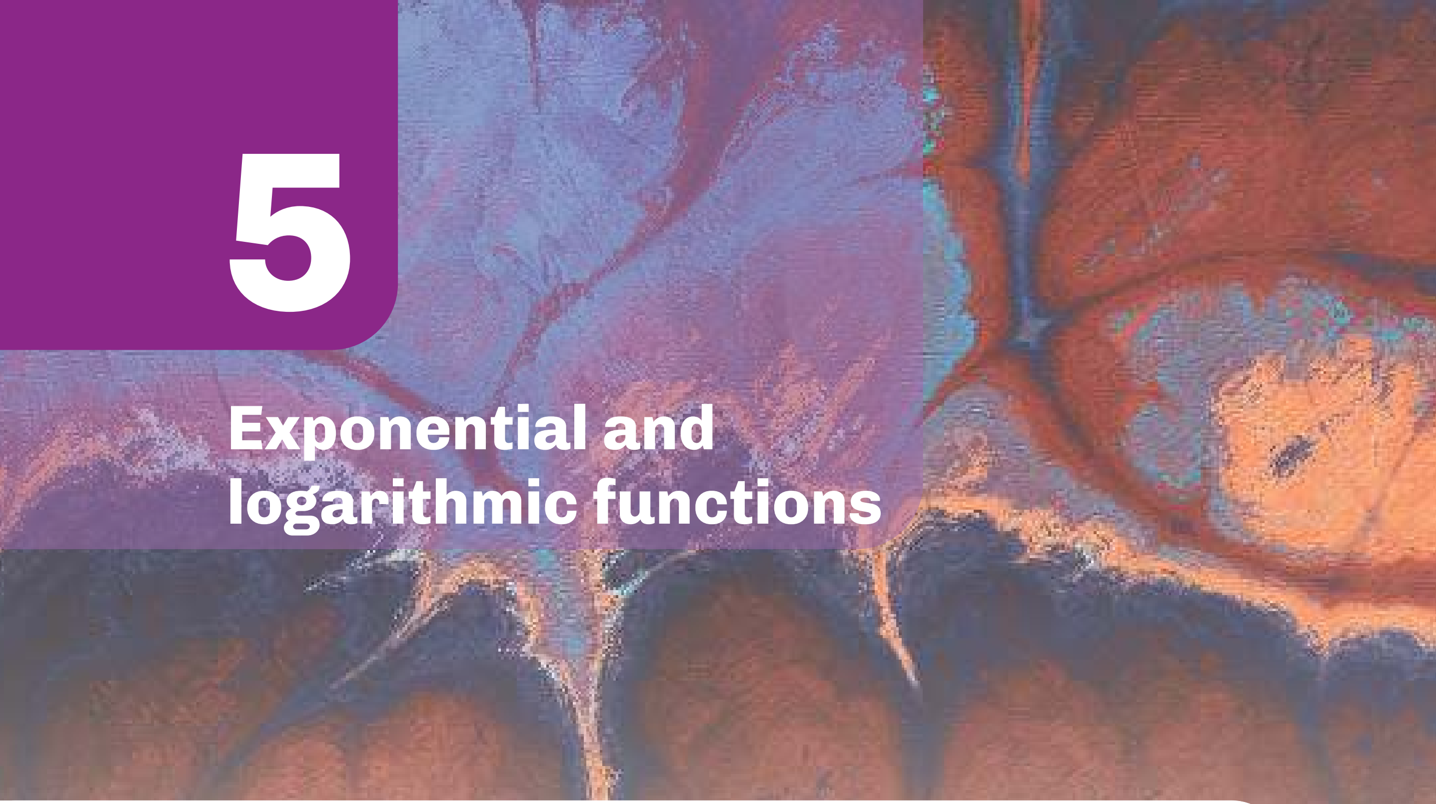

5

Exponential and logarithmic functions

Objectives

- ▶ To graph **exponential** and **logarithmic functions** and transformations of these functions.
- ▶ To introduce **Euler's number** *e*.
- ▶ To revise the **index** and **logarithm laws**.
- ▶ To solve **exponential** and **logarithmic equations**.
- ▶ To find rules for the graphs of exponential and logarithmic functions.
- ▶ To find inverses of exponential and logarithmic functions.
- ▶ To apply exponential functions in modelling growth and decay.

Our work on functions is continued in this chapter. Many of the concepts introduced in Chapters 1 and 3 – domain, range, transformations and inverse functions – are used in the context of exponential and logarithmic functions.

An **exponential function** has a rule of the form $f(x) = ka^x$, where k is a non-zero constant and the base a is a positive real number other than 1.

Exponential functions were introduced in Mathematical Methods Units 1 & 2 and it was shown that there are practical situations where these functions can be applied, including radioactive decay and population growth. Some of these applications are further investigated in this chapter.

We also introduce the exponential function $f(x) = e^x$, which has many interesting properties. In particular, this function is its own derivative. That is, $f'(x) = f(x)$.

Here we define the number e as

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

We will show that limits such as this arise in the consideration of compound interest.

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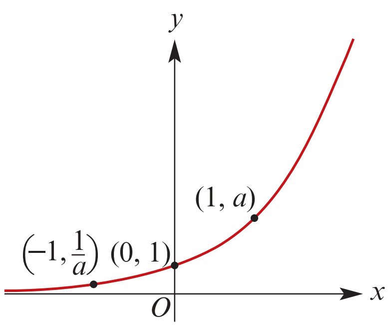
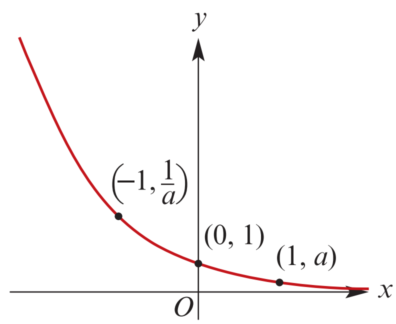
5A Exponential functions

The function $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$, is an **exponential function**. The shape of the graph depends on whether $a > 1$ or $0 < a < 1$.

Graph of $f(x) = a^x$ for $a > 1$



Graph of $f(x) = a^x$ for $0 < a < 1$



- Key values are $f(-1) = \dfrac{1}{a}$, $f(0) = 1$ and $f(1) = a$.
- The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+$.
- The x-axis is a horizontal asymptote.

An exponential function with $a > 1$ is strictly increasing, and an exponential function with $0 < a < 1$ is strictly decreasing. In both cases, the function is one-to-one.

Graphing transformations of $f(x) = a^x$

Translations

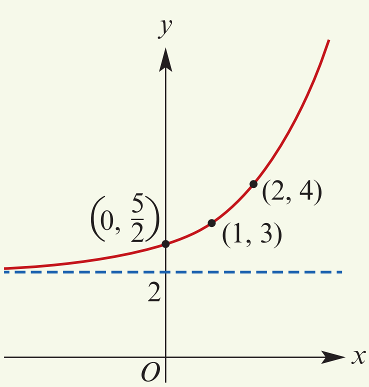
If the translation $(x, y) \to (x + h, y + k)$ is applied to the graph of $y = a^x$, then the image has equation $y = a^{x-h} + k$.

- The horizontal asymptote of the image has equation $y = k$.
- The range of the image is (k, ∞).

Example 1

Sketch the graph and state the range of $y = 2^{x-1} + 2$.

Solution



The range of the function is $(2, \infty)$.

Explanation

The graph of $y = 2^x$ is translated 1 unit in the positive direction of the x-axis and 2 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 1, y + 2)$.

Translation of key points:

- $(-1, \frac{1}{2}) \to (0, \frac{5}{2})$
- $(0, 1) \to (1, 3)$
- $(1, 2) \to (2, 4)$

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Reflections

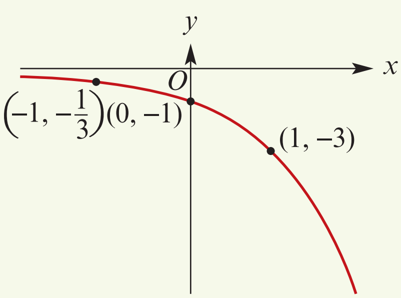
If a **reflection in the x-axis**, given by the mapping $(x, y) \to (x, -y)$, is applied to the graph of $y = a^x$, then the image has equation $y = -a^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(-\infty, 0)$.

Example 2

Sketch the graph of $y = -3^x$.

Solution



Explanation

The graph of $y = 3^x$ is reflected in the x-axis.

The mapping is $(x, y) \to (x, -y)$.

Reflection of key points:

- $(-1, \frac{1}{3}) \to (-1, -\frac{1}{3})$
- $(0, 1) \to (0, -1)$
- $(1, 3) \to (1, -3)$

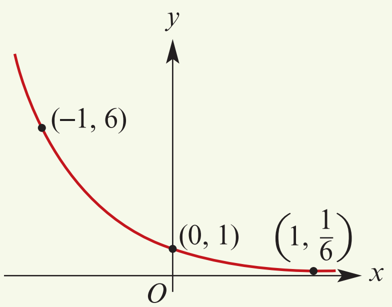
If a **reflection in the y-axis**, given by the mapping $(x, y) \to (-x, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{-x}$. This can also be written as $y = \dfrac{1}{a^x}$ or $y = \left(\dfrac{1}{a}\right)^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

Example 3

Sketch the graph of $y = 6^{-x}$.

Solution



Explanation

The graph of $y = 6^x$ is reflected in the y-axis.

The mapping is $(x, y) \to (-x, y)$.

Reflection of key points:

- $(-1, \frac{1}{6}) \to (1, \frac{1}{6})$
- $(0, 1) \to (0, 1)$
- $(1, 6) \to (-1, 6)$

Dilations

For $k > 0$, if a **dilation of factor k from the x-axis**, given by the mapping $(x, y) \to (x, ky)$, is applied to the graph of $y = a^x$, then the image has equation $y = ka^x$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

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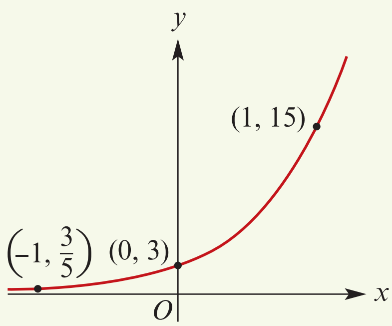
Example 4

Sketch the graph of each of the following:

a $y = 3 \times 5^x$ **b** $y = 0.2 \times 8^x$

Solution

a



Explanation

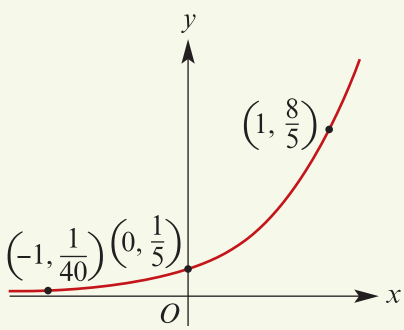
The graph of $y = 5^x$ is dilated by factor 3 from the x-axis.

The mapping is $(x, y) \to (x, 3y)$.

Dilation of key points:

- $(-1, \frac{1}{5}) \to (-1, \frac{3}{5})$
- $(0, 1) \to (0, 3)$
- $(1, 5) \to (1, 15)$

b



The graph of $y = 8^x$ is dilated by factor $\frac{1}{5}$ from the x-axis.

The mapping is $(x, y) \to (x, \frac{1}{5}y)$.

Dilation of key points:

- $(-1, \frac{1}{8}) \to (-1, \frac{1}{40})$
- $(0, 1) \to (0, \frac{1}{5})$
- $(1, 8) \to (1, \frac{8}{5})$

For $k > 0$, if a **dilation of factor k from the y-axis**, given by the mapping $(x, y) \to (kx, y)$, is applied to the graph of $y = a^x$, then the image has equation $y = a^{\frac{x}{k}}$.

- The horizontal asymptote of the image has equation $y = 0$.
- The range of the image is $(0, \infty)$.

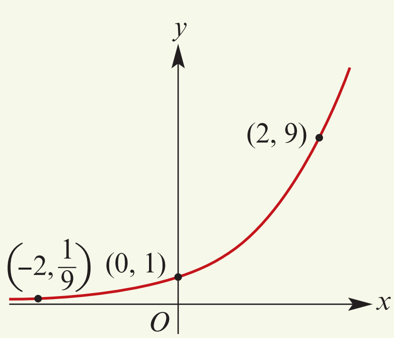
Example 5

Sketch the graph of each of the following:

a $y = 9^{\frac{x}{2}}$ **b** $y = 3^{2x}$

Solution

a



Explanation

The graph of $y = 9^x$ is dilated by factor 2 from the y-axis.

The mapping is $(x, y) \to (2x, y)$.

Dilation of key points:

- $(-1, \frac{1}{9}) \to (-2, \frac{1}{9})$
- $(0, 1) \to (0, 1)$
- $(1, 9) \to (2, 9)$

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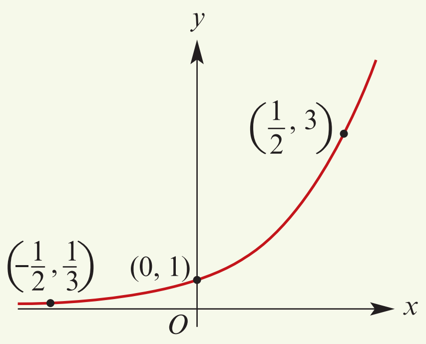
b



The graph of $y = 3^x$ is dilated by factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, y)$.

Dilation of key points:

- $(-1, \frac{1}{3}) \to (-\frac{1}{2}, \frac{1}{3})$
- $(0, 1) \to (0, 1)$
- $(1, 3) \to (\frac{1}{2}, 3)$

Note: Since $9^{\frac{x}{2}} = \left(9^{\frac{1}{2}}\right)^x = 3^x$, the graph of $y = 9^{\frac{x}{2}}$ is the same as the graph of $y = 3^x$.
Similarly, the graph of $y = 3^{2x}$ is the same as the graph of $y = 9^x$.

