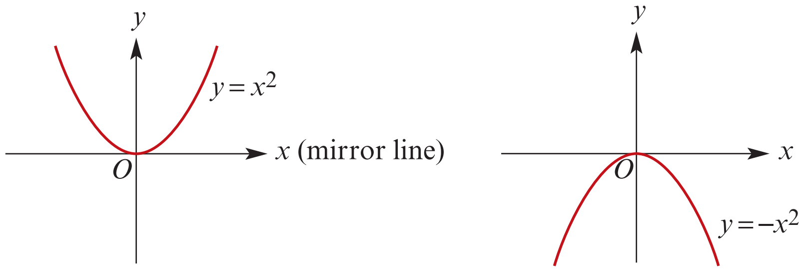
3

Transformations

Objectives

- To introduce a notation for considering transformations of the plane, including **translations**, **reflections** in an axis and **dilations** from an axis.
- To determine a sequence of transformations given the equation of a curve and its image.
- To use transformations to help with graph sketching.
- To consider transformations of power functions.
- To determine the rule for a function given sufficient information.

Many graphs of functions can be described as transformations of graphs of other functions, or 'movements' of graphs about the Cartesian plane. For example, the graph of the function $y = -x^2$ can be considered as a reflection in the x-axis of the graph of the function $y = x^2$.



A good understanding of transformations, combined with knowledge of the 'simplest' function and its graph in each family, provides an important tool with which to sketch graphs and identify rules of more complicated functions.

Transformations of the plane with rules of the form $(x, y) \to (ax + by, cx + dy)$ can be implemented through 2×2 matrices. An alternative approach to transformations using matrices is available on-line if you wish to pursue this method.

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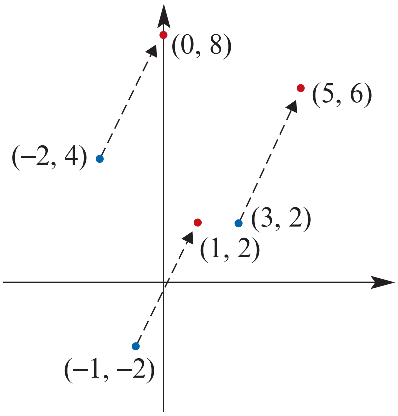
3A Translations

The **Cartesian plane** is represented by the set $\mathbb{R}^2$ of all ordered pairs of real numbers. That is, $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$. The transformations considered in this book associate each ordered pair of $\mathbb{R}^2$ with a unique ordered pair. We can refer to them as examples of **transformations of the plane**.

For example, the translation 3 units in the positive direction of the x-axis (to the right) associates with each ordered pair (x, y) a new ordered pair $(x + 3, y)$. This translation is a transformation of the plane. Each point in the plane is mapped to a unique second point. Furthermore, every point in the plane is an image of another point under this translation.

Notation

Consider the translation 2 units in the positive direction of the x-axis (to the right) and 4 units in the positive direction of the y-axis (up). This can be described by the rule $(x, y) \to (x + 2, y + 4)$. This reads as '$(x, y)$ maps to $(x + 2, y + 4)$'.



For example, $(3, 2) \to (3 + 2, 2 + 4)$.

In applying this translation, it is useful to think of every point (x, y) in the plane as being mapped to a new point (x', y'). This point (x, y) is the only point which maps to (x', y'). The following can be written for this translation:

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

- A translation of h units in the positive direction of the x-axis and k units in the positive direction of the y-axis is described by the rule

  $$(x, y) \to (x + h, y + k)$$

  or $\quad x' = x + h \quad$ and $\quad y' = y + k$

  where h and k are positive numbers.

- A translation of h units in the negative direction of the x-axis and k units in the negative direction of the y-axis is described by the rule

  $$(x, y) \to (x - h, y - k)$$

  or $\quad x' = x - h \quad$ and $\quad y' = y - k$

  where h and k are positive numbers.

Notes:

- Under a translation, if $(a', b') = (c', d')$, then $(a, b) = (c, d)$.
- For a translation $(x, y) \to (x + h, y + k)$, for each point $(a, b) \in \mathbb{R}^2$ there is a point (p, q) such that $(p, q) \to (a, b)$. (It is clear that $(p - h, q - k) \to (p, q)$ under this translation.)

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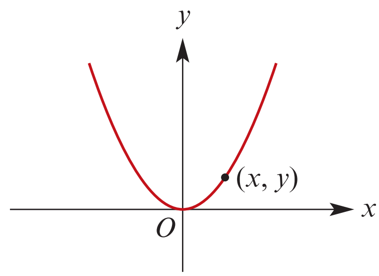
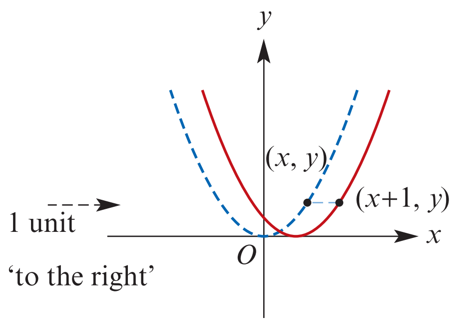
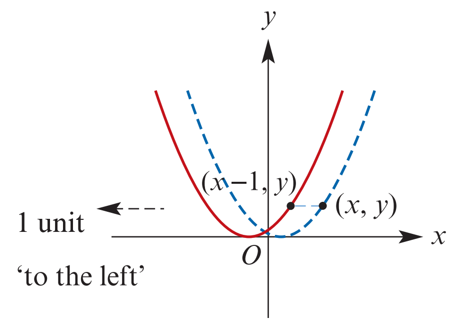
Applying translations to sketch graphs

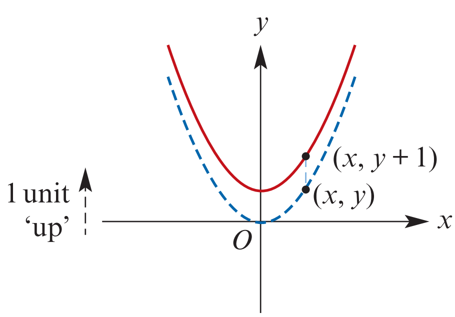
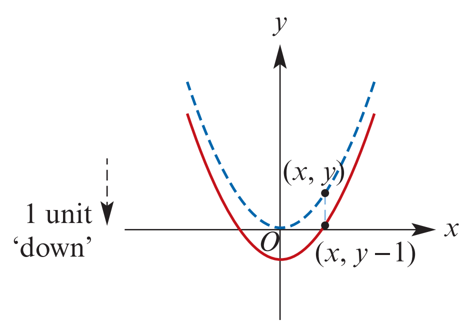
A translation moves every point on the graph the same distance in the same direction.

Translations parallel to an axis

We start by looking at the images of the graph of $y = x^2$ shown on the right under translations that are parallel to an axis.



Translation of 1 unit in the positive direction of the x-axis	Translation of 1 unit in the negative direction of the x-axis
 The point (x, y) is mapped onto $(x + 1, y)$, i.e. $(x, y) \to (x + 1, y)$. The image has equation $y = (x - 1)^2$.	 The point (x, y) is mapped onto $(x - 1, y)$, i.e. $(x, y) \to (x - 1, y)$. The image has equation $y = (x + 1)^2$.

Translation of 1 unit in the positive direction of the y-axis	Translation of 1 unit in the negative direction of the y-axis
 The point (x, y) is mapped onto $(x, y + 1)$, i.e. $(x, y) \to (x, y + 1)$. The image has equation $y = x^2 + 1$.	 The point (x, y) is mapped onto $(x, y - 1)$, i.e. $(x, y) \to (x, y - 1)$. The image has equation $y = x^2 - 1$.

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General translations of a curve

Every translation of the plane can be described by giving two components:

- a translation parallel to the x-axis and
- a translation parallel to the y-axis.

Consider a translation of 2 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis applied to the graph of $y = x^2$.

Translate the set of points defined by the function

$$\{ (x, y) : y = x^2 \}$$

by the translation defined by the rule

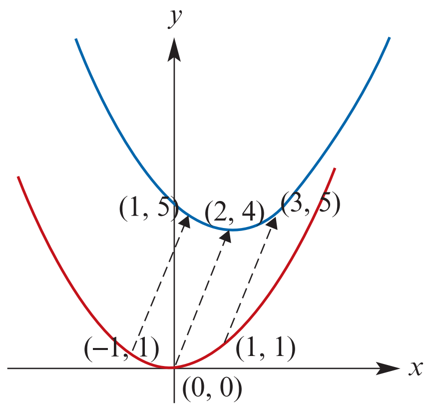
$$(x, y) \to (x + 2, y + 4)$$

$$x' = x + 2 \quad \text{and} \quad y' = y + 4$$

For each point (x, y) there is a unique point (x', y') and vice versa.

We have $x = x' - 2$ and $y = y' - 4$.

This means the points on the curve with equation $y = x^2$ are mapped to the curve with equation $y' - 4 = (x' - 2)^2$.



Hence $\{ (x, y) : y = x^2 \}$ maps to $\{ (x', y') : y' - 4 = (x' - 2)^2 \}$.

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Proof A point (a, b) is on the graph of $y = f(x)$

$$\Leftrightarrow \quad f(a) = b$$

$$\Leftrightarrow \quad f(a + h - h) = b$$

$$\Leftrightarrow \quad f(a + h - h) = b + k - k$$

$$\Leftrightarrow \quad (a + h, b + k) \text{ is a point on the graph of } y - k = f(x - h)$$

Note: The double arrows indicate that the steps are reversible.

Example 1

Find the equation for the image of the curve with equation $y = f(x)$, where $f(x) = \dfrac{1}{x}$, under a translation 3 units in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

Solution	Explanation
Let (x', y') be the image of the point (x, y), where (x, y) is a point on the graph of $y = f(x)$.	The rule is $(x, y) \to (x + 3, y - 2)$.
Then $x' = x + 3$ and $y' = y - 2$.	
Hence $x = x' - 3$ and $y = y' + 2$.	
The graph of $y = f(x)$ is mapped to the graph of $y' + 2 = f(x' - 3)$	Substitute $x = x' - 3$ and $y = y' + 2$ into $y = f(x)$.
i.e. $y = \frac{1}{x}$ is mapped to $y' + 2 = \frac{1}{x' - 3}$.	
The equation of the image can be written as $y = \frac{1}{x - 3} - 2$	

Recognising that a transformation has been applied makes it easy to sketch many graphs.

For example, in order to sketch the graph of $y = \sqrt{x - 2}$, note that it is of the form $y = f(x - 2)$ where $f(x) = \sqrt{x}$. That is, the graph of $y = \sqrt{x}$ is translated 2 units in the positive direction of the x-axis.

Examples of two other functions to which this translation is applied are:

$$f(x) = x^2 \qquad f(x - 2) = (x - 2)^2$$

$$f(x) = \frac{1}{x} \qquad f(x - 2) = \frac{1}{x - 2}$$

Summary 3A

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the translation $(x, y) \to (x + h, y + k)$ to the graph of $y = f(x)$.
- Replacing x with $x - h$ and y with $y - k$ in the equation to obtain $y - k = f(x - h)$ and graphing the result.

Exercise 3A

1 Find the image of the point $(-2, 5)$ after a mapping of a translation:

- **a** of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis
- **b** of 3 units in the negative direction of the x-axis and 5 units in the positive direction of the y-axis
- **c** of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis
- **d** defined by the rule $(x, y) \to (x - 3, y + 2)$
- **e** defined by the rule $(x, y) \to (x - 1, y + 1)$.

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Example 1

2 Find the equation for the image of the curve $y = f(x)$, where $f(x) = \frac{1}{x}$, under:

a a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

b a translation 2 units in the negative direction of the x-axis and 3 units in the positive direction of the y-axis

c a translation $\frac{1}{2}$ unit in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.

