

Solution

a $\Pr(6) \approx \dfrac{97}{1000} = 0.097$

b The most likely outcome is 3, since it has the highest relative frequency.

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Probabilities from area

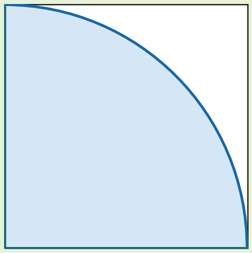
When we use the model of equally likely outcomes to determine probabilities, we count both the outcomes in the event and the outcomes in the sample space, and use the ratio to determine the probability of the event.

This idea can be extended to calculate probabilities when areas are involved, by assuming that the probabilities of all points in the region (which can be considered to be the sample space) are equally likely.

Example 5

A dartboard consists of a square of side length 2 metres containing a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.

If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.



Solution

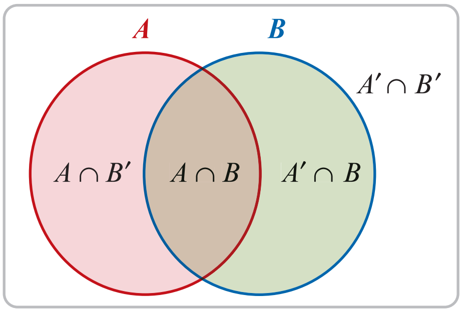
Area of blue region $= \frac{1}{4}\pi r^2 = \frac{1}{4}\pi \times 4 = \pi \text{ m}^2$

Area of dartboard $= 2 \times 2 = 4 \text{ m}^2$

$$\Pr(\text{hitting blue region}) = \frac{\text{area of blue region}}{\text{area of dartboard}} = \frac{\pi}{4}$$

Probability tables

A **probability table** is an alternative to a Venn diagram when illustrating a probability problem diagrammatically. Consider the Venn diagram which illustrates two intersecting sets A and B.



From the Venn diagram it can be seen that the sample space is divided by the sets into four disjoint regions: $A \cap B$, $A \cap B'$, $A' \cap B$ and $A' \cap B'$. These regions may be represented in a table as follows. Such a table is sometimes referred to as a **Karnaugh map**.

	B	B'
A	$A \cap B$	$A \cap B'$
A'	$A' \cap B$	$A' \cap B'$

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In a probability table, the entries give the probabilities of each of these events occurring.

	B	B'
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$

Summing the rows and columns, we can complete the table as shown.

	B	B'	
A	$\Pr(A \cap B)$	$\Pr(A \cap B')$	$\Pr(A)$
A'	$\Pr(A' \cap B)$	$\Pr(A' \cap B')$	$\Pr(A')$
	$\Pr(B)$	$\Pr(B')$	1

These tables can be useful when solving problems involving probability, as shown in the next example.

Example 6

Simone visits the dentist every 6 months for a checkup. The probability that she will need her teeth cleaned is 0.35, the probability that she will need a filling is 0.1 and the probability that she will need both is 0.05.

a What is the probability that she will not need her teeth cleaned on a visit, but will need a filling?

b What is the probability that she will not need either of these treatments?

Solution

The information in the question may be entered into a table as shown, where we use C to represent 'cleaning' and F to represent 'filling'.

	F	F'	
C	0.05		0.35
C'			
	0.1		1

All the empty cells in the table may now be filled in by subtraction:

	F	F'	
C	0.05	0.3	0.35
C'	0.05	0.6	0.65
	0.1	0.9	1

a The probability that she will not need her teeth cleaned but will need a filling is given by $\Pr(C' \cap F) = 0.05$.

b The probability that she will not need either of these treatments is $\Pr(C' \cap F') = 0.6$.

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Summary 13A

- The **sample space**, ε, for a random experiment is the set of all possible outcomes.
- An **event** is a subset of the sample space. The probability of an event A occurring is denoted by $\Pr(A)$.
- Equally likely outcomes If the sample space ε for an experiment contains n outcomes, all of which are equally likely to occur, we assign a probability of $\frac{1}{n}$ to each outcome. Then the probability of an event A is given by

$$\Pr(A) = \frac{\text{number of outcomes in } A}{\text{total number of outcomes}} = \frac{n(A)}{n(\varepsilon)}$$

- Estimates of probability When a probability is unknown, it can be estimated by the relative frequency obtained through repeated trials of the random experiment under consideration. In this case,

$$\Pr(A) \approx \frac{\text{number of times event } A \text{ occurs}}{\text{number of trials}} \qquad \text{for a large number of trials}$$

- Whichever method of determining probability is used, the rules of probability hold:
  - $\Pr(A) \geq 0$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - The sum of the probabilities of all outcomes of an experiment is 1.
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**
- If two events A and B are **mutually exclusive** (i.e. if A and B have no outcomes in common), then $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.

Exercise 13A

Example 1

1 An experiment consists of rolling a die and tossing a coin. Use a tree diagram to list the sample space for the experiment.

2 Two coins are tossed and a die is rolled. Use a tree diagram to show all the possible outcomes.

Example 2

3 If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

a a queen

b not a club

c a queen or a heart

d either a king or a queen?

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4 A six-sided die is marked with a 1 on two sides, a 2 on one side, and a 3 on the remaining three sides. Find the probability that when the die is rolled:

a a 3 shows **b** a 2 or a 3 shows.

5 Suppose that the probability that a student owns a smartphone is 0.7, the probability that they own a laptop is 0.6, and the probability that they own both is 0.5. What is the probability that a student owns either a smartphone or a laptop or both?

6 At a particular university, the probability that an Arts student studies a language is 0.3, literature is 0.6, and both is 0.25. What is the probability that an Arts student studies either a language or literature or both?

7 A computer manufacturer notes that 5% of their computers are returned owing to faulty disk drives, 2% are returned owing to faulty keyboards, and 0.3% are returned because both disk drives and keyboards are faulty. Find the probability that the next computer manufactured will be returned with:

a a faulty disk drive or a faulty keyboard

b a faulty disk drive and a working keyboard.

8 A new drug has been released and produces some minor side effects: 8% of users suffer only loss of sleep, 12% of users suffer only nausea, and 75% of users have no side effects at all. What percentage of users suffer from both loss of sleep and nausea?

9 In a particular town, the probability that an adult owns a car is 0.7, while the probability that an adult owns a car and is employed is 0.6. If a randomly selected adult is found to own a car, what is the probability that he or she is also employed?

Example 3 **10** An insurance company analysed the records of 500 drivers to determine the relationship between age and accidents in the last year.

	Accidents in the last year				
Age	0	1	2	3	Over 3
Under 20	19	35	25	17	10
20–29	30	45	33	39	17
30–39	40	33	15	6	2
40–49	18	15	10	3	1
Over 49	21	25	17	13	11

What is the probability that a driver chosen from this group at random:

a is under 20 years old and has had three accidents in the last year

b is from 40 to 49 years old and has had no accidents in the last year

c is from 20 to 29 years old

d has had more than three accidents in the last year?

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11 200 people were questioned and classified according to sex and whether or not they think private individuals should be allowed to carry guns. The results are shown in the table.

Do you think private individuals should be allowed to carry guns?

	Male	Female	Total
Yes	70	60	130
No	50	20	70
Total	120	80	200

One person is selected at random from these 200.

a What is the probability that the person thinks private individuals should be allowed to carry guns?

b What is the probability that the person is male and thinks private individuals should be allowed to carry guns?

Example 4 **12** Use the given data to estimate the probability of the specified event occurring:

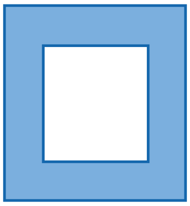
a Pr(head) if a coin is tossed 200 times and 114 heads observed

b Pr(ten) if a spinner is spun 380 times and lands on the 'ten' 40 times

c Pr(two heads) if two coins are tossed 200 times and two heads are observed on 54 occasions

d Pr(three sixes) if three dice are rolled 500 times and three sixes observed only twice

Example 5 **13** Suppose that a square dartboard consists of a white square of side length 30 cm inside a larger blue square of side length 50 cm, as shown. If a dart thrown at the board has equal chance of landing anywhere on the board, what is the probability it lands in the white area? (Ignore the possibility that it might land on the line or miss the board altogether.)

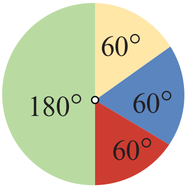


14 A spinner is as shown in the diagram. Find the probability that when spun the pointer will land on:

a the green section

b the yellow section

c any section except the yellow section.



Example 6 **15** In a particular country it has been established that the probability that a person drinks tea is 0.45, the probability that a person drinks coffee is 0.65, and the probability that a person drinks neither tea nor coffee is 0.22. Use the information to complete a probability table and hence determine the probability that a randomly selected person in that country:

a drinks tea but not coffee

b drinks tea and coffee.

