6

Circular functions

Objectives

- To measure angles in **degrees** and **radians**.
- To define the circular functions **sine**, **cosine** and **tangent**.
- To explore the **symmetry properties** of circular functions.
- To find **exact values** of circular functions.
- To **sketch graphs** of circular functions.
- To **solve equations** involving circular functions.
- To apply circular functions in modelling **periodic motion**.

Following on from our study of polynomial, exponential and logarithmic functions, we meet a further three important functions in this chapter. Again we use the notation developed in Chapter 1 for describing functions and their properties.

In this chapter we revise and extend our consideration of the functions sine, cosine and tangent. The first two of these functions have the real numbers as their domain, and the third the real numbers without the odd multiples of $\frac{\pi}{2}$.

An important property of these three functions is that they are periodic. That is, they each repeat their values in regular intervals or periods. In general, a function f is **periodic** if there is a positive constant a such that $f(x + a) = f(x)$. The sine and cosine functions each have period 2π, while the tangent function has period π.

The sine and cosine functions are used to model wave motion, and are therefore central to the application of mathematics to any problem in which periodic motion is involved – from the motion of the tides and ocean waves to sound waves and modern telecommunications.

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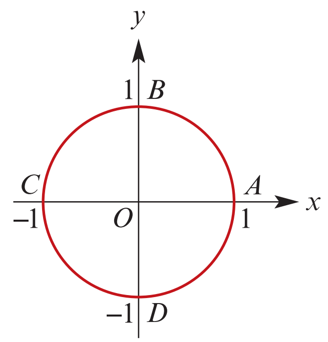
6A Measuring angles in degrees and radians

The diagram shows a **unit circle**, i.e. a circle of radius 1 unit.



The circumference of the unit circle $= 2\pi \times 1$

$= 2\pi$ units

Thus, the distance in an anticlockwise direction around the circle from

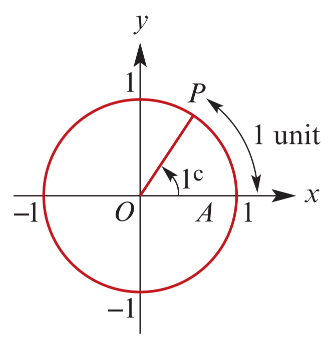
$$A \text{ to } B = \frac{\pi}{2} \text{ units}$$

$$A \text{ to } C = \pi \text{ units}$$

$$A \text{ to } D = \frac{3\pi}{2} \text{ units}$$

Definition of a radian

In moving around the circle a distance of 1 unit from A to P, the angle POA is defined. The measure of this angle is 1 radian.



One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.

Note: Angles formed by moving **anticlockwise** around the unit circle are defined as **positive**; those formed by moving **clockwise** are defined as **negative**.

Degrees and radians

The angle, in radians, swept out in one revolution of a circle is $2\pi^c$.

$$2\pi^c = 360^\circ$$

$$\therefore \quad \pi^c = 180^\circ$$

$$\therefore \quad \boxed{1^c = \frac{180^\circ}{\pi} \quad \text{or} \quad 1^\circ = \frac{\pi^c}{180}}$$

Example 1

Convert 30° to radians.

Solution	Explanation
$1^\circ = \dfrac{\pi^c}{180}$	
$\therefore \quad 30^\circ = \dfrac{30 \times \pi}{180} = \dfrac{\pi^c}{6}$	Multiply by $\dfrac{\pi}{180}$ and simplify by cancelling.

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Example 2

Convert $\dfrac{\pi^c}{4}$ to degrees.

Solution

$$1^c = \frac{180^\circ}{\pi}$$

$$\therefore \quad \frac{\pi^c}{4} = \frac{\pi \times 180}{4 \times \pi} = 45^\circ$$

Explanation

Multiply by $\dfrac{180}{\pi}$ and simplify by cancelling.

Note: Often the symbol for radians, ^{c}, is omitted.
For example, the angle 45° is written as $\dfrac{\pi}{4}$ rather than $\dfrac{\pi^c}{4}$.

Summary 6A

- One **radian** (written 1^c) is the angle subtended at the centre of the unit circle by an arc of length 1 unit.
- To convert:
  - degrees to radians, multiply by $\dfrac{\pi}{180}$
  - radians to degrees, multiply by $\dfrac{180}{\pi}$.

Exercise 6A

Example 1

1 Express the following angles in radian measure in terms of π:

a 50° **b** 136° **c** 250°
d 340° **e** 420° **f** 490°

Example 2

2 Express, in degrees, the angles with the following radian measures:

a $\dfrac{\pi}{3}$ **b** $\dfrac{5\pi}{6}$ **c** $\dfrac{4\pi}{3}$
d $\dfrac{7\pi}{9}$ **e** 3.5π **f** $\dfrac{7\pi}{5}$

3 Use a calculator to convert each of the following angles from radians to degrees:

a 0.8 **b** 1.64 **c** 2.5
d 3.96 **e** 4.18 **f** 5.95

4 Use a calculator to express each of the following in radian measure. (Give your answer correct to two decimal places.)

a 37° **b** 74° **c** 115°
d 122.25° **e** 340° **f** 132.5°

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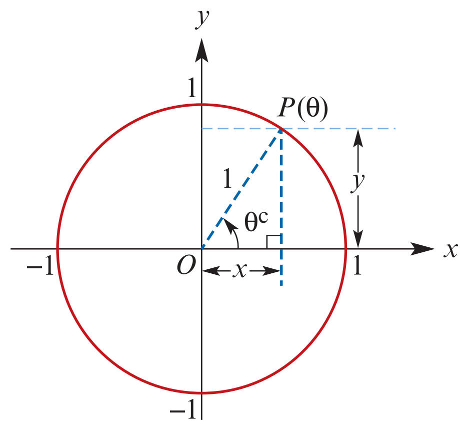
6B Defining circular functions: sine, cosine and tangent

The point P on the unit circle corresponding to an angle θ is written $P(\theta)$.



The x-coordinate of $P(\theta)$ is determined by the angle θ. Similarly, the y-coordinate of $P(\theta)$ is determined by the angle θ. So we can define two functions, called sine and cosine, as follows:

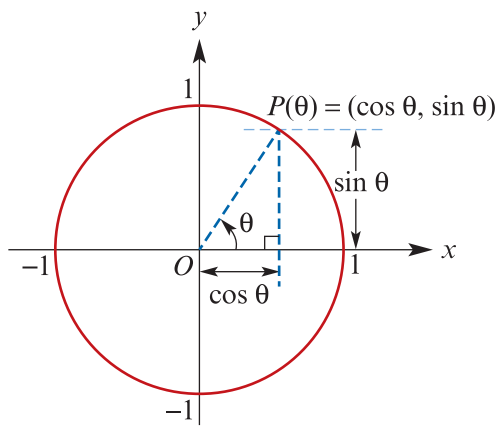
> The x-coordinate of $P(\theta)$ is given by
>
> $x = \text{cosine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$
>
> The y-coordinate of $P(\theta)$ is given by
>
> $y = \text{sine}\,\theta, \quad \text{for } \theta \in \mathbb{R}$

These functions are usually written in an abbreviated form as follows:

$x = \cos\theta$

$y = \sin\theta$

Hence the coordinates of $P(\theta)$ are $(\cos\theta, \sin\theta)$.



Note: Adding 2π to the angle results in a return to the same point on the unit circle. Thus $\cos(2\pi + \theta) = \cos\theta$ and $\sin(2\pi + \theta) = \sin\theta$.

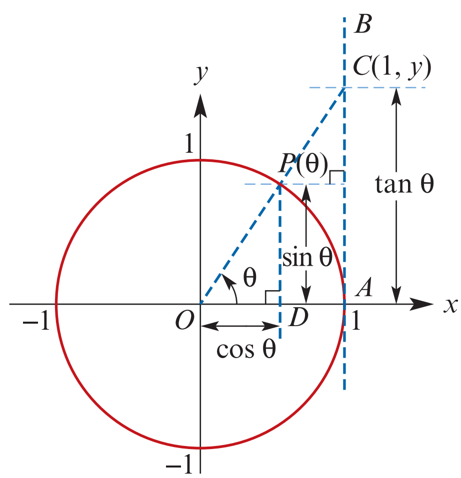
Again consider the unit circle.

If we draw a tangent to the unit circle at A, then the y-coordinate of C, the point of intersection of the line OP and the tangent, is called **tangent** θ (abbreviated to $\tan\theta$).

By considering the similar triangles OPD and OCA:

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \quad \boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$



Note that $\tan\theta$ is undefined when $\cos\theta = 0$. The domain of tan is $\mathbb{R} \setminus \{\theta : \cos\theta = 0\}$ and so $\tan\theta$ is undefined when $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots$

Note: Adding π to the angle does not change the line OP. Thus $\tan(\pi + \theta) = \tan\theta$.

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From the periodicity of the circular functions:

- $\sin(2k\pi + \theta) = \sin\theta$, for all integers k
- $\cos(2k\pi + \theta) = \cos\theta$, for all integers k
- $\tan(k\pi + \theta) = \tan\theta$, for all integers k.

Example 3

Evaluate each of the following:

a $\sin\left(\frac{3\pi}{2}\right)$ **b** $\sin\left(-\frac{3\pi}{2}\right)$ **c** $\cos\left(\frac{5\pi}{2}\right)$ **d** $\cos\left(-\frac{\pi}{2}\right)$

e $\cos\left(\frac{23\pi}{2}\right)$ **f** $\sin\left(\frac{55\pi}{2}\right)$ **g** $\tan(55\pi)$ **h** $\tan\left(\frac{15\pi}{2}\right)$

Solution	Explanation
a $\sin\left(\frac{3\pi}{2}\right) = -1$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
b $\sin\left(-\frac{3\pi}{2}\right) = 1$	since $P\left(-\frac{3\pi}{2}\right)$ has coordinates $(0, 1)$.
c $\cos\left(\frac{5\pi}{2}\right) = \cos\left(2\pi + \frac{\pi}{2}\right) = 0$	since $P\left(\frac{\pi}{2}\right)$ has coordinates $(0, 1)$.
d $\cos\left(-\frac{\pi}{2}\right) = 0$	since $P\left(-\frac{\pi}{2}\right)$ has coordinates $(0, -1)$.
e $\cos\left(\frac{23\pi}{2}\right) = \cos\left(10\pi + \frac{3\pi}{2}\right) = 0$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
f $\sin\left(\frac{55\pi}{2}\right) = \sin\left(26\pi + \frac{3\pi}{2}\right) = -1$	since $P\left(\frac{3\pi}{2}\right)$ has coordinates $(0, -1)$.
g $\tan(55\pi) = 0$	since $\tan(k\pi) = 0$, for any integer k.
h $\tan\left(\frac{15\pi}{2}\right)$ is undefined	since $\tan\left(\frac{k\pi}{2}\right)$ is undefined for any odd integer k.

Example 4

Evaluate using a calculator. (Give answers to two decimal places.)

a $\tan 1.3$ **b** $\sin 1.8$ **c** $\cos(-2.6)$ **d** $\sin 3.8$

e $\tan(-2.8)$ **f** $\tan 59^\circ$ **g** $\tan 138^\circ$

Solution

a $\tan 1.3 = 3.60$ **b** $\sin 1.8 = 0.97$

c $\cos(-2.6) = -0.86$ **d** $\sin 3.8 = -0.61$

e $\tan(-2.8) = 0.36$ **f** $\tan 59^\circ = 1.66$

g $\tan 138^\circ = -0.90$

Explanation

Your calculator should be in radian mode for **a**–**e** and in degree mode for **f** and **g**.

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Exact values of circular functions

A calculator can be used to find the values of the circular functions for different values of θ. For many values of θ the calculator gives an approximation. We consider some values of θ such that sin, cos and tan can be calculated exactly.

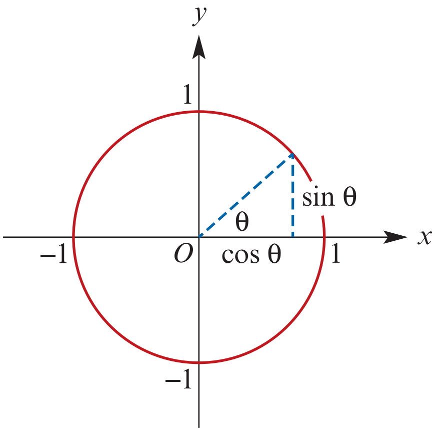
Exact values for 0 (0°) and $\frac{\pi}{2}$ (90°)

From the unit circle:

$\sin 0° = 0 \qquad \sin 90° = 1$

$\cos 0° = 1 \qquad \cos 90° = 0$

$\tan 0° = 0 \qquad \tan 90°$ is undefined



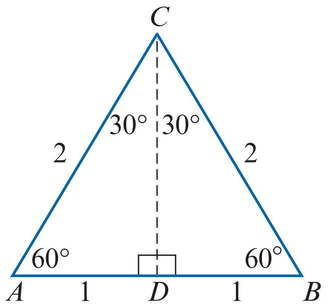
Exact values for $\frac{\pi}{6}$ (30°) and $\frac{\pi}{3}$ (60°)

Consider an equilateral triangle ABC of side length 2 units.
In $\triangle ACD$, by Pythagoras' theorem, $CD = \sqrt{AC^2 - AD^2} = \sqrt{3}$.

$$\sin 30° = \frac{AD}{AC} = \frac{1}{2} \qquad \sin 60° = \frac{CD}{AC} = \frac{\sqrt{3}}{2}$$

$$\cos 30° = \frac{CD}{AC} = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{AD}{AC} = \frac{1}{2}$$

$$\tan 30° = \frac{AD}{CD} = \frac{1}{\sqrt{3}} \qquad \tan 60° = \frac{CD}{AD} = \frac{\sqrt{3}}{1} = \sqrt{3}$$



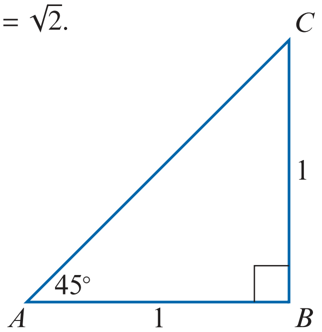
Exact values for $\frac{\pi}{4}$ (45°)

For the triangle ABC shown on the right, we have $AC = \sqrt{1^2 + 1^2} = \sqrt{2}$.

$$\sin 45° = \frac{BC}{AC} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{AB}{AC} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{BC}{AB} = 1$$



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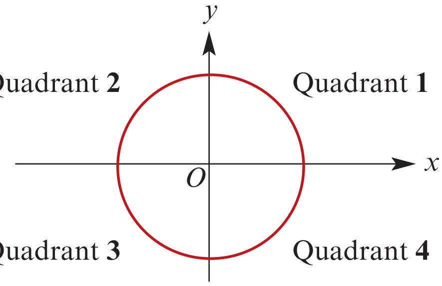
Symmetry properties of circular functions

The coordinate axes divide the unit circle into four quadrants. The quadrants can be numbered, anticlockwise from the positive direction of the x-axis, as shown.