3 Sketch the graph of each of the following. Label asymptotes and axis intercepts, and state the domain and range.

a $y = \frac{1}{x} + 3$ **b** $y = \frac{1}{x^2} - 3$ **c** $y = \frac{1}{(x+2)^2}$

d $y = \sqrt{x-2}$ **e** $y = \frac{1}{x-1}$ **f** $y = \frac{1}{x} - 4$

g $y = \frac{1}{x+2}$ **h** $y = \frac{1}{x-3}$ **i** $f(x) = \frac{1}{(x-3)^2}$

j $f(x) = \frac{1}{(x+4)^2}$ **k** $f(x) = \frac{1}{x-1} + 1$ **l** $f(x) = \frac{1}{x-2} + 2$

4 For $y = f(x) = \frac{1}{x}$, sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $y = f(x-1)$ **b** $y = f(x) + 1$ **c** $y = f(x+3)$

d $y = f(x) - 3$ **e** $y = f(x+1)$ **f** $y = f(x) - 1$

5 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = x^2, \quad f_1(x) = (x+5)^2$ **b** $f(x) = \frac{1}{x}, \quad f_1(x) = \frac{1}{x} + 2$

c $f(x) = \frac{1}{x^2}, \quad f_1(x) = \frac{1}{x^2} + 4$ **d** $f(x) = \frac{1}{x^2} - 3, \quad f_1(x) = \frac{1}{x^2}$

e $f(x) = \frac{1}{x-3}, \quad f_1(x) = \frac{1}{x}$

6 Write down the equation of the image when the graph of each of the functions below is transformed by:

i a translation of 7 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis

ii a translation of 2 units in the negative direction of the x-axis and 6 units in the negative direction of the y-axis

iii a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

iv a translation of 1 unit in the negative direction of the x-axis and 4 units in the positive direction of the y-axis.

a $y = x^{\frac{1}{4}}$ **b** $y = \sqrt[3]{x}$ **c** $y = \frac{1}{x^3}$ **d** $y = \frac{1}{x^4}$

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7 Find the equation for the image of the graph of each of the following under the stated translation:

a $y = (x-2)^2 + 3$ Translation: $(x, y) \to (x-3, y+2)$

b $y = 2(x+3)^2 + 3$ Translation: $(x, y) \to (x+3, y-3)$

c $y = \dfrac{1}{(x-2)^2} + 3$ Translation: $(x, y) \to (x+4, y-2)$

d $y = (x+2)^3 + 1$ Translation: $(x, y) \to (x-1, y+1)$

e $y = \sqrt[3]{x-3} + 2$ Translation: $(x, y) \to (x-1, y+1)$

8 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

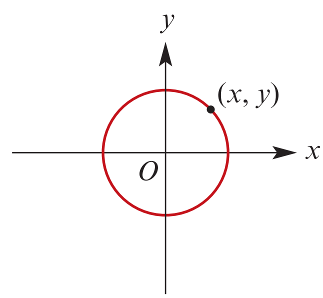
a $f(x) = \dfrac{1}{x^2}$, $f_1(x) = \dfrac{1}{(x-2)^2} + 3$ **b** $f(x) = \dfrac{1}{x}$, $f_1(x) = \dfrac{1}{x+2} - 3$

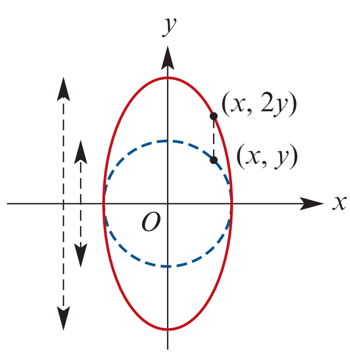
c $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{x+4} + 2$

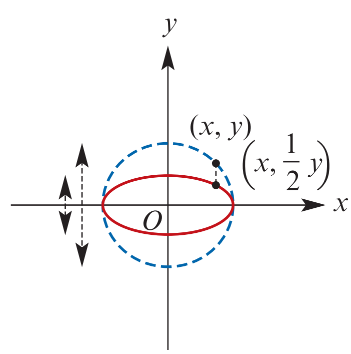
3B Dilations

We start with the example of a circle, as it is easy to visualise the effect of a dilation from an axis.

A dilation of a graph can be thought of as the graph 'stretching away from' or 'shrinking towards' an axis.



<table>
<tr><th>Dilation of factor 2 from the x-axis</th><th>Dilation of factor $\frac{1}{2}$ from the x-axis</th></tr>
<tr><td>

The graph is 'stretched' to twice the height.
The point (x, y) is mapped onto $(x, 2y)$, i.e. $(x, y) \to (x, 2y)$.</td>
<td>

The graph is 'shrunk' to half the height.
The point (x, y) is mapped onto $(x, \frac{1}{2}y)$, i.e. $(x, y) \to (x, \frac{1}{2}y)$.</td></tr>
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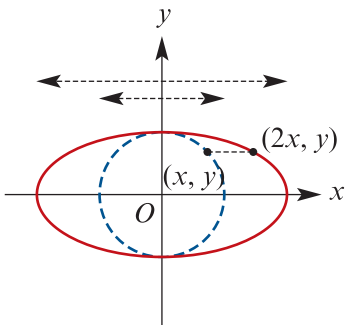
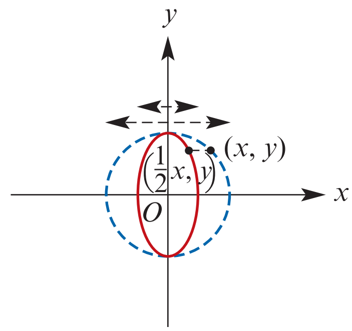
Dilation of factor 2 from the y-axis	Dilation of factor $\frac{1}{2}$ from the y-axis
	
The graph is 'stretched' to twice the width. The point (x, y) is mapped onto $(2x, y)$, i.e. $(x, y) \to (2x, y)$.	The graph is 'shrunk' to half the width. The point (x, y) is mapped onto $(\frac{1}{2}x, y)$, i.e. $(x, y) \to (\frac{1}{2}x, y)$.

Dilation from the x-axis

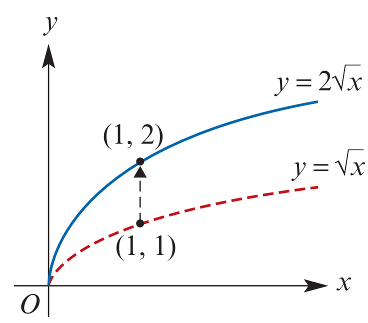
We can determine the equation of the image of a curve under a dilation by following the same approach used for translations.

A dilation of factor 2 from the x-axis is defined by the rule $(x, y) \to (x, 2y)$.

Hence the point with coordinates $(1, 1) \to (1, 2)$.

Consider the curve with equation $y = \sqrt{x}$ and the dilation of factor 2 from the x-axis.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = x$ and $y' = 2y$, and thus $x = x'$ and $y = \dfrac{y'}{2}$.
- Substituting for x and y, we see that the curve with equation $y = \sqrt{x}$ maps to the curve with equation $\dfrac{y'}{2} = \sqrt{x'}$, i.e. the curve with equation $y = 2\sqrt{x}$.



For b a positive constant, a dilation of factor b from the x-axis is described by the rule

$$(x, y) \to (x, by)$$

or $\quad x' = x \quad$ and $\quad y' = by$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation from the x-axis $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
- Replacing y with $\dfrac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.

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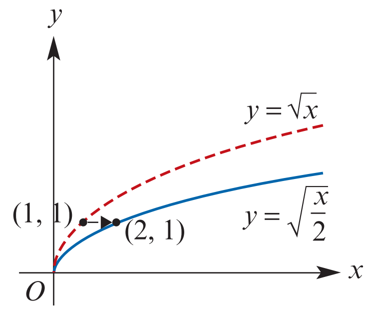
Dilation from the y-axis

A dilation of factor 2 from the y-axis is defined by the rule $(x, y) \to (2x, y)$. Hence the point with coordinates $(1, 1) \to (2, 1)$.

Again, consider the curve with equation $y = \sqrt{x}$.

- Let (x', y') be the image of the point with coordinates (x, y) on the curve.
- Hence $x' = 2x$ and $y' = y$, and thus $x = \dfrac{x'}{2}$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{\dfrac{x'}{2}}$.



For a a positive constant, a dilation of factor a from the y-axis is described by the rule

$$(x, y) \to (ax, y)$$

or $\quad x' = ax \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the dilation from the y-axis $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
- Replacing x with $\dfrac{x}{a}$ in the equation to obtain $y = f\left(\dfrac{x}{a}\right)$ and graphing the result.

Example 2

Determine the rule of the image when the graph of $y = \dfrac{1}{x^2}$ is dilated by a factor of 4:

a from the x-axis **b** from the y-axis.

Solution

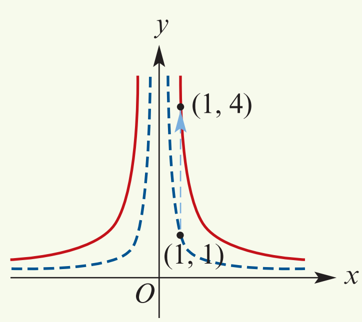
a $(x, y) \to (x, 4y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = x$, $y' = 4y$.

Rearranging gives $x = x'$, $y = \dfrac{y'}{4}$.

Therefore $y = \dfrac{1}{x^2}$ becomes $\dfrac{y'}{4} = \dfrac{1}{(x')^2}$.

The rule of the transformed function is $y = \dfrac{4}{x^2}$.



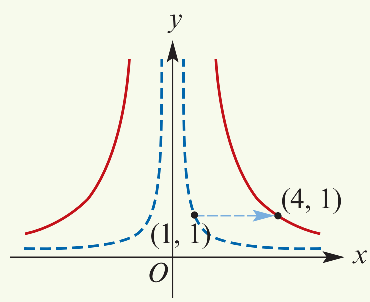
b $(x, y) \to (4x, y)$

Let (x', y') be the coordinates of the image of (x, y), so $x' = 4x$, $y' = y$.

Rearranging gives $x = \dfrac{x'}{4}$, $y = y'$.

Therefore $y = \dfrac{1}{x^2}$ becomes $y' = \dfrac{1}{(\frac{x'}{4})^2}$.

The rule of the transformed function is $y = \dfrac{16}{x^2}$.



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Example 3

Determine the factor of dilation when the graph of $y = \sqrt{3x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

Solution

a Note that a dilation from the y-axis 'changes' the x-values. So write the transformed function as

$$y' = \sqrt{3x'}$$

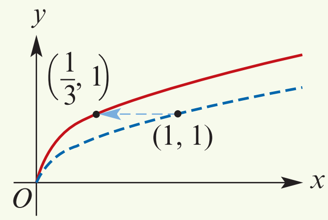
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = 3x'$ and $y = y'$ ('changed' x).

Rearranging gives $x' = \dfrac{x}{3}$ and $y' = y$.

So the mapping is given by $(x, y) \to \left(\dfrac{x}{3}, y\right)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\frac{1}{3}$ from the y-axis to produce the graph of $y = \sqrt{3x}$.



b Note that a dilation from the x-axis 'changes' the y-values. So write the transformed function as

$$\frac{y'}{\sqrt{3}} = \sqrt{x'}$$

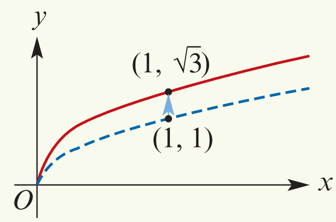
where (x', y') are the coordinates of the image of (x, y).

Therefore $x = x'$ and $y = \dfrac{y'}{\sqrt{3}}$ ('changed' y).

Rearranging gives $x' = x$ and $y' = \sqrt{3}\,y$.

So the mapping is given by $(x, y) \to (x, \sqrt{3}\,y)$.

The graph of $y = \sqrt{x}$ is dilated by a factor of $\sqrt{3}$ from the x-axis to produce the graph of $y = \sqrt{3x}$.



Summary 3B

For the graph of $y = f(x)$, we have the following two pairs of equivalent processes:

1. - Applying the **dilation from the x-axis** $(x, y) \to (x, by)$ to the graph of $y = f(x)$.
   - Replacing y with $\dfrac{y}{b}$ in the equation to obtain $y = bf(x)$ and graphing the result.
2. - Applying the **dilation from the y-axis** $(x, y) \to (ax, y)$ to the graph of $y = f(x)$.
   - Replacing x with $\dfrac{x}{a}$ in the equation to obtain $y = f\left(\dfrac{x}{a}\right)$ and graphing the result.

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Exercise 3B

Example 2

1 Determine the rule of the image when the graph of $y = \frac{1}{x}$ is dilated by a factor of 3:

a from the x-axis **b** from the y-axis.

2 Determine the rule of the image when the graph of $y = \frac{1}{x^2}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

3 Determine the rule of the image when the graph of $y = \sqrt{x}$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

4 Determine the rule of the image when the graph of $y = x^3$ is dilated by a factor of 2:

a from the x-axis **b** from the y-axis.

5 Sketch the graph of each of the following:

a $y = \frac{4}{x}$ **b** $y = \frac{1}{2x}$ **c** $y = \sqrt{3x}$ **d** $y = \frac{2}{x^2}$

6 For $y = f(x) = \frac{1}{x^2}$, sketch the graph of each of the following:

a $y = f(2x)$ **b** $y = 2f(x)$ **c** $y = f\left(\frac{x}{2}\right)$

d $y = 3f(x)$ **e** $y = f(5x)$ **f** $y = f\left(\frac{x}{4}\right)$

7 Sketch the graphs of each of the following on the one set of axes:

a $y = \frac{1}{x}$ **b** $y = \frac{3}{x}$ **c** $y = \frac{3}{2x}$

8 Sketch the graph of the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 3\sqrt{x}$.

Example 3

9 Determine the factor of dilation when the graph of $y = \sqrt{5x}$ is obtained by dilating the graph of $y = \sqrt{x}$:

a from the y-axis **b** from the x-axis.

10 For each of the following, state a transformation which maps the graph of $y = f(x)$ to the graph of $y = f_1(x)$:

a $f(x) = \frac{1}{x^2}$, $f_1(x) = \frac{5}{x^2}$ **b** $f(x) = \sqrt{x}$, $f_1(x) = 4\sqrt{x}$

c $f(x) = \sqrt{x}$, $f_1(x) = \sqrt{5x}$ **d** $f(x) = \sqrt{\frac{x}{3}}$, $f_1(x) = \sqrt{x}$

e $f(x) = \frac{1}{4x^2}$, $f_1(x) = \frac{1}{x^2}$

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11 Write down the equation of the image when the graph of each of the functions below is transformed by:

- **i** a dilation of factor 4 from the x-axis
- **ii** a dilation of factor $\frac{2}{3}$ from the x-axis
- **iii** a dilation of factor $\frac{1}{2}$ from the y-axis
- **iv** a dilation of factor 5 from the y-axis.

a $y = x^2$ **b** $y = \dfrac{1}{x^2}$ **c** $y = \sqrt[3]{x}$ **d** $y = \dfrac{1}{x^3}$

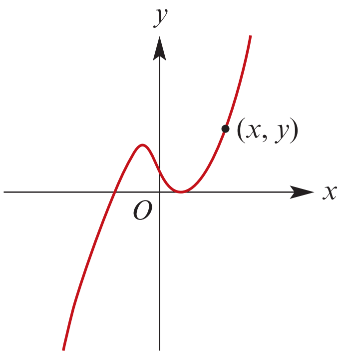
e $y = \dfrac{1}{x^4}$ **f** $y = \sqrt[4]{x}$ **g** $y = x^{\frac{1}{5}}$

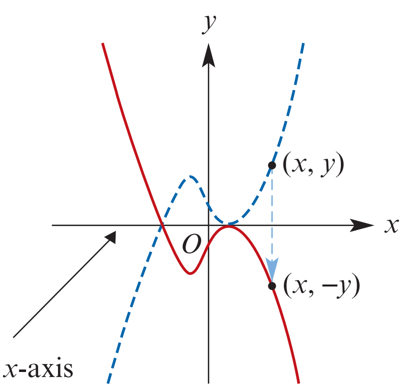
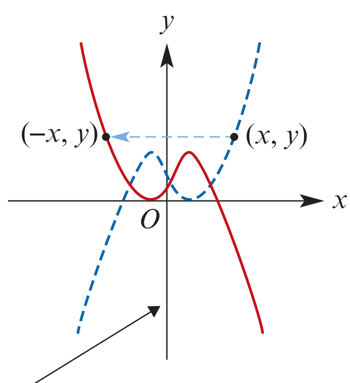
3C Reflections

The special case where the graph of a function is reflected in the line $y = x$ to produce the graph of the inverse relation is discussed separately in Section 1F.