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16 A chocolate is chosen at random from a box of chocolates. It is known that in this box:

- the probability that the chocolate is dark but not soft-centred is 0.15
- the probability that the chocolate is not dark but is soft-centred is 0.42
- the probability that the chocolate is not dark is 0.60.

Find the probability that the randomly chosen chocolate is:

a dark **b** soft-centred **c** not dark and not soft-centred.

17 Records indicate that, in Australia, 65% of secondary students participate in sport, and 71% of secondary students play computer games. They also show that 53% of students participate in sport and play computer games Use this information to find the probability that a student selected at random:

a does not participate in sport

b plays computer games and does not participate in sport

c does not play computer games and participates in sport

d does not play computer games and does not participate in sport.

18 If A and B are events such that $\Pr(A) = 0.42$, $\Pr(B) = 0.76$, and $\Pr(A \cup B) = 0.82$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cap B')$

13B Conditional probability and independence

The probability of an event A occurring when it is known that some event B has occurred is called conditional probability and is written $\Pr(A \mid B)$. This is usually read as 'the probability of A given B', and can be thought of as a means of adjusting probability in the light of new information.

Sometimes, the probability of an event is not affected by knowing that another event has occurred. For example, if two coins are tossed, then the probability of the second coin showing a head is independent of whether the first coin shows a head or a tail. Thus,

$$\begin{aligned}&\Pr(\text{head on second coin} \mid \text{head on first coin})\\&= \Pr(\text{head on second coin} \mid \text{tail on first coin})\\&= \Pr(\text{head on second coin})\end{aligned}$$

For other situations, however, a previous result may alter the probability. For example, the probability of rain today given that it rained yesterday will generally be different from the probability that it will rain today given that it didn't rain yesterday.

The **conditional probability** of an event A, given that event B has already occurred, is given by

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

This formula may be rearranged to give the **multiplication rule of probability**:

$$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

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The probabilities associated with a multi-stage experiment can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).

Example 7

In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

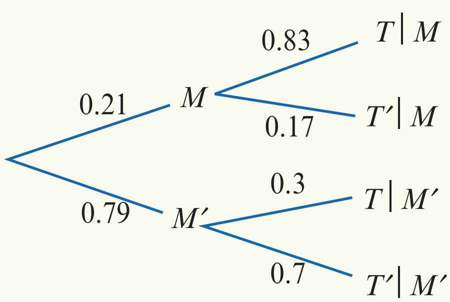
a on both Monday and Tuesday

b on Tuesday.

Solution

Let M represent the event 'rain on Monday' and T represent the event 'rain on Tuesday'.

The situation described in the question can be represented by a tree diagram. You can check that the probabilities are correct by seeing if they add to 1.



$\Pr(T \cap M) = 0.21 \times 0.83 = 0.1743$

$\Pr(T' \cap M) = 0.21 \times 0.17 = 0.0357$

$\Pr(T \cap M') = 0.79 \times 0.3 = 0.237$

$\Pr(T' \cap M') = 0.79 \times 0.7 = 0.553$

a The probability that it rains on both Monday and Tuesday is given by

$$\begin{aligned}\Pr(T \cap M) &= 0.21 \times 0.83\\ &= 0.1743\end{aligned}$$

b The probability that it rains on Tuesday is given by

$$\begin{aligned}\Pr(T) &= \Pr(T \cap M) + \Pr(T \cap M')\\ &= 0.1743 + 0.237\\ &= 0.4113\end{aligned}$$

The solution to part **b** of Example 7 is an application of a rule known as the law of total probability. This can be expressed in general terms as follows:

The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

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Example 8

Adrienne, Regan and Michael are doing the dishes. Since Adrienne is the oldest, she washes the dishes 40% of the time. Regan and Michael each wash 30% of the time. When Adrienne washes the probability of at least one dish being broken is 0.01, when Regan washes the probability is 0.02, and when Michael washes the probability is 0.03. Their parents don't know who is washing the dishes one particular night.

a What is the probability that at least one dish will be broken?

b Given that at least one dish is broken, what is the probability that the person washing was Michael?

Solution

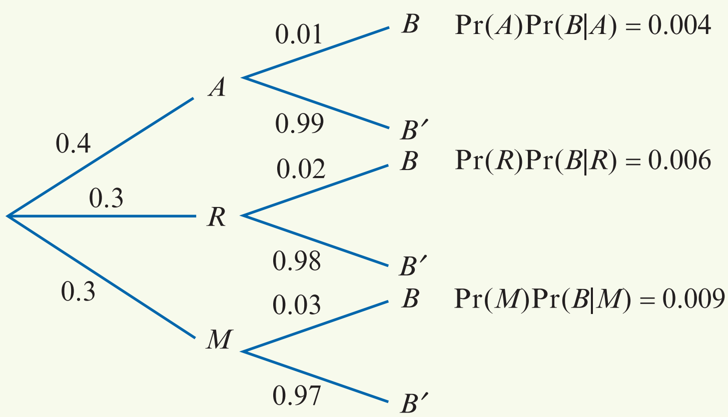
Let A be the event 'Adrienne washes the dishes', let R be the event 'Regan washes the dishes' and let M be the event 'Michael washes the dishes'. Then

$$\Pr(A) = 0.4, \qquad \Pr(R) = 0.3, \qquad \Pr(M) = 0.3$$

Let B be the event 'at least one dish is broken'. Then

$$\Pr(B \mid A) = 0.01, \qquad \Pr(B \mid R) = 0.02, \qquad \Pr(B \mid M) = 0.03$$

This information can be summarised in a tree diagram as shown:



a The probability of at least one dish being broken is

$$\begin{aligned}\Pr(B) &= \Pr(B \cap A) + \Pr(B \cap R) + \Pr(B \cap M)\\ &= \Pr(A)\Pr(B \mid A) + \Pr(R)\Pr(B \mid R) + \Pr(M)\Pr(B \mid M)\\ &= 0.004 + 0.006 + 0.009\\ &= 0.019\end{aligned}$$

b The required probability is

$$\begin{aligned}\Pr(M \mid B) &= \frac{\Pr(M \cap B)}{\Pr(B)}\\ &= \frac{0.009}{0.019} = \frac{9}{19}\end{aligned}$$

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Example 9

As part of an evaluation of the school canteen, all students at a Senior Secondary College (Years 10–12) were asked to rate the canteen as poor, good or excellent. The results are shown in the table.

	Year			
Rating	10	11	12	Total
Poor	30	20	10	60
Good	80	65	35	180
Excellent	60	65	35	160
Total	170	150	80	400

What is the probability that a student chosen at random from this college:

a is in Year 12

b is in Year 12 and rates the canteen as excellent

c is in Year 12, given that they rate the canteen as excellent

d rates the canteen as excellent, given that they are in Year 12?

Solution

Let T be the event 'the student is in Year 12' and let E be the event 'the rating is excellent'.

a $\Pr(T) = \frac{80}{400} = \frac{1}{5}$

From the table, we can see that there are 80 students in Year 12 and 400 students altogether.

b $\Pr(T \cap E) = \frac{35}{400} = \frac{7}{80}$

From the table, there are 35 students who are in Year 12 and also rate the canteen as excellent.

c $\Pr(T \mid E) = \frac{35}{160} = \frac{7}{32}$

From the table, a total of 160 students rate the canteen as excellent, and of these 35 are in Year 12.

d $\Pr(E \mid T) = \frac{35}{80} = \frac{7}{16}$

From the table, there are 80 students in Year 12, and of these 35 rate the canteen as excellent.

Note: The answers to parts **c** and **d** could also have been found using the rule for conditional probability, but here it is easier to determine the probability directly from the table.

Independent events

Two events A and B are **independent** if the probability of A occurring is the same, whether or not B has occurred.

Independent events

For events A and B with $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$, the following three conditions are all equivalent conditions for the independence of A and B:

- $\Pr(A \mid B) = \Pr(A)$
- $\Pr(B \mid A) = \Pr(B)$
- $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$

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Notes:

- Sometimes this definition of independence is referred to as **pairwise independence**.
- In the special case that $\Pr(A) = 0$ or $\Pr(B) = 0$, the condition $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$ holds since both sides are zero, and so we say that A and B are independent.

Example 10

The probability that Monica remembers to do her homework is 0.7, while the probability that Patrick remembers to do his homework is 0.4. If these events are independent, then what is the probability that:

a both will do their homework

b Monica will do her homework but Patrick forgets?