A translation parallel to the x-axis results in a dilation from the x-axis. For example, if the graph of $y = 5^x$ is translated 3 units in the positive direction of the x-axis, then the image is the graph of $y = 5^{x-3}$, which can be written $y = 5^{-3} \times 5^x$. Hence, a translation of 3 units in the positive direction of the x-axis is equivalent to a dilation of factor 5^{-3} from the x-axis.

Combinations of transformations

We have seen translations, reflections and dilations applied to exponential graphs. In the following example we consider combinations of these transformations.

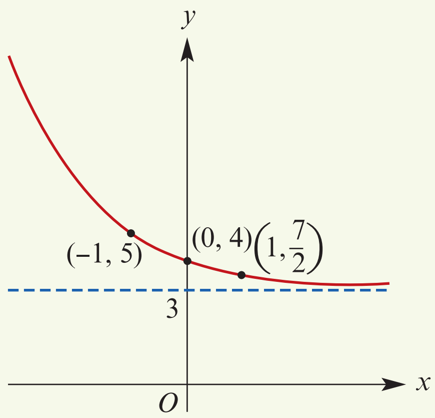
Example 6

Sketch the graph and state the range of each of the following:

a $y = 2^{-x} + 3$ **b** $y = 4^{3x} - 1$ **c** $y = -10^{x-1} - 2$

Solution

a



Graph of $y = 2^{-x} + 3$:

- The asymptote has equation $y = 3$.
- The y-axis intercept is $2^0 + 3 = 4$.
- The range of the function is $(3, \infty)$.

Explanation

The graph of $y = 2^{-x} + 3$ is obtained from the graph of $y = 2^x$ by a reflection in the y-axis followed by a translation 3 units in the positive direction of the y-axis.

The mapping is $(x, y) \to (-x, y + 3)$.

For example:

$(-1, \frac{1}{2}) \to (1, \frac{7}{2})$
$(0, 1) \to (0, 4)$
$(1, 2) \to (-1, 5)$

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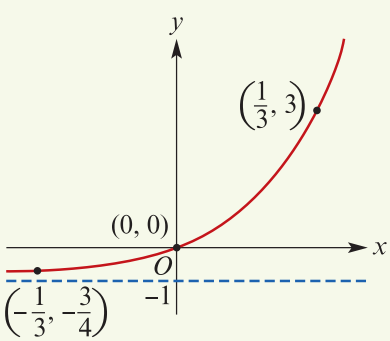
b



Graph of $y = 4^{3x} - 1$:

- The asymptote has equation $y = -1$.
- The y-axis intercept is $4^0 - 1 = 0$.
- The range of the function is $(-1, \infty)$.

The graph of $y = 4^{3x} - 1$ is obtained from the graph of $y = 4^x$ by a dilation of factor $\frac{1}{3}$ from the y-axis followed by a translation 1 unit in the negative direction of the y-axis.

The mapping is $(x, y) \to (\frac{1}{3}x, y - 1)$.

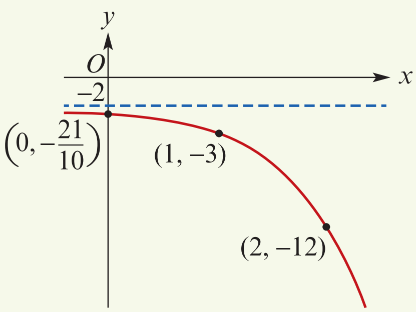
For example:

$(-1, \frac{1}{4}) \to (-\frac{1}{3}, -\frac{3}{4})$

$(0, 1) \to (0, 0)$

$(1, 4) \to (\frac{1}{3}, 3)$

c



Graph of $y = -10^{x-1} - 2$:

- The asymptote has equation $y = -2$.
- The y-axis intercept is $-10^{-1} - 2 = -\dfrac{21}{10}$.
- The range of the function is $(-\infty, -2)$.

The graph of $y = -10^{x-1} - 2$ is obtained from the graph of $y = 10^x$ by a reflection in the x-axis followed by a translation 1 unit in the positive direction of the x-axis and 2 units in the negative direction of the y-axis.

The mapping is $(x, y) \to (x + 1, -y - 2)$.

For example:

$(-1, \frac{1}{10}) \to (0, -\frac{21}{10})$

$(0, 1) \to (1, -3)$

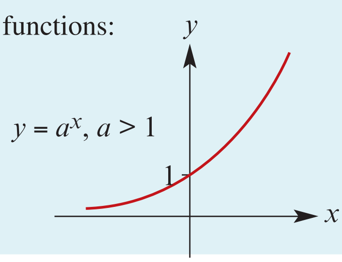
$(1, 10) \to (2, -12)$

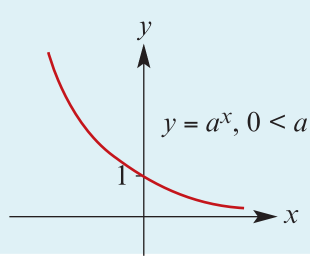
Note: We can use the method for determining transformations for each of the graphs in Example 6. Here we show the method for part **c**:

- Write the equation as $y' = -10^{x'-1} - 2$.
- Rearrange to $-y' - 2 = 10^{x'-1}$.
- We choose to write $y = -y' - 2$ and $x = x' - 1$.
- Hence $y' = -y - 2$ and $x' = x + 1$.

Summary 5A

- Graphs of exponential functions:





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- For $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of $y = a^x$ has the following properties:
  - The x-axis is an asymptote.
  - The y-axis intercept is 1.
  - The y-values are always positive.
  - There is no x-axis intercept.
- Transformations can be applied to exponential functions. For example, the graph of

  $$y = a^{b(x-h)} + k, \quad \text{where } b > 0$$

  can be obtained from the graph of $y = a^x$ by a dilation of factor $\frac{1}{b}$ from the y-axis followed by the translation $(x, y) \to (x + h, y + k)$.

Exercise 5A

Example 1

1 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 2^{x+1} - 2$ **b** $y = 2^{x-3} - 1$ **c** $y = 2^{x+2} - 1$ **d** $y = 2^{x-2} + 2$

Example 2 Example 3

2 For each of the following, use the one set of axes to sketch the two graphs (labelling asymptotes):

a $y = 2^x$ and $y = 3^x$ **b** $y = 2^{-x}$ and $y = 3^{-x}$

c $y = 5^x$ and $y = -5^x$ **d** $y = 1.5^x$ and $y = -1.5^x$

Example 4 Example 5

3 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3 \times 2^x$ **b** $y = \frac{1}{2} \times 5^x$ **c** $y = 2^{3x}$ **d** $y = 2^{\frac{x}{3}}$

Example 6

4 Sketch the graph and state the range of each of the following:

a $y = 3^{-x} + 2$ **b** $y = 2^{5x} - 4$ **c** $y = -10^{x-2} - 2$

5 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 3^x$ **b** $y = 3^x + 1$ **c** $y = 1 - 3^x$

d $y = (\frac{1}{3})^x$ **e** $y = 3^{-x} + 2$ **f** $y = (\frac{1}{3})^x - 1$

6 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = (\frac{1}{2})^{x-2}$ **b** $y = (\frac{1}{2})^x - 1$ **c** $y = (\frac{1}{2})^{x-2} + 1$

7 For $f(x) = 2^x$, sketch the graph of each of the following, labelling asymptotes where appropriate:

a $y = f(x + 1)$ **b** $y = f(x) + 1$ **c** $y = f(-x) + 2$

d $y = -f(x) - 1$ **e** $y = f(3x)$ **f** $y = f\left(\frac{x}{2}\right)$

g $y = 2f(x - 1) + 1$ **h** $y = f(x - 2)$

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8 For each of the following functions, sketch the graph (labelling the asymptote) and state the range:

a $y = 10^x - 1$ **b** $y = 10^{\frac{x}{10}} + 1$ **c** $y = 2 \times 10^x - 20$

d $y = 1 - 10^{-x}$ **e** $y = 10^{x+1} + 3$ **f** $y = 2 \times 10^{\frac{x}{10}} + 4$

9 A bank offers cash loans at 0.04% interest per day, compounded daily. A loan of \$10 000 is taken and the interest payable at the end of x days is given by $C_1 = 10\,000\,[(1.0004)^x - 1]$.

a Plot the graph of C_1 against x.

b Find the interest at the end of:

i 100 days **ii** 300 days.

c After how many days is the interest payable \$1000?

d A loan company offers \$10 000 and charges a fee of \$4.25 per day. The amount charged after x days is given by $C_2 = 4.25x$.

i Plot the graph of C_2 against x (using the same window as in part **a**).

ii Find the smallest value of x for which $C_2 < C_1$.

10 If you invest \$100 at an interest rate of 2% per day, compounded daily, then after x days the amount of money you have (in dollars) is given by $y = 100(1.02)^x$. For how many days would you have to invest to double your money?

11 **a** **i** Graph $y = 2^x$, $y = 3^x$ and $y = 5^x$ on the same set of axes.

ii For what values of x is $2^x > 3^x > 5^x$?

iii For what values of x is $2^x < 3^x < 5^x$?

iv For what values of x is $2^x = 3^x = 5^x$?

b Repeat part **a** for $y = (\frac{1}{2})^x$, $y = (\frac{1}{3})^x$ and $y = (\frac{1}{5})^x$.

c Use your answers to parts **a** and **b** to sketch the graph of $y = a^x$ for:

i $a > 1$ **ii** $a = 1$ **iii** $0 < a < 1$

5B The exponential function $f(x) = e^x$

In the previous section, we explored the family of exponential functions $f(x) = a^x$, where $a \in \mathbb{R}^+ \setminus \{1\}$. One particular member of this family is of great importance in mathematics.

This function has the rule $f(x) = e^x$, where e is Euler's number, named after the eighteenth century Swiss mathematician Leonhard Euler.

Euler's number is defined as follows.

Euler's number

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

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To see what the value of e might be, we could try large values of n and use a calculator to evaluate $(1 + \frac{1}{n})^n$, as shown in the table on the right.

As n is taken larger and larger, it can be seen that $(1 + \frac{1}{n})^n$ approaches a limiting value ($\approx 2.718\,28$).

n	$(1 + \frac{1}{n})^n$
100	$(1.01)^{100} = 2.704\,813\ldots$
1000	$(1.001)^{1000} = 2.716\,923\ldots$
10 000	$(1.0001)^{10\,000} = 2.718\,145\ldots$
100 000	$(1.000\,01)^{100\,000} = 2.718\,268\ldots$
1 000 000	$(1.000\,001)^{1\,000\,000} = 2.718\,280\ldots$

Like π, the number e is irrational:

$$e = 2.718\,281\,828\,459\,045\ldots$$

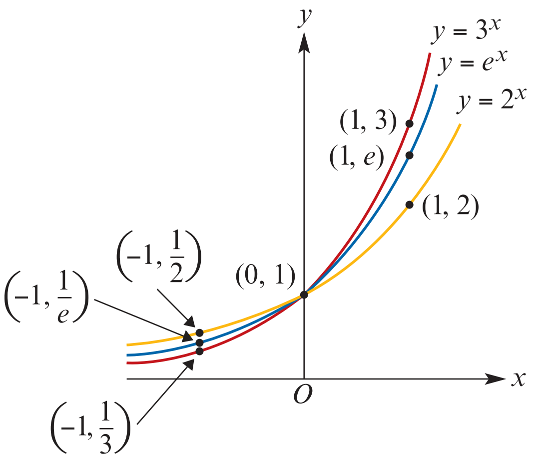
The function $f(x) = e^x$ is very important in mathematics. In Chapter 9 you will find that it has the remarkable property that $f'(x) = f(x)$. That is, the derivative of e^x is e^x.

Note: The function e^x can be found on your calculator.

Graphing $f(x) = e^x$

The graph of $y = e^x$ is as shown.

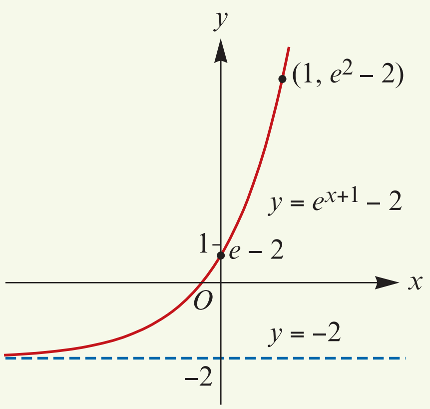
The graphs of $y = 2^x$ and $y = 3^x$ are shown on the same set of axes.



Example 7

Sketch the graph of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x+1} - 2$.

Solution



The asymptote has equation $y = -2$.
The y-axis intercept is $e - 2$.

Explanation

To find the transformation:

- Write the image as $y' + 2 = e^{x'+1}$.
- We can choose $y = y' + 2$ and $x = x' + 1$.
- Hence $y' = y - 2$ and $x' = x - 1$.

The mapping is

$$(x, y) \to (x - 1, y - 2)$$

which is a translation of 1 unit in the negative direction of the x-axis and 2 units in the negative direction of the y-axis.

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Compound interest

Assume that you invest $\$P$ at an annual interest rate r. If the interest is compounded only once per year, then the balance of your investment after t years is given by $A = P(1 + r)^t$.

Now assume that the interest is compounded n times per year. The interest rate in each period is $\frac{r}{n}$. The balance at the end of one year is $P\left(1 + \frac{r}{n}\right)^n$, and the balance at the end of t years is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(\left(1 + \frac{r}{n}\right)^{\frac{n}{r}}\right)^{rt}$$

We recognise that

$$\lim_{\frac{n}{r}\to\infty}\left(1 + \frac{r}{n}\right)^{\frac{n}{r}} = e$$

So, as $n \to \infty$, we can write $A = Pe^{rt}$.