In this chapter we study reflections in the x- or y-axis only.

First consider reflecting the graph of the function shown here in each axis, and observe the effect on a general point (x, y) on the graph.



Reflection in the x-axis	Reflection in the y-axis
 The x-axis acts as a 'mirror' line. The point (x, y) is mapped onto $(x, -y)$, i.e. $(x, y) \to (x, -y)$	 The y-axis acts as a 'mirror' line. The point (x, y) is mapped onto $(-x, y)$, i.e. $(x, y) \to (-x, y)$

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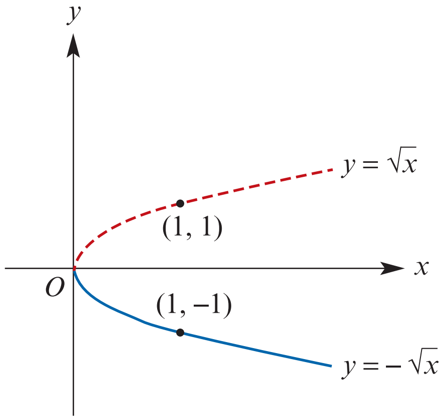
Reflection in the x-axis

A reflection in the x-axis can be defined by the rule $(x, y) \to (x, -y)$. Hence the point with coordinates $(1, 1) \to (1, -1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = x$ and $y' = -y$, which gives $x = x'$ and $y = -y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $-y' = \sqrt{x'}$, i.e. the curve with equation $y = -\sqrt{x}$.



A reflection in the x-axis is described by the rule

$$(x, y) \to (x, -y)$$

or $\quad x' = x \quad$ and $\quad y' = -y$

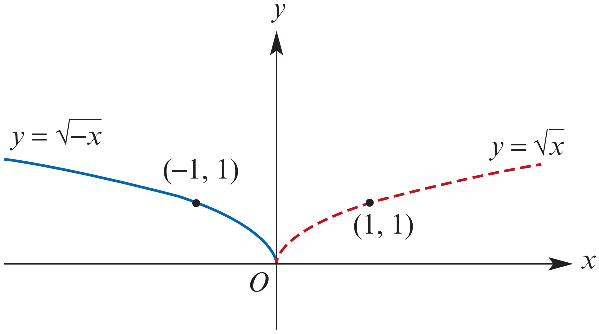
For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the x-axis $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

Reflection in the y-axis

A reflection in the y-axis can be defined by the rule $(x, y) \to (-x, y)$. Hence the point with coordinates $(1, 1) \to (-1, 1)$.

- Let (x', y') be the image of the point (x, y).
- Hence $x' = -x$ and $y' = y$, which gives $x = -x'$ and $y = y'$.
- The curve with equation $y = \sqrt{x}$ maps to the curve with equation $y' = \sqrt{-x'}$, i.e. the curve with equation $y = \sqrt{-x}$.



A reflection in the y-axis is described by the rule

$$(x, y) \to (-x, y)$$

or $\quad x' = -x \quad$ and $\quad y' = y$

For the graph of $y = f(x)$, the following two processes yield the same result:

- Applying the reflection in the y-axis $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

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Example 4

Find the equation of the image when the graph of $y = \sqrt{x}$ is reflected:

a in the x-axis **b** in the y-axis.

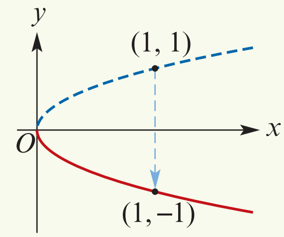
Solution

a Note that a reflection in the x-axis changes the y-values, and so $(x, y) \to (x, -y)$. Let (x', y') be the coordinates of the image of (x, y). Then $x' = x,\ y' = -y$.

Rearranging gives $x = x',\ y = -y'$.

Therefore $y = \sqrt{x}$ becomes $-y' = \sqrt{x'}$.

The rule of the transformed function is $y = -\sqrt{x}$.



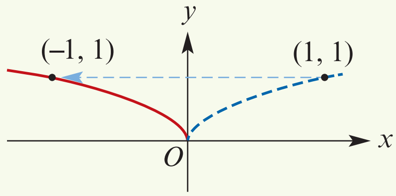
b Note that a reflection in the y-axis changes the x-values, and so $(x, y) \to (-x, y)$. Let (x', y') be the coordinates of the image of (x, y).

Then $x' = -x,\ y' = y$.

Rearranging gives $x = -x',\ y = y'$.

Therefore $y = \sqrt{x}$ becomes $y' = \sqrt{-x'}$.

The rule of the transformed function is $y = \sqrt{-x}$.



Summary 3C

For the graph of $y = f(x)$, we have the following two pairs of equivalent processes:

1
- Applying the **reflection in the x-axis** $(x, y) \to (x, -y)$ to the graph of $y = f(x)$.
- Replacing y with $-y$ in the equation to obtain $y = -f(x)$ and graphing the result.

2
- Applying the **reflection in the y-axis** $(x, y) \to (-x, y)$ to the graph of $y = f(x)$.
- Replacing x with $-x$ in the equation to obtain $y = f(-x)$ and graphing the result.

Exercise 3C

Example 4

1 Find the equation of the image when the graph of $y = (x - 1)^2$ is reflected:

a in the x-axis **b** in the y-axis.

2 Sketch the graph and state the domain of:

a $y = -\left(x^{\frac{1}{3}}\right)$ **b** $y = (-x)^3$

3 State a transformation which maps the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{-x}$.

4 Find the equation of the image when the graph of each of the functions below is transformed by: **i** a reflection in the x-axis **ii** a reflection in the y-axis.

a $y = x^3$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x^3}$ **d** $y = \dfrac{1}{x^4}$ **e** $y = x^{\frac{1}{3}}$ **f** $y = x^{\frac{1}{5}}$ **g** $y = x^{\frac{1}{4}}$

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3D Combinations of transformations

In the previous three sections, we considered three types of transformations separately. In the remainder of this chapter we look at situations where a graph may have been transformed by any combination of dilations, reflections and translations.

For example, first consider:

- a dilation of factor 2 from the x-axis
- followed by a reflection in the x-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x, -2y)$$

First the dilation is applied and then the reflection. For example, $(1, 1) \to (1, 2) \to (1, -2)$.

Another example is:

- a dilation of factor 2 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

The rule becomes

$$(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$$

First the dilation is applied and then the translation. For example, $(1, 1) \to (1, 2) \to (3, -1)$.

Example 5

Find the equation of the image of $y = \sqrt{x}$ under:

a a dilation of factor 2 from the x-axis followed by a reflection in the x-axis

b a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis.

Solution

a From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x, -2y)$.

If (x, y) maps to (x', y'), then $x' = x$ and $y' = -2y$. Thus $x = x'$ and $y = \dfrac{y'}{-2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y'}{-2} = \sqrt{x'}$$

and hence $y' = -2\sqrt{x'}$. The equation can be written as $y = -2\sqrt{x}$.

b From the discussion above, the rule is $(x, y) \to (x, 2y) \to (x + 2, 2y - 3)$.

If (x, y) maps to (x', y'), then $x' = x + 2$ and $y' = 2y - 3$. Thus $x = x' - 2$ and $y = \dfrac{y' + 3}{2}$.

So the image of $y = \sqrt{x}$ has equation

$$\frac{y' + 3}{2} = \sqrt{x' - 2}$$

and hence $y' = 2\sqrt{x' - 2} - 3$. The equation can be written as $y = 2\sqrt{x - 2} - 3$.

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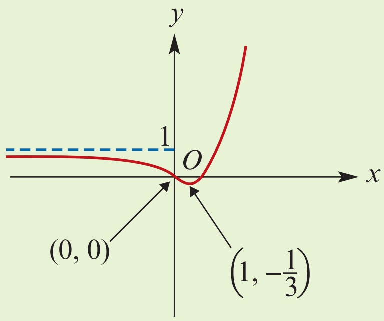
Example 6

Sketch the image of the graph shown under the following sequence of transformations:

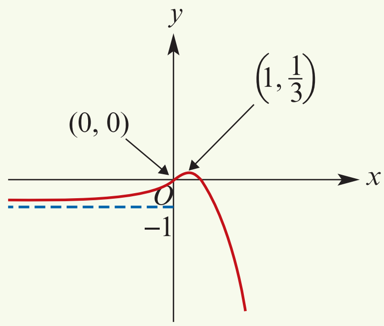
- a reflection in the x-axis
- a dilation of factor 3 from the x-axis
- a translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



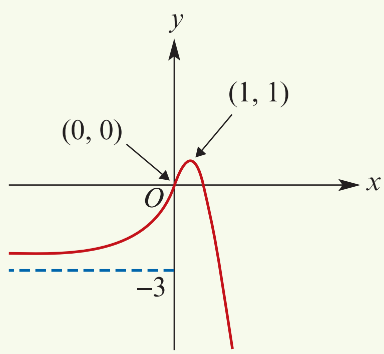
Solution

Consider each transformation in turn and sketch the graph at each stage.

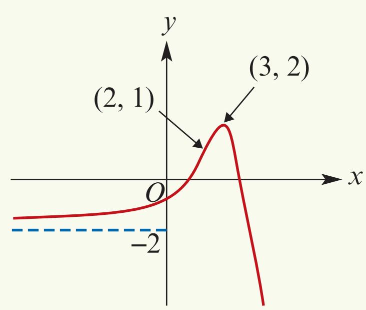
A reflection in the x-axis produces the graph shown on the right.



Next apply the dilation of factor 3 from the x-axis.



Finally, apply the translation 2 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.



Example 7

Find the equation of the image when the graph of $y = \sqrt{x}$ is translated 6 units in the negative direction of the x-axis, reflected in the y-axis and dilated by a factor of 2 from the x-axis.

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Solution

- The translation 6 units in the negative direction of the x-axis maps $(x, y) \to (x - 6, y)$.
- The reflection in the y-axis maps $(x - 6, y) \to (-(x - 6), y)$.
- The dilation of factor 2 from the x-axis maps $(-(x - 6), y) \to (-(x - 6), 2y)$.

In summary: $(x, y) \to (-(x - 6), 2y)$.

Let (x', y') be the coordinates of the image of (x, y). Then $x' = -(x - 6)$ and $y' = 2y$.

Rearranging gives $x = -x' + 6$ and $y = \dfrac{y'}{2}$.

Therefore $y = \sqrt{x}$ becomes $\dfrac{y'}{2} = \sqrt{-x' + 6}$.

The rule of the transformed function is $y = 2\sqrt{6 - x}$.

Example 8

For the graph of $y = x^2$:

a Sketch the graph of the image under the sequence of transformations:

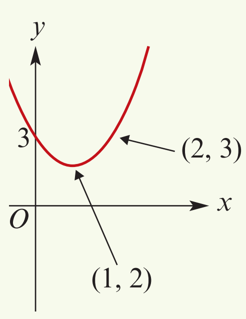
- a translation of 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis
- a dilation of factor 2 from the y-axis
- a reflection in the x-axis.

b State the rule of the image.

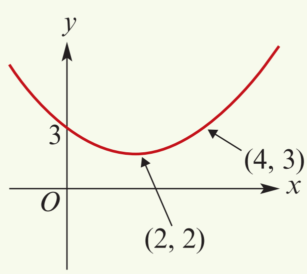
Solution

a Apply each transformation in turn and sketch the graph at each stage.

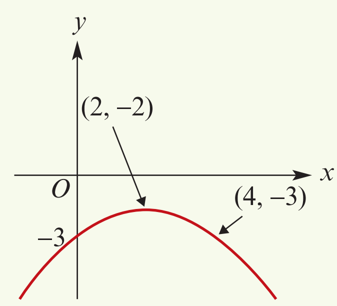
1 The translation:



2 The dilation of factor 2 from the y-axis:



3 The reflection in the x-axis:



b The mapping representing the sequence of transformations is

$$(x, y) \to (x + 1, y + 2) \to (2(x + 1), y + 2) \to (2(x + 1), -(y + 2))$$

Let (x', y') be the image of (x, y). Then $x' = 2(x + 1)$ and $y' = -(y + 2)$.

Rearranging gives $x = \frac{1}{2}(x' - 2)$ and $y = -y' - 2$.

Therefore $y = x^2$ becomes $-y' - 2 = \left(\frac{1}{2}(x' - 2)\right)^2$.

The rule of the image is $y = -\frac{1}{4}(x - 2)^2 - 2$.