Solution

Let M be the event 'Monica does her homework' and let P be the event 'Patrick does his homework'. Since these events are independent:

a
$$\begin{aligned}\Pr(M \cap P) &= \Pr(M) \times \Pr(P)\\ &= 0.7 \times 0.4\\ &= 0.28\end{aligned}$$

b
$$\begin{aligned}\Pr(M \cap P') &= \Pr(M) \times \Pr(P')\\ &= 0.7 \times 0.6\\ &= 0.42\end{aligned}$$

Summary 13B

- Conditional probability
  - The probability of an event A occurring when it is known that some event B has already occurred is called conditional probability and is written $\Pr(A \mid B)$.
  - In general, the **conditional probability** of an event A, given that event B has already occurred, is given by

    $$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

    This formula may be rearranged to give the **multiplication rule of probability**:

    $$\Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B)$$

- Law of total probability

  The **law of total probability** states that, in the case of two events A and B,

  $$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Independence

  Two events A and B are **independent** if the occurrence of one event has no effect on the probability of the occurrence of the other, that is, if

  $$\Pr(A \mid B) = \Pr(A)$$

  Events A and B are independent if and only if

  $$\Pr(A \cap B) = \Pr(A) \times \Pr(B)$$

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Skill-sheet

Exercise 13B

1 In a certain town, the probability that it rains on any Saturday is 0.25. If it rains on Saturday, then the probability of rain on Sunday is 0.8. If it does not rain on Saturday, then the probability of rain on Sunday is 0.1. For a given week, find the probability that:

Example 7

a it rains on both Saturday and Sunday
b it rains on neither day
c it rains on Sunday.

2 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(A \cap B) = 0.1$, find:

a $\Pr(B \mid A)$
b $\Pr(A \mid B)$

3 Given that for two events A and B, $\Pr(A) = 0.6$, $\Pr(B) = 0.3$ and $\Pr(B \mid A) = 0.1$, find:

a $\Pr(A \cap B)$
b $\Pr(A \mid B)$

4 In Alia's school, the probability that a student studies French is 0.5, and the probability that they study both French and Chinese is 0.3. Find the probability that a student studies Chinese, given that they study French.

Example 8

5 The chance that a harvest is poorer than average is 0.5 when there is no disease present, but if it is known that a certain disease D is present, this probability increases to 0.8. The disease D is present in 30% of harvests. Find the probability that, when a harvest is observed to be poorer than average, the disease D is present.

Example 9

6 A group of 1000 eligible voters were asked their age and their preference in an upcoming election, with the following results.

	Age			
Preference	18–25	26–40	Over 40	Total
Candidate A	200	100	85	385
Candidate B	250	230	50	530
No preference	50	20	15	85
Total	500	350	150	1000

What is the probability that a person chosen from this group at random:

a is 18–25 years of age
b prefers candidate A
c is 18–25 years of age, given that they prefer candidate A
d prefers candidate A, given that they are 18–25 years of age?

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7 The following data was derived from accident records on a highway noted for its above-average accident rate.

	Probable cause			
Type of accident	Speed	Alcohol	Other	Total
Fatal	42	61	12	115
Non-fatal	88	185	60	333
Total	130	246	72	448

Use the table to find:

a the probability that speed is the cause of the accident
b the probability that the accident is fatal
c the probability that the accident is fatal, given that speed is the cause
d the probability that the accident is fatal, given that alcohol is the cause.

Example 10

8 The probability of James winning a particular tennis match is independent of Sally winning another particular tennis match. If the probability of James winning is 0.8 and the probability of Sally winning is 0.3, find:

a the probability that they both win
b the probability that either or both of them win.

9 An experiment consists of drawing a number at random from $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Let $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 3, 5, 7, 9\}$ and $C = \{4, 6, 8, 9\}$.

a Are A and B independent? **b** Are A and C independent?
c Are B and C independent?

10 If A and B are independent events such that $\Pr(A) = 0.5$ and $\Pr(B) = 0.4$, find:

a $\Pr(A \mid B)$ **b** $\Pr(A \cap B)$ **c** $\Pr(A \cup B)$

11 Nathan knows that his probability of kicking more than four goals on a wet day is 0.3, while on a dry day it is 0.6. The probability that it will be wet on the day of the next game is 0.7. Calculate the probability that Nathan will kick more than four goals in the next game.

12 Two events A and B are independent. If $\Pr(B) = 3 \times \Pr(A)$, and $\Pr(A \cap B) = 0.1452$, find the value of $\Pr(A)$.

13 A and B are events such that $\Pr(A) = \frac{1}{2}\Pr(B)$. If A and B are independent, and $\Pr(A \cup B) = 0.28$, find the value of $\Pr(A)$.

14 Find the probability that, in three tosses of a fair coin, there are three heads, given that there is at least one head.

15 A die is tossed and the number showing recorded. It is then tossed again and the number again recorded. The sum of the two numbers is then calculated. Find the probability that the sum of the two recorded numbers is 8, given that the first recorded number is odd.

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16 The test used to determine if a person suffers from a particular disease is not perfect. The probability of a person with the disease returning a positive result is 0.95, while the probability of a person without the disease returning a positive result is 0.02. The probability that a randomly selected person has the disease is 0.03. What is the probability that a randomly selected person will return a positive result?

17 Anya goes through three sets of traffic lights when she cycles to school each morning. The probability she stops at the first set is 0.6. If she stops at any one set, the probability that she has to stop at the next is 0.9. If she doesn't have to stop at any one set, the probability that she doesn't have to stop at the next is 0.7. Use a tree diagram to find the probability that:

a she stops at all three sets of lights **b** she stops only at the second set of lights

c she stops at exactly one set of lights.

18 There are four red socks and two blue socks in a drawer. Two socks are removed at random. What is the probability of obtaining:

a two red socks **b** two blue socks **c** one of each colour?

19 A car salesperson was interested in the relationship between the size of the car a customer purchased and their marital status. From the sales records, the table on the right was constructed.
What is the probability that a person chosen at random from this group:

	Marital status		
Size of car	Married	Single	Total
Large	60	20	80
Medium	100	60	160
Small	90	70	160
Total	250	150	400

a drives a small car **b** is single and drives a small car

c is single, given that they drive a small car

d drives a small car, given that they are single?

20 Jenny has two boxes of chocolates. Box A contains three white chocolates and four dark chocolates. Box B contains two white chocolates and five dark chocolates. Jenny first chooses a box at random and then selects a chocolate at random from it. Find the probability that:

a Jenny selects a white chocolate

b given that Jenny selects a white chocolate, it was chosen from box A.

21 A bag contains three red, four white and five black balls. If three balls are taken without replacement, what is the probability that they are all the same colour?

22 At a particular petrol station, 30% of customers buy premium unleaded, 60% buy standard unleaded and 10% buy diesel. When a customer buys premium unleaded, there is a 25% chance they will fill the tank. Of the customers buying standard unleaded, 20%

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fill their tank. Of those buying diesel, 70% fill their tank.

a What is the probability that, when a car leaves the petrol station, it will not have a full tank?

b Given that a car leaving the petrol station has a full tank, what is the probability that the tank contains standard unleaded petrol?

13C Discrete random variables

Suppose that three balls are drawn at random from a jar containing four white and six black balls, with replacement (i.e. each selected ball is replaced before the next draw). The sample space for this random experiment is as follows:

$$\varepsilon = \{WWW, WWB, WBW, BWW, WBB, BWB, BBW, BBB\}$$

Suppose the variable of interest is the number of white balls in the sample. This corresponds to a simpler sample space whose outcomes are numbers.

If X represents the number of white balls in the sample, then the possible values of X are 0, 1, 2 and 3. Since the actual value that X will take is the result of a random experiment, we say that X is a random variable.

A **random variable** is a function that assigns a number to each outcome in the sample space ε.

A random variable can be discrete or continuous:

- A **discrete random variable** is one that can take only a countable number of values. For example, the number of white balls in a sample of size three is a discrete random variable which may take one of the values 0, 1, 2, 3. Other examples include the number of children in a family, and a person's shoe size. (Note that discrete random variables do not have to take only whole-number values.)
- A **continuous random variable** is one that can take any value in an interval of the real number line, and is usually (but not always) generated by measuring. Height, weight, and the time taken to complete a puzzle are all examples of continuous random variables.

In this chapter we are interested in understanding more about discrete random variables.

Consider again the sample space for the random experiment described above. Each outcome in the sample space is associated with a value of X:

Experiment outcome	Value of X
WWW	$X = 3$
WWB	$X = 2$
WBW	$X = 2$
BWW	$X = 2$

Experiment outcome	Value of X
WBB	$X = 1$
BWB	$X = 1$
BBW	$X = 1$
BBB	$X = 0$

Associated with each event is a probability. Since the individual draws of the ball from the jar are independent events, we can determine the probabilities by multiplying and adding appropriate terms.

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Example 11

A jar contains four white and six black balls. What is the probability that, if three balls are drawn at random from the jar, with replacement, a white ball will be drawn exactly once (i.e. the situations where $X = 1$ in the table)?

Solution

$X = 1$ corresponds to the outcomes WBB, BWB and BBW.