Using symmetry, we can determine relationships between the circular functions for angles in different quadrants:

Quadrant 2

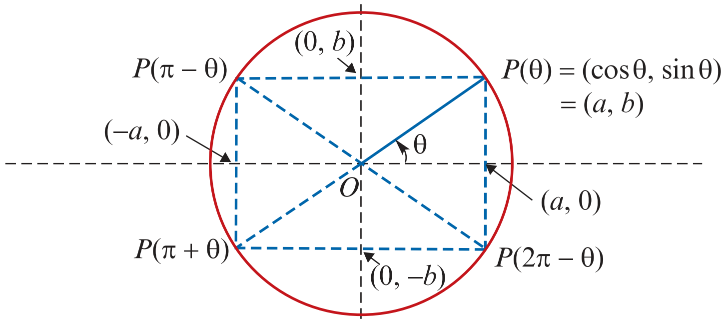
By symmetry:

$\sin(\pi - \theta) = b = \sin\theta$

$\cos(\pi - \theta) = -a = -\cos\theta$

$\tan(\pi - \theta) = \dfrac{b}{-a} = -\tan\theta$

Quadrant 1



Quadrant 3

$\sin(\pi + \theta) = -b = -\sin\theta$

$\cos(\pi + \theta) = -a = -\cos\theta$

$\tan(\pi + \theta) = \dfrac{-b}{-a} = \tan\theta$

Quadrant 4

$\sin(2\pi - \theta) = -b = -\sin\theta$

$\cos(2\pi - \theta) = a = \cos\theta$

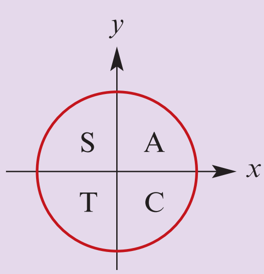
$\tan(2\pi - \theta) = \dfrac{-b}{a} = -\tan\theta$

Note: These relationships are true for all values of θ.

Signs of circular functions

Using the symmetry properties, the signs of sin, cos and tan for the four quadrants can be summarised as follows:

1st quadrant	all are positive	(A)
2nd quadrant	sin is positive	(S)
3rd quadrant	tan is positive	(T)
4th quadrant	cos is positive	(C)



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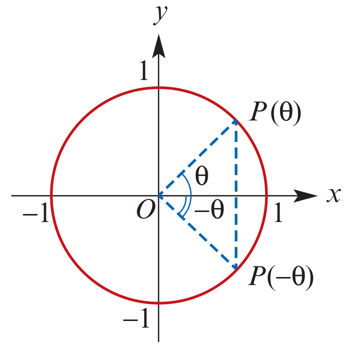
Negative of angles

By symmetry:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



Therefore:

- sin is an odd function
- cos is an even function
- tan is an odd function.

Example 5

Evaluate:

a $\cos\left(\frac{5\pi}{4}\right)$ **b** $\sin\left(\frac{11\pi}{6}\right)$ **c** $\cos\left(\frac{200\pi}{3}\right)$ **d** $\tan\left(\frac{52\pi}{6}\right)$

Solution

a
$$\begin{aligned}\cos\left(\frac{5\pi}{4}\right) &= \cos\left(\pi + \frac{\pi}{4}\right)\\ &= -\cos\left(\frac{\pi}{4}\right)\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

b
$$\begin{aligned}\sin\left(\frac{11\pi}{6}\right) &= \sin\left(2\pi - \frac{\pi}{6}\right)\\ &= -\sin\left(\frac{\pi}{6}\right)\\ &= -\frac{1}{2}\end{aligned}$$

c
$$\begin{aligned}\cos\left(\frac{200\pi}{3}\right) &= \cos\left(66\pi + \frac{2\pi}{3}\right)\\ &= \cos\left(\frac{2\pi}{3}\right)\\ &= \cos\left(\pi - \frac{\pi}{3}\right)\\ &= -\cos\left(\frac{\pi}{3}\right)\\ &= -\frac{1}{2}\end{aligned}$$

d
$$\begin{aligned}\tan\left(\frac{52\pi}{6}\right) &= \tan\left(8\pi + \frac{2\pi}{3}\right)\\ &= \tan\left(\frac{2\pi}{3}\right)\\ &= \tan\left(\pi - \frac{\pi}{3}\right)\\ &= -\tan\left(\frac{\pi}{3}\right)\\ &= -\sqrt{3}\end{aligned}$$

Example 6

If $\sin x = 0.6$, find the value of:

a $\sin(\pi - x)$ **b** $\sin(\pi + x)$ **c** $\sin(2\pi - x)$ **d** $\sin(-x)$

Solution

a
$$\begin{aligned}\sin(\pi - x) &= \sin x\\ &= 0.6\end{aligned}$$

b
$$\begin{aligned}\sin(\pi + x) &= -\sin x\\ &= -0.6\end{aligned}$$

c
$$\begin{aligned}\sin(2\pi - x) &= -\sin x\\ &= -0.6\end{aligned}$$

d
$$\begin{aligned}\sin(-x) &= -\sin x\\ &= -0.6\end{aligned}$$

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Example 7

If $\cos x° = 0.8$, find the value of:

a $\cos(180 - x)°$ **b** $\cos(180 + x)°$ **c** $\cos(360 - x)°$ **d** $\cos(-x)°$

Solution

a $\cos(180 - x)°$
$= -\cos x°$
$= -0.8$

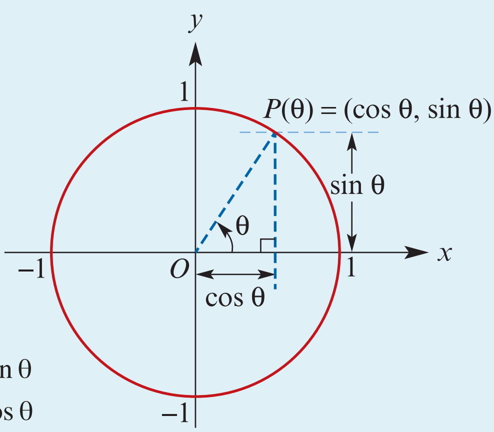
b $\cos(180 + x)°$
$= -\cos x°$
$= -0.8$

c $\cos(360 - x)°$
$= \cos x°$
$= 0.8$

d $\cos(-x)°$
$= \cos x°$
$= 0.8$

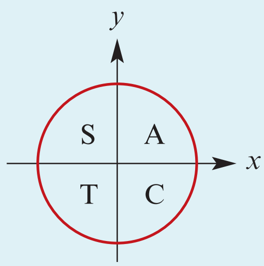
Summary 6B

- $P(\theta) = (\cos\theta, \sin\theta)$
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$
- The circular functions are periodic:
  - $\sin(2\pi + \theta) = \sin\theta$
  - $\cos(2\pi + \theta) = \cos\theta$
  - $\tan(\pi + \theta) = \tan\theta$



- Negative of angles:
  - sin is an odd function, i.e. $\sin(-\theta) = -\sin\theta$
  - cos is an even function, i.e. $\cos(-\theta) = \cos\theta$
  - tan is an odd function, i.e. $\tan(-\theta) = -\tan\theta$
- Memory aids:

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$ (30°)	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$ (45°)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$ (60°)	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$ (90°)	1	0	undefined



Exercise 6B

Example 3

1 Evaluate each of the following:

a $\sin(3\pi)$ **b** $\cos\left(-\frac{5\pi}{2}\right)$ **c** $\sin\left(\frac{7\pi}{2}\right)$ **d** $\cos(3\pi)$

e $\sin(-4\pi)$ **f** $\tan(-\pi)$ **g** $\tan(2\pi)$ **h** $\tan(-2\pi)$

i $\cos(23\pi)$ **j** $\cos\left(\frac{49\pi}{2}\right)$ **k** $\cos(35\pi)$ **l** $\cos\left(\frac{-45\pi}{2}\right)$

m $\tan(24\pi)$ **n** $\cos(20\pi)$

Example 4

2 Evaluate each of the following using a calculator. (Give answers correct to two decimal places.)

a $\sin 1.7$ **b** $\sin 2.6$ **c** $\sin 4.2$ **d** $\cos 0.4$

e $\cos 2.3$ **f** $\cos(-1.8)$ **g** $\sin(-1.7)$ **h** $\sin(-3.6)$

i $\tan 1.6$ **j** $\tan(-1.2)$ **k** $\tan 3.9$ **l** $\tan(-2.5)$

Example 5

3 Write down the exact values of:

a $\sin\left(\frac{3\pi}{4}\right)$ **b** $\cos\left(\frac{2\pi}{3}\right)$ **c** $\cos\left(\frac{7\pi}{6}\right)$ **d** $\sin\left(\frac{5\pi}{6}\right)$

e $\cos\left(\frac{4\pi}{3}\right)$ **f** $\sin\left(\frac{5\pi}{4}\right)$ **g** $\sin\left(\frac{7\pi}{4}\right)$ **h** $\cos\left(\frac{5\pi}{3}\right)$

i $\cos\left(\frac{11\pi}{3}\right)$ **j** $\sin\left(\frac{200\pi}{3}\right)$ **k** $\cos\left(-\frac{11\pi}{3}\right)$ **l** $\sin\left(\frac{25\pi}{3}\right)$

m $\sin\left(-\frac{13\pi}{4}\right)$ **n** $\cos\left(-\frac{20\pi}{3}\right)$ **o** $\sin\left(\frac{67\pi}{4}\right)$ **p** $\cos\left(\frac{68\pi}{3}\right)$

q $\tan\left(\frac{11\pi}{3}\right)$ **r** $\tan\left(\frac{200\pi}{3}\right)$ **s** $\tan\left(-\frac{11\pi}{6}\right)$ **t** $\tan\left(\frac{25\pi}{3}\right)$

u $\tan\left(-\frac{13\pi}{4}\right)$ **v** $\tan\left(-\frac{25\pi}{6}\right)$

Example 6

4 If $\sin\theta = 0.52$, $\cos x = 0.68$ and $\tan\alpha = 0.4$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

5 If $\sin\theta = 0.4$, $\cos x = 0.7$ and $\tan\alpha = 1.2$, find the value of:

a $\sin(\pi - \theta)$ **b** $\cos(\pi + x)$ **c** $\sin(2\pi + \theta)$ **d** $\tan(\pi + \alpha)$

e $\sin(\pi + \theta)$ **f** $\cos(2\pi - x)$ **g** $\tan(2\pi - \alpha)$ **h** $\cos(\pi - x)$

i $\sin(-\theta)$ **j** $\cos(-x)$ **k** $\tan(-\alpha)$

Example 7

6 Without using a calculator, evaluate the sin, cos and tan of each of the following:

a 150° **b** 225° **c** 405° **d** −120° **e** −315° **f** −30°

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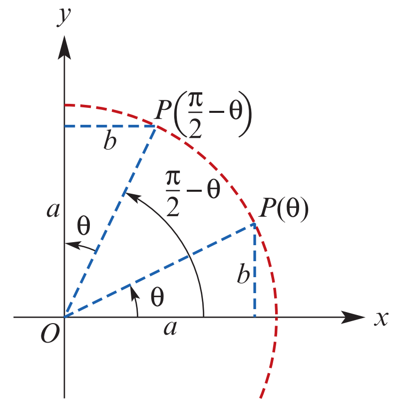
6C Further symmetry properties and the Pythagorean identity

Complementary relationships

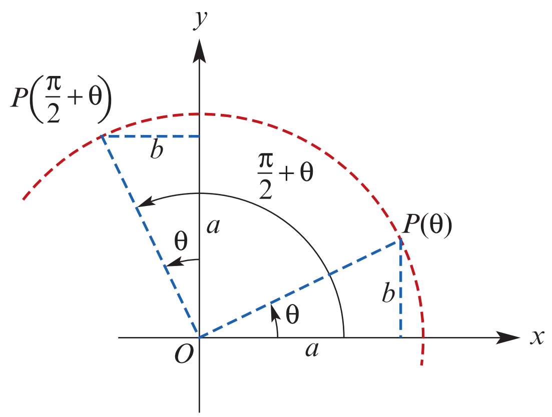
From the diagram to the right:

$$\sin\left(\frac{\pi}{2} - \theta\right) = a = \cos\theta$$
$$\cos\left(\frac{\pi}{2} - \theta\right) = b = \sin\theta$$



From the diagram to the right:

$$\sin\left(\frac{\pi}{2} + \theta\right) = a = \cos\theta$$
$$\cos\left(\frac{\pi}{2} + \theta\right) = -b = -\sin\theta$$



Example 8

If $\sin\theta = 0.3$ and $\cos\psi = 0.8$, find the value of:

a $\sin\left(\frac{\pi}{2} - \psi\right)$

b $\cos\left(\frac{\pi}{2} + \theta\right)$

Solution

a $\sin\left(\frac{\pi}{2} - \psi\right) = \cos\psi = 0.8$

b $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta = -0.3$

The Pythagorean identity

Consider a point, $P(\theta)$, on the unit circle.

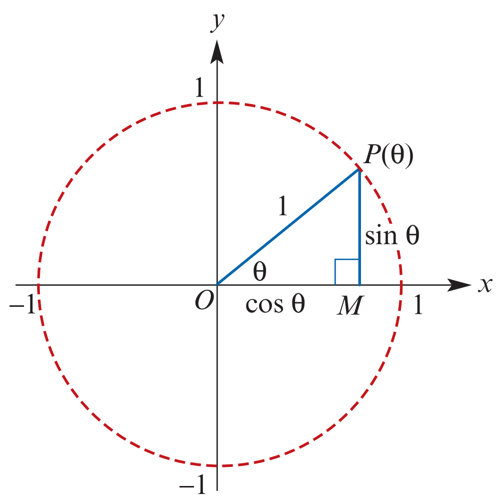
By Pythagoras' theorem,

$$OP^2 = OM^2 + MP^2$$
$$\therefore \quad 1 = (\cos\theta)^2 + (\sin\theta)^2$$

Now $(\cos\theta)^2$ and $(\sin\theta)^2$ may be written as $\cos^2\theta$ and $\sin^2\theta$. Thus we obtain:

$$\cos^2\theta + \sin^2\theta = 1$$

This holds for all values of θ, and is called the **Pythagorean identity**.



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Example 9

Given that $\sin x = \frac{3}{5}$ and $\frac{\pi}{2} < x < \pi$, find:

a $\cos x$

b $\tan x$

Solution

a Substitute $\sin x = \frac{3}{5}$ into the Pythagorean identity:

$$\cos^2 x + \sin^2 x = 1$$
$$\cos^2 x + \frac{9}{25} = 1$$
$$\cos^2 x = 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$

Therefore $\cos x = \pm\frac{4}{5}$. But x is in the 2nd quadrant, and so $\cos x = -\frac{4}{5}$.

b Using part **a**, we have

$$\tan x = \frac{\sin x}{\cos x}$$
$$= \frac{3}{5} \div \left(-\frac{4}{5}\right)$$
$$= \frac{3}{5} \times \left(-\frac{5}{4}\right)$$
$$= -\frac{3}{4}$$

Summary 6C

- **Complementary relationships**

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

Exercise 6C

Example 8

1 If $\sin x = 0.3$, $\cos\alpha = 0.6$ and $\tan\theta = 0.7$, find the value of:

a $\cos(-\alpha)$ **b** $\sin\left(\frac{\pi}{2} + \alpha\right)$ **c** $\tan(-\theta)$

d $\cos\left(\frac{\pi}{2} - x\right)$ **e** $\sin(-x)$ **f** $\tan\left(\frac{\pi}{2} - \theta\right)$

g $\cos\left(\frac{\pi}{2} + x\right)$ **h** $\sin\left(\frac{\pi}{2} - \alpha\right)$ **i** $\sin\left(\frac{3\pi}{2} + \alpha\right)$

j $\cos\left(\frac{3\pi}{2} - x\right)$ **k** $\tan\left(\frac{3\pi}{2} - \theta\right)$ **l** $\cos\left(\frac{5\pi}{2} - x\right)$

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Example 9

2 **a** Given that $\cos x = \frac{3}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

b Given that $\sin x = \frac{5}{13}$ and $\frac{\pi}{2} < x < \pi$, find $\cos x$ and $\tan x$.

c Given that $\cos x = \frac{1}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

d Given that $\sin x = -\frac{12}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

e Given that $\cos x = \frac{4}{5}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

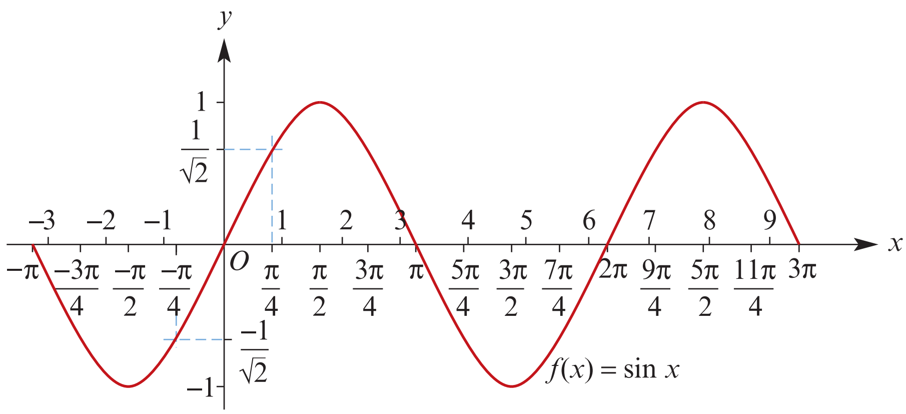
f Given that $\sin x = -\frac{5}{13}$ and $\pi < x < \frac{3\pi}{2}$, find $\cos x$ and $\tan x$.

g Given that $\cos x = \frac{8}{10}$ and $\frac{3\pi}{2} < x < 2\pi$, find $\sin x$ and $\tan x$.