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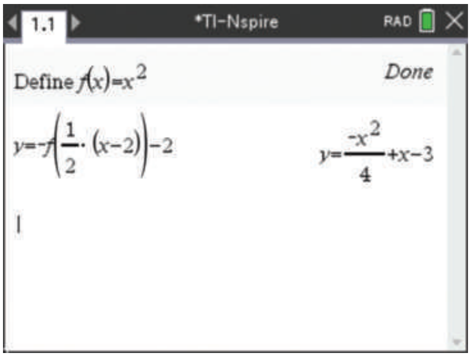
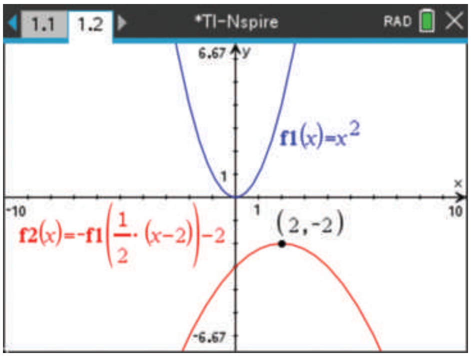
Using the TI-Nspire

- Define $f(x) = x^2$.
- The rule for the transformed function is $-f\left(\frac{1}{2}(x-2)\right) - 2$.
- The calculator gives the equation of the image of the graph under this sequence of transformations.



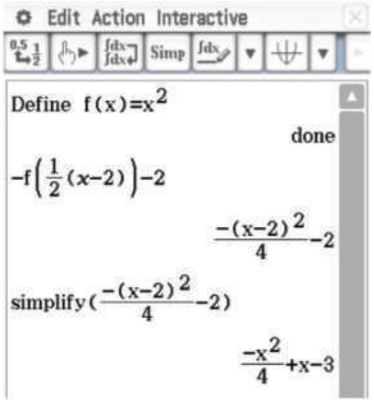
- The new function can also be entered in the transformation format in a **Graphs** page as shown.



Using the Casio ClassPad

- Define $f(x) = x^2$.
- Enter the rule for the transformed function as $-f\left(\frac{1}{2}(x-2)\right) - 2$.
- Highlight the resulting expression and select **Interactive** > **Transformation** > **simplify** to obtain the simplified form.

Note: This result can be achieved more simply by tapping on Simp at the top of the screen.



- To graph both functions, tap on the graph icon.
- Highlight each function and drag into the graph window.
- Use the window icon to adjust the window.

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Summary 3D

A sequence of transformations can be applied, and the rule for transforming points of the plane can be described. For example, the sequence

- a dilation of factor 3 from the x-axis
- followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis
- followed by a reflection in the x-axis

can be described by the rule $(x, y) \to (x, 3y) \to (x + 2, 3y - 3) \to (x + 2, 3 - 3y)$.

Let $x' = x + 2$ and $y' = 3 - 3y$. Then $x = x' - 2$ and $y = \dfrac{3 - y'}{3}$.

The graph of $y = f(x)$ maps to $\dfrac{3 - y'}{3} = f(x' - 2)$. That is, the graph of $y = f(x)$ maps to the graph of $y = 3 - 3f(x - 2)$.

Skill-sheet

Exercise 3D

1 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

Example 5

i a dilation of factor 2 from the x-axis, followed by a translation 2 units in the positive direction of the x-axis and 3 units in the negative direction of the y-axis

ii a dilation of factor 3 from the y-axis, followed by a translation 2 units in the negative direction of the x-axis and 4 units in the negative direction of the y-axis

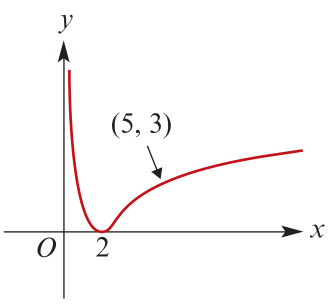
iii a dilation of factor 2 from the x-axis, followed by a reflection in the y-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x^2}$

Example 6

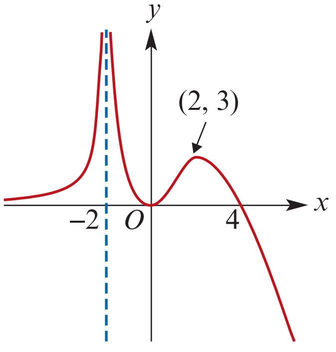
2 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the x-axis
- a dilation of factor 2 from the x-axis
- a translation 3 units in the positive direction of the x-axis and 4 units in the positive direction of the y-axis.



3 Sketch the image of the graph shown under the following sequence of transformations:

- a reflection in the y-axis
- a translation 2 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis
- a dilation of factor 2 from the y-axis.



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Example 7

4 Find the rule of the image when the graph of each of the functions listed below undergoes each of the following sequences of transformations:

i a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

ii a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis

iii a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

iv a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis

v a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis

vi a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

a $y = x^2$ **b** $y = \sqrt[3]{x}$ **c** $y = \dfrac{1}{x}$ **d** $y = x^4$

e $y = \dfrac{1}{x^3}$ **f** $y = \dfrac{1}{x^4}$ **g** $y = x^{-2}$

5 Find the rule of the image when the graph of $y = \sqrt{x}$ is translated 4 units in the negative direction of the x-axis, reflected in the x-axis and dilated by factor 3 from the y-axis.

Example 8

6 For the graph of $y = \dfrac{3}{x^2}$:

a Sketch the graph of the image under the sequence of transformations:
- a dilation of factor 2 from the x-axis
- a translation of 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis
- a reflection in the x-axis.

b State the rule of the image.

7 For the graph of $y = x^{\frac{1}{3}}$:

a Sketch the graph of the image under the sequence of transformations:
- a reflection in the y-axis
- a translation of 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis.

b State the rule of the image.

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3E Determining transformations

The method that has been used to find the effect of a transformation on a graph can be used in reverse to find a sequence of transformations that takes a graph to its image.

For example, to find a sequence of transformations which maps $y = \sqrt{x}$ to $y' = -2\sqrt{x'}$, work backwards through the steps in the solution of Example 5a:

- $y = \sqrt{x}$ maps to $\dfrac{y'}{-2} = \sqrt{x'}$.
- Hence $x = x'$ and $y = \dfrac{y'}{-2}$, and therefore $x' = x$ and $y' = -2y$.
- The transformation is a dilation of factor 2 from the x-axis followed by a reflection in the x-axis.

This can also be done by inspection, of course, if you recognise the form of the image. For the combinations of transformations in this course, it is often simpler to do this.

Example 9

a Find a sequence of transformations which takes the graph of $y = x^2$ to the graph of $y = 2(x - 2)^2 + 3$.

b Find a sequence of transformations which takes the graph of $y = \sqrt{x}$ to the graph of $y = \sqrt{5x - 2}$.

Solution

a The transformation can be found by inspection, but we shall use the method.

The graph of $y = x^2$ maps to $y' = 2(x' - 2)^2 + 3$. Rearranging this equation gives

$$\frac{y' - 3}{2} = (x' - 2)^2$$

We choose to write $y = \dfrac{y' - 3}{2}$ and $x = x' - 2$.

Solving for x' and y' gives

$$x' = x + 2 \quad \text{and} \quad y' = 2y + 3$$

So we can write the transformation as

$$(x, y) \to (x, 2y) \to (x + 2, 2y + 3)$$

This transformation is a dilation of factor 2 from the x-axis followed by a translation of 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b We have $y = \sqrt{x}$ and $y' = \sqrt{5x' - 2}$. We choose to write $y = y'$ and $x = 5x' - 2$. Hence

$$x' = \frac{x + 2}{5} = \frac{x}{5} + \frac{2}{5} \quad \text{and} \quad y' = y$$

The transformation is a dilation of factor $\frac{1}{5}$ from the y-axis followed by a translation of $\frac{2}{5}$ units in the positive direction of the x-axis.

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Example 10

a Find a sequence of transformations which takes the graph of $y = \dfrac{3}{(x-1)^2} + 6$ to the graph of $y = \dfrac{1}{x^2}$.

b Find a sequence of transformations which takes the graph of $y = (5x-1)^2 + 6$ to the graph of $y = x^2$.

Solution

a Write $\dfrac{y-6}{3} = \dfrac{1}{(x-1)^2}$ and $y' = \dfrac{1}{(x')^2}$. The points (x, y) satisfying $\dfrac{y-6}{3} = \dfrac{1}{(x-1)^2}$ are mapped to the points (x', y') satisfying $y' = \dfrac{1}{(x')^2}$.

Hence we choose to write

$$y' = \frac{y-6}{3} \quad \text{and} \quad x' = x - 1$$

We can write this transformation as

$$(x, y) \to (x-1, y-6) \to \left(x-1, \frac{y-6}{3}\right)$$

This is a translation of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis followed by a dilation of factor $\frac{1}{3}$ from the x-axis.

b Write $y - 6 = (5x-1)^2$ and $y' = (x')^2$. The points (x, y) satisfying $y - 6 = (5x-1)^2$ are mapped to the points (x', y') satisfying $y' = (x')^2$.

Hence we choose to write

$$y' = y - 6 \quad \text{and} \quad x' = 5x - 1$$

One transformation is a dilation of factor 5 from the y-axis followed by a translation of 1 unit in the negative direction of the x-axis and 6 units in the negative direction of the y-axis.

We note that the transformations we found are far from being the only possible answers. In fact there are infinitely many choices.

Summary 3E

The notation developed in this chapter can be used to help find the transformation that takes the graph of a function to its image.

For example, if the graph of $y = f(x)$ is mapped to the graph of $y' = 2f(x' - 3)$, we can see that the transformation

$$x' = x + 3 \quad \text{and} \quad y' = 2y$$

is a suitable choice. This is a translation of 3 units to the right followed by a dilation of factor 2 from the x-axis.

There are infinitely many transformations that take the graph of $y = f(x)$ to the graph of $y' = 2f(x' - 3)$. The one we chose is conventional.

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Skill-sheet

Exercise 3E

1 For each of the following, find a sequence of transformations that takes:

a the graph of $y = x^3$ to the graph of:

i $y = 2(x-1)^3 + 3$ **ii** $y = -(x+1)^3 + 2$ **iii** $y = (2x+1)^3 - 2$

Example 9

b the graph of $y = \dfrac{1}{x^2}$ to the graph of:

i $y = \dfrac{2}{(x+3)^2}$ **ii** $y = \dfrac{1}{(x+3)^2} + 2$ **iii** $y = \dfrac{1}{(x-3)^2} - 2$

c the graph of $y = \sqrt[3]{x}$ to the graph of:

i $y = \sqrt[3]{x+3} + 2$ **ii** $y = 2\sqrt[3]{3x}$ **iii** $y = -\sqrt[3]{x} + 2$

Example 10

2 **a** Find a sequence of transformations that takes the graph of $y = (x-1)^2 + 6$ to the graph of $y = x^2$.

b Find a sequence of transformations that takes the graph of $y = 2(x-1)^2 - 3$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = \dfrac{1}{(x-1)^2} - 6$ to the graph of $y = \dfrac{1}{x^2}$.

d Find a sequence of transformations that takes the graph of $y = \dfrac{2}{(x-1)^2} - 5$ to the graph of $y = \dfrac{1}{x^2}$.

e Find a sequence of transformations that takes the graph of $y = (2x-1)^2 + 6$ to the graph of $y = x^2$.

3 **a** Find a sequence of transformations that takes the graph of $y = \dfrac{5}{(x-3)^2} - 7$ to the graph of $y = \dfrac{1}{x^2}$.

b Find a sequence of transformations that takes the graph of $y = (3x+2)^2 + 5$ to the graph of $y = x^2$.

c Find a sequence of transformations that takes the graph of $y = -3(3x+1)^2 + 7$ to the graph of $y = x^2$.

d Find a sequence of transformations that takes the graph of $y = 2\sqrt{4-x}$ to the graph of $y = \sqrt{x}$.

e Find a sequence of transformations that takes the graph of $y = 2\sqrt{4-x} + 3$ to the graph of $y = -\sqrt{x} + 6$.

4 In each case below, state a sequence of transformations that transforms the graph of the first equation into the graph of the second equation:

a $y = \dfrac{1}{x},\ y = \dfrac{2}{x-1} + 3$ **b** $y = \dfrac{1}{x^2},\ y = \dfrac{2}{(x+4)^2} - 7$ **c** $y = \dfrac{1}{x^3},\ y = \dfrac{4}{(1-x)^3} - 5$

d $y = \sqrt[3]{x},\ y = 2 - \sqrt[3]{x+1}$ **e** $y = \dfrac{1}{\sqrt{x}},\ y = \dfrac{2}{\sqrt{-x}} + 3$ **f** $y = \dfrac{2}{3-x} + 4,\ y = \dfrac{1}{x}$

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3F Using transformations to sketch graphs

By considering a rule for a graph as a combination of transformations of a more 'simple' rule, we can readily sketch graphs of many apparently 'complicated' functions.

Example 11

Identify a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto the graph of $y = \frac{4}{x+5} - 3$. Use this to sketch the graph of $y = \frac{4}{x+5} - 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

Solution

Rearrange the equation of the transformed graph to have the same 'shape' as $y = \frac{1}{x}$:

$$\frac{y' + 3}{4} = \frac{1}{x' + 5}$$

where (x', y') are the coordinates of the image of (x, y).

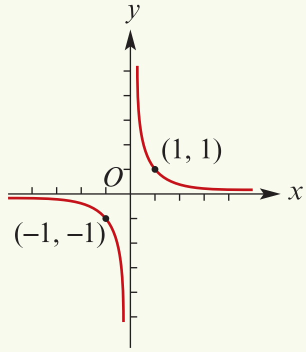
Therefore $x = x' + 5$ and $y = \frac{y' + 3}{4}$. Rearranging gives $x' = x - 5$ and $y' = 4y - 3$.