Since there are 10 balls in total, $\Pr(W) = \frac{4}{10} = 0.4$ and $\Pr(B) = \frac{6}{10} = 0.6$.

$$\begin{aligned}\text{Thus } \Pr(X = 1) &= \Pr(WBB) + \Pr(BWB) + \Pr(BBW)\\ &= (0.4 \times 0.6 \times 0.6) + (0.6 \times 0.4 \times 0.6) + (0.6 \times 0.6 \times 0.4)\\ &= 0.432\end{aligned}$$

Discrete probability distributions

The **probability distribution** for a discrete random variable consists of all the values that the random variable can take, together with the probability of each of these values. For example, if a fair die is rolled, then the probability distribution is:

x	1	2	3	4	5	6
$\Pr(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probability distribution of a discrete random variable X is described by a function

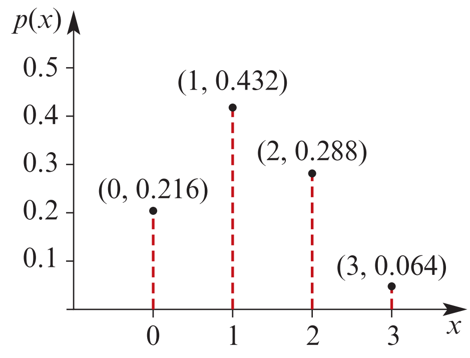
$$p(x) = \Pr(X = x)$$

This function is called a **discrete probability function** or a **probability mass function**.

Consider again the black and white balls from Example 11. The probability distribution for X, the number of white balls in the sample, is given by the following table:

x	0	1	2	3
$p(x)$	0.216	0.432	0.288	0.064

The probability distribution may also be given graphically, as shown on the right.



Note that the probabilities in the table sum to 1, which must occur if all values of the random variable have been listed.

We will use the following notation, which is discussed further in Appendix A:

- the sum of all the values of $p(x)$ is written as $\sum_{x} p(x)$
- the sum of the values of $p(x)$ for x between a and b inclusive is written as $\sum_{a \le x \le b} p(x)$

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For *any* discrete probability function $p(x)$, the following two conditions must hold:

1 Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

$$0 \le p(x) \le 1 \quad \text{for all } x$$

2 The sum of all the values of $p(x)$ must be 1. That is,

$$\sum_x p(x) = 1$$

To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Example 12

Consider the table shown.

x	0	1	2	3
$p(x)$	0.2	0.3	0.1	0.4

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \le 2)$.

Solution

a Yes, each value of $p(x)$ is between 0 and 1, and the values add to 1.

b
$$\begin{aligned}\Pr(X \le 2) &= p(0) + p(1) + p(2)\\ &= 0.2 + 0.3 + 0.1\\ &= 0.6\end{aligned}$$

Example 13

Let X be the number of heads showing when a fair coin is tossed three times.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more heads show.

c Find the probability that more than one head shows.

Solution

a The sample space is $\varepsilon = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$.

Now
$$\begin{aligned}p(0) &= \Pr(X = 0) = \Pr(\{TTT\}) &&= \frac{1}{8}\\ p(1) &= \Pr(X = 1) = \Pr(\{HTT, THT, TTH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}\\ p(2) &= \Pr(X = 2) = \Pr(\{HHT, HTH, THH\}) &&= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}\\ p(3) &= \Pr(X = 3) = \Pr(\{HHH\}) &&= \frac{1}{8}\end{aligned}$$

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Thus the probability distribution of X is:

x	0	1	2	3
$p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b The probability that one or more heads shows is

$$\Pr(X \geq 1) = p(1) + p(2) + p(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

c The probability that more than one head shows is

$$\Pr(X > 1) = \Pr(X \geq 2) = p(2) + p(3) = \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Example 14

The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6	7
$p(x)$	0.01	0.25	0.40	0.30	0.02	0.02

Find:

a $\Pr(X \geq 4)$ **b** $\Pr(X \geq 4 \mid X > 2)$ **c** $\Pr(X < 5 \mid X > 2)$

Solution

a
$$\begin{aligned}\Pr(X \geq 4) &= \Pr(X = 4) + \Pr(X = 5) + \Pr(X = 6) + \Pr(X = 7)\\ &= 0.4 + 0.3 + 0.02 + 0.02\\ &= 0.74\end{aligned}$$

b
$$\begin{aligned}\Pr(X \geq 4 \mid X > 2) &= \frac{\Pr(X \geq 4)}{\Pr(X > 2)}\\ &= \frac{0.74}{0.99} \qquad \text{since } \Pr(X > 2) = 1 - 0.01 = 0.99\\ &= \frac{74}{99}\end{aligned}$$

c
$$\begin{aligned}\Pr(X < 5 \mid X > 2) &= \frac{\Pr(2 < X < 5)}{\Pr(X > 2)}\\ &= \frac{\Pr(X = 3) + \Pr(X = 4)}{\Pr(X > 2)}\\ &= \frac{0.65}{0.99}\\ &= \frac{65}{99}\end{aligned}$$

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Summary 13C

- For *any* discrete probability function $p(x)$, the following two conditions must hold:
  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

     $$0 \le p(x) \le 1 \quad \text{for all } x$$

  2. The sum of all the values of $p(x)$ must be 1. That is,

     $$\sum_x p(x) = 1$$

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

  $$\Pr(a \le X \le b) = \sum_{a \le x \le b} p(x)$$

Skill-sheet

Exercise 13C

1 Which of the following random variables are discrete?

a the number of people in your family
b waist measurement
c shirt size
d the number of times a die is rolled before obtaining a six

2 Which of the following random variables are discrete?

a your age
b your height to the nearest centimetre
c the time you will wait to be served at the bank
d the number of people in the queue at the bank

Example 11

3 A fair coin is tossed three times and the number of heads noted.

a List the sample space.
b List the possible values of the random variable X, the number of heads, together with the corresponding outcomes.
c Find $\Pr(X \ge 2)$.

Example 12

4 Consider the following table:

x	0	1	2	3	4
$p(x)$	0.1	0.2	0.1	0.4	0.2

a Does this meet the conditions to be a discrete probability distribution?
b Use the table to find $\Pr(X \le 3)$.

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Example 13

5 A jar contains four red and five blue balls. A ball is withdrawn, its colour is observed, and it is then replaced. This is repeated three times. Let X be the number of red balls among the three balls withdrawn.

a Find the probability distribution of X and show that all the probabilities sum to 1.

b Find the probability that one or more red balls are obtained.

c Find the probability that more than one red ball is obtained.

Example 14

6 Two dice are rolled and the numbers noted.

a List the sample space.

b A random variable Y is defined as the total of the numbers showing on the two dice. List the possible values of Y, together with the corresponding outcomes.

c Find:

i $\Pr(Y < 5)$ **ii** $\Pr(Y = 3 \mid Y < 5)$ **iii** $\Pr(Y \leq 3 \mid Y < 7)$

iv $\Pr(Y \geq 7 \mid Y > 4)$ **v** $\Pr(Y = 7 \mid Y > 4)$ **vi** $\Pr(Y = 7 \mid Y < 8)$

7 A die is weighted as follows:

$$\Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = 0.2, \qquad \Pr(1) = \Pr(6) = 0.1$$

The die is rolled twice, and the smaller of the numbers showing is noted. Let Y represent this value.

a List the sample space.

b List the possible values of Y.

c Find $\Pr(Y = 1)$.

8 Suppose that three balls are selected at random, with replacement, from a jar containing four white and six black balls. If X is the number of white balls in the sample, find:

a $\Pr(X = 2)$ **b** $\Pr(X = 3)$

c $\Pr(X \geq 2)$ **d** $\Pr(X = 3 \mid X \geq 2)$

9 A fair die is rolled twice and the numbers noted. Define the following events:

A = 'a four on the first roll'

B = 'a four on the second roll'

C = 'the sum of the two numbers is at least eight'

D = 'the sum of the two numbers is at least 10'

a List the sample space obtained.

b Find $\Pr(A)$, $\Pr(B)$, $\Pr(C)$ and $\Pr(D)$.

c Find $\Pr(A \mid B)$, $\Pr(A \mid C)$ and $\Pr(A \mid D)$.

d Which of the following pairs of events are independent?

i A and B **ii** A and C **iii** A and D

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10 Consider the table shown on the right.

x	0	1	2	3
$p(x)$	0.1	0.4	0.2	0.3

a Does this meet the conditions to be a discrete probability distribution?

b Use the table to find $\Pr(X \geq 2)$.

11 Which of the following is *not* a probability distribution?

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.5	0.7

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0.25	0.5	0.75	1.0
$p(x)$	−0.5	−0.25	0.25	0.5

d

x	10	20	30	40
$p(x)$	10%	20%	30%	40%

12 Three balls are selected from a jar containing four black and six red balls. Find the probability distribution of the number of black balls in the sample:

a if the ball chosen is replaced after each selection

b if the ball chosen is not replaced after each selection.