6D Graphs of sine and cosine

Graph of the sine function

A calculator can be used to plot the graph of $f(x) = \sin x$ for $-\pi \le x \le 3\pi$. Note that radian mode must be selected.



Observations from the graph of $y = \sin x$

- The graph repeats itself after an interval of 2π units.
  A function which repeats itself regularly is called a **periodic** function, and the interval between the repetitions is called the **period** of the function (also called the wavelength). Thus $y = \sin x$ has a period of 2π units.
- The maximum and minimum values of $\sin x$ are 1 and -1 respectively.
  The distance between the mean position and the maximum position is called the **amplitude**. The graph of $y = \sin x$ has an amplitude of 1.

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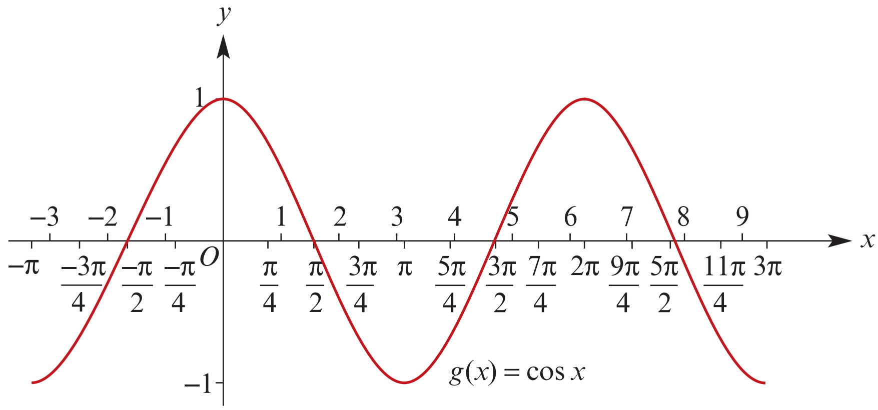
Graph of the cosine function

The graph of $g(x) = \cos x$ is shown below for $-\pi \le x \le 3\pi$.



Observations from the graph of $y = \cos x$

- The period is 2π.
- The amplitude is 1.
- The graph of $y = \cos x$ is the graph of $y = \sin x$ translated $\frac{\pi}{2}$ units in the negative direction of the x-axis.

Sketch graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

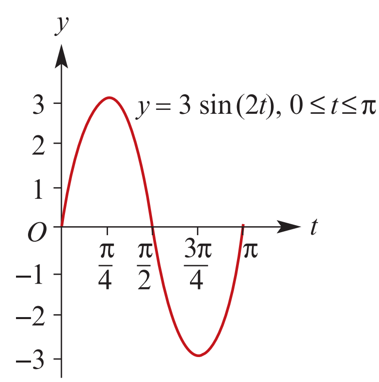
The graphs of functions of the forms $y = a\sin(nt)$ and $y = a\cos(nt)$ are transformations of the graphs of $y = \sin t$ and $y = \cos t$ respectively. We first consider the case where a and n are positive numbers.

Transformations: dilations

Graph of $y = 3\sin(2t)$ The image of the graph of $y = \sin t$ under a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{2}$ from the y-axis is $y = 3\sin(2t)$.

Note: Let $f(t) = \sin t$. Then the graph of $y = f(t)$ is transformed to the graph of $y = 3f(2t)$. The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{2}, 3y\right)$.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = 3\sin(2t)$	0	3	0	−3	0



We make the following observations about the graph of $y = 3\sin(2t)$:

- amplitude is 3
- period is π

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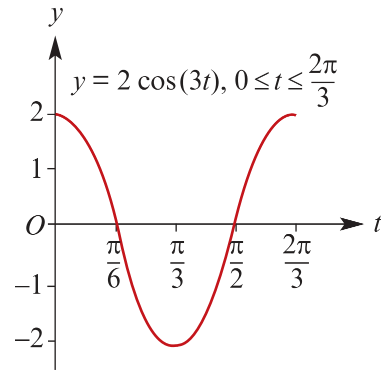
Graph of $y = 2\cos(3t)$ The image of the graph of $y = \cos t$ under a dilation of factor 2 from the t-axis and a dilation of factor $\frac{1}{3}$ from the y-axis is $y = 2\cos(3t)$.

t	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$
$y = 2\cos(3t)$	2	0	-2	0	2



We make the following observations about the graph of $y = 2\cos(3t)$:

- amplitude is 2
- period is $\frac{2\pi}{3}$

Amplitude and period Comparing these results with those for $y = \sin t$ and $y = \cos t$, the following general rules can be stated for a and n positive:

Function	Amplitude	Period
$y = a\sin(nt)$	a	$\frac{2\pi}{n}$
$y = a\cos(nt)$	a	$\frac{2\pi}{n}$

Example 10

For each of the following functions with domain $\mathbb{R}$, state the amplitude and period:

a $f(t) = 2\sin(3t)$ **b** $f(t) = -\frac{1}{2}\sin\left(\frac{t}{2}\right)$ **c** $f(t) = 4\cos(3\pi t)$

Solution

a Amplitude is 2

Period is $\frac{2\pi}{3}$

b Amplitude is $\frac{1}{2}$

Period is $2\pi \div \frac{1}{2} = 4\pi$

c Amplitude is 4

Period is $\frac{2\pi}{3\pi} = \frac{2}{3}$

Graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$

For a and n positive numbers, the graphs of $y = a\sin(nt)$ and $y = a\cos(nt)$ are obtained from the graphs of $y = \sin t$ and $y = \cos t$, respectively, by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The point with coordinates (t, y) is mapped to the point with coordinates $\left(\frac{t}{n}, ay\right)$.

The following are important properties of both of the functions $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Example 11

For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 3\sin(2x)$ **b** $g(x) = 4\sin\left(\frac{x}{2}\right)$

Solution

a The graph of $y = 3\sin(2x)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 3 from the x-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

The function $g(x) = 3\sin(2x)$ has amplitude 3 and period $\frac{2\pi}{2} = \pi$.

b The graph of $y = 4\sin\left(\frac{x}{2}\right)$ is obtained from the graph of $y = \sin x$ by a dilation of factor 4 from the x-axis and a dilation of factor 2 from the y-axis.

The function $g(x) = 4\sin\left(\frac{x}{2}\right)$ has amplitude 4 and period $2\pi \div \frac{1}{2} = 4\pi$.

Example 12

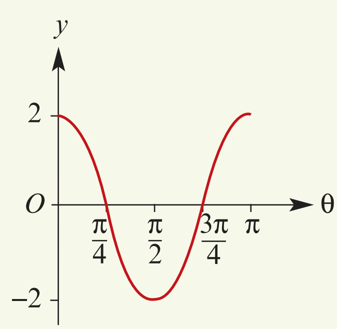
Sketch the graph of each of the following functions:

a $y = 2\cos(2\theta)$ **b** $y = \frac{1}{2}\sin\left(\frac{x}{2}\right)$

In each case, show one complete cycle.

Solution

a



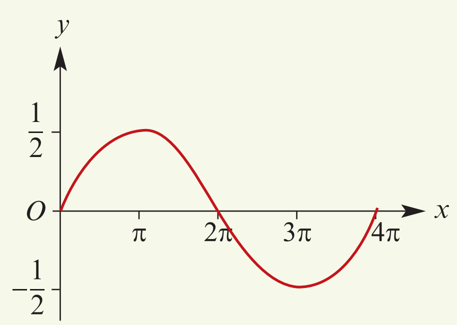
Explanation

The amplitude is 2.

The period is $\frac{2\pi}{2} = \pi$.

The graph of $y = 2\cos(2\theta)$ is obtained from the graph of $y = \cos\theta$ by a dilation of factor 2 from the θ-axis and a dilation of factor $\frac{1}{2}$ from the y-axis.

b



The amplitude is $\frac{1}{2}$.

The period is $2\pi \div \frac{1}{2} = 4\pi$.

The graph of $y = \frac{1}{2}\sin(\frac{x}{2})$ is obtained from the graph of $y = \sin x$ by a dilation of factor $\frac{1}{2}$ from the x-axis and a dilation of factor 2 from the y-axis.

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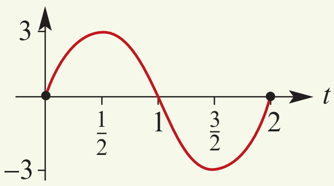
Example 13

Sketch the graph of $f\colon [0, 2] \to \mathbb{R}$, $f(t) = 3\sin(\pi t)$.

Solution



Explanation

The amplitude is 3.

The period is $2\pi \div \pi = 2$.

The graph of $f(t) = 3\sin(\pi t)$ is obtained from the graph of $y = \sin t$ by a dilation of factor 3 from the t-axis and a dilation of factor $\frac{1}{\pi}$ from the y-axis.

Transformations: reflections

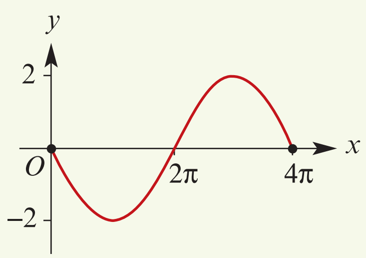
Example 14

Sketch the following graphs for $x \in [0, 4\pi]$:

a $f(x) = -2\sin\left(\frac{x}{2}\right)$ **b** $y = -\cos(2x)$

Solution

a

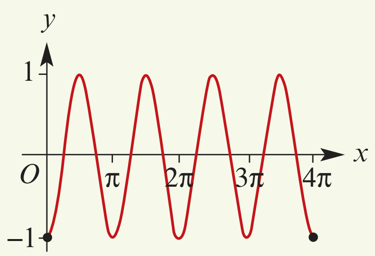


Explanation

The graph of $f(x) = -2\sin(\frac{x}{2})$ is obtained from the graph of $y = 2\sin(\frac{x}{2})$ by a reflection in the x-axis.

The amplitude is 2 and the period is 4π.

b



The graph of $y = -\cos(2x)$ is obtained from the graph of $y = \cos(2x)$ by a reflection in the x-axis.

The amplitude is 1 and the period is π.

Note: Recall that sin is an odd function and cos is an even function (i.e. $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$). When reflected in the y-axis, the graph of $y = \sin x$ transforms onto the graph of $y = -\sin x$, and the graph of $y = \cos x$ transforms onto itself.

Summary 6D

For positive numbers a and n, the graphs of $y = a\sin(nt)$, $y = -a\sin(nt)$, $y = a\cos(nt)$ and $y = -a\cos(nt)$ all have the following properties:

- The period is $\frac{2\pi}{n}$.
- The amplitude is a.
- The maximal domain is $\mathbb{R}$.
- The range is $[-a, a]$.

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Exercise 6D

Example 10

1 Write down **i** the period and **ii** the amplitude of each of the following:

a $3\sin\theta$ **b** $5\sin(3\theta)$ **c** $\frac{1}{2}\cos(2\theta)$ **d** $2\sin\left(\frac{1}{3}\theta\right)$

e $3\cos(4\theta)$ **f** $\frac{1}{2}\sin\theta$ **g** $3\cos\left(\frac{1}{2}\theta\right)$ **h** $2\sin\left(\frac{2\theta}{3}\right)$

Example 11

2 For each of the following, give a sequence of transformations which takes the graph of $y = \sin x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 4\sin(3x)$ **b** $g(x) = 5\sin\left(\frac{x}{3}\right)$

c $g(x) = 6\sin\left(\frac{x}{2}\right)$ **d** $g(x) = 4\sin(5x)$

3 For each of the following, give a sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = g(x)$, and state the amplitude and period of $g(x)$:

a $g(x) = 2\cos(3x)$ **b** $g(x) = 3\cos\left(\frac{x}{4}\right)$

c $g(x) = 6\cos\left(\frac{x}{5}\right)$ **d** $g(x) = 3\cos(7x)$

Example 12

4 Sketch the graph of each of the following, showing one complete cycle. State the amplitude and period.

a $y = 2\sin(3\theta)$ **b** $y = 2\cos(2\theta)$ **c** $y = 3\sin\left(\frac{1}{3}\theta\right)$

d $y = \frac{1}{3}\cos(2\theta)$ **e** $y = 3\sin(4\theta)$ **f** $y = 4\cos\left(\frac{1}{4}\theta\right)$

Example 13

5 Sketch the graph of $f\colon [0, 1] \to \mathbb{R}$, $f(t) = 3\sin(2\pi t)$.

6 Sketch the graph of $f\colon [0, 1] \to \mathbb{R}$, $f(t) = 3\sin\left(\frac{\pi t}{2}\right)$.

7 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5\cos(3x)$ for $0 \le x \le \pi$.

8 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{2}\sin(2x)$ for $-\pi \le x \le 2\pi$.

9 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2\cos\left(\frac{3x}{2}\right)$ for $0 \le x \le 2\pi$.

Example 14

10 Sketch the graph of $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = -3\cos\left(\frac{x}{2}\right)$ for $0 \le x \le 4\pi$.

11 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor 2 from the x-axis followed by a dilation of factor 3 from the y-axis.

12 Find the equation of the image of the graph of $y = \cos x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 3 from the y-axis.

13 Find the equation of the image of the graph of $y = \sin x$ under a dilation of factor $\frac{1}{2}$ from the x-axis followed by a dilation of factor 2 from the y-axis.

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6E Solution of trigonometric equations

In this section we revise methods for solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$.

Solving equations of the form $\sin t = b$ and $\cos t = b$

First we look at the techniques for solving equations of the form $\sin t = b$ and $\cos t = b$. These same techniques will be applied to solve more complicated trigonometric equations later in this section.

Example 15

Find all solutions of the equation $\sin\theta = \frac{1}{2}$ for $\theta \in [0, 4\pi]$.