The mapping is $(x, y) \to (x - 5, 4y - 3)$, and so a sequence of transformations is:

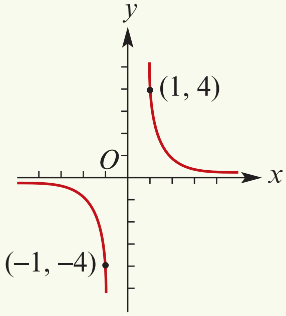
1. a dilation of factor 4 from the x-axis
2. a translation of 5 units in the negative direction of the x-axis
3. a translation of 3 units in the negative direction of the y-axis.

The original graph $y = \frac{1}{x}$ is shown on the right.

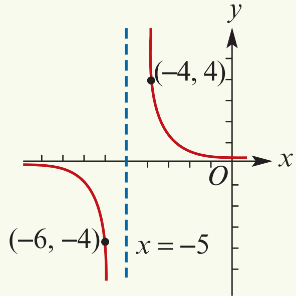
The effect of the transformations is shown below.



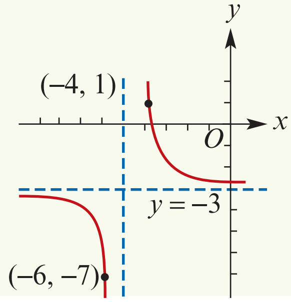
1 Dilation from x-axis:



2 Translation in negative direction of x-axis:



3 Translation in negative direction of y-axis:



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Find the axis intercepts in the usual way, as below. The transformed graph, with asymptotes and intercepts marked, is shown on the right.

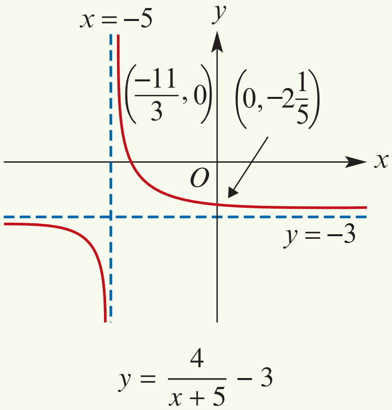
When $x = 0$, $y = \frac{4}{5} - 3 = -2\frac{1}{5}$

When $y = 0$, $\frac{4}{x+5} - 3 = 0$

$$4 = 3x + 15$$

$$3x = -11$$

$$\therefore \quad x = -\frac{11}{3}$$



Once you have done a few of these types of exercises, you can identify the transformations more quickly by carefully observing the rule of the transformed graph and relating it to the 'shape' of the simplest function in its family. Consider the following examples.

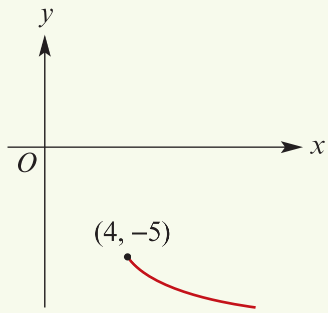
Example 12

Sketch the graph of $y = -\sqrt{x-4} - 5$.

Solution

The graph is obtained from the graph of $y = \sqrt{x}$ by:

- a reflection in the x-axis, followed by a translation of 5 units in the negative direction of the y-axis, and
- a translation of 4 units in the positive direction of the x-axis.



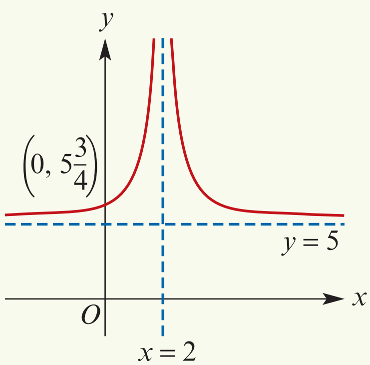
Example 13

Sketch the graph of $y = \frac{3}{(x-2)^2} + 5$.

Solution

This is obtained from the graph of $y = \frac{1}{x^2}$ by:

- a dilation of factor 3 from the x-axis, followed by a translation of 5 units in the positive direction of the y-axis, and
- a translation of 2 units in the positive direction of the x-axis.



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Summary 3F

In general, the function given by the equation

$$y = Af(n(x + c)) + b$$

where $b, c \in \mathbb{R}^+$ and $A, n \in \mathbb{R}^+$, represents a transformation of the graph of $y = f(x)$ by:

- a dilation of factor A from the x-axis, followed by a translation of b units in the positive direction of the y-axis, and
- a dilation of factor $\frac{1}{n}$ from the y-axis, followed by a translation of c units in the negative direction of the x-axis.

Similar statements can be made for $b, c \in \mathbb{R}^-$. The case where $A \in \mathbb{R}^-$ corresponds to a reflection in the x-axis and a dilation from the x-axis. The case where $n \in \mathbb{R}^-$ corresponds to a reflection in the y-axis and a dilation from the y-axis.

Exercise 3F

Example 11

1 Sketch the graph of each of the following. State the equations of asymptotes and the axis intercepts. State the range of each function.

a $f(x) = \dfrac{3}{x - 1}$ **b** $g(x) = \dfrac{2}{x + 1} - 1$ **c** $h(x) = \dfrac{3}{(x - 2)^2}$

d $f(x) = \dfrac{2}{(x - 1)^2} - 1$ **e** $h(x) = \dfrac{-1}{x - 3}$ **f** $f(x) = \dfrac{-1}{x + 2} + 3$

g $f(x) = \dfrac{2}{(x - 3)^2} + 4$

Example 12
Example 13

2 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = -\sqrt{x - 3}$ **b** $y = -\sqrt{x - 3} + 2$ **c** $y = \sqrt{2(x + 3)}$

d $y = \dfrac{1}{2x - 3}$ **e** $y = 5\sqrt{x + 2}$ **f** $y = -5\sqrt{x + 2} - 2$

g $y = \dfrac{-3}{x - 2}$ **h** $y = \dfrac{-2}{(x + 2)^2} - 4$ **i** $y = \dfrac{3}{2x} - 5$

j $y = \dfrac{5}{2x} + 5$ **k** $y = 2(x - 3)^2 + 5$

3 Show that $\dfrac{3x + 2}{x + 1} = 3 - \dfrac{1}{x + 1}$ and hence, without using your calculator, sketch the graph of

$$f : \mathbb{R} \setminus \{-1\} \to \mathbb{R},\ f(x) = \frac{3x + 2}{x + 1}$$

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4 Show that $\dfrac{4x-5}{2x+1} = 2 - \dfrac{7}{2x+1}$ and hence, without using your calculator, sketch the graph of

$$f\colon \mathbb{R} \setminus \{-\tfrac{1}{2}\} \to \mathbb{R},\ f(x) = \frac{4x-5}{2x+1}$$

Hint: $f(x) = 2 - \dfrac{7}{2(x+\frac{1}{2})}$

5 Sketch the graph of each of the following without using your calculator. State the range of each.

a $y = \dfrac{2}{x-3} + 4$ **b** $y = \dfrac{4}{3-x} + 4$ **c** $y = \dfrac{2}{(x-1)^2} + 1$

d $y = 2\sqrt{x-1} + 2$ **e** $y = -3\sqrt{x-4} + 1$ **f** $y = 5\sqrt{2x+4} + 1$

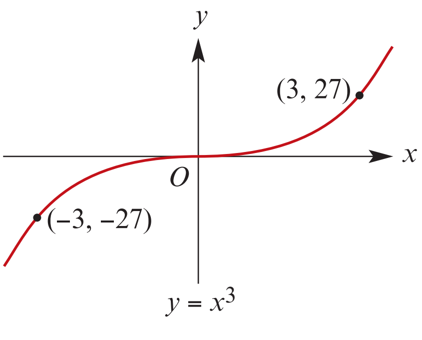
3G Transformations of power functions with positive integer index

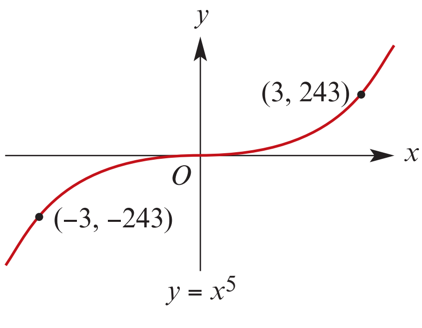
We recall that every quadratic polynomial function can be written in the turning point form $y = a(x-h)^2 + k$. This is not true for polynomials of higher degree. However, there are many polynomials that can be written as $y = a(x-h)^n + k$.

In Chapter 1 we introduced power functions, which include functions with rule $f(x) = x^n$, where n is a positive integer. In this section we continue our study of such functions and, in particular, we look at transformations of these functions.

The function $f(x) = x^n$ where n is an odd positive integer

The diagrams below show the graphs of $y = x^3$ and $y = x^5$.





Assume that n is an odd integer with $n \geq 3$. From Mathematical Methods Units 1 & 2, you will recall that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is odd and therefore $n - 1$ is even, we have $f'(x) = nx^{n-1} > 0$ for all non-zero x. That is, the gradient of the graph of $y = f(x)$ is positive for all non-zero x and is zero when $x = 0$. Recall that, for functions of this form, the stationary point at $(0, 0)$ is called a stationary point of inflection.

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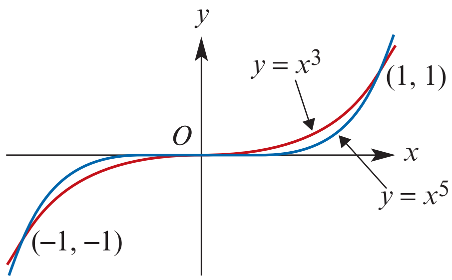
Comparing the graphs of $y = x^n$ and $y = x^m$ for n and m odd

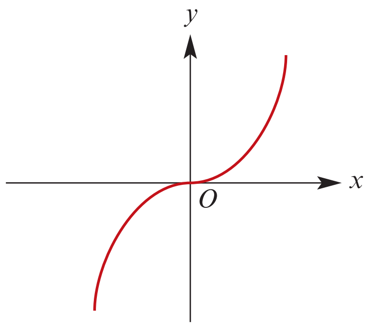
Assume that n and m are odd positive integers with $n > m$. Then:

- $x^n = x^m$ for $x = -1, 0, 1$
- $x^n > x^m$ for $-1 < x < 0$ and for $x > 1$
- $x^n < x^m$ for $0 < x < 1$ and for $x < -1$.



It should be noted that the appearance of graphs is dependent on the scales on the x- and y-axes.

Power functions of odd degree are often depicted as shown.



Transformations of $f(x) = x^n$ where n is an odd positive integer

Transformations of these functions result in graphs with rules of the form $y = a(x - h)^n + k$ where a, h and k are real constants.

Example 14

Sketch the graph of:

a $y = (x - 2)^3 + 1$ **b** $y = -(x - 1)^3 + 2$ **c** $y = 2(x + 1)^3 + 2$

Solution

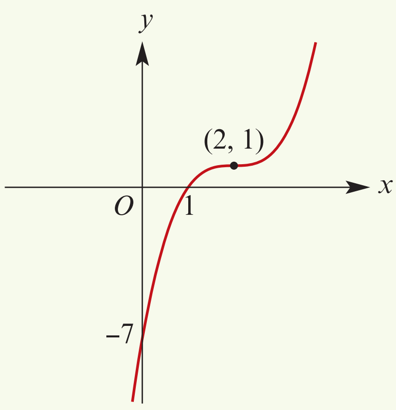
a The translation $(x, y) \to (x + 2, y + 1)$ maps the graph of $y = x^3$ onto the graph of $y = (x - 2)^3 + 1$.

So $(2, 1)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = (-2)^3 + 1 = -7$
- When $y = 0$,

$$\begin{aligned} 0 &= (x - 2)^3 + 1 \\ -1 &= (x - 2)^3 \\ -1 &= x - 2 \\ \therefore \quad x &= 1 \end{aligned}$$



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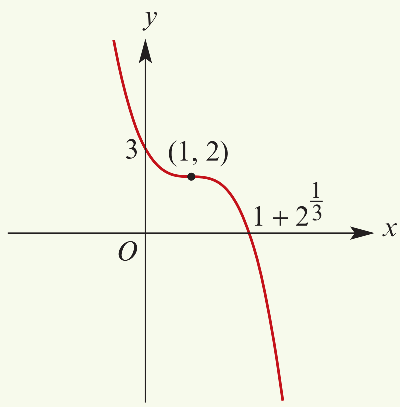
b A reflection in the x-axis followed by the translation $(x, y) \to (x + 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = -(x - 1)^3 + 2$.

So $(1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = -(-1)^3 + 2 = 3$
- When $y = 0$,

$$\begin{aligned} 0 &= -(x-1)^3 + 2 \\ (x-1)^3 &= 2 \\ x - 1 &= 2^{\frac{1}{3}} \\ \therefore \quad x &= 1 + 2^{\frac{1}{3}} \approx 2.26 \end{aligned}$$



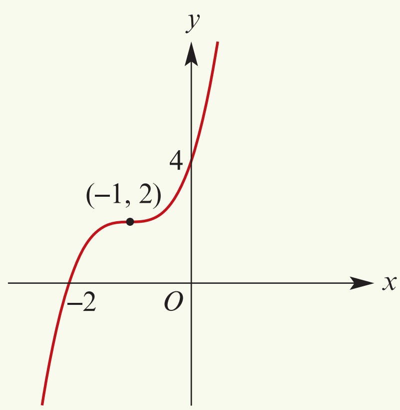
c A dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y + 2)$ maps the graph of $y = x^3$ onto the graph of $y = 2(x + 1)^3 + 2$.

So $(-1, 2)$ is a point of zero gradient.