13 A coin is known to be biased such that the probability of obtaining a head on any toss is 0.4. Find the probability distribution of X, the number of heads observed when the coin is tossed twice.

14 A spinner is numbered from 1 to 5, and each of the five numbers is equally likely to come up. Find:

a the probability distribution of X, the number showing on the spinner

b $\Pr(X \geq 3)$, the probability that the number showing on the spinner is three or more

c $\Pr(X \leq 3 \mid X \geq 3)$

15 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of X, the sum of the numbers showing on the two dice.

c Draw a graph of the probability distribution of X.

d Find $\Pr(X \geq 9)$, the probability that the sum of the two numbers showing is nine or more.

e Find $\Pr(X \leq 10 \mid X \geq 9)$.

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16 Two dice are rolled and the numbers noted.

a List the sample space for this experiment.

b Find the probability distribution of Y, the remainder when the larger number showing is divided by the smaller number. (Note that, if the two numbers are the same, then $Y = 0$.)

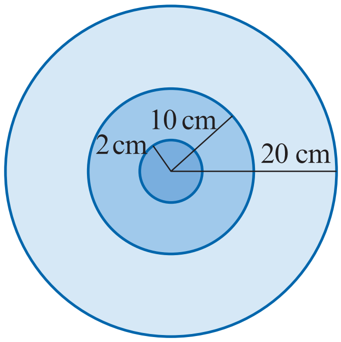
c Draw a graph of the probability distribution of Y.

17 Suppose that two socks are drawn without replacement from a drawer containing four red and six black socks. Let X represent the number of red socks obtained.

a Find the probability distribution for X.

b From the probability distribution, determine the probability that a pair of socks is obtained.

18 A dartboard consists of three circular sections, with radii of 2 cm, 10 cm and 20 cm respectively, as shown in the diagram.



When a dart lands in the centre circle the score is 100 points, in the middle circular section the score is 20 points and in the outer circular section the score is 10 points. Assume that all darts thrown hit the board, each dart is equally likely to land at any point on the dartboard, and none lands on the lines.

a Find the probability distribution for X, the number of points scored on one throw.

b Find the probability distribution for Y, the total score when two darts are thrown.

19 Erin and Nick are going to play a tennis match. Suppose that they each have an equal chance of winning any set (0.5) and that they plan to play until one player has won three sets. Let X be the number of sets played until the match is complete.

a Find $\Pr(X = 3)$.

b List the outcomes that correspond to $X = 4$, and use this to find $\Pr(X = 4)$.

c Hence, or otherwise, find $\Pr(X = 5)$.

20 In a particular game a player tosses two different coins. The probability of a head with Coin A is 0.6. The probability that coin B shows a head is 0.4. The player loses one point for each head thrown, and gains two points for each tail thrown. Let X be the sum of the points obtained from the two coin tosses. Find the probability distribution of X.

13D Expected value (mean), variance and standard deviation

From your studies of statistics, you may already be familiar with the mean as a measure of centre and with the variance and the standard deviation as measures of spread. When these are calculated from a set of data, they are termed 'sample statistics'. It is also possible to use the probability distribution to determine the theoretically 'true' values of the mean, variance and standard deviation. When they are calculated from the probability distribution, they are called 'population parameters'. Determining the values of these parameters is the topic for this section.

Expected value (mean)

When the mean of a random variable is determined from the probability distribution, it is generally called the **expected value** of the random variable. Expected value has a wide variety of applications. The concept of expected value first arose in gambling problems, where gamblers wished to know how much they could expect to win or lose in the long run, in order to decide whether or not a particular game was a good investment.

Example 15

A person may buy a lucky ticket for \$1. They have a 20% chance of winning \$2, a 5% chance of winning \$11, and otherwise they lose. How much could you expect to win (or lose) per game in the long run?

Solution

Let P be the amount the person will profit from each game. As it costs \$1 to play, the person can lose \$1 ($P = -1$), win \$1 ($P = 1$) or win \$10 ($P = 10$). Thus the amount that the person may win, \$$P$, has a probability distribution given by:

p	-1	1	10
$\Pr(P = p)$	0.75	0.20	0.05

Suppose you played the game 1000 times. You would expect to lose \$1 about 750 times, to win \$1 about 200 times and to win \$10 about 50 times. Thus, you would win about

$$\frac{-1 \times 750 + 1 \times 200 + 10 \times 50}{1000} = -\$0.05 \text{ per game}$$

Thus your 'expectation' is to lose 5 cents per game, and we write this as

$$\mathrm{E}(P) = -0.05$$

Note: This value gives an indication of the worth of the game: in the long run, you would expect to lose about 5 cents per game. This is called the **expected value** of P (or the **mean** of P). It is not the amount we expect to profit on any one game. (You cannot lose 5 cents in one game!) It is the amount that we expect to win on average per game in the long run.

Example 15 demonstrates how the expected value of a random variable X is determined.

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The **expected value** of a discrete random variable X is determined by summing the products of each value of X and the probability that X takes that value.

That is,

$$\begin{aligned}\mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$

The expected value $\mathrm{E}(X)$ may be considered as the long-run average value of X. It is generally denoted by the Greek letter μ (mu), and is also called the **mean** of X.

Example 16

A coin is biased in favour of heads such that the probability of obtaining a head on any single toss is 0.6. The coin is tossed three times and the results noted. If X is the number of heads obtained on the three tosses, find $\mathrm{E}(X)$, the expected value of X.

Solution

The following probability distribution can be found by listing the outcomes in the sample space and determining the value of X and the associated probability for each outcome.

x	0	1	2	3
$p(x)$	0.064	0.288	0.432	0.216

$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot p(x)\\ &= (0 \times 0.064) + (1 \times 0.288) + (2 \times 0.432) + (3 \times 0.216)\\ &= 0.288 + 0.864 + 0.648\\ &= 1.8\end{aligned}$$

Note: This means that, if the experiment were repeated many times, then an average of 1.8 heads per three tosses would be observed.

Sometimes we wish to find the expected value of a function of X. This is determined by calculating the value of the function for each value of X, and then summing the products of these values and the associated probabilities.

The expected value of $g(X)$ is given by

$$E[g(X)] = \sum_x g(x) \cdot p(x)$$

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Example 17

For the random variable X defined in Example 16, find:

a $\mathrm{E}(3X+1)$ **b** $\mathrm{E}(X^2)$

Solution

a $$\mathrm{E}(3X+1) = \sum_x (3x+1)\cdot p(x)$$
$$= (1\times 0.064) + (4\times 0.288) + (7\times 0.432) + (10\times 0.216)$$
$$= 6.4$$

b $$\mathrm{E}(X^2) = \sum_x x^2\cdot p(x)$$
$$= (0^2\times 0.064) + (1^2\times 0.288) + (2^2\times 0.432) + (3^2\times 0.216)$$
$$= 3.96$$

Let us compare the values found in Example 17 with the value of $\mathrm{E}(X)$ found in Example 16. In part **a**, we found that $\mathrm{E}(3X+1) = 6.4$. Since $3\mathrm{E}(X)+1 = 3\times 1.8+1 = 6.4$, we see that

$$\mathrm{E}(3X+1) = 3\mathrm{E}(X)+1$$

In part **b**, we found that $\mathrm{E}(X^2) = 3.96$. Since $[\mathrm{E}(X)]^2 = 1.8^2 = 3.24$, we see that

$$\mathrm{E}(X^2) \neq [\mathrm{E}(X)]^2$$

These two examples illustrate an important point concerning expected values.

In general, the expected value of a function of X is not equal to that function of the expected value of X. That is,

$$\mathrm{E}[g(X)] \neq g[\mathrm{E}(X)]$$

An exception is when the function is linear:

Expected value of $aX+b$

$$\mathrm{E}(aX+b) = a\mathrm{E}(X)+b \qquad \text{(for } a, b \text{ constant)}$$

This is illustrated in Example 18.

Example 18

For a \$5 monthly fee, a TV repair company guarantees customers a complete service. The company estimates the probability that a customer will require one service call in a month as 0.05, the probability of two calls as 0.01 and the probability of three or more calls as 0.00. Each call costs the repair company \$40. What is the TV repair company's expected monthly gain from such a contract?

Solution

We may summarise the given information in the following table.

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Calls	0	1	2	≥ 3
Gain, g	5	−35	−75	
$\Pr(G = g)$	0.94	0.05	0.01	0.00

$$\begin{aligned} \mathrm{E}(G) &= \sum_{g=0}^{2} g \cdot \Pr(G = g) \\ &= 5 \times 0.94 - 35 \times 0.05 - 75 \times 0.01 \\ &= 2.20 \end{aligned}$$

Thus, the company can expect to gain \$2.20 per month on each contract sold.