For example, if \$1000 is invested for one year at 5%, the resulting amount is \$1050. However, if the interest is compounded 'continuously', then the amount is given by

$$A = Pe^{rt} = 1000 \times e^{0.05} = 1000 \times 1.051\,271\ldots \approx 1051.27$$

That is, the balance after one year is \$1051.27.

Summary 5B

Euler's number is the natural base for exponential functions:

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$
$$= 2.718\,281\ldots$$

Skill-sheet

Exercise 5B

1 Sketch the graph of each of the following and state the range:

Example 7

a $f(x) = e^x + 1$
b $f(x) = 1 - e^x$
c $f(x) = 1 - e^{-x}$
d $f(x) = e^{-2x}$
e $f(x) = e^{x-1} - 2$
f $f(x) = 2e^x$
g $h(x) = 2(1 + e^x)$
h $h(x) = 2(1 - e^{-x})$
i $g(x) = 2e^{-x} + 1$
j $h(x) = 2e^{x-1}$
k $f(x) = 3e^{x+1} - 2$
l $h(x) = 2 - 3e^x$

2 For each of the following, give a sequence of transformations that maps the graph of $y = e^x$ to the graph of $y = f_1(x)$:

a $f_1(x) = e^{x+2} - 3$
b $f_1(x) = 3e^{x+1} - 4$
c $f_1(x) = 5e^{2x+1}$
d $f_1(x) = 2 - e^{x-1}$
e $f_1(x) = 3 - 2e^{x+2}$
f $f_1(x) = 4e^{2x} - 1$

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3 Find the rule of the image when the graph of $f(x) = e^x$ undergoes each of the following sequences of transformations:

a a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

b a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis

c a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis

d a reflection in the x-axis, followed by a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis

e a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a dilation of factor 2 from the x-axis, followed by a reflection in the x-axis

f a translation 3 units in the positive direction of the x-axis and 4 units in the negative direction of the y-axis, followed by a reflection in the x-axis, followed by a dilation of factor 2 from the x-axis.

4 For each of the following, give a sequence of transformations that maps the graph of $y = f_1(x)$ to the graph of $y = e^x$:

a $f_1(x) = e^{x+2} - 3$ **b** $f_1(x) = 3e^{x+1} - 4$

c $f_1(x) = 5e^{2x+1}$ **d** $f_1(x) = 2 - e^{x-1}$

e $f_1(x) = 3 - 2e^{x+2}$ **f** $f_1(x) = 4e^{2x} - 1$

5 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $e^x = x + 2$ **b** $e^{-x} = x + 2$

c $x^2 = e^x$ **d** $x^3 = e^x$

6 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = e^x$.

b Using the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f\left(\dfrac{x}{3}\right)$ **iii** $y = f(-x)$

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5C Exponential equations

One method for solving exponential equations is to use the one-to-one property of exponential functions:

$$a^x = a^y \text{ implies } x = y, \quad \text{for } a \in \mathbb{R}^+ \setminus \{1\}$$

Example 8

Find the value of x for which:

a $4^x = 256$

b $3^{x-1} = 81$

Solution

a $4^x = 256$

$4^x = 4^4$

$\therefore \quad x = 4$

b $3^{x-1} = 81$

$3^{x-1} = 3^4$

$\therefore \quad x - 1 = 4$

$x = 5$

When solving an exponential equation, you may also need to use the index laws.

Index laws

For all positive numbers a and b and all real numbers x and y:

- $a^x \times a^y = a^{x+y}$
- $a^x \div a^y = a^{x-y}$
- $(a^x)^y = a^{xy}$
- $(ab)^x = a^x b^x$
- $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$
- $a^{-x} = \dfrac{1}{a^x}$
- $a^x = \dfrac{1}{a^{-x}}$
- $a^0 = 1$

Note: More generally, each index law applies for real numbers a and b provided both sides of the equation are defined. For example: $a^m \times a^n = a^{m+n}$ for $a \in \mathbb{R}$ and $m, n \in \mathbb{Z}$.

Example 9

Find the value of x for which $5^{2x-4} = 25^{-x+2}$.

Solution	Explanation
$5^{2x-4} = 25^{-x+2}$	Express both sides of the equation as powers with base 5.
$= (5^2)^{-x+2}$	
$= 5^{-2x+4}$	
$\therefore \quad 2x - 4 = -2x + 4$	Use the fact that $5^a = 5^b$ implies $a = b$.
$4x = 8$	
$x = 2$	

To solve the equations in the next example, we must recognise that they will become quadratic equations once we make a substitution.

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Example 10

Solve for x:

a $9^x = 12 \times 3^x - 27$

b $3^{2x} = 27 - 6 \times 3^x$

Solution

a We can write the equation as

$$(3^x)^2 = 12 \times 3^x - 27$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 12y - 27 \\ y^2 - 12y + 27 &= 0 \\ (y-3)(y-9) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad y = 3 \quad &\text{or} \quad y = 9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = 3^2 \\ x = 1 \quad &\text{or} \quad x = 2 \end{aligned}$$

b We can write the equation as

$$(3^x)^2 = 27 - 6 \times 3^x$$

Let $y = 3^x$. The equation becomes

$$\begin{aligned} y^2 &= 27 - 6y \\ y^2 + 6y - 27 &= 0 \\ (y-3)(y+9) &= 0 \end{aligned}$$

$$\begin{aligned} \therefore \quad y = 3 \quad &\text{or} \quad y = -9 \\ 3^x = 3 \quad &\text{or} \quad 3^x = -9 \end{aligned}$$

The only solution is $x = 1$, since $3^x > 0$ for all x.

Summary 5C

- One method for solving an exponential equation, without using a calculator, is first to express both sides of the equation as powers with the same base and then to equate the indices (since $a^x = a^y$ implies $x = y$, for any base $a \in \mathbb{R}^+ \setminus \{1\}$).

  For example: $2^{x+1} = 8 \Leftrightarrow 2^{x+1} = 2^3 \Leftrightarrow x + 1 = 3 \Leftrightarrow x = 2$
- Equations such as $3^{2x} - 6 \times 3^x - 27 = 0$ can be solved by making a substitution. In this case, substitute $y = 3^x$ to obtain a quadratic equation in y.

Exercise 5C

1 Simplify the following expressions:

a $3x^2y^3 \times 2x^4y^6$ **b** $\dfrac{12x^8}{4x^2}$ **c** $18x^2y^3 \div (3x^4y)$

d $(4x^4y^2)^2 \div (2(x^2y)^4)$ **e** $(4x^0)^2$ **f** $15(x^5y^{-2})^4 \div (3(x^4y)^{-2})$

g $\dfrac{3(2x^2y^3)^4}{2x^3y^2}$ **h** $(8x^3y^6)^{\frac{1}{3}}$ **i** $\dfrac{x^2 + y^2}{x^{-2} + y^{-2}}$

Example 8

2 Solve for x in each of the following:

a $3^x = 81$ **b** $81^x = 9$ **c** $2^x = 256$

d $625^x = 5$ **e** $32^x = 8$ **f** $5^x = 125$

g $16^x = 1024$ **h** $2^{-x} = \dfrac{1}{64}$ **i** $5^{-x} = \dfrac{1}{625}$

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Example 9

3 Solve for n in each of the following:

a $5^{2n} \times 25^{2n-1} = 625$ **b** $4^{2n-2} = 1$ **c** $4^{2n-1} = \dfrac{1}{256}$

d $\dfrac{3^{n-2}}{9^{2-n}} = 27$ **e** $2^{2n-2} \times 4^{-3n} = 64$ **f** $2^{n-4} = 8^{4-n}$

g $27^{n-2} = 9^{3n+2}$ **h** $8^{6n+2} = 8^{4n-1}$ **i** $125^{4-n} = 5^{6-2n}$

j $2^{n-1} \times 4^{2n+1} = 16$ **k** $(27 \times 3^n)^n = 27^n \times 3^{\frac{1}{4}}$

Example 10

4 Solve for x:

a $3^{2x} - 2(3^x) - 3 = 0$ **b** $5^{2x} - 23(5^x) - 50 = 0$ **c** $5^{2x} - 10(5^x) + 25 = 0$

d $2^{2x} = 6(2^x) - 8$ **e** $8(3^x) - 6 = 2(3^{2x})$ **f** $2^{2x} - 20(2^x) = -64$

g $4^{2x} - 5(4^x) = -4$ **h** $3(3^{2x}) = 28(3^x) - 9$ **i** $7(7^{2x}) = 8(7^x) - 1$

5D Logarithms

Consider the statement

$$2^3 = 8$$

This may be written in an alternative form:

$$\log_2 8 = 3$$

which is read as ‘the logarithm of 8 to the base 2 is equal to 3’.

For $a \in \mathbb{R}^+ \setminus \{1\}$, the **logarithm function** with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

Note: Since a^x is positive, the expression $\log_a y$ is only defined when y is positive.

Further examples:

- $3^2 = 9$ is equivalent to $\log_3 9 = 2$
- $10^4 = 10\,000$ is equivalent to $\log_{10} 10\,000 = 4$
- $a^0 = 1$ is equivalent to $\log_a 1 = 0$

Example 11

Without the aid of a calculator, evaluate the following:

a $\log_2 32$ **b** $\log_3 81$

Solution

a Let $\log_2 32 = x$

Then $2^x = 32$

$2^x = 2^5$

Therefore $x = 5$, giving $\log_2 32 = 5$.

b Let $\log_3 81 = x$

Then $3^x = 81$

$3^x = 3^4$

Therefore $x = 4$, giving $\log_3 81 = 4$.

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Note: To find $\log_2 32$, we ask 'What power of 2 gives 32?'
To find $\log_3 81$, we ask 'What power of 3 gives 81?'

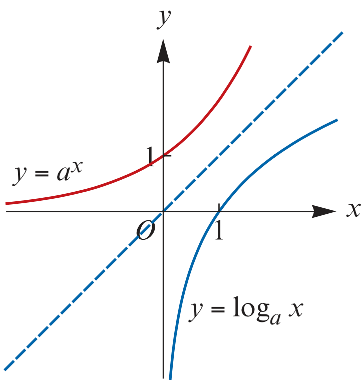
Inverse functions

For each base $a \in \mathbb{R}^+ \setminus \{1\}$, the exponential function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is one-to-one and so has an inverse function.

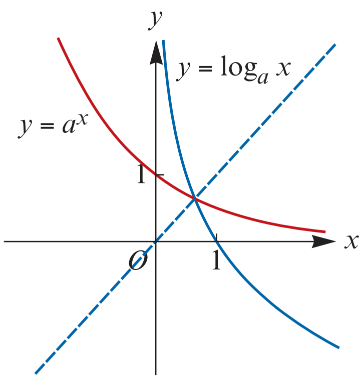
Let $a \in \mathbb{R}^+ \setminus \{1\}$. The inverse of the exponential function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is the logarithmic function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.

- $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
- $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

Because they are inverse functions, the graphs of $y = a^x$ and $y = \log_a x$ are reflections of each other in the line $y = x$.



$a > 1$



$0 < a < 1$

The natural logarithm

Earlier in the chapter we defined the number e and the important function $f(x) = e^x$. The inverse of this function is $f^{-1}(x) = \log_e x$. Because the logarithm function with base e is known as the **natural logarithm**, the expression $\log_e x$ is also written as $\ln x$.

The common logarithm

The function $f(x) = \log_{10} x$ has both historical and practical importance. When logarithms were used as a calculating device, it was often base 10 that was used. By simplifying calculations, logarithms contributed to the advancement of science, and especially of astronomy. In schools, books of tables of logarithms were provided for calculations and this was done up to the 1970s.

Base 10 logarithms are used for scales in science such as the Richter scale, decibels and pH.

You can understand the practicality of base 10 by observing:

- $\log_{10} 10 = 1$
- $\log_{10} 100 = 2$
- $\log_{10} 1000 = 3$
- $\log_{10} 10\,000 = 4$
- $\log_{10} 0.1 = -1$
- $\log_{10} 0.01 = -2$
- $\log_{10} 0.001 = -3$
- $\log_{10} 0.0001 = -4$

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Laws of logarithms

The index laws are used to establish rules for computations with logarithms.

Law 1: Logarithm of a product

The logarithm of a product is the sum of their logarithms:

$$\log_a(mn) = \log_a m + \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then $a^x = m$ and $a^y = n$, and therefore

$$mn = a^x \times a^y = a^{x+y} \qquad \text{(using the first index law)}$$

Hence $\log_a(mn) = x + y = \log_a m + \log_a n$.

For example:

$$\begin{aligned}\log_{10} 200 + \log_{10} 5 &= \log_{10}(200 \times 5)\\ &= \log_{10} 1000 = 3\end{aligned}$$

Law 2: Logarithm of a quotient

The logarithm of a quotient is the difference of their logarithms:

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Proof Let $\log_a m = x$ and $\log_a n = y$, where m and n are positive real numbers. Then as before $a^x = m$ and $a^y = n$, and therefore

$$\frac{m}{n} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{(using the second index law)}$$

Hence $\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n$.

For example:

$$\begin{aligned}\log_2 32 - \log_2 8 &= \log_2\left(\frac{32}{8}\right)\\ &= \log_2 4 = 2\end{aligned}$$

Law 3: Logarithm of a power

$$\log_a(m^p) = p\log_a m$$

Proof Let $\log_a m = x$. Then $a^x = m$, and therefore

$$m^p = (a^x)^p = a^{xp} \qquad \text{(using the third index law)}$$

Hence $\log_a(m^p) = xp = p\log_a m$.

For example:

$$\log_2 32 = \log_2(2^5) = 5$$

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Law 4: Logarithm of $\frac{1}{m}$

$$\log_a(m^{-1}) = -\log_a m$$

Proof Use logarithm law 3 with $p = -1$.