Find the axis intercepts:

- When $x = 0$, $y = 2 + 2 = 4$
- When $y = 0$,

$$\begin{aligned} 0 &= 2(x+1)^3 + 2 \\ -1 &= (x+1)^3 \\ -1 &= x + 1 \\ \therefore \quad x &= -2 \end{aligned}$$



Example 15

The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(1, 1)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(1, 1)$ is the point of zero gradient,

$$h = 1 \quad \text{and} \quad k = 1$$

So $y = a(x - 1)^3 + 1$ and, since the graph passes through $(0, 4)$,

$$\begin{aligned} 4 &= -a + 1 \\ \therefore \quad a &= -3 \end{aligned}$$

Example 16

a Find the rule for the image of the graph of $y = x^5$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^5$ to the graph of $y = 6 - 2(x + 5)^5$.

Solution

a $(x, y) \to (-x, y) \to (-2x, y) \to (-2x + 2, y + 3)$

Let (x', y') be the image of (x, y) under this transformation.

Then $x' = -2x + 2$ and $y' = y + 3$. Hence $x = \dfrac{x' - 2}{-2}$ and $y = y' - 3$.

Therefore the graph of $y = x^5$ maps to the graph of

$$y' - 3 = \left(\frac{x' - 2}{-2}\right)^5$$

i.e. to the graph of

$$y = -\frac{1}{32}(x - 2)^5 + 3$$

b Rearrange $y' = 6 - 2(x' + 5)^5$ to $\dfrac{y' - 6}{-2} = (x' + 5)^5$.

Therefore $y = \dfrac{y' - 6}{-2}$ and $x = x' + 5$, which gives $y' = -2y + 6$ and $x' = x - 5$.

The sequence of transformations is:

- reflection in the x-axis
- dilation of factor 2 from the x-axis
- translation 5 units in the negative direction of the x-axis and 6 units in the positive direction of the y-axis.

The function $f(x) = x^n$ where n is an even positive integer

Assume that n is an even integer with $n \geq 2$. From Mathematical Methods Units 1 & 2, you will recall that the derivative function of $f(x) = x^n$ has rule

$$f'(x) = nx^{n-1}$$

Hence the gradient is zero when $x = 0$. Since n is even and therefore $n - 1$ is odd, we have $f'(x) = nx^{n-1} > 0$ for $x > 0$, and $f'(x) = nx^{n-1} < 0$ for $x < 0$. Thus the graph of $y = f(x)$ has a turning point at $(0, 0)$; this point is a local minimum.

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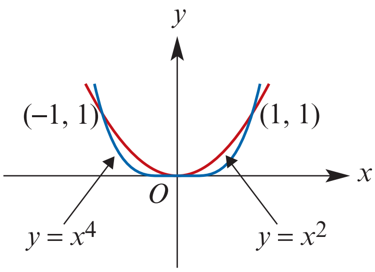
Comparing the graphs of $y = x^n$ and $y = x^m$ for n and m even

Assume that n and m are even positive integers with $n > m$. Then:

- $x^n = x^m$ for $x = -1, 0, 1$
- $x^n > x^m$ for $x < -1$ and for $x > 1$
- $x^n < x^m$ for $-1 < x < 0$ and for $0 < x < 1$.



Example 17

The graph of $y = a(x-h)^4 + k$ has a turning point at $(2, 2)$ and passes through the point $(0, 4)$. Find the values of a, h and k.

Solution

Since $(2, 2)$ is the turning point,

$$h = 2 \quad \text{and} \quad k = 2$$

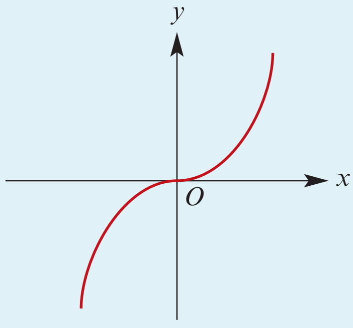
So $y = a(x-2)^4 + 2$ and, since the graph passes through $(0, 4)$,

$$4 = 16a + 2$$

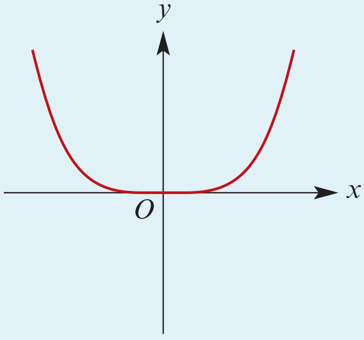
$$\therefore \quad a = \frac{1}{8}$$

Summary 3G

- A graph with rule of the form $y = a(x-h)^n + k$ can be obtained as a transformation of the graph of $y = x^n$.
- **Odd index** If n is an odd integer with $n \geq 3$, then the graph of $y = x^n$ has a shape similar to the one shown below; there is a point of zero gradient at $(0, 0)$.



- **Even index** If n is an even integer with $n \geq 2$, then the graph of $y = x^n$ has a shape similar to the one shown below; there is a turning point at $(0, 0)$.



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Exercise 3G

Example 14

1 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = 2x^3$
b $g(x) = -2x^3$
c $h(x) = x^5 + 1$
d $f(x) = x^3 - 4$
e $f(x) = (x + 1)^3 - 8$
f $f(x) = 2(x - 1)^3 - 2$
g $g(x) = -2(x - 1)^3 + 2$
h $h(x) = 3(x - 2)^3 - 4$
i $f(x) = 2(x - 1)^3 + 2$
j $h(x) = -2(x - 1)^3 - 4$
k $f(x) = (x + 1)^5 - 32$
l $f(x) = 2(x - 1)^5 - 2$

Example 15

2 The graph of $y = a(x - h)^3 + k$ has a point of zero gradient at $(0, 4)$ and passes through the point $(1, 1)$. Find the values of a, h and k.

3 Find the equation of the image of $y = x^3$ under each of the following transformations:

a a dilation of factor 3 from the x-axis
b a translation with rule $(x, y) \to (x - 1, y + 1)$
c a reflection in the x-axis followed by the translation $(x, y) \to (x + 2, y - 3)$
d a dilation of factor 2 from the x-axis followed by the translation $(x, y) \to (x - 1, y - 2)$
e a dilation of factor 3 from the y-axis.

Example 16

4 **a** Find the rule for the image of the graph of $y = x^3$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 3 from the y-axis
- translation 3 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^3$ to the graph of $y = 4 - 3(x + 1)^3$.

5 Find the rule for the image of the graph of $y = x^4$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 2 units in the negative direction of the x-axis and 1 unit in the negative direction of the y-axis.

6 Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 5 - 3(x + 1)^4$.

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7 By applying suitable transformations to $y = x^4$, sketch the graph of each of the following:

a $y = 3(x-1)^4 - 2$ **b** $y = -2(x+2)^4$ **c** $y = (x-2)^4 - 6$
d $y = 2(x-3)^4 - 1$ **e** $y = 1 - (x+4)^4$ **f** $y = -3(x-2)^4 - 3$

Example 17

8 The graph of $y = a(x-h)^4 + k$ has a turning point at $(-2, 3)$ and passes through the point $(0, -6)$. Find the values of a, h and k.

9 The graph of $y = a(x-h)^4 + k$ has a turning point at $(1, 7)$ and passes through the point $(0, 23)$. Find the values of a, h and k.

3H Determining the rule for a function from its graph

Given sufficient information about a curve, we can determine its rule. For example, if we know the coordinates of two points on a hyperbola of the form

$$y = \frac{a}{x} + b$$

then we can find the rule for the hyperbola, i.e. we can find the values of a and b.

Sometimes the rule has a more specific form. For example, the curve may be a dilation of $y = \sqrt{x}$. Then we know its rule is of the form $y = a\sqrt{x}$, and the coordinates of one point on the curve (with the exception of the origin) will be enough to determine the value of a.

Example 18

The points $(1, 5)$ and $(4, 2)$ lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

Solution

When $x = 1$, $y = 5$ and so

$$5 = a + b \qquad (1)$$

When $x = 4$, $y = 2$ and so

$$2 = \frac{a}{4} + b \qquad (2)$$

Subtract (2) from (1):

$$3 = \frac{3a}{4}$$

$$\therefore \quad a = 4$$

Substitute in (1) to find b:

$$5 = 4 + b$$

$$\therefore \quad b = 1$$

The equation of the curve is $y = \frac{4}{x} + 1$.

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Example 19

The points (2, 1) and (10, 6) lie on a curve with equation $y = a\sqrt{x-1} + b$. Find the values of a and b.

Solution

When $x = 2$, $y = 1$ and so

$$1 = a\sqrt{1} + b$$

i.e. $1 = a + b$ (1)

When $x = 10$, $y = 6$ and so

$$6 = a\sqrt{9} + b$$

i.e. $6 = 3a + b$ (2)

Subtract (1) from (2):

$$5 = 2a$$

$$\therefore \quad a = \frac{5}{2}$$

Substitute in (1) to find $b = -\frac{3}{2}$.

The equation of the curve is $y = \frac{5}{2}\sqrt{x-1} - \frac{3}{2}$.

Skill-sheet

Exercise 3H

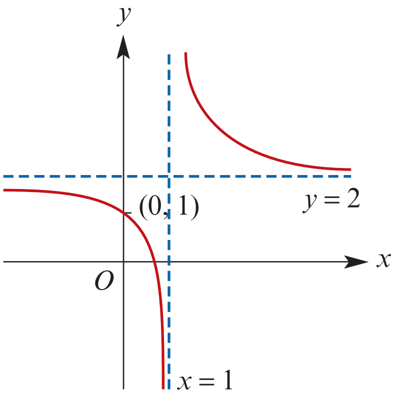
Example 18

1 The points (1, 4) and (3, 1) lie on a curve with equation $y = \frac{a}{x} + b$. Find the values of a and b.

2 The graph shown has the rule

$$y = \frac{A}{x+b} + B$$

Find the values of A, b and B.



Example 19

3 The points (3, 1) and (11, 6) lie on a curve with equation $y = a\sqrt{x-2} + b$. Find the values of a and b.

4 The points with coordinates (1, 5) and (16, 11) lie on a curve which has a rule of the form $y = A\sqrt{x} + B$. Find A and B.

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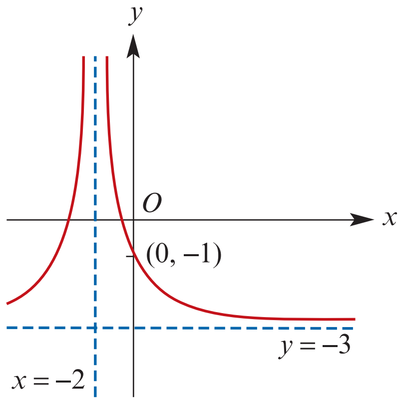
5 The points with coordinates $(1, 1)$ and $(0.5, 7)$ lie on a curve which has a rule of the form $y = \dfrac{A}{x^2} + B$. Find the values of A and B.

6 The graph shown has the rule

$$y = \frac{A}{(x+b)^2} + B$$

Find the values of A, b and B.



7 The points with coordinates $(1, -1)$ and $(2, \frac{3}{4})$ lie on a curve which has a rule of the form $y = \dfrac{a}{x^3} + b$. Find the values of a and b.

8 The points with coordinates $(-1, 4)$ and $(1, -8)$ lie on a curve which has a rule of the form $y = a\sqrt[3]{x} + b$. Find the values of a and b.

3I A notation for transformations

The following table gives a summary of some basic transformations of the plane. In each row of the table, the point (x', y') is the image of the point (x, y) under the mapping.

Mapping	Rule
Reflection in the x-axis	$x' = x$ $y' = -y$
Reflection in the y-axis	$x' = -x$ $y' = y$
Dilation of factor k from the y-axis	$x' = kx$ $y' = y$
Dilation of factor k from the x-axis	$x' = x$ $y' = ky$
• h units in the positive direction of the x-axis • k units in the positive direction of the y-axis	$x' = x + h$ $y' = y + k$

The transformations in the table and combinations of these transformations all have rule of the form

$$(x, y) \to (ax + h, by + k) \text{ where } a \neq 0 \text{ and } b \neq 0$$

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We can consider transformations of the plane as functions with two variables and use a similar notation to that we have used for functions of one variable.

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (ax + h, by + k), \quad a \neq 0, b \neq 0$$

We note that both the maximal domain and range of transformations of this type is $\mathbb{R}^2$

Linear transformations

Dilations, reflection in the x-axis and reflection in the y-axis and combinations of these are examples of **linear transformations**. Linear transformations can be defined by having a rule of the form

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (ax + by, cx + dy)$$

A further example of a linear transformation is reflection in the line $y = x$.

Mapping	Rule	The graph of $y = f(x)$ maps to
Reflection in the line $y = x$	$x' = y,\ y' = x$	$x = f(y)$

You have met this transformation in the study of inverse functions.

We note that linear transformations of the form $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (ax + h, by + k)$ are not linear unless $h = k = 0$

Example 20

Let $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (3x + 2, 4y - 6)$

a Evaluate

i $T(4, 5)$ **ii** $T(-1, 2)$

b Find the equation of the image of the graph $y = f(x)$, where $f(x) = 2^x$, under this transformation.

Solution

a **i** $T(4, 5) = (3 \times 4 + 2, 4 \times 5 - 6 = (14, 24)$

ii $T(-1, 2) = (3 \times (-1) + 2, 4 \times 2 - 6 = (-1, 2)$

b $T(x, y) = (3x + 2, 4y - 6)$ an let $(x', y') = (3x + 2, 4y - 6)$.

Then:

$x' = 3x + 2 \Rightarrow x = \dfrac{x' - 2}{3}$

$y' = 4y - 6 \Rightarrow y = \dfrac{y' + 6}{4}$

The image of $y = 2^x$ is $\dfrac{y' + 6}{4} = 2^{\frac{x'-2}{3}}$

That is, the image has equation $y = 4 \times 2^{\frac{x-2}{3}} - 6$

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Example 21

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $\quad T(x, y) = (ax + h, by + k)$. Given that $T(1, 6) = (-7, -3)$ and $T(-2, 3) = (11, 12)$ determine the values of a, b, h and k.