Alternative solution

An alternative method of solution uses the formula for the expected value of $aX + b$, as follows.

Let X be the number of calls received. Then

$$G = 5 - 40X$$

and so $\quad \mathrm{E}(G) = 5 - 40 \times \mathrm{E}(X)$

Since $\quad \mathrm{E}(X) = 1 \times 0.05 + 2 \times 0.01 = 0.07$

we have $\quad \mathrm{E}(G) = 5 - 40 \times 0.07 = 2.20 \quad$ as previously determined.

Another useful property of expectation is that the expected value of the sum of two random variables is equal to the sum of their expected values. That is, if X and Y are two random variables, then

$$\mathrm{E}(X + Y) = \mathrm{E}(X) + \mathrm{E}(Y)$$

Measures of variability: variance and standard deviation

As well as knowing the long-run average value of a random variable (the mean), it is also useful to have a measure of how close to this mean are the possible values of the random variable — that is, a measure of how spread out the probability distribution is. The most useful measures of variability for a discrete random variable are the variance and the standard deviation.

The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean or expected value μ. It is defined as

$$\mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

and may be considered as the long-run average value of the square of the distance from X to μ. The variance is usually denoted by σ^2, where σ is the lowercase Greek letter *sigma*.

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From the definition,

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\ &= \sum_x (x-\mu)^2 \cdot \Pr(X=x)\end{aligned}$$

Since the variance is determined by squaring the distance from X to μ, it is no longer in the units of measurement of the original random variable X. A measure of spread in the appropriate unit is found by taking the square root of the variance.

> The **standard deviation** of X is defined as
>
> $$\mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$
>
> The standard deviation is usually denoted by σ.

Using the definition is not always the easiest way to calculate the variance.

> An alternative (computational) formula for variance is
>
> $$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

Proof We already know that

$$\mathrm{E}(aX+b) = a\mathrm{E}(X) + b \qquad (1)$$

and $$\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y) \qquad (2)$$

Hence

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}[(X-\mu)^2] \\ &= \mathrm{E}(X^2 - 2\mu X + \mu^2) \\ &= \mathrm{E}(X^2) + \mathrm{E}(-2\mu X + \mu^2) && \text{using (2)} \\ &= \mathrm{E}(X^2) - 2\mu\mathrm{E}(X) + \mu^2 && \text{using (1)} \\ &= \mathrm{E}(X^2) - 2\mu^2 + \mu^2 && \text{since } \mu = \mathrm{E}(X) \\ &= \mathrm{E}(X^2) - \mu^2\end{aligned}$$

Example 19

For the probability distribution shown, find $\mathrm{E}(X^2)$ and $[\mathrm{E}(X)]^2$ and hence find the variance and standard deviation of X.

x	0	1	2	3
$\Pr(X = x)$	0.08	0.18	0.4	0.34

Solution

We have

$$\mathrm{E}(X) = 1 \times 0.18 + 2 \times 0.4 + 3 \times 0.34 = 2$$

$$[\mathrm{E}(X)]^2 = \mu^2 = 4$$

$$\mathrm{E}(X^2) = 1 \times 0.18 + 4 \times 0.4 + 9 \times 0.34 = 4.84$$

Hence $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mu^2 = 4.84 - 4 = 0.84$ and $\mathrm{sd}(X) = \sqrt{0.84} = 0.9165$

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It is often useful to determine the variance of the linear function of a random variable, for which we can use the following rule.

Variance of $aX + b$

$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X) \qquad \text{(for } a, b \text{ constant)}$$

Example 20

If X is a random variable such that $\mathrm{Var}(X) = 9$, find:

a $\mathrm{Var}(3X + 2)$

b $\mathrm{Var}(-X)$

Solution

a
$$\begin{aligned}\mathrm{Var}(3X + 2) &= 3^2\mathrm{Var}(X)\\ &= 9 \times 9\\ &= 81\end{aligned}$$

b
$$\begin{aligned}\mathrm{Var}(-X) &= \mathrm{Var}(-1 \times X)\\ &= (-1)^2\mathrm{Var}(X)\\ &= \mathrm{Var}(X)\\ &= 9\end{aligned}$$

Summary 13D

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,
$$\begin{aligned}\mu = \mathrm{E}(X) &= \sum_x x \cdot \Pr(X = x)\\ &= \sum_x x \cdot p(x)\end{aligned}$$
- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as
$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$
An alternative (computational) formula for variance is
$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$
- The **standard deviation** of a random variable X is defined as
$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

Exercise 13D

Example 15

1 Tickets in a game of chance can be purchased for \$2. Each ticket has a 30% chance of winning \$2, a 10% chance of winning \$20, and otherwise loses. How much might you expect to win or lose if you play the game 100 times?

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Example 16

2 For each of the following probability distributions, find the mean (expected value):

a

x	1	3	5	7
$p(x)$	0.1	0.3	0.3	0.3

b

x	−1	0	1	2
$p(x)$	0.25	0.25	0.25	0.25

c

x	0	1	2	3	4	5	6	7
$p(x)$	0.09	0.22	0.26	0.21	0.13	0.06	0.02	0.01

d

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$p(x)$	0.08	0.13	0.09	0.19	0.20	0.03	0.10	0.18

e

x	2	3	4	5	6	7	8	9	10	11	12
$p(x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

f

x	−3	−2	−1	0	1	2	3
$p(x)$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$

3 A business consultant evaluates a proposed venture as follows. A company stands to make a profit of $10 000 with probability 0.15, to make a profit of $5000 with probability 0.45, to break even with probability 0.25, and to lose $5000 with probability 0.15. Find the expected profit.

4 A spinner is numbered from 0 to 5, and each of the six numbers has an equal chance of coming up. A player who bets $1 on any number wins $5 if that number comes up; otherwise the $1 is lost. What is the player's expected profit on the game?

5 Suppose that the probability of having a female child is not as high as that of having a male child, and that the probability distribution for the number of male children in a three-child family has been determined from past records as follows.

Number of males, x	0	1	2	3
$p(x)$	0.12	0.36	0.38	0.14

What is the mean number of males in a three-child family?

6 A player throws a die with faces numbered from 1 to 6 inclusive. If the player obtains a 6, she throws the die a second time, and in this case her score is the sum of 6 and the second number; otherwise her score is the number first obtained. The player has no more than two throws.

Let X be the random variable denoting the player's score. Write down the probability

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distribution of X, and determine the mean of X.

Example 17

7 The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6
$p(x)$	0.01	0.25	0.40	0.30	0.04

Calculate:

a $\mathrm{E}(X)$

b $\mathrm{E}(X^3)$

c $\mathrm{E}(5X - 4)$

d $\mathrm{E}\left(\dfrac{1}{X}\right)$

Example 18

8 Manuel is a car salesperson. In any week his probability of making sales is as follows:

Number of cars sold, x	2	3	4	5	6
$\Pr(X = x)$	0.45	0.25	0.20	0.08	0.02

If he is paid \$2000 commission on each car sold, what is his expected weekly income?

Example 19

9 A discrete random variable X takes values 0, 1, 2, 4, 8 with probabilities as shown in the table.

x	0	1	2	4	8
$\Pr(X = x)$	p	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$

a Find p.

b Find $\mathrm{E}(X)$.

c Find $\mathrm{Var}(X)$.

10 A biased die is such that the probability of any face landing uppermost is proportional to the number on that face. Thus, if X denotes the score obtained in one throw of this die, then $\Pr(X = r) = kr$ for $r = 1, 2, 3, 4, 5, 6$, where k is a constant.

a Find the value of k.

b Find $\mathrm{E}(X)$.

c Find $\mathrm{Var}(X)$.

11 An unbiased die is in the form of a regular tetrahedron and has its faces numbered 1, 2, 3 and 4. When the die is thrown onto a horizontal table, the number on the face in contact with the table is noted. Two such dice are thrown and the score, X, is found by multiplying these numbers together.

a Give the probability distribution of X.

b Determine the values of:

i $\Pr(X > 8)$ **ii** $\mathrm{E}(X)$ **iii** $\mathrm{Var}(X)$

12 A coin and a six-sided die are thrown simultaneously. The random variable X is defined as follows: If the coin shows a head, then X is the score on the die; if the coin shows a tail, then X is twice the score on the die.

a Find the expected value, μ, of X.

b Find $\Pr(X < \mu)$.

c Find $\mathrm{Var}(X)$.