For example:

$$\log_a(\tfrac{1}{2}) = \log_a(2^{-1}) = -\log_a 2$$

Law 5

$$\log_a 1 = 0 \quad \text{and} \quad \log_a a = 1$$

Proof Since $a^0 = 1$, we have $\log_a 1 = 0$.

Since $a^1 = a$, we have $\log_a a = 1$.

Example 12

Express the following as the logarithm of a single term:

$$2\log_e 3 + \log_e 16 - 2\log_e\left(\frac{6}{5}\right)$$

Solution

$$\begin{aligned} 2\log_e 3 + \log_e 16 - 2\log_e\left(\frac{6}{5}\right) &= \log_e(3^2) + \log_e 16 - \log_e\left(\frac{6}{5}\right)^2 \\ &= \log_e 9 + \log_e 16 - \log_e\left(\frac{36}{25}\right) \\ &= \log_e\left(9 \times 16 \times \frac{25}{36}\right) \\ &= \log_e 100 \end{aligned}$$

Logarithmic equations

Example 13

Solve each of the following equations for x:

a $\log_2 x = 5$ **b** $\log_2(2x - 1) = 4$ **c** $\log_e(3x + 1) = 0$

Solution

a
$$\begin{aligned} \log_2 x &= 5 \\ x &= 2^5 \\ \therefore \quad x &= 32 \end{aligned}$$

b
$$\begin{aligned} \log_2(2x - 1) &= 4 \\ 2x - 1 &= 2^4 \\ 2x &= 17 \\ \therefore \quad x &= \frac{17}{2} \end{aligned}$$

c
$$\begin{aligned} \log_e(3x + 1) &= 0 \\ 3x + 1 &= e^0 \\ 3x &= 1 - 1 \\ \therefore \quad x &= 0 \end{aligned}$$

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Example 14

Solve each of the following equations for x:

a $\log_e(x-1)+\log_e(x+2)=\log_e(6x-8)$ **b** $\log_2 x-\log_2(7-2x)=\log_2 6$

Solution

a
$$\begin{aligned}\log_e(x-1)+\log_e(x+2)&=\log_e(6x-8)\\ \log_e((x-1)(x+2))&=\log_e(6x-8)\\ x^2+x-2&=6x-8\\ x^2-5x+6&=0\\ (x-3)(x-2)&=0\\ \therefore\quad x=3 \text{ or } x&=2\end{aligned}$$

Note: The solutions must satisfy $x-1>0$, $x+2>0$ and $6x-8>0$. Therefore both of these solutions are allowable.

b
$$\begin{aligned}\log_2 x-\log_2(7-2x)&=\log_2 6\\ \log_2\left(\frac{x}{7-2x}\right)&=\log_2 6\\ \frac{x}{7-2x}&=6\\ x&=42-12x\\ 13x&=42\\ \therefore\quad x&=\frac{42}{13}\end{aligned}$$

Using the TI-Nspire

- Use **solve** from the **Algebra** menu as shown.
- Note that $\ln(x)=\log_e(x)$. The logarithm with base e is available on the keypad by pressing ctrl e^x.

Note: Logarithms with other bases are obtained by pressing the **log** key (ctrl 10^x) and completing the template.

Using the Casio ClassPad

- For a logarithm with base e, use ln from the Math1 keyboard. Note that $\ln(x)=\log_e(x)$.
- Enter and highlight the equation $\ln(x-1)+\ln(x+2)=\ln(6x-8)$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set to x.

Note: For logarithms with other bases, tap $\log_{\blacksquare}\square$ and complete the template.

Example 15

Solve each of the following equations for x:

a $\log_e(2x+1)-\log_e(x-1)=4$ **b** $\log_e(x-1)+\log_e(x+1)=1$

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Solution

a $\log_e(2x+1) - \log_e(x-1) = 4$

$$\log_e\left(\frac{2x+1}{x-1}\right) = 4$$

$$\frac{2x+1}{x-1} = e^4$$

$$2x+1 = e^4(x-1)$$

$$(2-e^4)x = -(e^4+1)$$

$$\therefore \quad x = \frac{e^4+1}{e^4-2}$$

b $\log_e(x-1) + \log_e(x+1) = 1$

$$\log_e((x-1)(x+1)) = 1$$

$$\log_e(x^2-1) = 1$$

$$x^2 - 1 = e$$

$$\therefore \quad x = \pm\sqrt{e+1}$$

But the original equation is not defined for $x = -\sqrt{e+1}$ and so the only solution is $x = \sqrt{e+1}$.

Example 16

Solve the equation $\log_x 27 = \frac{3}{2}$ for x.

Solution

$\log_x 27 = \frac{3}{2}$ is equivalent to $x^{\frac{3}{2}} = 27$

$$(\sqrt{x})^3 = 3^3$$

$$\sqrt{x} = 3$$

$$\therefore \quad x = 9$$

Summary 5D

- For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function base a is defined as follows:

  $a^x = y$ is equivalent to $\log_a y = x$
- To evaluate $\log_a y$ ask the question: 'What power of a gives y?'
- For $a \in \mathbb{R}^+ \setminus \{1\}$, the inverse of the exponential function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is the logarithmic function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.
  - $\log_a(a^x) = x$ for all x
  - $a^{\log_a x} = x$ for all positive values of x
- Laws of logarithms
  1. $\log_a(mn) = \log_a m + \log_a n$
  2. $\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$
  3. $\log_a(m^p) = p\log_a m$
  4. $\log_a(m^{-1}) = -\log_a m$
  5. $\log_a 1 = 0$ and $\log_a a = 1$

Skill-sheet

Exercise 5D

1 Evaluate each of the following:

Example 11

a $\log_{10} 1000$ **b** $\log_2\left(\frac{1}{16}\right)$ **c** $\log_{10} 0.001$

d $\log_2 64$ **e** $\log_{10} 1\,000\,000$ **f** $\log_2\left(\frac{1}{128}\right)$

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Example 12

2 Express each of the following as the logarithm of a single term:

a $\log_e 2 + \log_e 3$ **b** $\log_e 32 - \log_e 8$

c $\log_e 10 + \log_e 100 + \log_e 1000$ **d** $\log_e\left(\frac{1}{2}\right) + \log_e 14$

e $\log_e\left(\frac{1}{3}\right) + \log_e\left(\frac{1}{4}\right) + \log_e\left(\frac{1}{5}\right)$ **f** $\log_e(uv) + \log_e(uv^2) + \log_e(uv^3)$

g $2\log_e x + 5\log_e x$ **h** $\log_e(x+y) + \log_e(x-y) - \log_e(x^2-y^2)$

Example 13

3 Solve each of the following equations for x:

a $\log_{10} x = 2$ **b** $2\log_2 x = 8$ **c** $\log_e(x-5) = 0$

d $\log_2 x = 6$ **e** $2\log_e(x+5) = 6$ **f** $\log_e(2x) = 0$

g $\log_e(2x+3) = 0$ **h** $\log_{10} x = -3$ **i** $2\log_2(x-4) = 10$

Example 14

4 Solve each of the following equations for x:

a $\log_{10} x = \log_{10} 3 + \log_{10} 5$ **b** $\log_e x = \log_e 15 - \log_e 3$

c $\log_e x = \frac{2}{3}\log_e 8$ **d** $\log_e x + \log_e(2x-1) = 0$

e $2\log_e x - \log_e(x-1) = \log_e(x+3)$

5 Express each of the following as the logarithm of a single term:

a $\log_{10} 9 + \log_{10} 3$ **b** $\log_2 24 - \log_2 6$ **c** $\frac{1}{2}\log_{10} a - \frac{1}{2}\log_{10} b$

d $1 + \log_{10} a - \frac{1}{3}\log_{10} b$ **e** $\frac{1}{2}\log_{10} 36 - \frac{1}{3}\log_{10} 27 - \frac{2}{3}\log_{10} 64$

6 Without using your calculator, evaluate each of the following:

a $\log_{10} 5 + \log_{10} 2$ **b** $\log_{10} 5 + 3\log_{10} 2 - \log_{10} 4$

c $\log_2 \sqrt{2} + \log_2 1 + 2\log_2 2$ **d** $2\log_{10} 5 + 2\log_{10} 2 + 1$

e $4\log_{10} 2 - \log_{10} 16$

7 Simplify the following expressions:

a $\log_3\left(\frac{1}{3^x}\right)$ **b** $\log_2 x - 2\log_2 y + \log_2(xy^2)$ **c** $\log_e(x^2-y^2) - \log_e(x-y) - \log_e(x+y)$

Example 15

8 Solve each of the following equations for x:

a $\log_e(x^2 - 2x + 8) = 2\log_e x$ **b** $\log_e(5x) - \log_e(3-2x) = 1$

9 Solve each of the following equations for x:

a $\log_e x + \log_e(3x+1) = 1$ **b** $8e^{-x} - e^x = 2$

Example 16

10 Solve each of the following equations for x:

a $\log_x 81 = 4$ **b** $\log_x\left(\frac{1}{32}\right) = 5$

11 Solve $2\log_e x + \log_e 4 = \log_e(9x-2)$.

12 Given that $\log_a N = \frac{1}{2}(\log_a 24 - \log_a 0.375 - 6\log_a 3)$, find the value of N.

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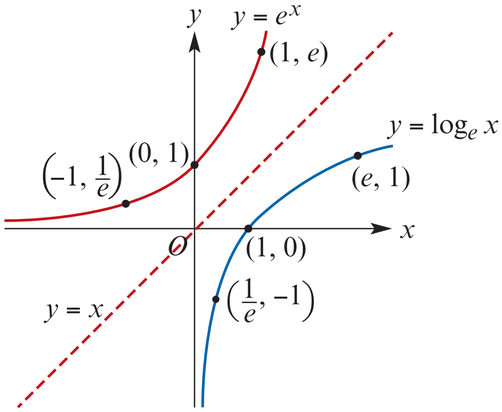
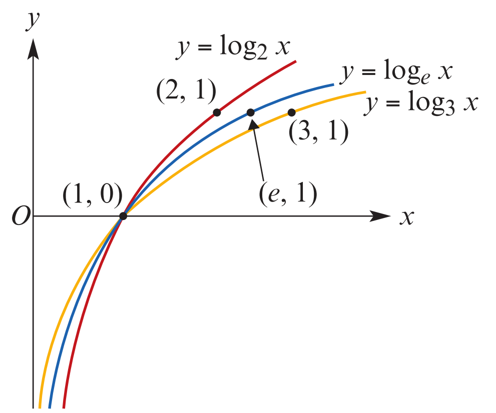
5E Graphing logarithmic functions

The graphs of $y = e^x$ and its inverse function $y = \log_e x$ are shown on the one set of axes.



The graphs of $y = \log_2 x$, $y = \log_e x$ and $y = \log_3 x$ are shown on the one set of axes.



For each base $a \in \mathbb{R}^+ \setminus \{1\}$, the graph of the logarithmic function $f(x) = \log_a x$ has the following features:

- Key values are $f\left(\frac{1}{a}\right) = -1$, $f(1) = 0$ and $f(a) = 1$.
- The maximal domain is $\mathbb{R}^+$ and the range is $\mathbb{R}$.
- The y-axis is a vertical asymptote.

A logarithmic function with $a > 1$ is strictly increasing, and a logarithmic function with $0 < a < 1$ is strictly decreasing. In both cases, the function is one-to-one.

Graphing transformations of $f(x) = \log_a x$

We now look at transformations applied to the graph of $f(x) = \log_a x$ where $a > 1$. We make the following general observations:

- The graph of $y = \log_a(mx - n)$, where $m > 0$, has a vertical asymptote $x = \frac{n}{m}$ and implied domain $\left(\frac{n}{m}, \infty\right)$. The x-axis intercept is $\frac{1+n}{m}$.

Example 17

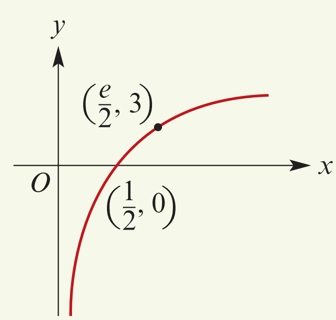
Sketch the graph of $y = 3\log_e(2x)$.

Solution

This is obtained from the graph of $y = \log_e x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The mapping is $(x, y) \to (\frac{1}{2}x, 3y)$.

- $(1, 0) \to (\frac{1}{2}, 0)$
- $(e, 1) \to (\frac{1}{2}e, 3)$



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Example 18

Sketch the graph and state the implied domain of each of the following:

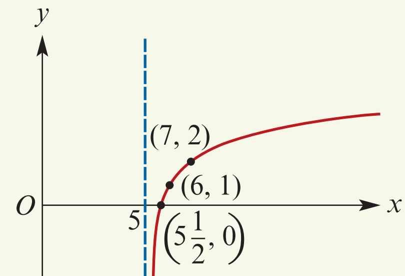
a $y = \log_2(x - 5) + 1$ **b** $y = -\log_3(x + 4)$

Solution

a The graph of $y = \log_2(x-5)+1$ is obtained from the graph of $y = \log_2 x$ by a translation of 5 units in the positive direction of the x-axis and 1 unit in the positive direction of the y-axis.

The mapping is $(x, y) \to (x + 5, y + 1)$.

- $(1, 0) \to (6, 1)$
- $(2, 1) \to (7, 2)$



The asymptote has equation $x = 5$.