Solution

Since $T(1, 6) = (-7, -3)$ we have the equations:

$a + h = -7 \ldots (1)$ and $6b + k = -3 \ldots (2)$

Since $T(-2, 3) = (11, 12)$ we have the equations:

$-2a + h = 11 \ldots (3)$ and $3b + k = 12 \ldots (4)$

Subtract (3) from (1):

$$3a = -18$$
$$a = -6$$
$$\therefore h = -1$$

Subtract (4) from (2):

$$3b = -15$$
$$b = -5$$
$$\therefore k = 27$$

$a = -6, b = -5, h = -1, k = 27$

Composition of transformations

In section 7D we looked at combinations of transformations. This can be formalised with our new notation by turning to **composition of transformations** which is the same idea that we implemented in our study of functions.

Example 22

Let $T_1(x, y) = (x + 3, 2y)$ and $T_2(x, y) = (3x + 2, y - 3)$. Find the rule for;

a $T_2(T_1(x, y))$

b $T_1(T_2(x, y))$

Solution

a
$$\begin{aligned} T_2(T_1(x, y)) &= T_2(x + 3, 2y) \\ &= (3(x + 3) + 2, 2y - 3) \\ &= (3x + 11, 2y - 3) \end{aligned}$$

b
$$\begin{aligned} T_1(T_2(x, y)) &= T_1(3x + 2, y - 3) \\ &= (3x + 2 + 3, 2(y - 3)) \\ &= (3x + 5, 2y - 6) \end{aligned}$$

Notes:

- Note that two transformatons of the plane T_1 and T_2 are equal if $T_1(x, y) = T_2(x, y)$ for all $(x, y) \in \mathbb{R}^2$.
- The composition of transformations T and S can be written as $T \circ S$.
- In general, for transformations T and S, $T(S(x, y)) \neq S(T(x, y))$. That is, $T \circ S \neq S \circ T$.
- Some familes of transformations do commute. For example if S and T are translations then, $T(S(x, y)) = S(T(x, y))$.

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Inverses of transformations

A transformation T is **one-to-one** if $T(x_1, y_1) = T(x_2, y_2)$ implies $x_1 = x_2$ and $y_1 = y_2$. All of the transformations that we are considering in this section are one-to-one transformations.

Given a transformation T we can define a new transformation T^{-1}, the **inverse** of T, by defining:

$$T^{-1}(x', y') = (x, y) \text{ if } T(x, y) = (x', y')$$

Note: The function T^{-1} is also a one-to one function, and T is the inverse of T^{-1}.

Furthermore we have:

$$T \circ T^{-1}(x, y) = (x, y) \text{ for all } (x, y) \in \mathbb{R}^2$$
$$T^{-1} \circ T(x, y) = (x, y) \text{ for all } (x, y) \in \mathbb{R}^2$$

Example 23

Find the inverse of the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (3x - 2, -5y + 3)$$

Solution

We know $T \circ T^{-1}(x, y) = (x, y)$

Let $T^{-1}(x, y) = (w, z)$. Then

$$T(T^{-1}(x, y)) = (x, y)$$
$$T(w, z) = (x, y)$$
$$(3w - 2, -5z + 3) = (x, y)$$
$$\therefore 3w - 2 = x \text{ and } -5z + 3 = y$$
$$\therefore w = \frac{x + 2}{3} \text{ and } z = \frac{3 - y}{5}$$
$$\text{Hence, } T^{-1}(x, y) = \left(\frac{x + 2}{3}, \frac{3 - y}{5}\right)$$

You can check that $T \circ T^{-1}(x, y) = T^{-1} \circ T(x, y) = (x, y)$.

Transformations on subsets of $\mathbb{R}^2$

When you deal with functions for which the domain is not $\mathbb{R}$ it is of interest to see how the domain (and range) are transformed.

Example 24

Consider the function $f : [0, 3] \to \mathbb{R},\ f(x) = -x^2 + 2x$.

a Find the range of f

b Find the image of f under the transformation with rule $T(x, y) = (2x, -2y + 3)$. State the domain and range of this image.

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Solution

a The graph of $y = f(x)$ has a local maximum at $(1, 1)$. The endpoints have coordinates $(0, 0)$ and $(3, -3)$. Therefore range is $[-3, 1]$

b Let $T(x, y) = (x', y')$

Therefore, $x' = 2x$ and $y' = -2y + 3$.

Thus, $x = \frac{x'}{2}$ and $y = \frac{y' - 3}{-2}$

Hence $y = -x^2 + 2x$ is mapped to $\frac{y' - 3}{-2} = -\left(\frac{x'}{2}\right)^2 + 2\left(\frac{x'}{2}\right)$

Simplifying,

The image has equation $y = \frac{x^2}{2} - 2x + 3$

The domain is calculated as $[2 \times 0, 2 \times 3] = [0, 6]$.

The turning point of the image, which is a local minimum, has coordinates

$$(2 \times 1, -2 \times 1 + 3) = (2, 1)$$

The range can be calculated from the domain of the image and the equation of the image as $[1, 9]$ or you can consider the transformation of the range of the original function.

Note: There is a reflection in the x-axis and so care must be taken with the end points of the range.

Exercise 3I

Example 20

1 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (x - 2, 2y + 3)$

a Evaluate

i $T(-2, 5)$ **ii** $T(4, 2)$

b Find the equation of the image of the graph $y = f(x)$, where $f(x) = 2^x$, under this transformation.

Example 21

2 Let $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (ax + h, by + k)$. Given that $T(-1, 7) = (-7, -3)$ and $T(-2, -3) = (4, 6)$ determine the values of a, b, h and k.

Example 22

3 Let $T_1(x, y) = (2x, 2y - 3)$ and $T_2(x, y) = (-x + 2, -y - 3)$. Find the rule for;

a $T_2(T_1(x, y))$ **b** $T_1(T_2(x, y))$

Example 23

4 Find the inverse of the transformation

$$T : \mathbb{R}^2 \to \mathbb{R}^2, \quad T(x, y) = (-2x + 2, y - 3)$$

5 Determine the set which is the image of the set $\{(x, y) : x \in [2, 5], y \in [-3, 7], x, y \in \mathbb{R}\}$ under the transformation with rule $T(x, y) = (2x + 6, y - 3)$

Example 24

6 Consider the function $f : [0, 4] \to \mathbb{R}$, $f(x) = x^2$.

a Find the range of f

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b Find the image of f under the transformation with rule $T(x, y) = (-2x, 2y + 4)$. State the domain and range of this image.

7 Let $T_1(x, y) = \left(\frac{1}{2}x, y - 3\right)$, $T_2(x, y) = (-x, y + 3)$ and $T_3(x, y) = (-2x, y - 3)$. Find the rule for;

a $T_2(T_1(x, y))$ **b** $T_1(T_2(x, y))$ **c** $T_3(T_1(x, y))$

d $T_1(T_3(x, y))$ **e** $T_2(T_3(x, y))$ **f** $T_3(T_2(x, y))$

8 Find the inverse of each of the following transformations assuming domain $\mathbb{R}^2$

a $T(x, y) = (-x + 2, -y - 3)$ **b** $S(x, y) = (x + 2, y - 3)$

c $T(x, y) = (-3x - 2, 6 - y)$ **d** $S(x, y) = (-2x + 3, 4 - y)$

9 Consider the function $f : [-1, 2] \to \mathbb{R}, \quad f(x) = x^3$.

a Find the range of f

b Find the image of f under the transformation with rule $T(x, y) = (-x + 3, -2y + 4)$. State the domain and range of this image.

10 Let $T_1(x, y) = (x - 5, y + 2)$ and $T_2(x, y) = \left(-x, \frac{1}{2}y\right)$

a Determine the rules for

i $T_1(T_2(x, y))$ **ii** $T_2(T_1(x, y))$ **iii** $T_1(T_1(x, y))$

b Describe each of resulting transformations in words.

11 Let $T_1(x, y) = (3x, 2y)$, $T_2(x, y) = (x + 3, y - 2)$ and $T_3(x, y) = (-x, y)$.

a Determine the rules for

i $T_1(T_2(T_3(x, y)))$ **ii** $T_2(T_1(T_3(x, y)))$ **iii** $T_3(T_1(T_2(x, y)))$

b Describe each of resulting transformations in words.

12 A transformation of the form $T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + h, by + k)$ maps the graph of $f : [c, d] \to \mathbb{R}, f(x) = \sqrt{x}$ to $f : [4, 8] \to \mathbb{R}, f(x) = -3\sqrt{2x - 5} + 6$. State a possible set of values of a, h, b and k and the corresponding values of c and d

13 Let $f : \mathbb{R} \backslash \left\{-\frac{2}{5}\right\} \to \mathbb{R}, \quad f(x) = \frac{1}{5x + 2}$.

a Find f^{-1}

b Determine the rule for a transformation of the form,

$$T : \mathbb{R}^2 \to \mathbb{R}^2, T(x, y) = (ax + h, by + k)$$

which maps the graph of $y = f(x)$ to the graph of $y = f^{-1}(x)$

14 Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2, T_1(x, y) = (a_1x + h_1, b_1y + k_1)$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2, T_2(x, y) = (a_2x + h_2, b_2y + k_2)$.

a Give the rules for $T_1 \circ T_2$ and $(T_1 \circ T_2)^{-1}$

b Give the rules for T_1^{-1}, T_2^{-1} and $T_2^{-1} \circ T_1^{-1}$

c Prove that $(T_1 \circ T_2)^{-1} = T_2^{-1} \circ T_1^{-1}$

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Review

Chapter summary

Assign-ment

Nrich

In the following table, the rule for each transformation is given along with the rule for the image of the graph of $y = f(x)$.

Mapping	Rule	The graph of $y = f(x)$ maps to
Reflection in the x-axis	$x' = x$ $y' = -y$	$y = -f(x)$
Reflection in the y-axis	$x' = -x$ $y' = y$	$y = f(-x)$
Dilation of factor a from the y-axis	$x' = ax$ $y' = y$	$y = f\left(\frac{x}{a}\right)$
Dilation of factor b from the x-axis	$x' = x$ $y' = by$	$y = bf(x)$
Reflection in the line $y = x$	$x' = y$ $y' = x$	$x = f(y)$
• h units in the positive direction of the x-axis • k units in the positive direction of the y-axis	$x' = x + h$ $y' = y + k$	$y - k = f(x - h)$

Technology-free questions

1 Sketch the graph of each of the following. Label any asymptotes and axis intercepts. State the range of each function.

a $f : \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \dfrac{1}{x} - 3$

b $f : (2, \infty) \to \mathbb{R},\ f(x) = \dfrac{1}{x - 2}$

c $f : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = \dfrac{2}{x - 1} - 3$

d $f : (2, \infty) \to \mathbb{R},\ f(x) = \dfrac{-3}{2 - x} + 4$

e $f : \mathbb{R} \setminus \{1\} \to \mathbb{R},\ f(x) = 1 - \dfrac{1}{x - 1}$

2 Sketch the graph of each of the following:

a $f(x) = 2\sqrt{x - 3} + 1$

b $g(x) = \dfrac{3}{(x - 2)^2} - 1$

c $h(x) = \dfrac{-3}{(x - 2)^2} - 1$

3 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = -2(x + 1)^3$

b $g(x) = -2(x - 1)^5 + 8$

c $h(x) = 2(x - 2)^5 + 1$

d $f(x) = 4(x - 1)^3 - 4$

4 The points with coordinates $(1, 6)$ and $(16, 12)$ lie on a curve which has a rule of the form $y = a\sqrt{x} + b$. Find a and b.

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5 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has rule

$$T(x, y) = (x - 4, -2y - 1)$$

Find the image of the curve with equation $y = \sqrt{x}$ under this transformation.

6 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ has rule

$$T(x, y) = (3x - 4, -y - \frac{1}{2})$$

Find the image of the curve with equation $y = 2\sqrt{x - 4} + 3$ under this transformation.

7 The points with coordinates $(1, 3)$ and $(3, 7)$ lie on a curve with equation of the form $y = \frac{a}{x} + b$. Find the values of a and b.

8 **a** Find the rule for the image of the graph of $y = -x^2$ under the following sequence of transformations:

- reflection in the y-axis
- dilation of factor 2 from the y-axis
- translation 4 units in the positive direction of the x-axis and 6 units in the positive direction of the y-axis.

b Find a sequence of transformations which takes the graph of $y = x^4$ to the graph of $y = 6 - 4(x + 1)^4$.

9 Identify a sequence of transformations that maps the graph of $y = \frac{1}{x^2}$ onto the graph of $y = \frac{3}{(x-5)^2} + 3$. Use this to sketch the graph of $y = \frac{3}{(x-5)^2} + 3$, stating the equations of asymptotes and the coordinates of axis intercepts.