Solution

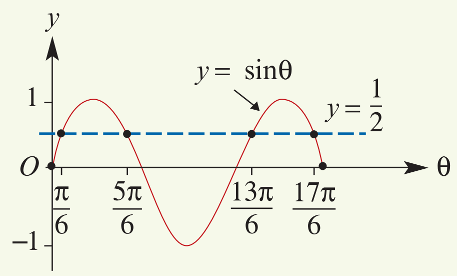
The solution for $\theta \in \left[0, \frac{\pi}{2}\right]$ is $\theta = \frac{\pi}{6}$.

The second solution is $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$.

The third solution is $\theta = 2\pi + \frac{\pi}{6} = \frac{13\pi}{6}$.

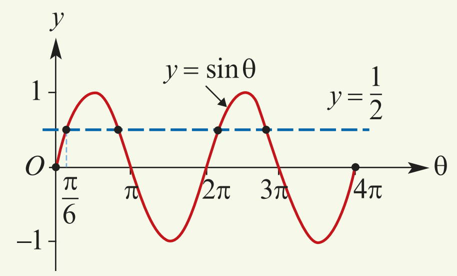
The fourth solution is $\theta = 2\pi + \frac{5\pi}{6} = \frac{17\pi}{6}$.

These four solutions are shown on the graph below.



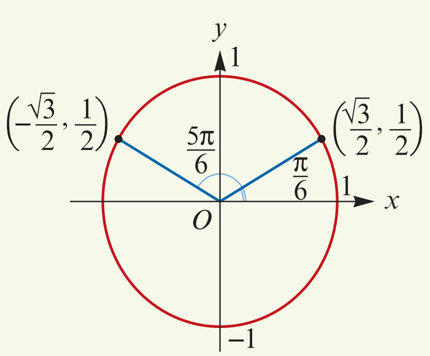
Explanation

By sketching a graph, we can see that there are four solutions in the interval $[0, 4\pi]$.



The first solution can be obtained from a knowledge of exact values or by using $\sin^{-1}$ on your calculator.

The second solution is obtained using symmetry. The sine function is positive in the 2nd quadrant and $\sin(\pi - \theta) = \sin\theta$.



Further solutions are found by adding 2π, since $\sin\theta = \sin(2\pi + \theta)$.

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Example 16

Find two values of x:

a $\sin x = -0.3$ with $0 \le x \le 2\pi$

b $\cos x^\circ = -0.7$ with $0^\circ \le x^\circ \le 360^\circ$

Solution

a Consider $\sin \alpha = 0.3$ with $\alpha \in \left[0, \frac{\pi}{2}\right]$.
The solution is $\alpha = 0.30469\ldots$

The value of $\sin x$ is negative for $P(x)$ in the 3rd and 4th quadrants.

3rd quadrant:

$$\begin{aligned} x &= \pi + 0.30469\ldots \\ &= 3.446 \qquad \text{(to 3 d.p.)} \end{aligned}$$

4th quadrant:

$$\begin{aligned} x &= 2\pi - 0.30469\ldots \\ &= 5.978 \qquad \text{(to 3 d.p.)} \end{aligned}$$

The solutions of $\sin x = -0.3$ in $[0, 2\pi]$ are $x = 3.446$ and $x = 5.978$.

b Consider $\cos \alpha^\circ = 0.7$ with $\alpha^\circ \in [0^\circ, 90^\circ]$.
The solution is $\alpha^\circ = 45.57^\circ$.

The value of $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants.

2nd quadrant:

$$\begin{aligned} x^\circ &= 180^\circ - 45.57^\circ \\ &= 134.43^\circ \end{aligned}$$

3rd quadrant:

$$\begin{aligned} x^\circ &= 180^\circ + 45.57^\circ \\ &= 225.57^\circ \end{aligned}$$

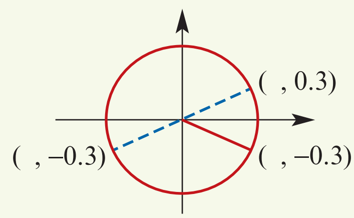
The solutions of $\cos x^\circ = -0.7$ in $[0^\circ, 360^\circ]$ are $x^\circ = 134.43^\circ$ and $x^\circ = 225.57^\circ$.

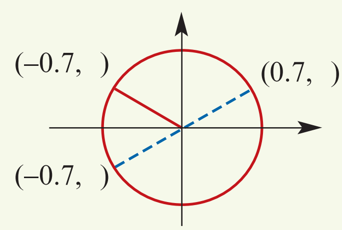
Explanation

First consider the corresponding equation for the 1st quadrant. Use your calculator to find the solution for α.

Decide which quadrants will contain a solution for x.

Find the solutions using the symmetry relationships (or the graph of $y = \sin x$).





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Example 17

Find all the values of $\theta°$ between $0°$ and $360°$ for which:

a $\cos\theta° = \dfrac{\sqrt{3}}{2}$ **b** $\sin\theta° = -\dfrac{1}{2}$ **c** $\cos\theta° - \dfrac{1}{\sqrt{2}} = 0$

Solution

a $\cos\theta° = \dfrac{\sqrt{3}}{2}$

$\theta° = 30°$ or $\theta° = 360° - 30°$

$\theta° = 30°$ or $\theta° = 330°$

Explanation

$\cos\theta°$ is positive, and so $P(\theta°)$ lies in the 1st or 4th quadrant.

$\cos(360° - \theta°) = \cos\theta°$

Solution

b $\sin\theta° = -\dfrac{1}{2}$

$\theta° = 180° + 30°$ or $\theta° = 360° - 30°$

$\theta° = 210°$ or $\theta° = 330°$

Explanation

$\sin\theta°$ is negative, and so $P(\theta°)$ lies in the 3rd or 4th quadrant.

$\sin(180° + \theta°) = -\sin\theta°$

$\sin(360° - \theta°) = -\sin\theta°$

Solution

c $\cos\theta° - \dfrac{1}{\sqrt{2}} = 0$

$\therefore \quad \cos\theta° = \dfrac{1}{\sqrt{2}}$

$\theta° = 45°$ or $\theta° = 360° - 45°$

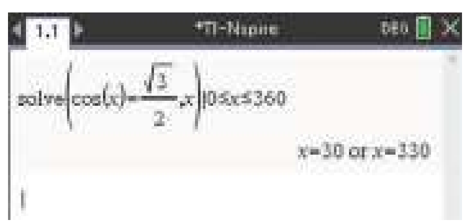
$\theta° = 45°$ or $\theta° = 315°$

Explanation

$\cos\theta°$ is positive, and so $P(\theta°)$ lies in the 1st or 4th quadrant.

Using the TI-Nspire

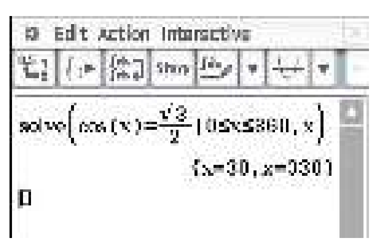
For Example 17a, make sure the calculator is in degree mode and complete as shown.



Using the Casio ClassPad

- Ensure your calculator is in degree mode.
- Use the Math1 and Math3 keyboards to enter

  $\cos(x) = \dfrac{\sqrt{3}}{2} \mid 0 \le x \le 360$
- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



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Solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$

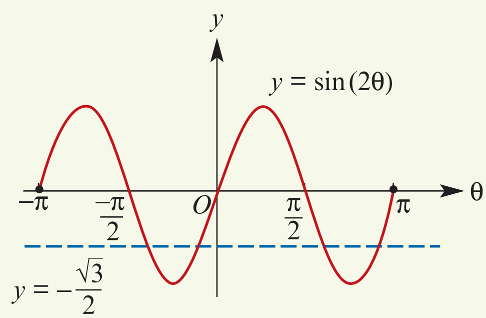
The techniques introduced above can be applied in a more general situation. This is achieved by a simple substitution, as shown in the following example.

Example 18

Solve the equation $\sin(2\theta) = -\dfrac{\sqrt{3}}{2}$ for $\theta \in [-\pi, \pi]$.

Solution

It is clear from the graph that there are four solutions.



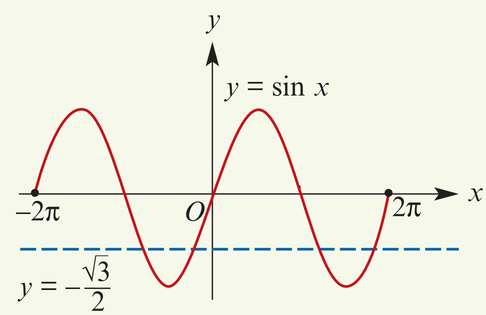
To solve the equation, let $x = 2\theta$.

Note: If $\theta \in [-\pi, \pi]$, then we have $x = 2\theta \in [-2\pi, 2\pi]$.

Now consider the equation

$$\sin x = -\frac{\sqrt{3}}{2} \quad \text{for } x \in [-2\pi, 2\pi]$$

The 1st quadrant solution to the equation $\sin\alpha = \dfrac{\sqrt{3}}{2}$ is $\alpha = \dfrac{\pi}{3}$.



Using symmetry, the solutions to $\sin x = -\dfrac{\sqrt{3}}{2}$ for $x \in [0, 2\pi]$ are

$x = \pi + \dfrac{\pi}{3}$ and $x = 2\pi - \dfrac{\pi}{3}$

i.e. $x = \dfrac{4\pi}{3}$ and $x = \dfrac{5\pi}{3}$

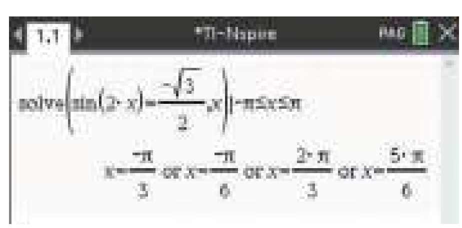
The other two solutions (obtained by subtracting 2π) are $x = \dfrac{4\pi}{3} - 2\pi$ and $x = \dfrac{5\pi}{3} - 2\pi$.

$\therefore$ The required solutions for x are $-\dfrac{2\pi}{3}, -\dfrac{\pi}{3}, \dfrac{4\pi}{3}$ and $\dfrac{5\pi}{3}$.

$\therefore$ The required solutions for θ are $-\dfrac{\pi}{3}, -\dfrac{\pi}{6}, \dfrac{2\pi}{3}$ and $\dfrac{5\pi}{6}$.

Using the TI-Nspire

Ensure that the calculator is in radian mode and complete as shown.



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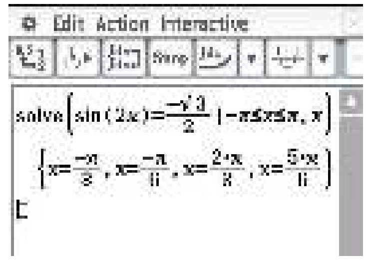
Using the Casio ClassPad

- Ensure your calculator is in radian mode.
- Use the Math1 and Math3 keyboards to enter

  $\sin(2x) = \dfrac{-\sqrt{3}}{2} \;\Big|\; -\pi \le x \le \pi$
- Highlight the equation and domain. Then select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to x.



Summary 6E

- For solving equations of the form $\sin t = b$ and $\cos t = b$:
  - First find the solutions in the interval $[0, 2\pi]$. This can be done using your knowledge of exact values and symmetry properties, or with the aid of a calculator.
  - Further solutions can be found by adding and subtracting multiples of 2π.
- For solving equations of the form $a\sin(nt) = b$ and $a\cos(nt) = b$:
  - First substitute $x = nt$. Work out the interval in which solutions for x are required. Then proceed as in the case above to solve for x.
  - Once the solutions for x are found, the solutions for t can be found.

  For example: To solve $\sin(3t) = \frac{1}{2}$ for $t \in [0, 2\pi]$, first let $x = 3t$. The equation becomes $\sin x = \frac{1}{2}$ and the required solutions for x are in the interval $[0, 6\pi]$.

Skill-sheet

Exercise 6E

1 Solve each of the following for $x \in [0, 4\pi]$:

Example 15

a $\sin x = \dfrac{1}{\sqrt{2}}$ **b** $\cos x = \dfrac{\sqrt{3}}{2}$ **c** $\sin x = \dfrac{-\sqrt{3}}{2}$

d $\cos x = \dfrac{1}{\sqrt{2}}$ **e** $\sin x = 1$ **f** $\cos x = -1$

2 Solve each of the following for $x \in [-\pi, \pi]$:

a $\sin x = -\dfrac{1}{2}$ **b** $\cos x = \dfrac{\sqrt{3}}{2}$ **c** $\cos x = \dfrac{-\sqrt{3}}{2}$

3 Solve each of the following for $x \in [0, 2\pi]$:

a $\sqrt{2}\sin x - 1 = 0$ **b** $\sqrt{2}\cos x + 1 = 0$ **c** $2\cos x + \sqrt{3} = 0$

d $2\sin x + 1 = 0$ **e** $1 - \sqrt{2}\cos x = 0$ **f** $4\cos x + 2 = 0$

Example 16

4 Find all values of x between 0 and 2π for which:

a $\sin x = 0.6$ **b** $\cos x = 0.8$ **c** $\sin x = -0.45$ **d** $\cos x = -0.2$

5 Find all values of $\theta°$ between 0° and 360° for which:

a $\sin\theta° = 0.3$ **b** $\cos\theta° = 0.4$ **c** $\sin\theta° = -0.8$ **d** $\cos\theta° = -0.5$

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Example 17

6 Without using a calculator, find all the values of θ between 0 and 360 for each of the following:

a $\cos\theta° = \frac{1}{2}$ **b** $\sin\theta° = \frac{\sqrt{3}}{2}$ **c** $\sin\theta° = -\frac{1}{\sqrt{2}}$

d $2\cos\theta° + 1 = 0$ **e** $2\sin\theta° = \sqrt{3}$ **f** $2\cos\theta° = -\sqrt{3}$

Example 18

7 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -\frac{1}{2}$ **b** $\cos(2\theta) = \frac{\sqrt{3}}{2}$ **c** $\sin(2\theta) = \frac{1}{2}$

d $\sin(3\theta) = -\frac{1}{\sqrt{2}}$ **e** $\cos(2\theta) = -\frac{\sqrt{3}}{2}$ **f** $\sin(2\theta) = -\frac{1}{\sqrt{2}}$

8 Find, without using a calculator, all the values of x between 0 and 2π for each of the following:

a $2\cos(3x) + \sqrt{3} = 0$ **b** $4\sin(2x) - 2 = 0$ **c** $\sqrt{2}\cos(3x) - 1 = 0$

d $10\sin(3x) - 5 = 0$ **e** $2\sin(2x) = \sqrt{2}$ **f** $4\cos(3x) = -2\sqrt{3}$

g $-2\sin(3x) = \sqrt{2}$ **h** $4\cos(2x) = -2$ **i** $-2\cos(2x) = \sqrt{2}$

9 Solve the following equations for $\theta \in [0, 2\pi]$:

a $\sin(2\theta) = -0.8$ **b** $\sin(2\theta) = -0.6$ **c** $\cos(2\theta) = 0.4$ **d** $\cos(3\theta) = 0.6$

6F Sketch graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$

In this section, we consider translations of graphs of functions of the form $f(t) = a\sin(nt)$ and $g(t) = a\cos(nt)$ in the direction of the t-axis.