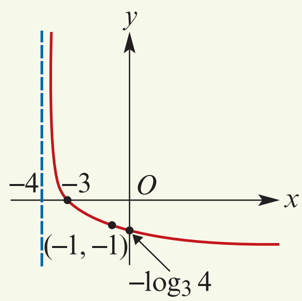
When $y = 0$,
$$\begin{aligned}\log_2(x-5)+1 &= 0\\ \log_2(x-5) &= -1\\ x-5 &= 2^{-1}\\ \therefore\quad x &= 5\tfrac{1}{2}\end{aligned}$$

The domain of the function is $(5, \infty)$.

b The graph of $y = -\log_3(x+4)$ is obtained from the graph of $y = \log_3 x$ by a reflection in the x-axis and a translation of 4 units in the negative direction of the x-axis.

The mapping is $(x, y) \to (x - 4, -y)$.

- $(1, 0) \to (-3, 0)$
- $(3, 1) \to (-1, -1)$



The asymptote has equation $x = -4$.

When $x = 0$,
$$\begin{aligned}y &= -\log_3(0+4)\\ &= -\log_3 4\end{aligned}$$

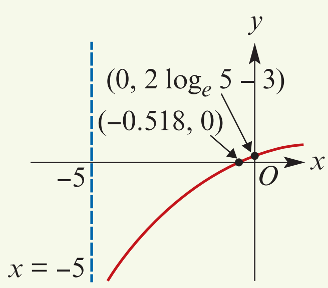
The domain of the function is $(-4, \infty)$.

Example 19

Sketch the graph of $y = 2\log_e(x + 5) - 3$ and state the implied domain.

Solution

The graph of $y = 2\log_e(x+5) - 3$ is obtained from the graph of $y = \log_e x$ by a dilation of factor 2 from the x-axis followed by a translation of 5 units in the negative direction of the x-axis and 3 units in the negative direction of the y-axis.



The equation of the asymptote is $x = -5$.

The domain of the function is $(-5, \infty)$.

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Axis intercepts

When $x = 0$, $\quad y = 2\log_e(0+5) - 3$

$\qquad\qquad\quad = 2\log_e 5 - 3$

When $y = 0$,

$$\begin{aligned} 2\log_e(x+5) - 3 &= 0 \\ \log_e(x+5) &= \frac{3}{2} \\ x + 5 &= e^{\frac{3}{2}} \\ \therefore \quad x &= e^{\frac{3}{2}} - 5 \end{aligned}$$

Exponential and logarithmic graphs with different bases

It is often useful to know how to go from one base to another.

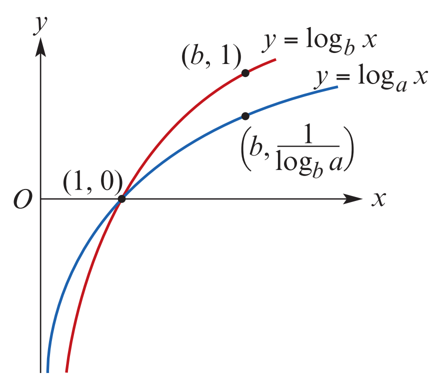
To change the base of $\log_a x$ from a to b (where $a, b > 0$ and $a, b \neq 1$), we use the definition that $y = \log_a x$ implies $a^y = x$. Taking $\log_b$ of both sides:

$$\begin{aligned} \log_b(a^y) &= \log_b x \\ y\log_b a &= \log_b x \\ y &= \frac{\log_b x}{\log_b a} \end{aligned}$$

Since $y = \log_a x$, this gives:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

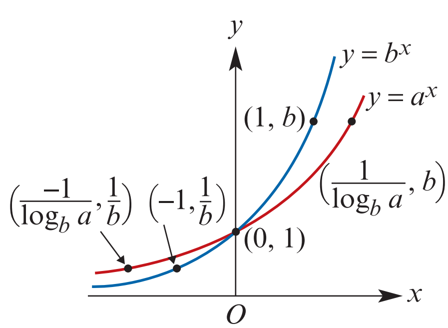
Hence the graph of $y = \log_a x$ can be obtained from the graph of $y = \log_b x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the x-axis.



Using properties of inverses, we can write $a = b^{\log_b a}$. This gives:

$$a^x = b^{(\log_b a)x}$$

Hence the graph of $y = a^x$ can be obtained from the graph of $y = b^x$ by a dilation of factor $\dfrac{1}{\log_b a}$ from the y-axis.



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Example 20

Find a transformation that takes the graph of $y = 2^x$ to the graph of $y = e^x$.

Solution

We can write $e = 2^{\log_2 e}$ and so

$$\begin{aligned} e^x &= (2^{\log_2 e})^x \\ &= 2^{(\log_2 e)x} \end{aligned}$$

The graph of $y = e^x$ is the image of the graph of $y = 2^x$ under a dilation of factor $\dfrac{1}{\log_2 e}$ from the y-axis.

Exercise 5E

Example 17

1 Sketch the graph of each of the following:

a $y = 2\log_e(3x)$

b $y = 4\log_e(5x)$

c $y = 2\log_e(4x)$

d $y = 3\log_e\left(\dfrac{x}{2}\right)$

Example 18

2 For each of the following functions, sketch the graph (labelling axis intercepts and asymptotes) and state the maximal domain and range:

a $y = 2\log_e(x - 3)$

b $y = \log_e(x + 3) - 2$

c $y = 2\log_e(x + 1) - 1$

d $y = 2 + \log_e(3x - 2)$

e $y = -2\log_e(x + 2)$

f $y = -2\log_e(x - 2)$

g $y = 1 - \log_e(x + 1)$

h $y = \log_e(2 - x)$

i $y + 1 = \log_e(4 - 3x)$

Example 19

3 Sketch the graph of each of the following. Label the axis intercepts and asymptotes. State the implied domain of each function.

a $y = \log_2(2x)$

b $y = \log_{10}(x - 5)$

c $y = -\log_{10} x$

d $y = \log_{10}(-x)$

e $y = \log_{10}(5 - x)$

f $y = 2\log_2(2x) + 2$

g $y = -2\log_2(3x)$

h $y = \log_{10}(-x - 5) + 2$

i $y = 4\log_2(-3x)$

j $y = 2\log_2(2 - x) - 6$

k $y = \log_e(2x - 1)$

l $y = -\log_e(3 - 2x)$

4 Solve each of the following equations using a calculator. Give answers correct to three decimal places.

a $-x + 2 = \log_e x$

b $\dfrac{1}{3}\log_e(2x + 1) = -\dfrac{1}{2}x + 1$

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5 **a** Using a calculator, plot the graph of $y = f(x)$ where $f(x) = \log_e x$.

b Using the same screen, plot the graphs of:

i $y = f(-x)$ **ii** $y = -f(x)$ **iii** $y = f\left(\frac{x}{3}\right)$ **iv** $y = f(3x)$

Example 20

6 Find a transformation that takes the graph of $y = 3^x$ to the graph of $y = e^x$.

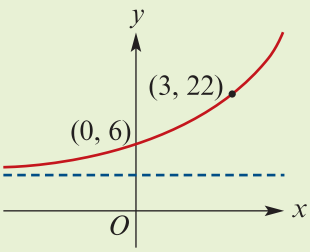
7 Find a transformation that takes the graph of $y = e^x$ to the graph of $y = 2^x$.

5F Determining rules for graphs of exponential and logarithmic functions

In previous chapters, we have determined the rules for graphs of various types of functions, including polynomial functions. In this chapter, we consider similar questions for exponential and logarithmic functions.

Example 21

The rule for the function with the graph shown is of the form $y = ae^x + b$. Find the values of a and b.



Solution

When $x = 0$, $y = 6$ and when $x = 3$, $y = 22$:

$$6 = ae^0 + b \qquad (1)$$
$$22 = ae^3 + b \qquad (2)$$

Subtract (1) from (2):

$$16 = a(e^3 - e^0)$$
$$16 = a(e^3 - 1)$$
$$\therefore \quad a = \frac{16}{e^3 - 1}$$

From equation (1):

$$b = 6 - a$$
$$= 6 - \frac{16}{e^3 - 1}$$
$$= \frac{6e^3 - 22}{e^3 - 1}$$

The function has rule $y = \left(\frac{16}{e^3 - 1}\right)e^x + \frac{6e^3 - 22}{e^3 - 1}$.

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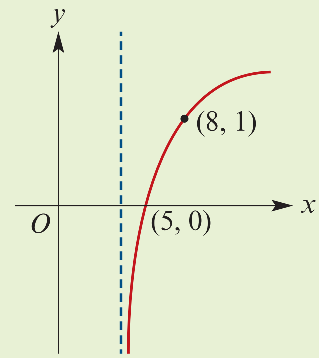
Example 22

The rule for the function with the graph shown is of the form $y = a\log_e(x + b)$. Find the values of a and b.



Solution

$$0 = a\log_e(5 + b) \qquad (1)$$
$$1 = a\log_e(8 + b) \qquad (2)$$

From (1): $\log_e(5 + b) = 0$

$$5 + b = e^0$$
$$\therefore \quad b = -4$$

Substitute in (2): $1 = a\log_e 4$

$$\therefore \quad a = \frac{1}{\log_e 4}$$

The rule is $y = \dfrac{1}{\log_e 4}\log_e(x - 4)$.

Explanation

Form two equations in a and b by substituting into the rule $y = a\log_e(x + b)$:

- $y = 0$ when $x = 5$
- $y = 1$ when $x = 8$

Example 23

Given that $y = Ae^{bt}$ with $y = 6$ when $t = 1$ and $y = 8$ when $t = 2$, find A and b.

Solution

$$6 = Ae^b \qquad (1)$$
$$8 = Ae^{2b} \qquad (2)$$

Divide (2) by (1): $\dfrac{4}{3} = e^b$

$$\therefore \quad b = \log_e \frac{4}{3}$$

Substitute in (1): $6 = Ae^{\log_e \frac{4}{3}}$

$$6 = \frac{4}{3}A$$
$$\therefore \quad A = \frac{18}{4} = \frac{9}{2}$$

The rule is $y = \dfrac{9}{2}e^{(\log_e \frac{4}{3})t}$.

Explanation

Form two equations in A and b by substituting into the rule $y = Ae^{bt}$:

- $y = 6$ when $t = 1$
- $y = 8$ when $t = 2$

$e^{\log_e \frac{4}{3}} = \dfrac{4}{3}$ since $e^{\log_e a} = a$ for all $a > 0$

Note that $y = \dfrac{9}{2}\left(e^{\log_e \frac{4}{3}}\right)^t = \dfrac{9}{2}\left(\dfrac{4}{3}\right)^t$

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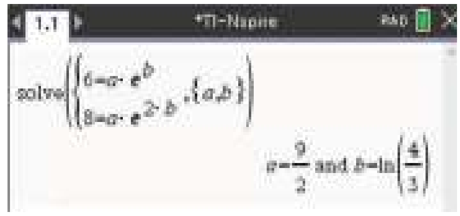
Using the TI-Nspire

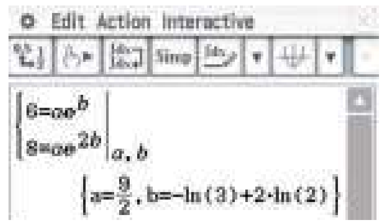
Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations** and complete as shown.

Note: Do not use the 'e' from the alpha keys; it will be treated as a variable.



Using the Casio ClassPad

- Select the simultaneous equations template [template].
- Enter the equations as shown: select [$e^{\blacksquare}$] from the [Math1] keyboard and select the parameters $\boldsymbol{a}, \boldsymbol{b}$ from the [Var] keyboard.



Exercise 5F

Example 21

1 An exponential function has rule $y = a \times e^x + b$ and the points with coordinates $(0, 5)$ and $(4, 11)$ are on the graph of the function. Find the values of a and b.

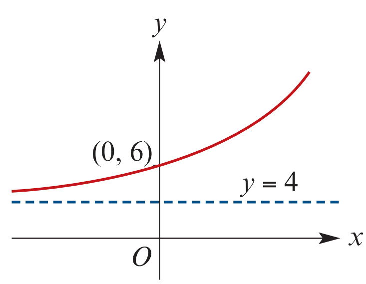
Example 22

2 A logarithmic function has rule $y = a \log_e(x + b)$ and the points with coordinates $(5, 0)$ and $(10, 2)$ are on the graph of the function. Find the values of a and b.

3 The graph shown has rule

$$y = ae^x + b$$

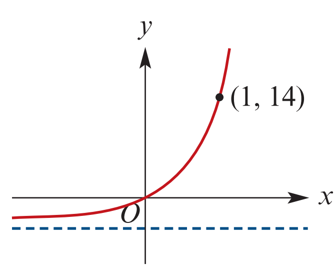
Find the values of a and b.



4 The rule for the function for which the graph is shown is of the form

$$y = ae^x + b$$

Find the values of a and b.



Example 23

5 Find the values of a and b such that the graph of $y = ae^{-bx}$ goes through the points $(3, 50)$ and $(6, 10)$.

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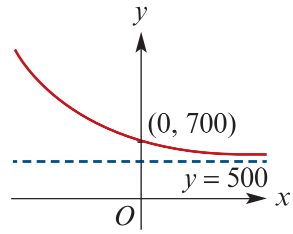
6 The rule for the function f is of the form

$$f(x) = ae^{-x} + b$$

Find the values of a and b.

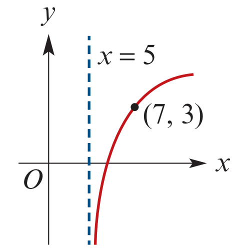


7 Find the values of a and b such that the graph of $y = a\log_2 x + b$ goes through the points $(8, 10)$ and $(32, 14)$.

8 The rule of the graph shown is of the form

$$y = a\log_2(x - b)$$

Find the values of a and b.



9 Find the values of a and b such that the graph of $y = ae^{bx}$ goes through the points $(3, 10)$ and $(6, 50)$.

10 Find the values of a and b such that the graph of $y = a\log_2(x - b)$ passes through the points $(5, 2)$ and $(7, 4)$.