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Example 20

13 If $\text{Var}(X) = 16$, find:

a $\text{Var}(2X)$ **b** $\text{Var}(X + 2)$ **c** $\text{Var}(1 - X)$ **d** $\text{sd}(3X)$

14 A random variable X has the probability distribution shown. Find:

x	1	2	3	4	5
$\Pr(X = x)$	c	0.3	0.1	0.2	0.05

a the constant c

b $\text{E}(X)$, the mean of X

c $\text{Var}(X)$, the variance of X, and hence the standard deviation of X

15 A random variable X has the probability distribution shown.

x	1	2	3	4	5
$\Pr(X = x)$	k	$2k$	$3k$	$4k$	$5k$

Find:

a the constant k **b** $\text{E}(X)$, the expectation of X

c $\text{Var}(X)$, the variance of X

16 Two dice are rolled. If X is the sum of the numbers showing on the two dice, find:

a $\text{E}(X)$, the mean of X

b $\text{Var}(X)$, the variance of X

17 The number of heads, X, obtained when a fair coin is tossed six times has the following probability distribution.

x	0	1	2	3	4	5	6
$p(x)$	0.0156	0.0937	0.2344	0.3126	0.2344	0.0937	0.0156

Find:

a $\text{E}(X)$, the mean of X

b $\text{Var}(X)$, the variance of X

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Chapter summary

Assignment

Nrich

Probability

- Probability is a numerical measure of the chance of a particular event occurring and may be determined experimentally or by symmetry.
- Whatever method is used to determine the probability, the following rules will hold:
  - $0 \le \Pr(A) \le 1$ for all events $A \subseteq \varepsilon$
  - $\Pr(\varnothing) = 0$ and $\Pr(\varepsilon) = 1$
  - $\Pr(A') = 1 - \Pr(A)$, where A' is the complement of A
  - $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$, the **addition rule**.
- Probabilities associated with combined events are sometimes able to be calculated more easily from a probability table.
- Two events A and B are **mutually exclusive** if $A \cap B = \varnothing$. In this case, we have $\Pr(A \cap B) = 0$ and therefore $\Pr(A \cup B) = \Pr(A) + \Pr(B)$.
- The **conditional probability** of event A occurring, given that event B has already occurred, is

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)} \qquad \text{if } \Pr(B) \neq 0$$

giving $\quad \Pr(A \cap B) = \Pr(A \mid B) \times \Pr(B) \qquad$ (the **multiplication rule**)

- The probabilities associated with multi-stage experiments can be calculated by constructing an appropriate tree diagram and multiplying along the relevant branches (from the multiplication rule).
- The **law of total probability** states that, in the case of two events A and B,

$$\Pr(A) = \Pr(A \mid B)\Pr(B) + \Pr(A \mid B')\Pr(B')$$

- Two events A and B are **independent** if

$$\Pr(A \mid B) = \Pr(A)$$

so whether or not B has occurred has no effect on the probability of A occurring.

- Events A and B are independent if and only if $\Pr(A \cap B) = \Pr(A) \times \Pr(B)$.

Discrete random variables

- A **discrete** random variable X is one which can take only a countable number of values. Often these values are whole numbers, but not necessarily.
- The **probability distribution** of X is a function $p(x) = \Pr(X = x)$ that assigns a probability to each value of X. It can be represented by a rule, a table or a graph, and must give a probability $p(x)$ for every value x that X can take.
- For *any* discrete probability distribution, the following two conditions must hold:

  1. Each value of $p(x)$ belongs to the interval $[0, 1]$. That is,

  $$0 \le p(x) \le 1 \quad \text{for all } x$$

  2. The sum of all the values of $p(x)$ must be 1. That is,

  $$\sum_x p(x) = 1$$

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Review

- To determine the probability that X takes a value in the interval from a to b (including the values a and b), add the values of $p(x)$ from $x = a$ to $x = b$:

$$\Pr(a \leq X \leq b) = \sum_{a \leq x \leq b} p(x)$$

- The **expected value** (or **mean**) of a discrete random variable X may be considered as the long-run average value of X. It is found by summing the products of each value of X and the probability that X takes that value. That is,

$$\mu = \mathrm{E}(X) = \sum_x x \cdot \Pr(X = x)$$
$$= \sum_x x \cdot p(x)$$

- The expected value of a function of X is given by

$$\mathrm{E}[g(X)] = \sum_x g(x) \cdot p(x)$$

- The **variance** of a random variable X is a measure of the spread of the probability distribution about its mean μ. It is defined as

$$\sigma^2 = \mathrm{Var}(X) = \mathrm{E}[(X - \mu)^2]$$

An alternative (computational) formula for variance is

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$$

- The **standard deviation** of a random variable X is defined as

$$\sigma = \mathrm{sd}(X) = \sqrt{\mathrm{Var}(X)}$$

- Linear function of a discrete random variable:

$$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$$
$$\mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$$

Technology-free questions

1 A box contains five black and four white balls. Find the probability that two balls drawn at random are of different colour if:

a the first ball drawn is replaced before the second is drawn.

b the balls are drawn without replacement.

2 A box contains m chocolates, q of which are milk chocolate, and the remainder are dark chocolate.

a If one chocolate is taken randomly from the box, what is the probability that it is dark?

b If two chocolates are taken randomly from the box without replacement, find an expression for the probability that they are both dark.

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3 A gambler has two coins, A and B; the probabilities of their turning up heads are 0.8 and 0.4 respectively. One coin is selected at random and tossed twice, and a head and a tail are observed. Find the probability that the coin selected was A.

4 A factory has two machines, Machine I and Machine II. Machine I produces 60% of the items each day, and Machine II produced 40%. The factory owner knows that 3% of the items from Machine I are faulty, and 2% of the items from Machine II are faulty.

a What is the probability that an item randomly selected from the day's production is faulty?

b If the selected item is faulty, what is the probability that it was produced by Machine II?

5 The probability distribution of a discrete random variable X is given by the following table. Show that $p = 0.5$ or $p = 1$.

x	0	1	2	3
$\Pr(X = x)$	$0.4p^2$	0.1	0.1	$1 - 0.6p$

6 A random variable X has the following probability distribution.

x	−1	0	1	2	3	4
$\Pr(X = x)$	k	$2k$	$3k$	$2k$	k	k

Find:

a the constant k **b** $\mathrm{E}(X)$, the mean of X **c** $\mathrm{Var}(X)$, the variance of X

7 If X has a probability function given by

$$p(x) = \frac{1}{4}, \quad x = 2, 4, 16, 64$$

find:

a $\mathrm{E}(X)$ **b** $\mathrm{Var}(X)$ **c** $\mathrm{sd}(X)$

8 A manufacturer sells cylinders for \$$x$ each; the cost of the manufacture of each cylinder is \$2. If a cylinder is defective, it is returned and the purchase money refunded. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is $\frac{1}{5}$.

a Let P be the profit per cylinder. Find the probability function of P.

b Find the mean of P in terms of x.

c How much should the manufacturer sell the cylinders for in order to make a profit in the long term?

9 A group of 1000 drivers were classified according to their age and the number of accidents they had been involved in during the previous year. The results are shown in the table.

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	Age < 30	Age ≥ 30
At most one accident	130	170
More than one accident	470	230

a Calculate the probability that, if a driver is chosen at random from this group, the driver is aged less than 30 and has had more than one accident.

b Calculate the probability that a randomly chosen driver is aged less than 30, given that he or she has had more than one accident.

10 This year, 70% of the population have been immunised against a certain disease. Records indicate that an immunised person has a 5% chance of contracting the disease, whereas for a non-immunised person the chance is 60%.

a Calculate the overall percentage of the population who are expected to contract the disease.

b Given that a person has been diagnosed with the disease, what is the probability that they had been immunised?

11 Given $\Pr(A) = \frac{1}{2}$, $\Pr(B) = \frac{1}{4}$ and $\Pr(A \mid B) = \frac{1}{6}$, find:

a $\Pr(A \cap B)$ **b** $\Pr(A \cup B)$ **c** $\Pr(A' \mid B)$ **d** $\Pr(A \mid B')$

Multiple-choice questions

1 Two event A and B are independent, where $\Pr(A) = \frac{\Pr(B)}{2}$, and $\Pr(A \cup B) = 0.405$. $\Pr(A)$ is equal to

A 0.15 **B** 0.27 **C** 0.3 **D** 0.45 **E** 0.135

2 A box contains four red balls and three yellow balls. Two balls are selected at random from the box without replacement. The probability that they are the same colour is

A $\frac{1}{7}$ **B** $\frac{2}{7}$ **C** $\frac{18}{49}$ **D** $\frac{12}{49}$ **E** $\frac{3}{7}$