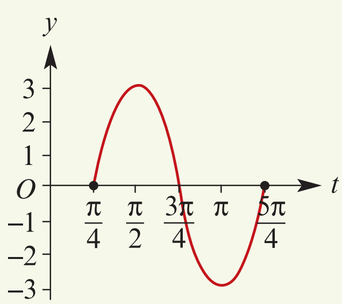
Example 19

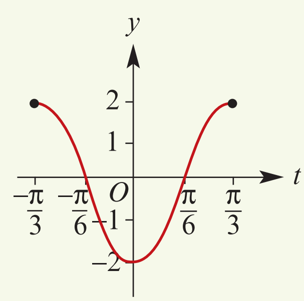
On separate axes, draw the graphs of the following functions. Use a calculator to help establish the shape. Set the window appropriately by noting the range and period.

a $y = 3\sin 2\left(t - \frac{\pi}{4}\right), \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right), \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a The range is $[-3, 3]$ and the period is π. **b** The range is $[-2, 2]$ and the period is $\frac{2\pi}{3}$.





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Observations from the example

a The graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$ is the same shape as $y = 3\sin(2t)$, but is translated $\frac{\pi}{4}$ units in the positive direction of the t-axis.

b The graph of $y = 2\cos 3\left(t + \frac{\pi}{3}\right)$ is the same shape as $y = 2\cos(3t)$, but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis.

The effect of $\pm\varepsilon$ is to translate the graph parallel to the t-axis. (Here $\pm\varepsilon$ is called the phase.)

Note: The techniques of Chapter 3 can be used to find the sequence of transformations. The graph of $y = \sin t$ is transformed to the graph of $y = 3\sin 2\left(t - \frac{\pi}{4}\right)$.

- Rearrange the second equation as $\frac{y'}{3} = \sin 2\left(t' - \frac{\pi}{4}\right)$.
- We can choose to write $y = \frac{y'}{3}$ and $t = 2\left(t' - \frac{\pi}{4}\right)$. Hence $y' = 3y$ and $t' = \frac{t}{2} + \frac{\pi}{4}$.
- The transformation is a dilation of factor 3 from the t-axis, followed by a dilation of factor $\frac{1}{2}$ from the y-axis, and then by a translation of $\frac{\pi}{4}$ units in the positive direction of the t-axis.

Example 20

For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

Solution

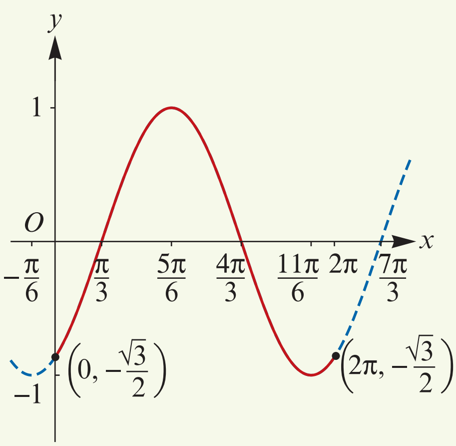
a $f(0) = \sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$ and $f(2\pi) = \sin\left(2\pi - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

b The graph of $y = \sin\left(x - \frac{\pi}{3}\right)$ is the graph of $y = \sin x$ translated $\frac{\pi}{3}$ units to the right.

The period of f is 2π and the amplitude is 1.

The endpoints are $\left(0, -\frac{\sqrt{3}}{2}\right)$ and $\left(2\pi, -\frac{\sqrt{3}}{2}\right)$.

There are many ways to proceed from here.



Method 1

- Start at $x = \frac{\pi}{3}$ and start to draw one cycle stopping at $x = 2\pi$.
- Each 'loop' of the graph is of length π and each 'half loop' is of length $\frac{\pi}{2}$.
- The 'half loop' going back from $x = \frac{\pi}{3}$ would end at $\frac{\pi}{3} - \frac{\pi}{2} = -\frac{\pi}{6}$.

Method 2

Find the x-axis intercepts by solving the equation $\sin\left(x - \frac{\pi}{3}\right) = 0$. Use symmetry to find the coordinates of the maximum and minimum points.

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Summary 6F

The graphs of $y = a \sin n(t \pm \varepsilon)$ and $y = a \cos n(t \pm \varepsilon)$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase.

Exercise 6F

Example 19

1 Sketch the graph of each of the following, showing one complete cycle. State the period and amplitude, and the greatest and least values of y.

a $y = 3\sin\left(\theta - \frac{\pi}{2}\right)$ **b** $y = \sin 2(\theta + \pi)$ **c** $y = 2\sin 3\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 2\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 3\sin(2x - \pi)$ **f** $y = 2\cos 3\left(\theta + \frac{\pi}{4}\right)$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{3}\right)$ **h** $y = -3\sin\left(2x + \frac{\pi}{3}\right)$ **i** $y = -3\cos 2\left(\theta + \frac{\pi}{2}\right)$

Example 20

2 For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \cos\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

3 For the function $f\colon [0, 2\pi] \to \mathbb{R}$, $f(x) = \sin 2\left(x - \frac{\pi}{3}\right)$:

a find $f(0)$ and $f(2\pi)$

b sketch the graph of f.

4 For the function $f\colon [-\pi, \pi] \to \mathbb{R}$, $f(x) = \sin 3\left(x + \frac{\pi}{4}\right)$:

a find $f(-\pi)$ and $f(\pi)$

b sketch the graph of f.

5 Find the equation of the image of $y = \sin x$ for each of the following transformations:

a dilation of factor 2 from the y-axis followed by dilation of factor 3 from the x-axis

b dilation of factor $\frac{1}{2}$ from the y-axis followed by dilation of factor 3 from the x-axis

c dilation of factor 3 from the y-axis followed by dilation of factor 2 from the x-axis

d dilation of factor $\frac{1}{2}$ from the y-axis followed by translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis

e dilation of factor 2 from the y-axis followed by translation of $\frac{\pi}{3}$ units in the negative direction of the x-axis.

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6G Sketch graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$

In general, the effect of $\pm b$ is to translate the graph $\pm b$ units parallel to the y-axis.

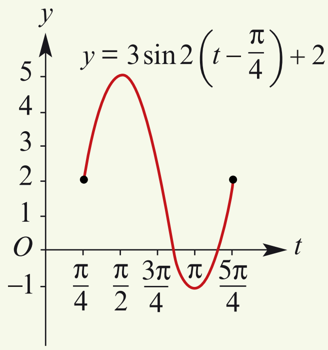
Example 21

Sketch each of the following graphs. Use a calculator to help establish the shape.

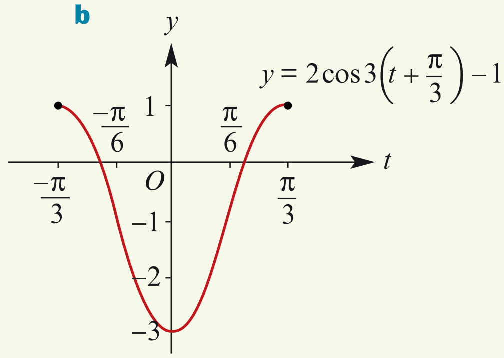
a $y = 3\sin 2\left(t - \frac{\pi}{4}\right) + 2, \quad \frac{\pi}{4} \le t \le \frac{5\pi}{4}$ **b** $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1, \quad -\frac{\pi}{3} \le t \le \frac{\pi}{3}$

Solution

a



b



Finding axis intercepts

Example 22

Sketch the graph of each of the following for $x \in [0, 2\pi]$. Clearly indicate axis intercepts.

a $y = \sqrt{2}\sin(x) + 1$ **b** $y = 2\cos(2x) - 1$ **c** $y = 2\sin 2\left(x - \frac{\pi}{3}\right) - \sqrt{3}$

Solution

a To determine the x-axis intercepts, solve the equation $\sqrt{2}\sin(x) + 1 = 0$.

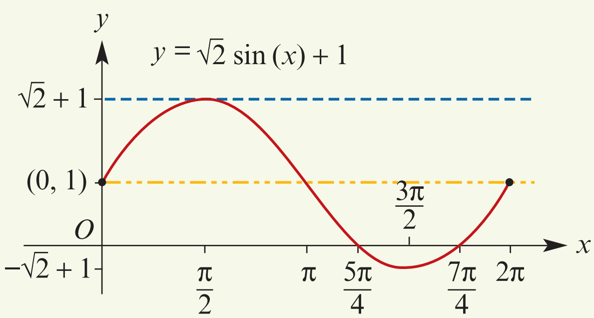
$$\sqrt{2}\sin(x) + 1 = 0$$

$$\therefore \quad \sin x = -\frac{1}{\sqrt{2}}$$

$$\therefore x = \pi + \frac{\pi}{4} \text{ or } 2\pi - \frac{\pi}{4}$$

$$\therefore x = \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The x-axis intercepts are $\frac{5\pi}{4}$ and $\frac{7\pi}{4}$.



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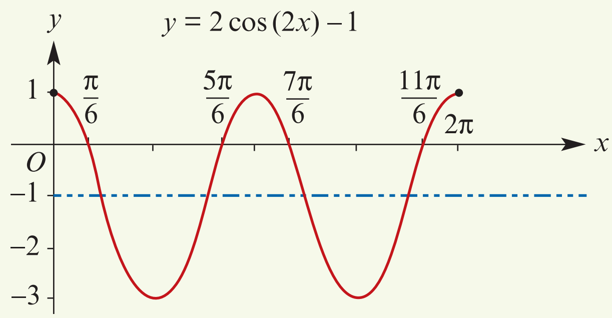
b $2\cos(2x) - 1 = 0$

$\therefore \quad \cos(2x) = \dfrac{1}{2}$

$\therefore 2x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$ or $\dfrac{11\pi}{3}$

$\therefore \quad x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$

The x-axis intercepts are $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$.



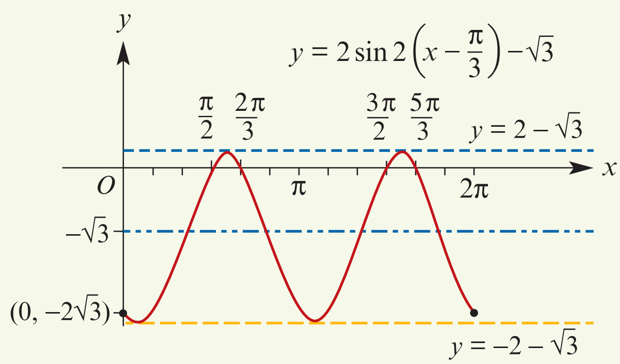
c $\sin 2\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

$\therefore 2\left(x - \dfrac{\pi}{3}\right) = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{7\pi}{3}$ or $\dfrac{8\pi}{3}$

$\therefore \quad x - \dfrac{\pi}{3} = \dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{7\pi}{6}$ or $\dfrac{4\pi}{3}$

$\therefore \quad x = \dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{2}$ or $\dfrac{5\pi}{3}$

The x-axis intercepts are $\dfrac{\pi}{2}, \dfrac{2\pi}{3}, \dfrac{3\pi}{2}$ and $\dfrac{5\pi}{3}$.



Summary 6G

The graphs of $y = a \sin n(t \pm \varepsilon) \pm b$ and $y = a \cos n(t \pm \varepsilon) \pm b$ are translations of the graphs of $y = a \sin(nt)$ and $y = a \cos(nt)$ respectively.

The graphs are translated $\mp\varepsilon$ units parallel to the t-axis, where $\pm\varepsilon$ is called the phase. They are also translated $\pm b$ units parallel to the y-axis.

Skill-sheet

Exercise 6G

1 Sketch each of the following graphs. Use a calculator to help establish the shape. Label the endpoints with their coordinates.

Example 21

a $y = 2\sin 2\left(t - \dfrac{\pi}{3}\right) + 2, \quad \dfrac{\pi}{3} \le t \le \dfrac{4\pi}{3}$ **b** $y = 2\cos 3\left(t + \dfrac{\pi}{4}\right) - 1, \quad -\dfrac{\pi}{4} \le t \le \dfrac{5\pi}{12}$

Example 22

2 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\sin(x) + 1$ **b** $y = 2\sin(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos(x) + 1$

d $y = 2\sin(2x) - 2$ **e** $y = \sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right) + 1$

3 Sketch the graph of each of the following for $x \in [-\pi, 2\pi]$:

a $y = 2\sin(3x) - 2$ **b** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$ **c** $y = 2\sin(2x) - 3$

d $y = 2\cos(2x) + 1$ **e** $y = 2\cos 2\left(x - \frac{\pi}{3}\right) - 1$ **f** $y = 2\sin 2\left(x + \frac{\pi}{6}\right) + 1$

4 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin 2\left(x + \frac{\pi}{3}\right) + 1$ **b** $y = -2\sin 2\left(x + \frac{\pi}{6}\right) + 1$ **c** $y = 2\cos 2\left(x + \frac{\pi}{4}\right) + \sqrt{3}$

5 Sketch the graph of each of the following, showing one complete cycle. State the period, amplitude and range in each case.

a $y = 2\sin\left(\theta - \frac{\pi}{3}\right)$ **b** $y = \sin 2(\theta - \pi)$ **c** $y = 3\sin 2\left(\theta + \frac{\pi}{4}\right)$

d $y = \sqrt{3}\sin 3\left(\theta - \frac{\pi}{2}\right)$ **e** $y = 2\sin(3x) + 1$ **f** $y = 3\cos 2\left(x + \frac{\pi}{2}\right) - 1$

g $y = \sqrt{2}\sin 2\left(\theta - \frac{\pi}{6}\right) + 2$ **h** $y = 3 - 4\sin(2x)$ **i** $y = 2 - 3\cos 2\left(\theta - \frac{\pi}{2}\right)$

6 Find the equation of the image of the graph of $y = \cos x$ under:

a a dilation of factor $\frac{1}{2}$ from the x-axis, followed by a dilation of factor 3 from the y-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

b a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis

c a dilation of factor $\frac{1}{3}$ from the x-axis, followed by a reflection in the x-axis, then followed by a translation of $\frac{\pi}{3}$ units in the positive direction of the x-axis.

7 Give a sequence of transformations that takes the graph of $y = \sin x$ to the graph of:

a $y = -3\sin(2x)$ **b** $y = -3\sin 2\left(x - \frac{\pi}{3}\right)$

c $y = 3\sin 2\left(x - \frac{\pi}{3}\right) + 2$ **d** $y = 5 - 2\sin 2\left(x - \frac{\pi}{3}\right)$

8 Sketch the graph of each of the following for $x \in [0, 2\pi]$. List the x-axis intercepts of each graph for this interval.

a $y = 2\cos x + 1$ **b** $y = 2\cos(2x) - \sqrt{3}$ **c** $y = \sqrt{2}\cos x - 1$

d $y = 2\cos x - 2$ **e** $y = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) + 1$

9 Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

a $y = 2\sin\left(x - \frac{\pi}{4}\right) + 1$ **b** $y = 1 - 2\sin x$ **c** $y = 2\cos 3\left(x - \frac{\pi}{4}\right)$

d $y = 2\cos\left(3x - \frac{\pi}{4}\right)$ **e** $y = 1 - \cos(2x)$ **f** $y = -1 - \sin x$

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6H Addition of ordinates for circular functions

Sums of trigonometric functions play an important role in mathematics and have many applications, such as audio compression. We recall the following from Chapter 1:

Key points to consider when sketching $y = (f + g)(x)$

- When $f(x) = 0$, $(f + g)(x) = g(x)$.
- When $g(x) = 0$, $(f + g)(x) = f(x)$.
- If $f(x)$ and $g(x)$ are positive, then $(f + g)(x) > g(x)$ and $(f + g)(x) > f(x)$.
- If $f(x)$ and $g(x)$ are negative, then $(f + g)(x) < g(x)$ and $(f + g)(x) < f(x)$.
- If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f + g)(x) < f(x)$.
- Look for values of x for which $f(x) + g(x) = 0$.