11 The points $(3, 10)$ and $(5, 12)$ lie on the graph of $y = a\log_e(x - b) + c$. The graph has a vertical asymptote with equation $x = 1$. Find the values of a, b and c.

12 The graph of the function with rule $f(x) = a\log_e(-x) + b$ passes though the points $(-2, 6)$ and $(-4, 8)$. Find the values of a and b.

5G Solution of exponential equations using logarithms

Example 24

If $\log_2 6 = k\log_2 3 + 1$, find the value of k.

Solution

$$\begin{aligned}
\log_2 6 &= k\log_2 3 + 1\\
&= \log_2(3^k) + \log_2 2\\
&= \log_2(2 \times 3^k)\\
\therefore\quad 6 &= 2 \times 3^k\\
3 &= 3^k\\
k &= 1
\end{aligned}$$

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Example 25

Solve for x if $2^x = 11$, expressing the answer to two decimal places.

Solution

$$2^x = 11 \;\Leftrightarrow\; x = \log_2 11$$
$$= 3.45943\ldots$$

Therefore $x \approx 3.46$ correct to two decimal places.

Example 26

Solve $3^{2x-1} = 28$, expressing the answer to three decimal places.

Solution

$$3^{2x-1} = 28 \;\Leftrightarrow\; 2x - 1 = \log_3 28$$

Thus $2x - 1 = \log_3 28$

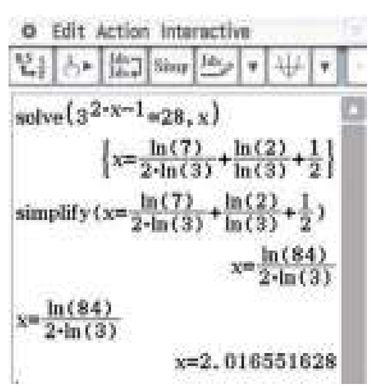
$$2x = \log_3 28 + 1$$
$$x = \frac{1}{2}(\log_3 28 + 1)$$
$$\approx 2.017 \qquad \text{correct to three decimal places}$$

Using the TI-Nspire

- Use menu > **Algebra** > **Solve** and complete as shown.
- Convert to a decimal answer using ctrl enter or menu > **Number** > **Convert to Decimal**.
- Round to three decimal places as required: $x = 2.017$.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the equation $3^{2x-1} = 28$.
- Go to **Interactive** > **Equation/Inequality** > **solve** and tap OK.
- Copy and paste the answer into the next entry line and go to **Interactive** > **Transformation** > **simplify** to obtain a simplified exact answer.
- Highlight the answer and tap $\frac{0.5}{}\leftrightarrow\frac{1}{2}$ to obtain the decimal approximation.



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Example 27

Solve the inequality $0.7^x \geq 0.3$.

Solution

Taking $\log_{10}$ of both sides:

$$\log_{10}(0.7^x) \geq \log_{10} 0.3$$
$$x \log_{10} 0.7 \geq \log_{10} 0.3$$
$$\therefore \quad x \leq \frac{\log_{10} 0.3}{\log_{10} 0.7} \qquad \text{(direction of inequality reversed since } \log_{10} 0.7 < 0\text{)}$$

Alternatively, we can solve the inequality $0.7^x \geq 0.3$ directly as follows:

Note that $0 < 0.7 < 1$ and thus $y = 0.7^x$ is strictly decreasing. Therefore the inequality $0.7^x \geq 0.3$ holds for $x \leq \log_{0.7} 0.3$.

Summary 5G

- If $a \in \mathbb{R}^+ \setminus \{1\}$ and $x \in \mathbb{R}$, then the statements $a^x = b$ and $\log_a b = x$ are equivalent. This defining property of logarithms may be used in the solution of exponential equations and inequalities. For example:
  - $2^x = 5 \Leftrightarrow x = \log_2 5$
  - $2^x \geq 5 \Leftrightarrow x \geq \log_2 5$
  - $0.3^x = 5 \Leftrightarrow x = \log_{0.3} 5$
  - $0.3^x \geq 5 \Leftrightarrow x \leq \log_{0.3} 5$
- An exponential inequality may also be solved by taking $\log_a$ of both sides. For $a > 1$, the direction of the inequality stays the same (as $y = \log_a x$ is strictly increasing). For $0 < a < 1$, the direction of the inequality reverses (as $y = \log_a x$ is strictly decreasing).

Exercise 5G

Example 24

1 **a** If $\log_2 8 = k \log_2 7 + 2$, find the value of k.

b If $\log_2 7 - x \log_2 7 = 4$, find the value of x.

c If $\log_e 7 - x \log_e 14 = 1$, find the value of x.

Example 25
Example 26

2 Use your calculator to solve each of the following equations, correct to two decimal places:

a $2^x = 6$ **b** $3^x = 0.7$ **c** $3^x = 11$

d $4^x = 5$ **e** $2^{-x} = 5$ **f** $0.2^x = 3$

g $5^x = 3^{x-1}$ **h** $8^x = 2005^{x+1}$ **i** $3^{x-1} = 8$

j $0.3^{x+2} = 0.7$ **k** $2^{x-1} = 3^{x+1}$ **l** $1.4^{x+2} = 25(0.9^x)$

m $5^x = 2^{2x-2}$ **n** $2^{\frac{1}{2}(x+2)} = 3^{x-1}$ **o** $2^{x+1} \times 3^{x-1} = 100$

3 Solve for x using a calculator. Express your answer correct to two decimal places.

a $2^x < 7$ **b** $3^x > 6$ **c** $0.2^x > 3$

d $3^{x-2} \leq 8$ **e** $0.2^x \leq 0.4$

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4 Solve each of the following equations for x. Give exact answers.

a $2^x = 5$ **b** $3^{2x-1} = 8$ **c** $7^{3x+1} = 20$
d $3^x = 7$ **e** $3^x = 6$ **f** $5^x = 6$
g $3^{2x} - 3^{x+2} + 8 = 0$ **h** $5^{2x} - 4 \times 5^x - 5 = 0$

Example 27

5 Solve each of the following inequalities for x. Give exact answers.

a $7^x > 52$ **b** $3^{2x-1} < 40$ **c** $4^{3x+1} \geq 5$
d $3^{x-5} \leq 30$ **e** $3^x < 106$ **f** $5^x < 0.6$

6 **a** If $a \log_2 7 = 3 - \log_6 14$, find the value of a, correct to three significant figures.
b If $\log_3 18 = \log_{11} k$, find the value of k, correct to one decimal place.

7 Prove that if $\log_r p = q$ and $\log_q r = p$, then $\log_q p = pq$.

8 If $u = \log_9 x$, find in terms of u:

a x **b** $\log_9(3x)$ **c** $\log_x 81$

9 Solve the equation $\log_5 x = 16 \log_x 5$.

10 Given that $q^p = 25$, find $\log_5 q$ in terms of p.

5H Inverses

We have observed that $f(x) = \log_a x$ and $g(x) = a^x$ are inverse functions. In this section, this observation is used to find inverses of related functions and to transform equations. An important consequence is the following:

$$\log_a(a^x) = x \qquad \text{for all } x \in \mathbb{R}$$
$$a^{\log_a x} = x \qquad \text{for all } x \in \mathbb{R}^+$$

Example 28

Find the inverse of the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^x + 2$ and state the domain and range of the inverse function.

Solution

Recall that the transformation 'reflection in the line $y = x$' is given by the mapping $(x, y) \to (y, x)$. Consider

$$x = e^y + 2$$
$$x - 2 = e^y$$
$$\therefore \quad y = \log_e(x - 2)$$

Thus the inverse function has rule $f^{-1}(x) = \log_e(x - 2)$.

Domain of f^{-1} = range of $f = (2, \infty)$.

Range of f^{-1} = domain of $f = \mathbb{R}$.

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Example 29

Rewrite the equation $y = 2\log_e(x) + 3$ with x as the subject.

Solution

$$y = 2\log_e(x) + 3$$

$$\frac{y-3}{2} = \log_e x$$

$$\therefore \quad x = e^{\frac{y-3}{2}}$$

Example 30

Find the inverse of the function $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 2\log_e(x-1) + 3$. State the domain and range of the inverse.

Solution

Solve $x = 2\log_e(y-1) + 3$ for y:

$$\frac{x-3}{2} = \log_e(y-1)$$

$$y - 1 = e^{\frac{x-3}{2}}$$

$$\therefore \quad y = e^{\frac{x-3}{2}} + 1$$

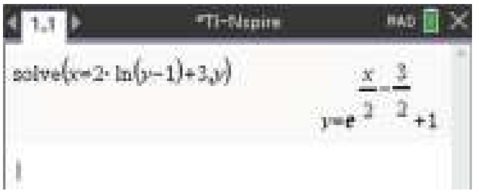
Hence $f^{-1}(x) = e^{\frac{x-3}{2}} + 1$.

Domain of f^{-1} = range of $f = \mathbb{R}$.

Range of f^{-1} = domain of $f = (1, \infty)$.

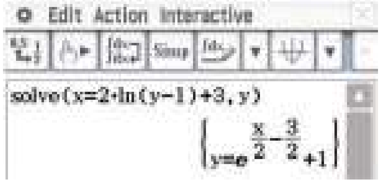
Using the TI-Nspire

Use **solve** from the **Algebra** menu as shown.



Using the Casio ClassPad

- Enter and highlight $x = 2\ln(y-1) + 3$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to y.



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Example 31

Rewrite the equation $P = Ae^{kt}$ with t as the subject.

Solution

$$P = Ae^{kt}$$

Take logarithms with base e of both sides:

$$\begin{aligned}\log_e P &= \log_e(Ae^{kt})\\ &= \log_e A + \log_e(e^{kt})\\ &= \log_e A + kt\\ \therefore \quad t &= \frac{1}{k}(\log_e P - \log_e A)\\ &= \frac{1}{k}\log_e\left(\frac{P}{A}\right)\end{aligned}$$

Summary 5H

Let $a \in \mathbb{R}^+ \setminus \{1\}$. The functions $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ and $g : \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_a x$ are inverse functions. That is, $g = f^{-1}$.

- $\log_a(a^x) = x$ for all x
- $a^{\log_a x} = x$ for all positive values of x

Exercise 5H

Example 28 **1** Find the inverse of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - 2$ and state the domain and range of the inverse function.

2 On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{-x} + 3$.

3 On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$, where $f : (1, \infty) \to \mathbb{R}$, $f(x) = \log_e(x - 1)$.

Example 29 **4** Rewrite the equation $y = 3\log_e(x) - 4$ with x as the subject.

Example 30 **5** Find the inverse of each of the following functions and state the domain and range in each case:

a $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(2x)$

b $f : \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 3\log_e(2x) + 1$

c $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x + 2$

d $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x+2}$

e $f : (-\frac{1}{2}, \infty) \to \mathbb{R}$, $f(x) = \log_e(2x + 1)$

f $f : (-\frac{2}{3}, \infty) \to \mathbb{R}$, $f(x) = 4\log_e(3x + 2)$

g $f : (-1, \infty) \to \mathbb{R}$, $f(x) = \log_{10}(x + 1)$

h $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2e^{x-1}$

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6 The function f has the rule $f(x) = 1 - e^{-x}$.

a Sketch the graph of f.

b Find the domain of f^{-1} and find $f^{-1}(x)$.

c Sketch the graph of f^{-1} on the same set of axes as the graph of f.

7 Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 5e^{2x} - 3$.

a Sketch the graph of f.

b Find the inverse function f^{-1}.

c Sketch the graph of f^{-1} on the same set of axes as the graph of f.

8 Let $f: \mathbb{R}^+ \to \mathbb{R}$ where $f(x) = 2\log_e(x) + 1$.

a Sketch the graph of f.

b Find the inverse function f^{-1} and state the range.

c Sketch the graph of f^{-1} on the same set of axes as the graph of f.

Example 31

9 Rewrite the equation $P = Ae^{-kt} + b$ with t as the subject.

10 For each of the following formulas, make the pronumeral in brackets the subject:

a $y = 2\log_e(x) + 5$ (x)

b $P = Ae^{-6x}$ (x)

c $y = ax^n$ (n)

d $y = 5 \times 10^x$ (x)

e $y = 5 - 3\log_e(2x)$ (x)

f $y = 6x^{2n}$ (n)

g $y = \log_e(2x - 1)$ (x)

h $y = 5(1 - e^{-x})$ (x)

11 For $f: \mathbb{R} \to \mathbb{R}$, $f(x) = 2e^x - 4$:

a Find the inverse function f^{-1}.

b Find the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

12 For $f: (-3, \infty) \to \mathbb{R}$, $f(x) = 2\log_e(x + 3) + 4$:

a Find the inverse function f^{-1}.

b Find the coordinates of the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

13 **a** Using a calculator, for each of the following plot the graphs of $y = f(x)$ and $y = g(x)$, together with the line $y = x$, on the one set of axes:

i $f(x) = \log_e x$ and $g(x) = e^x$

ii $f(x) = 2\log_e(x) + 3$ and $g(x) = e^{\frac{x-3}{2}}$

iii $f(x) = \log_{10} x$ and $g(x) = 10^x$

b Use your answers to part **a** to comment on the relationship in general between $f(x) = a\log_b(x) + c$ and $g(x) = b^{\frac{x-c}{a}}$.

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5I Exponential growth and decay

We will show in Chapter 11 that, if the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is the **rate constant**.

If $k > 0$, the model represents **growth**:

- growth of cells
- population growth
- continuously compounded interest

If $k < 0$, the model represents **decay**:

- radioactive decay
- cooling of materials

An equivalent way to write this model is as $A = A_0 b^t$, where we take $b = e^k$. In this form, growth corresponds to $b > 1$ and decay corresponds to $b < 1$.