3 Suppose that the random variable X has the probability distribution given in the following table:

x	-3	-2	-1	0	1	2	3
$\Pr(X = x)$	0.07	0.15	0.22	0.22	0.17	0.12	0.05

$\Pr(-3 \leq X < 0)$ is equal to

A 0.59 **B** 0.37 **C** 0.22 **D** 0.44 **E** 0.66

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4 Consider the following table:

x	−1	0	1	2	3
$\Pr(X = x)$	$2k$	$3k$	0.1	$3k$	$2k$

The mean of X is

A 0.09 **B** 0.27 **C** 1.0 **D** 1.27 **E** 1.45

5

x	1	2	3	4	5
$\Pr(X = x)$	0.46	0.24	0.14	0.09	0.07

$\Pr(X < \mu)$, where μ is the mean of X, is

A 0.24 **B** 0.46 **C** 0.38 **D** 0.70 **E** 0.84

6 The number times per week that students in a large school report they attend after school activities is a random variable with the following discrete probability distribution:

x	0	1	2	3	4	5
$\Pr(X = x)$	0.1	0.25	0.3	0.2	0.1	.05

If two students are selected at random, the probability that they attend after school activities the same number of times per week is

A 0.155 **B** 0.205 **C** 0.215 **D** 0.81 **E** 0.9

7 A random variable X is such that $E(X) = 1.20$ and $E(X^2) = 1.69$. The standard deviation of X is equal to

A 1.3 **B** $\sqrt{3.13}$ **C** 0.25 **D** 0.7 **E** 0.5

8 Suppose that a random variable X is such that $E(X) = 100$ and $\text{Var}(X) = 100$. Suppose further that Y is a random variable such that $Y = 3X + 10$. Then

A $E(Y) = 300$ and $\text{Var}(Y) = 900$ **B** $E(Y) = 310$ and $\text{Var}(Y) = 300$
C $E(Y) = 310$ and $\text{Var}(Y) = 900$ **D** $E(Y) = 300$ and $\text{Var}(Y) = 30$
E $E(Y) = 310$ and $\text{Var}(Y) = 100\sqrt{3}$

9 The random variable X has the probability distribution shown, where $0 < p < \frac{1}{3}$.
The mean of X is

x	−1	0	1
$\Pr(X = x)$	p	$2p$	$1 - 3p$

A 1 **B** 0 **C** $1 - 4p$ **D** $4p$ **E** $1 + 4p$

10 The random variable X has the probability distribution shown on the right.
If the mean of X is 0.2, then

x	−2	0	2
$\Pr(X = x)$	a	b	0.2

A $a = 0.2,\ b = 0.6$ **B** $a = 0.1,\ b = 0.7$ **C** $a = 0.4,\ b = 0.4$
D $a = 0.7,\ b = 0.1$ **E** $a = 0.5,\ b = 0.3$

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Extended-response questions

1 Given the following probability function:

x	1	2	3	4	5	6	7
$\Pr(X = x)$	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

a Find c.

b Evaluate $\Pr(X \geq 5)$.

c If $\Pr(X \leq k) > 0.5$, find the minimum value of k.

2 Janet and Alan are going to play a tennis match. The probability of Janet winning the first set is 0.3. After that, Janet's probability of winning a set is 0.6 if she has won the previous set, but only 0.4 if she has lost it. The match will continue until either Janet or Alan has won two sets.

a Construct a tree diagram to show the possible course of the match.

b Find the probability that:

i Janet will win

ii Alan will win.

c Let X be the number of sets played until the match is complete.

i Find the probability distribution of X.

ii Find the expected number of sets that the match will take, $E(X)$.

d Given that the match lasted three sets, find the probability that Alan won.

3 Five identical cards are placed face down on the table. Three of the cards are marked \$5 and the remaining two are marked \$10. A player picks two cards at random (without replacement) and is paid an amount equal to the sum of the values on the two cards. How much should the player pay to play if this is to be a fair game? (A fair game is considered to be one for which $E(X) = 0$, where X is the profit from the game.)

4 A manufacturing company has three assembly lines: A, B and C. It has been found that 95% of the products produced on assembly line A will be free from faults, 98% from assembly line B will be free from faults and 99% from assembly line C will be free from faults. Assembly line A produces 50% of the day's output, assembly line B produces 30% of the day's output, and the rest is produced on assembly line C. If an item is chosen at random from the company's stock, find the probability that it:

a was produced on assembly line A

b is defective, given that it came from assembly line A

c is defective

d was produced on assembly line A, given that it was found to be defective.

5 A recent study found that P, the number of passengers per car entering a city on the freeway on a workday morning, is given by:

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p	0	1	2	3	4	5
$\Pr(P = p)$	0.39	0.27	0.16	0.12	0.04	0.02

a **i** Compute $E(P)$, the mean number of passengers per car.

ii Compute $\mathrm{Var}(P)$ and hence find the standard deviation of P.

iii Find $\Pr(\mu - 2\sigma \le P \le \mu + 2\sigma)$.

b The fees for cars at a toll booth on the freeway are as follows:

- Cars carrying no passengers pay \$1 toll.
- Cars carrying one passenger pay \$0.40 toll.
- Cars carrying two or more passengers pay no toll.

Let T be the toll paid by a randomly selected car on the freeway.

i Construct the probability distribution of T.

ii Find $E(T)$, the mean toll paid per car.

iii Find $\Pr(\mu - 2\sigma \le T \le \mu + 2\sigma)$.

6 The random variable Y, the number of cars sold in a week by a car salesperson, has the following probability distribution:

y	0	1	2	3	4	5	6	7	8
$\Pr(Y = y)$	0.135	0.271	0.271	0.180	0.090	0.036	0.012	0.003	0.002

a Compute $E(Y)$, the mean number of sales per week.

b Compute $\mathrm{Var}(Y)$ and hence find the standard deviation of Y.

c The car salesperson is given a bonus as follows: If fewer than three cars are sold in the week, no bonus is given; if three or four cars are sold, a \$100 bonus is given; for more than four cars, the bonus is \$200. Let B be the bonus paid to the salesperson.

i Construct the probability distribution for B.

ii Find $E(B)$, the mean bonus paid.

7 A given investment scheme is such that there is a 10% chance of receiving a profit of 40% of the amount invested, a 15% chance of a 30% profit, a 25% chance of a 20% profit, a 20% chance of a 10% profit, a 15% chance of breaking even, a 10% chance of a 10% loss and a 5% chance of 20% loss.

a Find the mean and standard deviation of the percentage return on the amount invested.

b Find the probability that the percentage return on the amount invested is within two standard deviations of the mean.

c An investor investing in the scheme pays a brokerage fee of 2% on the amount invested and a tax of 40% on the return (= profit − brokerage) of the investment. (Assume that a loss results in a tax refund for this investment.)
Express the percentage gain in terms of the percentage return on the amount invested, and hence find the mean and standard deviation of the percentage gain.

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8 A concert featuring a popular singer is scheduled to be held in a large open-air theatre. The promoter is concerned that rain will cause people to stay away. A weather forecaster predicts that the probability of rain on any day at that particular time of the year is 0.33. If it does not rain, the promoter will make a profit of \$250 000 on the concert. If it does rain, the profit will be reduced to \$20 000. An insurance company agrees to insure the concert for \$250 000 against rain for a premium of \$60 000. Should the promoter buy the insurance?

9 A game is devised as follows: On two rolls of a single die, you will lose \$10 if the sum showing is 7, and win \$11 if the sum showing is either 11 or 12. How much should you win or lose if any other sum comes up in order for the game to be fair?

10 A new machine is to be developed by a manufacturing company. Prototypes are to be made until one satisfies the specifications of the company. Only then will it go into production. However, if after three prototypes are made none is satisfactory, then the project will be abandoned.
It is estimated that the probability a prototype will fail to produce a satisfactory model is 0.35, independent of any other already tested.

a Find the probability that:

- **i** the first prototype is successful
- **ii** the first is not successful but the second is
- **iii** the first two are not successful but the third is
- **iv** the project is abandoned.

b It is estimated that the cost of developing and testing the first prototype is \$7 million and that each subsequent prototype developed costs half of the one before. Find the expected cost of the project.

c If a machine is developed, then it is estimated that the income will be \$20 million. (If the project is abandoned, there is no income.) Find the expected profit.

11 Alfred and Bertie play a game, each starting with \$100. Two dice are thrown. If the total score is 5 or more, then Alfred pays $\$x$ to Bertie, where $0 < x \le 8$. If the total score is 4 or less, then Bertie pays $\$(x + 8)$ to Alfred.

a Find the expected value of Alfred's cash after the first game.

b Find the value of x for the game to be fair.

c Given that $x = 3$, find the variance of Alfred's cash after the first game.

12 A die is loaded such that the chance of throwing a 1 is $\frac{x}{4}$, the chance of a 2 is $\frac{1}{4}$ and the chance of a 6 is $\frac{1}{4}(1 - x)$. The chance of a 3, 4 or 5 is $\frac{1}{6}$. The die is thrown twice.

a Prove that the chance of throwing a total of 7 is $\dfrac{9x - 9x^2 + 10}{72}$.

b Find the value of x which will make this chance a maximum and find this maximum probability.

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