Example 23

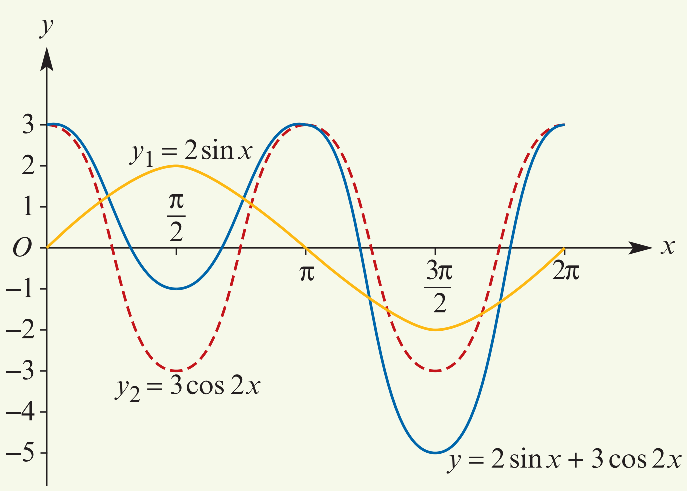
Using the same scale and axes, sketch the graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$ for $0 \le x \le 2\pi$. Use addition of ordinates to sketch the graph of $y = 2\sin x + 3\cos(2x)$.

Solution

The graphs of $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$ are shown. To obtain points on the graph of $y = 2\sin x + 3\cos(2x)$, the process of addition of ordinates is used.

Let $y = y_1 + y_2$ where $y_1 = 2\sin x$ and $y_2 = 3\cos(2x)$. A table of values is shown on the right.

x	y_1	y_2	y
0	0	3	3
$\frac{\pi}{4}$	$\sqrt{2}$	0	$\sqrt{2}$
$\frac{\pi}{2}$	2	−3	−1
π	0	3	3
$\frac{3\pi}{2}$	−2	−3	−5



Exercise 6H

Example 23

1 Use addition of ordinates to sketch the graph of each of the following for $\theta \in [-\pi, \pi]$:

a $y = \sin\theta + 2\cos\theta$ **b** $y = 2\cos(2\theta) + 3\sin(2\theta)$ **c** $y = \frac{1}{2}\cos(2\theta) - \sin\theta$

d $y = 3\cos\theta + \sin(2\theta)$ **e** $y = 2\sin\theta - 4\cos\theta$

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6I Determining rules for graphs of circular functions

In previous chapters, we introduced procedures for finding the rule for a graph known to come from a polynomial, exponential or logarithmic function. In this section, we find rules for graphs of functions known to be of the form $f(t) = A\sin(nt + \varepsilon) + b$.

Example 24

A function has rule $f(t) = A\sin(nt)$. The amplitude is 6; the period is 10. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 10$.

Solution

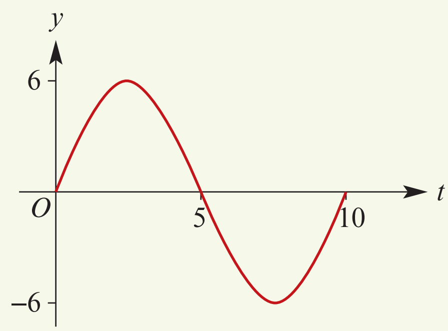
$$\text{Period} = \frac{2\pi}{n} = 10$$

$$\therefore \quad n = \frac{\pi}{5}$$

The amplitude is 6 and therefore $A = 6$.

The function has rule

$$f(t) = 6\sin\left(\frac{\pi t}{5}\right)$$

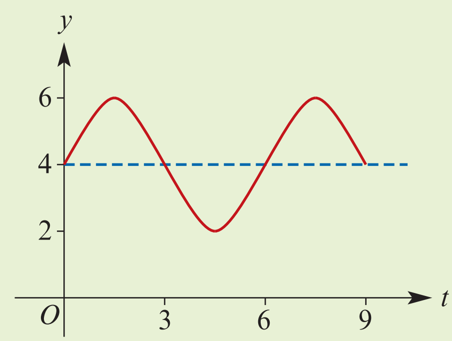


Example 25

The graph shown is that of a function with rule

$$y = A\sin(nt) + b$$

Find A, n and b.



Solution

The amplitude is 2 and so $A = 2$.

The period is 6. Therefore $\frac{2\pi}{n} = 6$ and so $n = \frac{\pi}{3}$.

The ‘centreline’ has equation $y = 4$ and so $b = 4$.

Hence the rule is

$$y = 2\sin\left(\frac{\pi t}{3}\right) + 4$$

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Example 26

A function with rule $y = A\sin(nt) + b$ has range $[-2, 4]$ and period 3. Find A, n and b.

Solution

The amplitude $A = \frac{1}{2}(4 - (-2)) = 3$. The 'centreline' has equation $y = 1$ and so $b = 1$.

The period is 3. Therefore $\dfrac{2\pi}{n} = 3$, which implies $n = \dfrac{2\pi}{3}$.

Hence the rule is $y = 3\sin\left(\dfrac{2\pi}{3}t\right) + 1$.

Example 27

A function with rule $y = A\sin(nt + \varepsilon)$ has the following properties:

- range $= [-2, 2]$
- period $= 6$
- when $t = 4$, $y = 0$.

Find values for A, n and ε.

Solution

Since the range is $[-2, 2]$, the amplitude $A = 2$.

Since the period is 6, we have $\dfrac{2\pi}{n} = 6$, which implies $n = \dfrac{\pi}{3}$.

Hence $y = 2\sin\left(\dfrac{\pi}{3}t + \varepsilon\right)$. When $t = 4$, $y = 0$ and so

$$2\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

$$\sin\left(\frac{4\pi}{3} + \varepsilon\right) = 0$$

Therefore $\dfrac{4\pi}{3} + \varepsilon = 0$ or $\pm\pi$ or $\pm 2\pi$ or $\dots$

We choose the simplest solution, which is $\varepsilon = -\dfrac{4\pi}{3}$.

The rule $y = 2\sin\left(\dfrac{\pi}{3}t - \dfrac{4\pi}{3}\right)$ satisfies the three properties.

Exercise 6I

Example 24

1 **a** A function has rule $f(t) = A\sin(nt)$. The amplitude is 4; the period is 6. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 6$.

b A function has rule $f(t) = A\sin(nt)$. The amplitude is 2; the period is 7. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 7$.

c A function has rule $f(t) = A\cos(nt)$. The amplitude is 3; the period is 5. Find A and n and sketch the graph of $y = f(t)$ for $0 \le t \le 5$.

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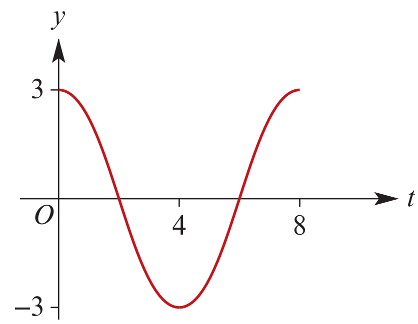
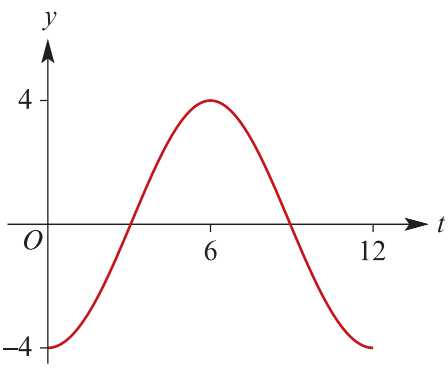
Example 25

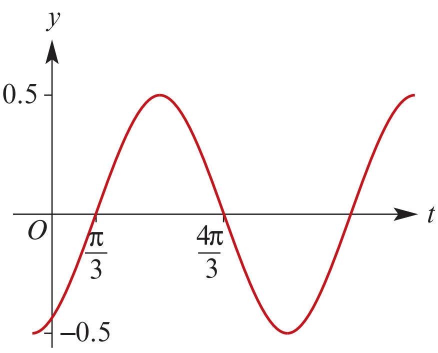
2 The graph shown has rule of the form
$y = A\cos(nt)$.
Find the values of A and n.



3 The graph shown has rule of the form
$y = A\cos(nt)$.
Find the values of A and n.



4 The graph shown has rule of the form
$y = A\sin(t + \varepsilon)$.
Find possible values for A and ε.



Example 26

5 A function with rule $y = A\sin(nt) + b$ has range $[2, 8]$ and period $\frac{2\pi}{3}$. Find the values of A, n and b.

Example 27

6 A function with rule $y = A\sin(nt + \varepsilon)$ has the following three properties:
- range $= [-4, 4]$
- period $= 8$
- when $t = 2$, $y = 0$.

Find values for A, n and ε.

7 A function with rule $y = A\sin(nt + \varepsilon)$ has range $[-2, 2]$ and period 6, and when $t = 1$, $y = 1$. Find possible values for A, n and ε.

8 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[-2, 6]$ and period 8, and when $t = 2$, $y = 2$. Find possible values for A, n, d and ε.

9 A function with rule $y = A\sin(nt + \varepsilon) + d$ has range $[0, 4]$ and period 6, and when $t = 1$, $y = 3$. Find possible values for A, n, d and ε.

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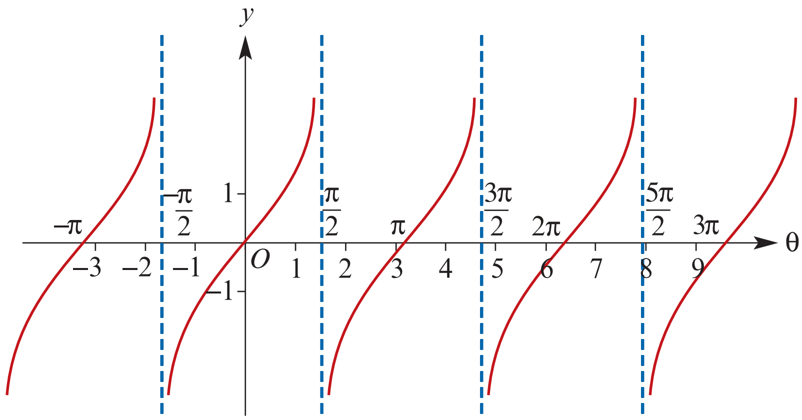
6J The tangent function

The tangent function is given by

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

A table of values for $y = \tan\theta$ is given below:

θ	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π
y	0	1	ud	-1	0	1	ud	-1	0	1	ud	-1	0	1	ud	-1	0



Note: There are vertical asymptotes at $\theta = \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$ and $\frac{5\pi}{2}$.

Observations from the graph of $y = \tan\theta$

- The graph repeats itself every π units, i.e. the period of tan is π.
- The range of tan is $\mathbb{R}$.
- The vertical asymptotes have equations $\theta = \frac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.

Graph of $y = a\tan(nt)$

For a and n positive numbers, the graph of $y = a\tan(nt)$ is obtained from the graph of $y = \tan t$ by a dilation of factor a from the t-axis and a dilation of factor $\frac{1}{n}$ from the y-axis.

The following are important properties of $f(t) = a\tan(nt)$:

- The period is $\frac{\pi}{n}$.
- The range is $\mathbb{R}$.
- The vertical asymptotes have equations $t = \frac{(2k+1)\pi}{2n}$ where $k \in \mathbb{Z}$.
- The axis intercepts are at $t = \frac{k\pi}{n}$ where $k \in \mathbb{Z}$.

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Example 28

Sketch the graph of each of the following for $x \in [-\pi, \pi]$:

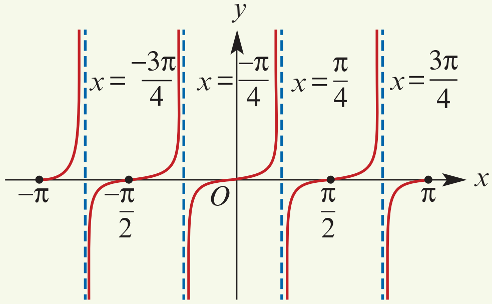
a $y = 3\tan(2x)$ **b** $y = -2\tan(3x)$

Solution

a Period $= \dfrac{\pi}{n} = \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}$

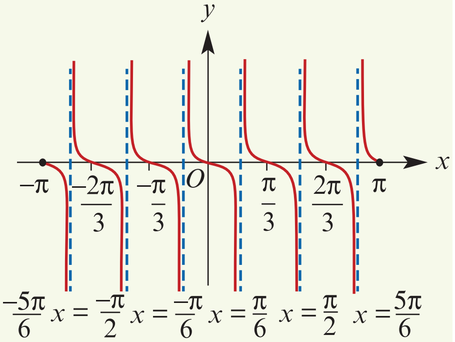
Axis intercepts: $x = \dfrac{k\pi}{2}$, $k \in \mathbb{Z}$



b Period $= \dfrac{\pi}{n} = \dfrac{\pi}{3}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{6}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{3}$, $k \in \mathbb{Z}$



Example 29

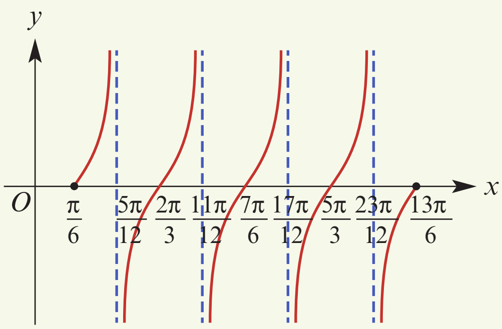
Sketch the graph of $y = 3\tan\left(2x - \dfrac{\pi}{3}\right)$ for $\dfrac{\pi}{6} \le x \le \dfrac{13\pi}{6}$.

Solution

Consider $y = 3\tan 2\left(x - \dfrac{\pi}{6}\right)$.

The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\dfrac{\pi}{6}$ units in the positive direction of the x-axis.



Period $= \dfrac{\pi}{2}$

Asymptotes: $x = \dfrac{(2k+1)\pi}{4} + \dfrac{\pi}{6} = \dfrac{(6k+5)\pi}{12}$, $k \in \mathbb{Z}$

Axis intercepts: $x = \dfrac{k\pi}{2} + \dfrac{\pi}{6} = \dfrac{(3k+1)\pi}{6}$, $k \in \mathbb{Z}$

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Solution of equations involving the tangent function

We now consider the solution of equations involving the tangent function. We will then apply this to finding the x-axis intercepts for graphs of the tangent function which have been translated parallel to the y-axis.

We recall the following exact values:

$$\tan 0 = 0, \qquad \tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}, \qquad \tan\left(\frac{\pi}{4}\right) = 1, \qquad \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$$

and the symmetry properties:

- $\tan(\pi + \theta) = \tan\theta$
- $\tan(-\theta) = -\tan\theta$

Example 30

Solve the equation $3\tan(2x) = \sqrt{3}$ for $x \in (0, 2\pi)$.

Solution

$$3\tan(2x) = \sqrt{3}$$

$$\tan(2x) = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$

$$\therefore \quad 2x = \frac{\pi}{6} \text{ or } \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6}$$

$$x = \frac{\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12}$$

Explanation

Since we want solutions for x in $(0, 2\pi)$, we find solutions for $2x$ in $(0, 4\pi)$.

Once we have found one solution for $2x$, we can obtain all other solutions by adding and subtracting multiples of π.

Example 31

Solve the equation $\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$ for $x \in [-2\pi, 2\pi]$.