10 Find a sequence of transformations that takes the graph of $y = 2x^2 - 3$ to the graph of $y = x^2$.

11 Find a sequence of transformations that takes the graph of $y = 2(x - 3)^3 + 4$ to the graph of $y = x^3$.

Multiple-choice questions

1 The point $P(3, -4)$ lies on the graph of a function f. The graph of f is translated 3 units up (parallel to the y-axis) and reflected in the x-axis. The coordinates of the final image of P are

A $(6, 4)$ **B** $(3, 1)$ **C** $(3, -1)$ **D** $(-3, 1)$ **E** $(3, 7)$

2 The graph of $y = x^3 + 4$ is translated 3 units 'down' and 2 units 'to the right'. The resulting graph has equation

A $y = (x - 2)^3 + 2$ **B** $y = (x - 2)^3 + 1$ **C** $y = (x - 2)^3 + 5$
D $y = (x + 2)^3 + 1$ **E** $y = (x + 2)^3 + 6$

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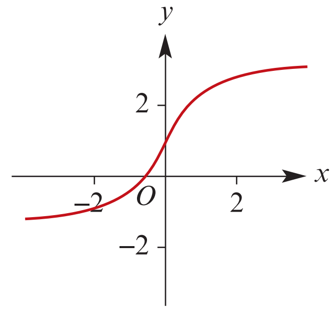
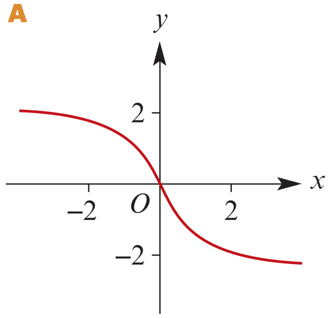
3 The graph of $y = f(x)$ is shown on the right.

Which one of the following could be the graph of $y = f(-x)$?



A



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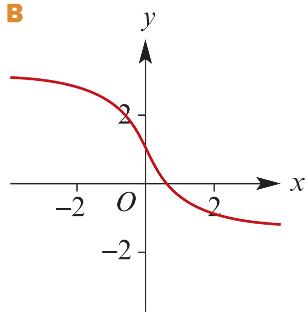

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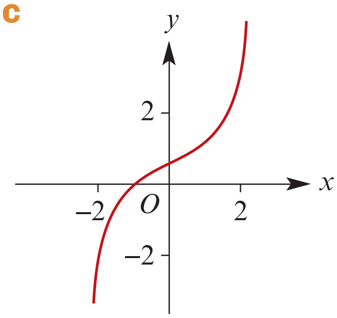
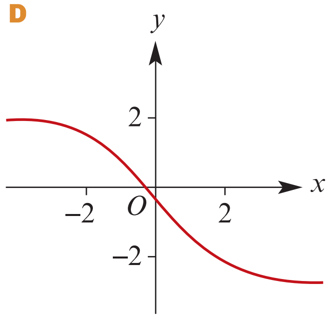
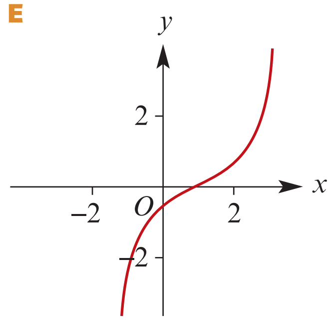


D



E



4 The graph of the function with rule $y = x^2$ is reflected in the x-axis and then translated 4 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis. The rule for the new function is

A $y = (-x + 4)^2 - 3$ **B** $y = -(x - 4)^2 + 3$ **C** $y = -(x - 3)^2 + 4$

D $y = (-x - 4)^2 + 3$ **E** $y = -(x + 4)^2 - 3$

5 The graph of $y = \dfrac{a}{x + b} + c$ is shown on the right. A possible set of values for a, b and c respectively is

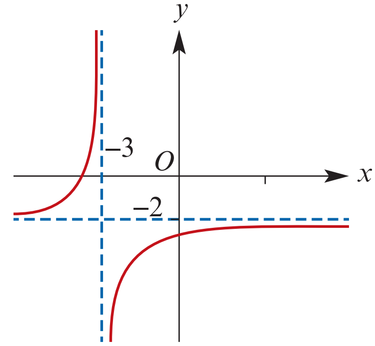
A $-1, 3, 2$

B $1, 2, -3$

C $-1, -3, -2$

D $-1, 3, -2$

E $1, 2, -3$



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Review

6 The graph of the function f is obtained from the graph of $y = x^{\frac{1}{3}}$ by a reflection in the y-axis followed by a dilation of factor 5 from the x-axis. The rule for f is

A $f(x) = -5x^{\frac{1}{3}}$ **B** $f(x) = \frac{1}{5}(-x)^{\frac{1}{3}}$ **C** $f(x) = (-5x)^{\frac{1}{3}}$

D $f(x) = -\frac{1}{5}x^{\frac{1}{3}}$ **E** $f(x) = -5(-x)^{\frac{1}{3}}$

7 The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with rule $T(x, y) = (3x - 2, -y - 1)$ maps the curve with equation $y = \sqrt[3]{x}$ to the curve with equation

A $y = 1 + \sqrt[3]{\frac{x-2}{3}}$ **B** $y = \sqrt[3]{\frac{x+2}{3}} - 1$ **C** $y = 2\sqrt[3]{\frac{x-3}{3}} - 1$

D $y = -1 - \sqrt[3]{\frac{x+2}{3}}$ **E** $y = -1 + \sqrt[3]{\frac{3x+2}{3}}$

8 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the graph of $y = \frac{1}{x}$ to the graph of $y = \frac{3}{2x+1} - 4$ is given by

A $T(x, y) = \left(\frac{1}{2}x - \frac{1}{2}, 3y - 4\right)$ **B** $T(x, y) = \left(\frac{1}{2}x - 2, 3y - 1\right)$

C $T(x, y) = \left(\frac{1}{2}x - 2, 3y - 4\right)$ **D** $T(x, y) = \left(3x + 2, \frac{1}{2}y - 4\right)$

E $T(x, y) = \left(3x - \frac{1}{2}, \frac{1}{2}y - 2\right)$

9 A transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that maps the graph of $y = -\frac{5}{2x-1} + 3$ to the graph of $y = \frac{1}{x}$ is given by

A $T(x, y) = \left(2x - 1, -\frac{1}{5}y + \frac{3}{5}\right)$ **B** $T(x, y) = \left(\frac{1}{2}x - 2, -\frac{1}{5}y + \frac{3}{5}\right)$

C $T(x, y) = \left(\frac{1}{2}x + 1, -\frac{1}{5}y - \frac{3}{5}\right)$ **D** $T(x, y) = \left(2x - 1, 5y - \frac{3}{5}\right)$

E $T(x, y) = \left(5x - \frac{1}{2}, \frac{1}{2}y - \frac{3}{5}\right)$

10 Let $f(x) = 3x - 2$ and $g(x) = x^2 - 4x + 2$. A sequence of transformations that takes the graph of $y = g(x)$ to the graph of $y = g(f(x))$ is

A a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{2}{3}$ units in the positive direction of the x-axis

B a dilation of factor 3 from the y-axis followed by a translation 2 units in the negative direction of the x-axis

C a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation $\frac{1}{2}$ unit in the positive direction of the x-axis

D a dilation of factor 3 from the y-axis followed by a translation 2 units in the positive direction of the x-axis

E a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 2 units in the positive direction of the x-axis

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Review

Extended-response questions

1 Consider the function $f\colon D \to \mathbb{R}$ with rule $f(x) = \dfrac{24}{x+2} - 6$, where D is the maximal domain for this rule.

a Find D.

b Describe a sequence of transformations which, when applied to the graph of $y = \dfrac{1}{x}$, produces the graph of $y = f(x)$. Specify the order in which these transformations are to be applied.

c Find the coordinates of the points where the graph of f cuts the axes.

Let $g\colon (-2, \infty) \to \mathbb{R}$, $g(x) = f(x)$.

d Find the rule for g^{-1}, the inverse of g.

e Write down the domain of g^{-1}.

f Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the one set of axes.

g Find the value(s) of x for which $g(x) = x$ and hence the value(s) of x for which $g(x) = g^{-1}(x)$.

2 Consider the function $f\colon D \to \mathbb{R}$ with rule $f(x) = 4 - 2\sqrt{2x+6}$, where D is the maximal domain for this rule.

a Find D.

b Describe a sequence of transformations which, when applied to the graph of $y = \sqrt{x}$, produces the graph of $y = f(x)$. Specify the order in which these transformations are to be applied.

c Find the coordinates of the points where the graph of f cuts the axes.

d Find the rule for f^{-1}, the inverse of f.

e Find the domain of f^{-1}.

f Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the one set of axes.

g Find the value(s) of x for which $f(x) = x$ and hence the value(s) of x for which $f(x) = f^{-1}(x)$.

3 **a** **i** Find the dilation from the x-axis which takes $y = x^2$ to the parabola with its vertex at the origin that passes through the point $(25, 15)$.

ii State the rule which reflects this dilated parabola in the x-axis.

iii State the rule which takes the reflected parabola of part **ii** to a parabola with x-axis intercepts $(0, 0)$ and $(50, 0)$ and vertex $(25, 15)$.

iv State the rule which takes the curve $y = x^2$ to the parabola defined in part **iii**.

b The plans for the entrance of a new building involve twin parabolic arches as shown in the diagram.

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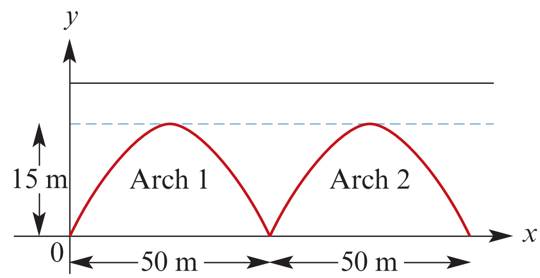
i From the results of part **a**, give the equation for the curve of arch 1.

ii Find the translation which maps the curve of arch 1 to the curve of arch 2.

iii Find the equation of the curve of arch 2.



c The architect wishes to have flexibility in her planning and so wants to develop an algorithm for determining the equations of the curves when each arch has width m metres and height n metres.

i Find the rule for the transformation which takes the graph of $y = x^2$ to the current arch 1 with these new dimensions.

ii Find the equation for the curve of arch 1.

iii Find the equation for the curve of arch 2.

4 Consider the function $g\colon D \to \mathbb{R}$ with rule $g(x) = \dfrac{3}{(3x-4)^2} + 6$, where D is the maximal domain for this rule.

a Find D.

b Find the smallest value of a such that $f\colon (a, \infty) \to \mathbb{R}$, $f(x) = g(x)$ is a one-to-one function.

c Find the inverse function of f.

d Find the value of x for which $f(x) = f^{-1}(x)$.

e On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

5 **a** Sketch the curve with equation $f(x) = \dfrac{50}{20-x}$, for $x \neq 20$.

b For $g(x) = \dfrac{50x}{20-x}$:

i Show that $g(x) = \dfrac{1000}{20-x} - 50$.

ii Sketch the graph of $y = g(x)$.

iii Show that $g(x) = 20f(x) - 50$.

c Find the rule for the function g^{-1}.

6 When the transformation with rule $(x, y) \to (y, x)$ (a reflection in the line $y = x$) is applied to the graph of a one-to-one function f, the resulting image has rule $y = f^{-1}(x)$, i.e. the graph of the inverse function is obtained.

a For the graph of $y = f(x)$, find the rule for the image of f, in terms of $f^{-1}(x)$, for each of the following sequences of transformations:

i
- a translation of 3 units in the positive direction of the x-axis
- a translation of 5 units in the positive direction of the y-axis
- a reflection in the line $y = x$

Review

ii
- a reflection in the line $y = x$
- a translation of 3 units in the positive direction of the x-axis
- a translation of 5 units in the positive direction of the y-axis

iii
- a dilation of factor 3 from the x-axis
- a dilation of factor 5 from the y-axis
- a reflection in the line $y = x$

iv
- a reflection in the line $y = x$
- a dilation of factor 5 from the y-axis
- a dilation of factor 3 from the x-axis.

b Find the image of the graph of $y = f(x)$, in terms of $f^{-1}(x)$, under the transformation with rule $(x, y) \to (ay + b, cx + d)$, where a, b, c and d are positive constants, and describe this transformation in words.

7 Let $f(x) = x^2 - 9$ and

$$g(x) = \begin{cases} \dfrac{1}{2}f(x+6) & \text{if } -10 \le x < -3 \\ f(x) & \text{if } -3 \le x < 3 \\ -\dfrac{2}{3}f(x-6) & \text{if } 3 \le x \le 10 \end{cases}$$

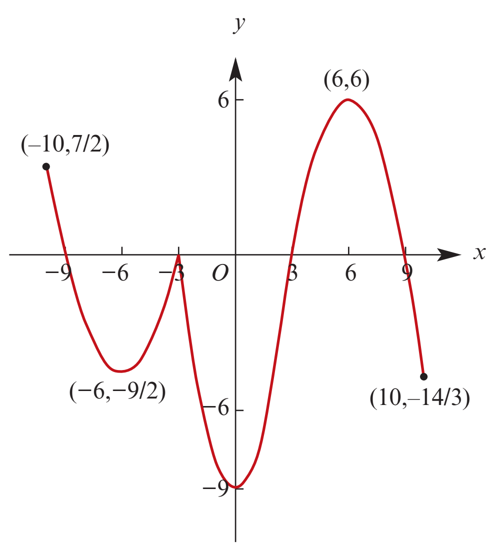
A graph of $y = g(x)$ is shown below right.

a State the range of $y = g(x)$

b Find the values of k such that the equation $y = k$ has

- **i** 0 solutions
- **ii** 1 solution
- **iii** 2 solutions
- **iv** 3 solutions
- **v** 4 solutions
- **vi** 5 solutions.

c Write the rule for $g(x)$ with the components written in polynomial form.



d State the range of the function with rule $y = -2g(3x)$.

e The transformation T with rule $T(x, y) = (-x + 2, 4 - 2y)$ is applied to the graph of the function g. State the domain and range of the image of g.

f **i** Determine the rule for a transformation S, with rule of the form $(x, y) \to (ax + h, cy + k)$, that takes the graph of $y = f(x)$ to the parabola with equation $y = -2x^2 + 12x + 2$.

ii Find the equation of the image of the graph of $y = g(x)$ under this transformation.

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