Cell growth

Suppose a particular type of bacteria cell divides into two new cells every T_D minutes. Let N_0 be the initial number of cells of this type. After t minutes the number of cells, N, is given by

$$N = N_0 2^{\frac{t}{T_D}}$$

where T_D is called the **generation time**.

Example 32

What is the generation time of a bacterial population that increases from 5000 cells to 100 000 cells in four hours of growth?

Solution

In this example, $N_0 = 5000$ and $N = 100\,000$ when $t = 240$.

Hence $100\,000 = 5000 \times 2^{\frac{240}{T_D}}$

$$20 = 2^{\frac{240}{T_D}}$$

Thus $T_D = \dfrac{240}{\log_2 20} \approx 55.53$ (correct to two decimal places).

The generation time is approximately 55.53 minutes.

Radioactive decay

Radioactive materials decay such that the amount of radioactive material, A, present at time t (in years) is given by

$$A = A_0 e^{-kt}$$

where A_0 is the initial amount and k is a positive constant that depends on the type of material. A radioactive substance is often described in terms of its **half-life**, which is the time required for half the material to decay.

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Example 33

After 1000 years, a sample of radium-226 has decayed to 64.7% of its original mass. Find the half-life of radium-226.

Solution

We use the formula $A = A_0 e^{-kt}$. When $t = 1000$, $A = 0.647A_0$. Thus

$$\begin{aligned} 0.647A_0 &= A_0 e^{-1000k} \\ 0.647 &= e^{-1000k} \\ -1000k &= \log_e 0.647 \\ k &= \frac{-\log_e 0.647}{1000} \approx 0.000435 \end{aligned}$$

To find the half-life, we consider when $A = \frac{1}{2}A_0$:

$$\begin{aligned} A_0 e^{-kt} &= \tfrac{1}{2}A_0 \\ e^{-kt} &= \tfrac{1}{2} \\ -kt &= \log_e(\tfrac{1}{2}) \\ t &= -\frac{\log_e(\frac{1}{2})}{k} \approx 1591.95 \end{aligned}$$

The half-life of radium-226 is approximately 1592 years.

Population growth

It is sometimes possible to model population growth through exponential models.

Example 34

The population of a town was 8000 at the beginning of 2007 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2016.

b In what year will the population be double that of 2014?

Solution

Let P be the population at time t years (measured from 1 January 2007). Then

$$P = 8000e^{kt}$$

At the end of 2014, $t = 8$ and $P = 15\,000$. Therefore

$$\begin{aligned} 15\,000 &= 8000e^{8k} \\ \frac{15}{8} &= e^{8k} \\ k &= \frac{1}{8}\log_e\left(\frac{15}{8}\right) \approx 0.079 \end{aligned}$$

The rate of increase is 7.9% per annum.

Note: The approximation 0.079 was not used in the calculations which follow. The value for k was held in the calculator.

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a When $t = 10$, $P = 8000e^{10k}$

$\approx 17\,552.6049$

$\approx 17\,550$

The population is approximately 17 550.

b When does $P = 30\,000$? Consider the equation

$$30\,000 = 8000e^{kt}$$

$$\frac{30\,000}{8000} = e^{kt}$$

$$\frac{15}{4} = e^{kt}$$

$$\therefore \quad t = \frac{1}{k}\log_e\left(\frac{15}{4}\right)$$

$$\approx 16.82$$

The population reaches 30 000 approximately 16.82 years after the beginning of 2007, i.e. during the year 2023.

Example 35

There are approximately ten times as many red kangaroos as grey kangaroos in a certain area. If the population of grey kangaroos increases at a rate of 11% per annum while that of the red kangaroos decreases at 5% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

Solution

Let G_0 be the population of grey kangaroos at the start.

Then the number of grey kangaroos after n years is $G = G_0(1.11)^n$, and the number of red kangaroos after n years is $R = 10G_0(0.95)^n$.

When the proportions are reversed:

$$G = 10R$$

$$G_0(1.11)^n = 10 \times 10G_0(0.95)^n$$

$$(1.11)^n = 100(0.95)^n$$

Taking $\log_e$ of both sides:

$$\log_e((1.11)^n) = \log_e(100(0.95)^n)$$

$$n\log_e 1.11 = \log_e 100 + n\log_e 0.95$$

$$\therefore \quad n = \frac{\log_e 100}{\log_e 1.11 - \log_e 0.95}$$

$$\approx 29.6$$

i.e. the proportions of the kangaroo populations will be reversed after 30 years.

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Summary 5I

There are many situations in which a varying quantity can be modelled by an exponential function. Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

Skill-sheet

Exercise 5I

1 A population of 1000 E. coli bacteria doubles every 15 minutes.

Example 32

a Determine the formula for the number of bacteria at time t minutes.

b How long will it take for the population to reach 10 000? (Give your answer to the nearest minute.)

2 In the initial period of its life a particular species of tree grows in the manner described by the rule $d = d_0 10^{mt}$ where d is the diameter (in cm) of the tree t years after the beginning of this period. The diameter is 52 cm after 1 year, and 80 cm after 3 years. Calculate the values of the constants d_0 and m.

3 The number of people, N, who have a particular disease at time t years is given by $N = N_0 e^{kt}$.

a If the number is initially 20 000 and the number decreases by 20% each year, find:

i the value of N_0 **ii** the value of k.

b How long does it take until only 5000 people are infected?

Example 33

4 Polonium-210 is a radioactive substance. The decay of polonium-210 is described by the formula $M = M_0 e^{-kt}$, where M is the mass in grams of polonium-210 left after t days, and M_0 and k are constants. At $t = 0$, $M = 10$ g and at $t = 140$, $M = 5$ g.

a Find the values of M_0 and k.

b What will be the mass of the polonium-210 after 70 days?

c After how many days is the mass remaining 2 g?

5 A quantity A of radium at time t years is given by $A = A_0 e^{-kt}$, where k is a positive constant and A_0 is the amount of radium at time $t = 0$.

a Given that $A = \frac{1}{2}A_0$ when $t = 1690$ years, calculate k.

b After how many years does only 20% of the original amount remain? Give your answer to the nearest year.

6 The half-life of plutonium-239 is 24 000 years. If 20 grams are present now, how long will it take until only 20% of the original sample remains? (Give your answer to the nearest year.)

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7 Carbon-14 is a radioactive substance with a half-life of 5730 years. It is used to determine the age of ancient objects. A Babylonian cloth fragment now has 40% of the carbon-14 that it contained originally. How old is the fragment of cloth?

Example 34

8 The population of a town was 10 000 at the beginning of 2002 and 15 000 at the end of 2014. Assume that the growth is exponential.

a Find the population at the end of 2017.

b In what year will the population be double that of 2014?

Example 35

9 There are approximately five times as many magpies as currawongs in a certain area. If the population of currawongs increases at a rate of 12% per annum while that of the magpies decreases at 6% per annum, find how many years must elapse before the proportions are reversed, assuming the same rates continue to apply.

10 The pressure in the Earth's atmosphere decreases exponentially as you rise above the surface. The pressure in millibars at a height of h kilometres is given approximately by the function $P(h) = 1000 \times 10^{-0.05428h}$.

a Find the pressure at a height of 4 km. (Give your answer to the nearest millibar.)

b Find the height at which the pressure is 450 millibars. (Give your answer to the nearest metre.)

11 A biological culture contains 500 000 bacteria at 12 p.m. on Sunday. The culture increases by 10% every hour. At what time will the culture exceed 4 million bacteria?

12 When a liquid is placed into a refrigerator, its temperature T°C at time t minutes is given by the formula $T = T_0 e^{-kt}$. The temperature is initially 100°C and drops to 40°C in 5 minutes. Find the temperature of the liquid after 15 minutes.

13 The number of bacteria in a certain culture at time t weeks is given by the rule $N = N_0 e^{kt}$. If when $t = 2$, $N = 101$ and when $t = 4$, $N = 203$, calculate the values of N_0 and k.

14 Five kilograms of sugar is gradually dissolved in a vat of water. After t hours, the amount, S kg, of undissolved sugar remaining is given by $S = 5 \times e^{-kt}$.

a Calculate k given that $S = 3.2$ when $t = 2$.

b At what time will there be 1 kg of sugar remaining?

15 The number of bacteria, N, in a culture increases exponentially with time according to the rule $N = a \times b^t$, where time t is measured in hours. When observation started, there were 1000 bacteria, and 5 hours later there were 15 000 bacteria.

a Find the values of a and b.

b Find, to the nearest hour, when there were 5000 bacteria.

c Find, to the nearest hour, when the number of bacteria first exceeds 1 000 000.

d How many bacteria would there be 12 hours after the first observation?

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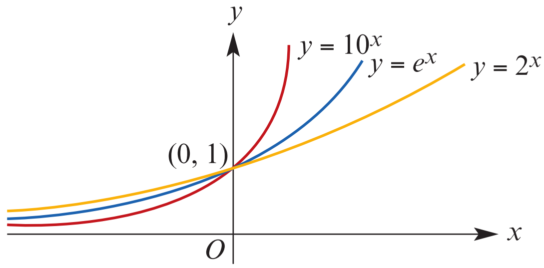
Chapter summary

Assign-ment

Nrich

- Sketch graphs of the form $y = a^x$ and transformations of these graphs.



- Index laws

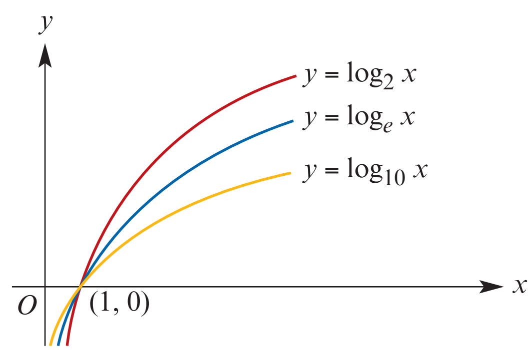
$$a^m \times a^n = a^{m+n} \qquad a^m \div a^n = a^{m-n} \qquad (a^m)^n = a^{mn}$$

- Logarithms

For $a \in \mathbb{R}^+ \setminus \{1\}$, the logarithm function with base a is defined as follows:

$$a^x = y \quad \text{is equivalent to} \quad \log_a y = x$$

- Sketch graphs of the form $y = \log_a x$ and transformations of these graphs.



- Logarithm laws

$$\log_a(mn) = \log_a m + \log_a n \qquad \log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

$$\log_a\left(\frac{1}{n}\right) = -\log_a n \qquad \log_a(m^p) = p\log_a m$$

- Change of base

$$\log_a x = \frac{\log_b x}{\log_b a} \qquad \text{and} \qquad a^x = b^{(\log_b a)x}$$

- Inverse functions

The inverse function of $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x$ is $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = \log_a x$.

  - $\log_a(a^x) = x$ for all $x \in \mathbb{R}$
  - $a^{\log_a x} = x$ for all $x \in \mathbb{R}^+$

- Law of exponential change

Assume that the rate at which the quantity A increases or decreases is proportional to its current value. Then the value of A at time t is given by

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant. Growth corresponds to $k > 0$, and decay corresponds to $k < 0$.

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Technology-free questions

1 Sketch the graph of each of the following. Label asymptotes and axis intercepts.

a $f(x) = e^x - 2$ **b** $g(x) = 10^{-x} + 1$ **c** $h(x) = \frac{1}{2}(e^x - 1)$

d $f(x) = 2 - e^{-x}$ **e** $f(x) = \log_e(2x + 1)$ **f** $h(x) = \log_e(x - 1) + 1$

g $g(x) = -\log_e(x - 1)$ **h** $f(x) = -\log_e(1 - x)$

2 **a** For $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = e^{2x} - 1$, find f^{-1}.

b For $f\colon (2, \infty) \to \mathbb{R}$, $f(x) = 3\log_e(x - 2)$, find f^{-1}.

c For $f\colon (-1, \infty) \to \mathbb{R}$, $f(x) = \log_{10}(x + 1)$, find f^{-1}.

d For $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = 2^x + 1$, find f^{-1}.

3 For each of the following, find y in terms of x:

a $\log_e y = \log_e x + 2$ **b** $\log_{10} y = \log_{10} x + 1$ **c** $\log_2 y = 3\log_2 x + 4$

d $\log_{10} y = -1 + 5\log_{10} x$ **e** $\log_e y = 3 - \log_e x$ **f** $\log_e y = 2x - 3$

4 Solve each of the following equations for x, expressing your answers in terms of logarithms with base e:

a $3^x = 11$ **b** $2^x = 0.8$ **c** $2^x = 3^{x+1}$

5 Solve each of the following for x:

a $2^{2x} - 2^x - 2 = 0$ **b** $\log_e(3x - 1) = 0$

c $\log_{10}(2x) + 1 = 0$ **d** $10^{2x} - 7 \times 10^x + 12 = 0$

e $6e^t + 5 - 6e^{-t} = 0$ **f** $\log_2(30 - x) - \log_2(5 - x) = 3$

6 The graph of the function with rule $y = 3\log_2(x + 1) + 2$ intersects the axes at the points $(a, 0)$ and $(0, b)$. Find the exact values of a and b.

7 The graph of $y = 5\log_{10}(x + 1)$ passes through the point $(k, 6)$. Find the value of k.

8 Find the exact value of x for which $4e^{3x} = 287$.

9 Find the value of x in terms of a, where $3\log_a x = 3 + \log_a 8$.

10 Show that, if $3^x = 4^y = 12^z$, then $z = \dfrac{xy}{x + y}$.

11 Evaluate $2\log_2 12 + 3\log_2 5 - \log_2 15 - \log_2 150$.

12 **a** Given that $\log_p 7 + \log_p k = 0$, where $p > 1$, find k.

b Given that $4\log_q 3 + 2\log_q 2 - \log_q 144 = 2$, find q.