Solution

$$\tan\left(\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right) = -1$$

implies

$$\frac{1}{2}\left(x - \frac{\pi}{4}\right) = \frac{-\pi}{4} \text{ or } \frac{3\pi}{4}$$

$$x - \frac{\pi}{4} = \frac{-\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\therefore \quad x = \frac{-\pi}{4} \text{ or } \frac{7\pi}{4}$$

Explanation

Note that

$$x \in [-2\pi, 2\pi] \quad \Leftrightarrow \quad x - \frac{\pi}{4} \in \left[-\frac{9\pi}{4}, \frac{7\pi}{4}\right]$$

$$\Leftrightarrow \quad \frac{1}{2}\left(x - \frac{\pi}{4}\right) \in \left[-\frac{9\pi}{8}, \frac{7\pi}{8}\right]$$

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Example 32

Sketch the graph of $y = 3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3}$ for $\frac{\pi}{6} \le x \le \frac{13\pi}{6}$.

Solution

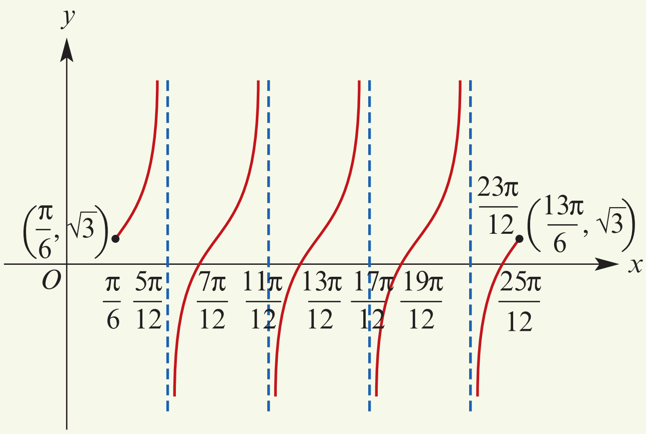
First write the equation as

$$y = 3\tan 2\left(x - \frac{\pi}{6}\right) + \sqrt{3}$$

The graph is the image of $y = \tan x$ under:

- a dilation of factor 3 from the x-axis
- a dilation of factor $\frac{1}{2}$ from the y-axis
- a translation of $\frac{\pi}{6}$ units in the positive direction of the x-axis
- a translation of $\sqrt{3}$ units in the positive direction of the y-axis.

The graph can be obtained from the graph in Example 29 by a translation $\sqrt{3}$ units in the positive direction of the y-axis.



To find the x-axis intercepts, solve the equation:

$$3\tan\left(2x - \frac{\pi}{3}\right) + \sqrt{3} = 0 \quad \text{for} \quad \frac{\pi}{6} \le x \le \frac{13\pi}{6}$$

$$\therefore \quad \tan\left(2x - \frac{\pi}{3}\right) = -\frac{\sqrt{3}}{3} = \frac{-1}{\sqrt{3}}$$

$$\therefore \quad 2x - \frac{\pi}{3} = \frac{5\pi}{6} \text{ or } \frac{11\pi}{6} \text{ or } \frac{17\pi}{6} \text{ or } \frac{23\pi}{6}$$

$$2x = \frac{7\pi}{6} \text{ or } \frac{13\pi}{6} \text{ or } \frac{19\pi}{6} \text{ or } \frac{25\pi}{6}$$

$$x = \frac{7\pi}{12} \text{ or } \frac{13\pi}{12} \text{ or } \frac{19\pi}{12} \text{ or } \frac{25\pi}{12}$$

Hence the x-axis intercepts are $\frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$ and $\frac{25\pi}{12}$.

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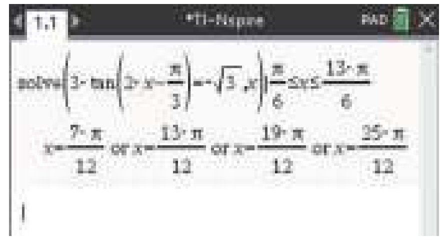
Using the TI-Nspire

To find the x-axis intercepts, enter:

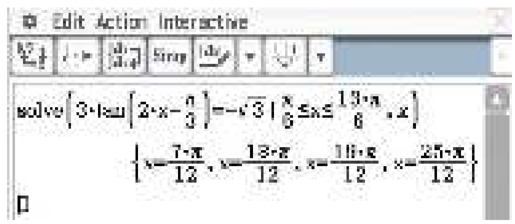
$$\text{solve}\left(3\tan\left(2x - \frac{\pi}{3}\right) = -\sqrt{3},\ x\right) \mid \frac{\pi}{6} \le x \le \frac{13\pi}{6}$$



Using the Casio ClassPad

- Make sure that the calculator is in radian mode.
- To find the x-axis intercepts, enter

  $3\tan\left(2x - \frac{\pi}{3}\right) = -\sqrt{3} \mid \frac{\pi}{6} \le x \le \frac{13\pi}{6}$
- Highlight and then select **Interactive** > **Equation/Inequality** > **solve**.



Solution of equations of the form $\sin(nx) = k\cos(nx)$

We can find the coordinates of the points of intersection of certain sine and cosine graphs by using the following observation:

If $\sin(nx) = k\cos(nx)$, then $\tan(nx) = k$.

This is obtained by dividing both sides of the equation $\sin(nx) = k\cos(nx)$ by $\cos(nx)$, for $\cos(nx) \ne 0$.

Example 33

On the same set of axes, sketch the graphs of $y = \sin x$ and $y = \cos x$ for $x \in [0, 2\pi]$ and find the coordinates of the points of intersection.

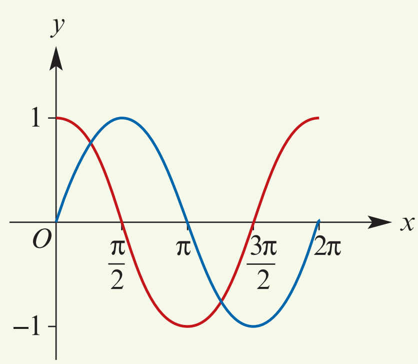
Solution

$\sin x = \cos x$ implies $\tan x = 1$

$$\therefore \quad x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

The coordinates of the points of intersection are

$$\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(\frac{5\pi}{4}, -\frac{1}{\sqrt{2}}\right)$$



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Example 34

Solve the equation $\sin(2x) = \cos(2x)$ for $x \in [0, 2\pi]$.

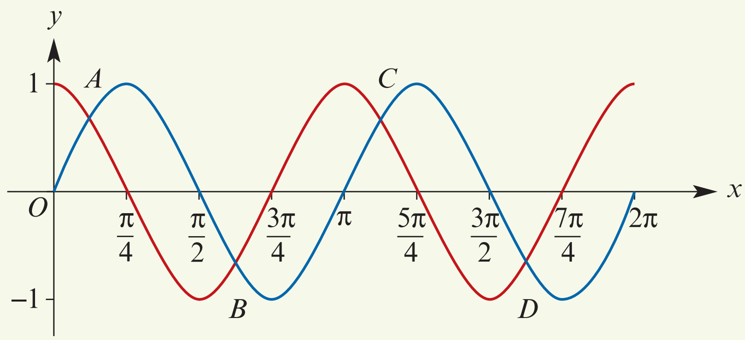
Solution

$\sin(2x) = \cos(2x)$ implies $\tan(2x) = 1$

$$\therefore \quad 2x = \frac{\pi}{4} \text{ or } \frac{5\pi}{4} \text{ or } \frac{9\pi}{4} \text{ or } \frac{13\pi}{4}$$

$$\therefore \quad x = \frac{\pi}{8} \text{ or } \frac{5\pi}{8} \text{ or } \frac{9\pi}{8} \text{ or } \frac{13\pi}{8}$$

This can be shown graphically.



The points of intersection A, B, C and D occur when $x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}$ and $\frac{13\pi}{8}$ respectively.

Summary 6J

- The tangent function is given by $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ for $\cos\theta \neq 0$.
  - The period is π.
  - The vertical asymptotes have equations $\theta = \dfrac{(2k+1)\pi}{2}$ where $k \in \mathbb{Z}$.
  - The axis intercepts are at $\theta = k\pi$ where $k \in \mathbb{Z}$.
- Useful symmetry properties:
  - $\tan(\pi + \theta) = \tan\theta$
  - $\tan(-\theta) = -\tan\theta$

Exercise 6J

1 State the period for each of the following:

a $\tan(3\theta)$ **b** $\tan\left(\frac{\theta}{2}\right)$ **c** $\tan\left(\frac{3\theta}{2}\right)$ **d** $\tan(\pi\theta)$ **e** $\tan\left(\frac{\pi\theta}{2}\right)$

Example 28 **2** Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = \tan(2x)$ **b** $y = 2\tan(3x)$ **c** $y = -2\tan(2x)$

Example 29 **3** Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 2\tan\left(x + \frac{\pi}{4}\right)$ **b** $y = 2\tan 3\left(x + \frac{\pi}{2}\right)$ **c** $y = 3\tan 2\left(x - \frac{\pi}{4}\right)$

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Example 30

4 **a** Solve the equation $\tan(2x) = 1$ for $x \in (0, 2\pi)$.

b Solve the equation $\tan(2x) = -1$ for $x \in (-\pi, \pi)$.

c Solve the equation $\tan(2x) = -\sqrt{3}$ for $x \in (-\pi, \pi)$.

d Solve the equation $\tan(2x) = \sqrt{3}$ for $x \in (-\pi, \pi)$.

e Solve the equation $\tan(2x) = \dfrac{1}{\sqrt{3}}$ for $x \in (-\pi, \pi)$.

Example 31

5 Solve the equation $\tan 2\left(x - \dfrac{\pi}{3}\right) = 1$ for $x \in [0, 2\pi]$.

6 Solve the equation $\tan\left(x - \dfrac{\pi}{4}\right) = \sqrt{3}$ for $x \in [0, 2\pi]$.

Example 32

7 Sketch the graph of each of the following for $x \in (0, 2\pi)$:

a $y = 3\tan x + 1$ **b** $y = 2\tan\left(x + \dfrac{\pi}{2}\right) + 1$ **c** $y = 3\tan 2\left(x - \dfrac{\pi}{4}\right) - 2$

8 Sketch the graph of $y = -2\tan(\pi\theta)$ for $-2 \le \theta \le 2$.

9 Sketch the graph of $y = \tan(-2\theta)$ for $\dfrac{-\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

Example 33

10 **a** On the same set of axes, sketch the graphs of $y = \cos(2x)$ and $y = -\sin(2x)$ for $x \in [-\pi, \pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes, sketch the graph of $y = \cos(2x) - \sin(2x)$.

Example 34

11 Solve each of the following equations for $x \in [0, 2\pi]$:

a $\sqrt{3}\sin x = \cos x$ **b** $\sin(4x) = \cos(4x)$ **c** $\sqrt{3}\sin(2x) = \cos(2x)$

d $-\sqrt{3}\sin(2x) = \cos(2x)$ **e** $\sin(3x) = -\cos(3x)$ **f** $\sin x = 0.5\cos x$

g $\sin x = 2\cos x$ **h** $\sin(2x) = -\cos(2x)$ **i** $\cos(3x) = \sqrt{3}\sin(3x)$

j $\sin(3x) = \sqrt{3}\cos(3x)$

12 **a** On the same set of axes, sketch the graphs of $y = \cos x$ and $y = \sqrt{3}\sin x$ for $x \in [0, 2\pi]$.

b Find the coordinates of the points of intersection.

c On the same set of axes, sketch the graph of $y = \cos x + \sqrt{3}\sin x$.

13 Solve each of the following equations for $0 \le x \le 2\pi$:

a $\tan\left(2x - \dfrac{\pi}{4}\right) = \sqrt{3}$ **b** $3\tan(2x) = -\sqrt{3}$ **c** $\tan\left(3x - \dfrac{\pi}{6}\right) = -1$

14 A function with rule $y = A\tan(nt)$ has the following properties:

- the asymptotes have equations $t = \dfrac{(2k+1)\pi}{6}$ where $k \in \mathbb{Z}$
- when $t = \dfrac{\pi}{12}$, $y = 5$.

Find values for A and n.

15 A function with rule $y = A\tan(nt)$ has period 2 and, when $t = \frac{1}{2}$, $y = 6$. Find values for A and n.

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6K General solution of trigonometric equations

We have seen how to solve equations involving circular functions over a restricted domain. We now consider the general solutions of such equations over the maximal domain for each function.

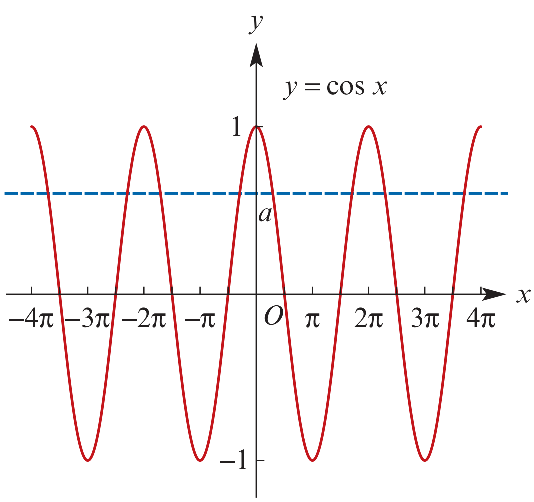
By convention:

- $\cos^{-1}$ has range $[0, \pi]$
- $\sin^{-1}$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
- $\tan^{-1}$ has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

For example:

$$\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}, \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}, \qquad \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}, \qquad \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

If an equation involving a circular function has one or more solutions in one 'cycle', then it will have corresponding solutions in each 'cycle' of its domain, i.e. there will be infinitely many solutions.



For example, consider the equation

$$\cos x = a$$

for some fixed $a \in [-1, 1]$. The solution in the interval $[0, \pi]$ is given by

$$x = \cos^{-1}(a)$$

By the symmetry properties of the cosine function, the other solutions are given by

$$-\cos^{-1}(a),\ \pm 2\pi + \cos^{-1}(a),\ \pm 2\pi - \cos^{-1}(a),\ \pm 4\pi + \cos^{-1}(a),\ \pm 4\pi - \cos^{-1}(a),\ \dots$$

In general, we have the following:

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
  $$x = 2n\pi \pm \cos^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$
- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
  $$x = n\pi + \tan^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$
- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
  $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}(a), \qquad \text{where } n \in \mathbb{Z}$$

Note: An alternative and more concise way to express the general solution of $\sin x = a$ is $x = n\pi + (-1)^n \sin^{-1}(a)$, where $n \in \mathbb{Z}$.

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Example 35

Find the general solution of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a $\cos x = 0.5$

$$x = 2n\pi \pm \cos^{-1}(0.5)$$
$$= 2n\pi \pm \frac{\pi}{3}$$
$$= \frac{(6n \pm 1)\pi}{3}, \quad n \in \mathbb{Z}$$

b $\tan(3x) = \dfrac{1}{\sqrt{3}}$

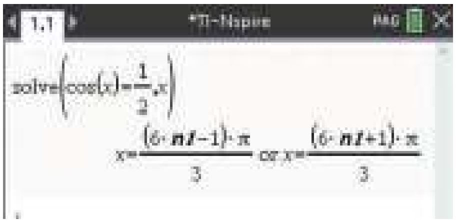
$$3x = n\pi + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= n\pi + \frac{\pi}{6}$$
$$= \frac{(6n + 1)\pi}{6}$$
$$x = \frac{(6n + 1)\pi}{18}, \quad n \in \mathbb{Z}$$

c $\sin x = \dfrac{\sqrt{2}}{2} = \dfrac{1}{\sqrt{2}}$

$$x = 2n\pi + \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$= 2n\pi + \frac{\pi}{4} \qquad\qquad = (2n + 1)\pi - \frac{\pi}{4}$$
$$= \frac{(8n + 1)\pi}{4}, \quad n \in \mathbb{Z} \qquad\qquad = \frac{(8n + 3)\pi}{4}, \quad n \in \mathbb{Z}$$

Using the TI-Nspire

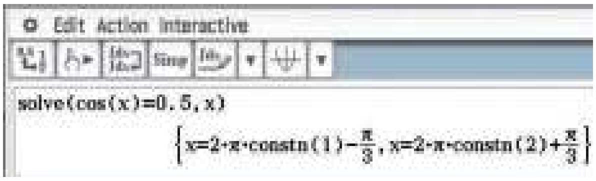
- Make sure the calculator is in radian mode.
- Use **solve** from the **Algebra** menu and complete as shown. Note the use of $\frac{1}{2}$ rather than 0.5 to ensure that the answer is exact.