13 Given that $\log_e y = a + b\log_e x$, where a and b are constants, find y in terms of x.

14 For $f\colon (4, \infty) \to \mathbb{R}$, $f(x) = \log_3(x - 4)$, state the domain of the inverse function f^{-1}.

15 The graph of the function with rule $f(x) = e^{2x} - 3ke^x + 5$ intersects the axes at $(0, 0)$ and $(a, 0)$ and has a horizontal asymptote at $y = b$. Find the exact values of a, b and k.

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16 $f(x) = 3^x$ and $g(x) = e^{kx}$. Find k such that $f(x) = g(x)$

17 Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = e^{3x} - 4$.

a Find the rule and domain of the inverse function f^{-1}. **b** Find $f(-f^{-1}(3x))$.

18 Let $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = k \log_3 x,\ k \in \mathbb{R}$. If $f(27) = 27$ find the value of k.

19 **a** Solve the polynomial equation $x^3 - 3x^2 - 6x + 8 = 0$ for x.

b Hence solve the equation $e^{3x} - 3e^{2x} - 6e^x + 8 = 0$ for x.

20 Let $f : \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_e x,\ g : \mathbb{R} \to \mathbb{R},\ g(x) = 2x^2 + 4$ and $h : \mathbb{R}^- \cup \{0\} \to \mathbb{R},\ h(x) = \log_e(2x^2 + 4)$.

a Find the rule, domain and range for $f \circ g$.

b Find the rule, domain and range fo h^{-1}.

21 Let $g(x) = 2^x$ and $f(x) = x^2 - 12x + 32$. Solve each of the equations for x.

a $f(g(x)) = 0$ **b** $g(f(x)) = 1$ **c** $f(g^{-1}(x)) = 0$

22 The solution of the equation $e^x - e^{-x} + 1 = 0$ is $x = \log_e\left(\frac{a}{2}\right)$ where $a \in \mathbb{R}^+$. Find a.

23 Let a and b be positive integers. Use change of base to show $\log_{ab} x = \dfrac{\log_a x}{1 + \log_a b}$. Use this result to show

$$\log_2 7 = \frac{1 - \log_{14} 2}{\log_{14} 2}.$$

24 Let $f(x) = e^x + e^{-x}$ and $g(x) = e^x - e^{-x}$. Solve the equation

$$[f(x)]^2 + [g(x)]^2 = 5 \text{ for } x.$$

25 Let $f: \mathbb{R} \to \mathbb{R}$ where $f(x) = 3^{x+2} - 2$.
The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T(x, y) = (x + c, y + d)$ maps the graph of $y = 3^x$ onto the graph of f.

a Find the values of c and d.

b Determine the coordinates of the point of intersection of the two graphs.

c Find f^{-1}

26 Let $f(x) = e^x + e^{-x}$ and $g(x) = e^x - e^{-x}$.

a Show that f is an even function. **b** Find $f(u) + f(-u)$.

c Find $f(u) - f(-u)$. **d** Find $[f(u)]^2 - 2$.

e Show that g is an odd function.

f Find $f(x) + g(x)$, $f(x) - g(x)$ and $f(x) \cdot g(x)$.

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Review

Multiple-choice questions

1 If $4\log_b(x^2) = \log_b 16 + 8$, then x is equal to

A b^4 **B** ± 6 **C** $\pm\sqrt{2}\,b$ **D** 2^6 **E** $2b^4$

2 The expression $\log_e(4e^{3x})$ is equal to

A $\log_e(e^{12x})$ **B** $\log_e(12) + x$ **C** $3x\log_e(4)$ **D** $\log_e(4) + 3x$ **E** $12x$

3 The expression $3^{\log_3(x-4)}$ is equal to

A $\dfrac{x}{4}$ **B** $x - 4$ **C** $3(x-4)$ **D** $3^x - 3^4$ **E** $\log_3 x - \log_3 4$

4 Consider the three functions

$$f\colon A \to \mathbb{R},\ f(x) = e^{2x}, \quad g\colon B \to \mathbb{R},\ g(x) = \frac{1}{x+1}, \quad h\colon C \to \mathbb{R},\ h(x) = e^{2x} + \frac{1}{x+1}$$

where A, B and C are the largest domains for which f, g and h respectively are defined. Which one of the following statements is true?

A $A \neq C$ and $\operatorname{ran}(g) = \operatorname{ran}(h)$
B $A = B$ and $\operatorname{ran}(f) \neq \operatorname{ran}(h)$
C $A \neq C$ and $\operatorname{ran}(f) = \operatorname{ran}(h)$
D $B = C$ and $\operatorname{ran}(g) = \operatorname{ran}(h)$
E $B = C$ and $\operatorname{ran}(g) \neq \operatorname{ran}(h)$

5 If $x = 5$ is a solution of the equation $\log_{10}(kx - 3) = 2$, then the exact value of k is

A $\dfrac{103}{5}$ **B** $\dfrac{\log_{10} 2 + 3}{5}$ **C** 2 **D** 5 **E** 21

6 $3^{4\log_3(x)+\log_3(4x)}$ is equal to

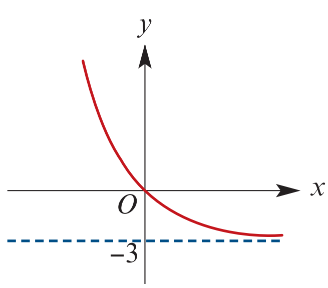
A $8x$ **B** $x^4 + 4x$ **C** $4x^5$ **D** 3^{8x} **E** $\log_3(4x^5)$

7 The solution of the equation $3x = 10^{-0.3x}$ is closest to

A 0.83 **B** 0.28 **C** 0 **D** 0.30 **E** 0.91

8 The graph of the function with rule $y = ae^{-x} + b$ is shown on the right.
The values of a and b respectively are

A $3,\ -3$
B $-3,\ 3$
C $-3,\ -3$
D $0,\ -3$
E $-3,\ 0$



9 Which one of the following statements is not true of the graph of the function $f\colon \mathbb{R}^+ \to \mathbb{R},\ f(x) = \log_5 x$?

A The domain is $\mathbb{R}^+$.
B The range is $\mathbb{R}$.
C It passes through the point $(5, 0)$.
D It has a vertical asymptote with equation $x = 0$.
E The slope of the tangent at any point on the graph is positive.

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10 If $3\log_2 x - 7\log_2(x-1) = 2 + \log_2 y$, then y is equal to

A $\dfrac{3x}{28(x-1)}$ **B** $\dfrac{1}{4x^4}$ **C** $3 - 4x$

D $\dfrac{x^3}{4(x-1)^7}$ **E** $x^3 - (x-1)^7 - 4$

11 The graph of the function $f(x) = e^{2x} - 12$ intersects the graph of $g(x) = -e^x$ where

A $x = \log_e 3$ **B** $x = \log_e 2$ **C** $x = \log_e 7$ **D** $x = \log_e 4$ **E** $x = \log_e 5$

12 Let the rule for a function g be $g(x) = \log_e((x-4)^2)$. For the function g, the maximal domain and range are

A $\mathbb{R}$, $\mathbb{R}$ **B** $(4, \infty)$, $\mathbb{R}^+$ **C** $\mathbb{R} \setminus \{4\}$, $\mathbb{R}$

D $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{0\}$ **E** $(-\infty, 4)$, $\mathbb{R}$

13 The maximal domain D of the function $f\colon D \to \mathbb{R}$, $f(x) = \log_e((x-3)^2) + 6$ is

A $(3, \infty)$ **B** $[3, \infty)$ **C** $\mathbb{R} \setminus \{3\}$ **D** $\mathbb{R}^+$ **E** $(\infty, 3)$

14 The function $f\colon [a, \infty) \to \mathbb{R}$, $f(x) = \log_e(x^2)$ will have an inverse function if

A $a \in (-\infty, 0)$ **B** $a \in (-\infty, -1)$ **C** $a \in (-\infty, 1)$

D $a \in (0, \infty)$ **E** $a \in (-1, \infty)$

15 The inverse of the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = e^{3x+4}$ is

A $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = -3\log_e(3x-4)$ **B** $f^{-1}\colon (e^4, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{\log_e(x) - 4}{3}$

C $f^{-1}\colon \mathbb{R}^+ \to \mathbb{R}$, $f^{-1}(x) = -3\log_e\left(\dfrac{x-4}{3}\right)$ **D** $f^{-1}\colon (e^4, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_e(3x-4)$

E $f^{-1}\colon (-\frac{4}{3}, \infty) \to \mathbb{R}$, $f^{-1}(x) = \log_e(3x) - 4$

16 If $f(x) = 2\log_e(3x)$ and $f(6x) = \log_e(y)$, then

A $y = 18x$ **B** $y = \dfrac{x}{3}$ **C** $y = 6x^2$ **D** $y = 324x^2$ **E** $y = 36x^2$

Extended-response questions

1 A liquid cools from its original temperature of 90°C to a temperature of T°C in x minutes. Given that $T = 90(0.98)^x$, find:

a the value of T when $x = 10$

b the value of x when $T = 27$.

2 The population of a village at the beginning of the year 1800 was 240. The population increased so that, after a period of n years, the new population was $240(1.06)^n$. Find:

a the population at the beginning of 1820

b the year in which the population first reached 2500.

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3 The value, $\$V$, of a particular car can be modelled by the equation $V = ke^{-\lambda t}$, where t years is the age of the car. The car's original price was \$22 497, and after 1 year it is valued at \$18 000.

a State the value of k and calculate λ, giving your answer to two decimal places.

b Find the value of the car when it is 3 years old.

4 The value, $\$M$, of a particular house during the period 1988 to 1994 can be modelled by the equation $M = Ae^{-pt}$, where t is the time in years after 1 January 1988. The value of the house on 1 January 1988 was \$65 000 and its value on 1 January 1989 was \$61 000.

a State the value of A and calculate the value of p, correct to two significant figures.

b What was the value of the house in 1993? Give your answer to the nearest \$100.

5 There are two species of insects living in a suburb: the *Asla bibla* and the *Cutus pius*. The number of *Asla bibla* alive at time t days after 1 January 2000 is given by

$$N_A(t) = 10\,000 + 1000t, \quad 0 \le t \le 15$$

The number of *Cutus pius* alive at time t days after 1 January 2000 is given by

$$N_C(t) = 8000 + 3 \times 2^t, \quad 0 \le t \le 15$$

a With a calculator, plot the graphs of $y = N_A(t)$ and $y = N_C(t)$ on the one screen.

b **i** Find the coordinates of the point of intersection of the two graphs.

ii At what time is $N_A(t) = N_C(t)$?

iii What is the number of each species of insect at this time?

c **i** Show that $N_A(t) = N_C(t)$ if and only if $t = 3\log_2 10 + \log_2\left(\dfrac{2+t}{3}\right)$.

ii Plot the graphs of $y = x$ and $y = 3\log_2 10 + \log_2\left(\dfrac{2+x}{3}\right)$ and find the coordinates of the point of intersection.

d It is found by observation that the model for *Cutus pius* does not quite work. It is known that the model for the population of *Asla bibla* is satisfactory. The form of the model for *Cutus pius* is $N_C(t) = 8000 + c \times 2^t$. Find the value of c, correct to two decimal places, if it is known that $N_A(15) = N_C(15)$.

6 The number of a type of bacteria is modelled by the formula $n = A(1 - e^{-Bt})$, where n is the size of the population at time t hours, and A and B are positive constants.

a When $t = 2$, $n = 10\,000$ and when $t = 4$, $n = 15\,000$.

i Show that $2e^{-4B} - 3e^{-2B} + 1 = 0$.

ii Use the substitution $a = e^{-2B}$ to show that $2a^2 - 3a + 1 = 0$.

iii Solve this equation for a.

iv Find the exact value of B.

v Find the exact value of A.

b Sketch the graph of n against t.

c After how many hours is the population of bacteria 18 000?

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7 The barometric pressure P (in centimetres of mercury) at a height h km above sea level is given by $P = 75(10^{-0.15h})$. Find:

a P when $h = 0$ **b** P when $h = 10$ **c** h when $P = 60$.

8 A radioactive substance is decaying such that the amount, A g, at time t years is given by the formula $A = A_0 e^{kt}$. If when $t = 1$, $A = 60.7$ and when $t = 6$, $A = 5$, find the values of the constants A_0 and k.

9 In a chemical reaction the amount, x g, of a substance that has reacted is given by $x = 8(1 - e^{-0.2t})$, where t is the time in minutes from the beginning of the reaction.

a Sketch the graph of x against t.

b Find the amount of substance that has reacted after:

i 0 minutes **ii** 2 minutes **iii** 10 minutes.

c Find the time when exactly 7 g of the substance has reacted.

10 Newton's law of cooling for an object in a medium of constant temperature states

$$T - T_s = (T_0 - T_s)\,e^{-kt}$$

where:

- T is the temperature (in °C) of the object at time t (in minutes)
- T_s is the temperature of the surrounding medium
- T_0 is the initial temperature of the object.

An egg at 96°C is placed to cool in a sink of water at 15°C. After 5 minutes the egg's temperature is 40°C. (Assume that the temperature of the water does not change.)

a Find the value of k.

b Find the temperature of the egg when $t = 10$.

c How long does it take for the egg to reach a temperature of 30°C?

11 The population of a colony of small, interesting insects is modelled by the following function:

$$N(t) = \begin{cases} 20e^{0.2t} & \text{for } 0 \le t \le 50 \\ 20e^{10} & \text{for } 50 < t \le 70 \\ 10e^{10}(e^{70-t} + 1) & \text{for } t > 70 \end{cases}$$

where t is the number of days.

a Sketch the graph of $N(t)$ against t.

b Find:

i $N(10)$ **ii** $N(40)$ **iii** $N(60)$ **iv** $N(80)$

c Find the number of days for the population to reach:

i 2968 **ii** 21 932

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