Using the Casio ClassPad

- Check that the calculator is in radian mode.
- In Main $\sqrt{\alpha}$, enter and highlight the equation $\cos(x) = 0.5$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Then tap EXE.
- To view the entire solution, rotate the screen by selecting Rotate.

Note: Replace constn(1) and constn(2) with n in the written answer.



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Example 36

Find the first three positive solutions of each of the following equations:

a $\cos x = 0.5$ **b** $\sqrt{3}\tan(3x) = 1$ **c** $2\sin x = \sqrt{2}$

Solution

a The general solution (from Example 35a) is given by $x = \dfrac{(6n \pm 1)\pi}{3}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \pm\dfrac{\pi}{3}$, and when $n = 1$, $x = \dfrac{5\pi}{3}$ or $x = \dfrac{7\pi}{3}$.

Thus the first three positive solutions of $\cos x = 0.5$ are $x = \dfrac{\pi}{3}, \dfrac{5\pi}{3}, \dfrac{7\pi}{3}$.

b The general solution (from Example 35b) is given by $x = \dfrac{(6n + 1)\pi}{18}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{18}$, and when $n = 1$, $x = \dfrac{7\pi}{18}$, and when $n = 2$, $x = \dfrac{13\pi}{18}$.

Thus the first three positive solutions of $\sqrt{3}\tan(3x) = 1$ are $x = \dfrac{\pi}{18}, \dfrac{7\pi}{18}, \dfrac{13\pi}{18}$.

c The general solution (from Example 35c) is $x = \dfrac{(8n + 1)\pi}{4}$ or $x = \dfrac{(8n + 3)\pi}{4}$, $n \in \mathbb{Z}$.

When $n = 0$, $x = \dfrac{\pi}{4}$ or $x = \dfrac{3\pi}{4}$, and when $n = 1$, $x = \dfrac{9\pi}{4}$ or $x = \dfrac{11\pi}{4}$.

Thus the first three positive solutions of $2\sin x = \sqrt{2}$ are $x = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}$.

Example 37

Find the general solution for each of the following:

a $\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$ **b** $\tan\left(2x - \dfrac{\pi}{3}\right) = 1$

Solution

a $\sin\left(x - \dfrac{\pi}{3}\right) = \dfrac{\sqrt{3}}{2}$

$$x - \frac{\pi}{3} = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\therefore \quad x = n\pi + (-1)^n\left(\frac{\pi}{3}\right) + \frac{\pi}{3}, \quad n \in \mathbb{Z}$$

The solutions are $x = \dfrac{(3n + 2)\pi}{3}$ for n even and $x = n\pi$ for n odd.

b $\tan\left(2x - \dfrac{\pi}{3}\right) = 1$

$$2x - \frac{\pi}{3} = n\pi + \frac{\pi}{4}$$

$$2x = n\pi + \frac{\pi}{4} + \frac{\pi}{3}$$

$$\therefore \quad x = \frac{1}{2}\left(n\pi + \frac{7\pi}{12}\right)$$

$$= \frac{(12n + 7)\pi}{24}, \quad n \in \mathbb{Z}$$

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Summary 6K

- For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is

$$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is

$$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

- For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is

$$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n + 1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Exercise 6K

1 Evaluate each of the following for:

i $n = 1$ **ii** $n = 2$ **iii** $n = -2$

a $2n\pi \pm \cos^{-1}(1)$ **b** $2n\pi \pm \cos^{-1}\left(-\frac{1}{2}\right)$

Example 35

2 Find the general solution of each of the following equations:

a $\cos x = \frac{\sqrt{3}}{2}$ **b** $2\sin(3x) = \sqrt{3}$ **c** $\sqrt{3}\tan x = 3$

Example 36

3 Find the first two positive solutions of each of the following equations:

a $\sin x = 0.5$ **b** $2\cos(2x) = \sqrt{3}$ **c** $\sqrt{3}\tan(2x) = -3$

4 Given that a trigonometric equation has general solution $x = n\pi + (-1)^n \sin^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-2\pi, 2\pi]$.

5 Given that a trigonometric equation has general solution $x = 2n\pi \pm \cos^{-1}\left(\frac{1}{2}\right)$, where $n \in \mathbb{Z}$, find the solutions of the equation in the interval $[-\pi, 2\pi]$.

Example 37

6 Find the general solution for each of the following:

a $\cos 2\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$ **b** $2\tan 2\left(x + \frac{\pi}{4}\right) = 2\sqrt{3}$ **c** $2\sin\left(x + \frac{\pi}{3}\right) = -1$

7 Find the general solution of $2\cos\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$ and hence find all the solutions for x in the interval $(-2\pi, 2\pi)$.

8 Find the general solution of $\sqrt{3}\tan\left(\frac{\pi}{6} - 3x\right) - 1 = 0$ and hence find all the solutions for x in the interval $[-\pi, 0]$.

9 Find the general solution of $2\sin(4\pi x) + \sqrt{3} = 0$ and hence find all the solutions for x in the interval $[-1, 1]$.

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6L Applications of circular functions

A **sinusoidal function** has a rule of the form $y = a\sin(nt + \varepsilon) + b$ or, equivalently, of the form $y = a\cos(nt + \varepsilon) + b$. Such functions can be used to model periodic motion.

Example 38

A wheel is mounted on a wall and rotates such that the distance, d cm, of a particular point P on the wheel from the ground is given by the rule

$$d = 100 - 60\cos\left(\frac{4\pi}{3}t\right)$$

where t is the time in seconds.

a How far is the point P above the ground when $t = 0$?

b How long does it take for the wheel to rotate once?

c Find the maximum and minimum distances of the point P above the ground.

d Sketch the graph of d against t.

e In the first rotation, find the intervals of time when the point P is less than 70 cm above the ground.

Solution

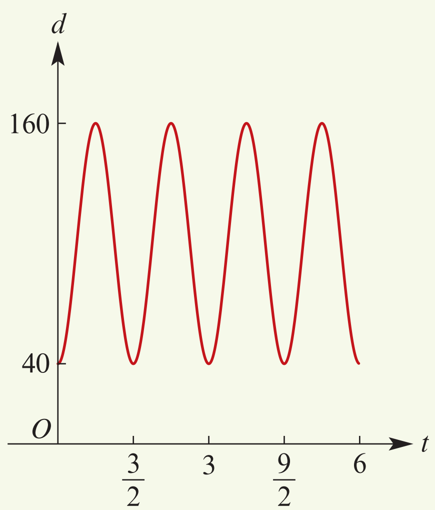
a When $t = 0$, $d = 100 - 60 \times 1 = 40$. The point is 40 cm above the ground.

b The period is $2\pi \div \frac{4\pi}{3} = \frac{3}{2}$. The wheel takes $\frac{3}{2}$ seconds to rotate once.

c The minimum occurs when $\cos\left(\frac{4\pi}{3}t\right) = 1$, which gives $d = 100 - 60 = 40$. Hence the minimum distance is 40 cm.

The maximum occurs when $\cos\left(\frac{4\pi}{3}t\right) = -1$, which gives $d = 100 + 60 = 160$. Hence the maximum distance is 160 cm.

d



e
$$100 - 60\cos\left(\frac{4\pi}{3}t\right) = 70$$
$$-60\cos\left(\frac{4\pi}{3}t\right) = -30$$
$$\cos\left(\frac{4\pi}{3}t\right) = \frac{1}{2}$$
$$\therefore \quad \frac{4\pi}{3}t = \frac{\pi}{3} \text{ or } \frac{5\pi}{3}$$
$$t = \frac{1}{4} \text{ or } \frac{5}{4}$$

From the graph, the distance is less than 70 cm for $t \in \left(0, \frac{1}{4}\right)$ and for $t \in \left(\frac{5}{4}, \frac{3}{2}\right)$.

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Example 39

It is suggested that the height, $h(t)$ metres, of the tide above mean sea level on 1 January at Warnung is given approximately by the rule $h(t) = 4\sin\left(\frac{\pi}{6}t\right)$, where t is the number of hours after midnight.

a Draw the graph of $y = h(t)$ for $0 \le t \le 24$.

b When was high tide?

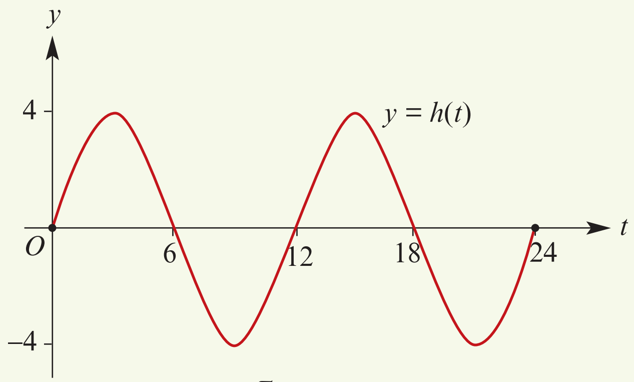
c What was the height of the high tide?

d What was the height of the tide at 8 a.m.?

e A boat can only cross the harbour bar when the tide is at least 1 metre above mean sea level. When could the boat cross the harbour bar on 1 January?

Solution

a



Note: Period $= 2\pi \div \frac{\pi}{6} = 12$

b High tide occurs when $h(t) = 4$:

$$4\sin\left(\frac{\pi}{6}t\right) = 4$$

$$\sin\left(\frac{\pi}{6}t\right) = 1$$

$$\frac{\pi}{6}t = \frac{\pi}{2}, \frac{5\pi}{2}$$

$$\therefore \quad t = 3, 15$$

i.e. high tide occurs at 03:00 and 15:00 (3 p.m.).

c The high tide has height 4 metres above the mean height.

d $h(8) = 4\sin\left(\frac{8\pi}{6}\right) = 4\sin\left(\frac{4\pi}{3}\right) = 4 \times \frac{-\sqrt{3}}{2} = -2\sqrt{3}$

At 8 a.m. the water is $2\sqrt{3}$ metres below the mean height.

e We first consider $4\sin\left(\frac{\pi}{6}t\right) = 1$:

$$\sin\left(\frac{\pi}{6}t\right) = \frac{1}{4}$$

$$\therefore \quad \frac{\pi}{6}t = 0.2526, 2.889, 6.5358, 9.172$$

$$\therefore \quad t = 0.4824, 5.5176, 12.4824, 17.5173$$

i.e. the water is at height 1 metre at 00:29, 05:31, 12:29, 17:31.
Thus the boat can pass across the harbour bar between 00:29 and 05:31, and between 12:29 and 17:31.

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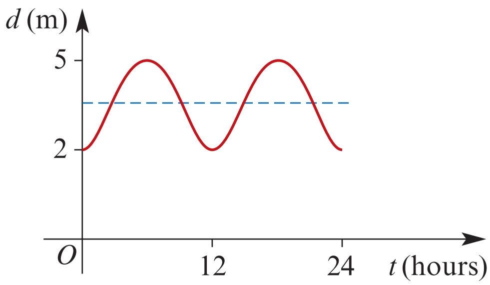
Exercise 6L

Example 38

1 The graph shows the distance, $d(t)$, of the tip of the hour hand of a large clock from the ceiling at time t hours.



a The function d is sinusoidal. Find:

i the amplitude

ii the period

iii the rule for $d(t)$

iv the length of the hour hand.

b At what times is the distance less than 3.5 metres from the ceiling?

Example 39

2 The water level on a beach wall is given by

$$d(t) = 6 + 4\cos\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$$

where t is the number of hours after midnight and d is the depth of the water in metres.

a Sketch the graph of $d(t)$ for $0 \le t \le 24$.

b What is the earliest time of day at which the water is at its highest?

c When is the water 2 m up the wall?

3 In a tidal river, the time between high tides is 12 hours. The average depth of water at a point in the river is 5 m; at high tide the depth is 8 m. Assume that the depth of water, $h(t)$ m, at this point is given by

$$h(t) = A\sin(nt + \varepsilon) + b$$

where t is the number of hours after noon. At noon there is a high tide.

a Find the values of A, n, b and ε.

b At what times is the depth of the water 6 m?

c Sketch the graph of $y = h(t)$ for $0 \le t \le 24$.

4 A particle moves along a straight line. Its position, x metres, relative to a fixed point O on the line is given by $x = 3 + 2\sin(3t)$, where t is the time in seconds.

a Find its greatest distance from O.

b Find its least distance from O.

c Find the times at which it is 5 m from O for $0 \le t \le 5$.

d Find the times at which it is 3 m from O for $0 \le t \le 3$.

e Describe the motion of the particle.

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5 The temperature, A°C, inside a house at t hours after 4 a.m. is given by the rule $A = 21 - 3\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$. The temperature, B°C, outside the house at the same time is given by $B = 22 - 5\cos\left(\frac{\pi t}{12}\right)$, for $0 \le t \le 24$.

- **a** Find the temperature inside the house at 8 a.m.
- **b** Write down an expression for $D = A - B$, the difference between the inside and outside temperatures.
- **c** Sketch the graph of D for $0 \le t \le 24$.
- **d** Determine when the inside temperature is less than the outside temperature.

6 Passengers on a ferris wheel access their seats from a platform 5 m above the ground. As each seat is filled, the ferris wheel moves around so that the next seat can be filled. Once all seats are filled, the ride begins and lasts for 6 minutes. The height, h m, of Isobel's seat above the ground t seconds after the ride has begun is given by $h = 15\sin(10t - 45)^\circ + 16.5$.

- **a** Use a calculator to sketch the graph of h against t for the first 2 minutes of the ride.
- **b** How far above the ground is Isobel's seat at the commencement of the ride?
- **c** After how many seconds does Isobel's seat pass the access platform?
- **d** How many times will her seat pass the access platform in the first 2 minutes?
- **e** How many times will her seat pass the access platform during the entire ride?

Due to a malfunction, the ferris wheel stops abruptly 1 minute 40 seconds into the ride.

- **f** How far above the ground is Isobel stranded?
- **g** If Isobel's brother Hamish had a seat 1.5 m above the ground at the commencement of the ride, how far above the ground is Hamish stranded?

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Review

Chapter summary

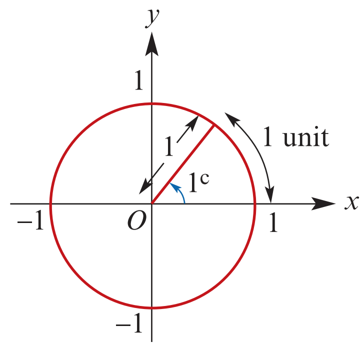
Assign-ment

Nrich

- **Definition of a radian**

One radian (written 1^c) is the angle formed at the centre of the unit circle by an arc of length 1 unit.

$$1^c = \frac{180^\circ}{\pi} \qquad 1^\circ = \frac{\pi^c}{180}$$



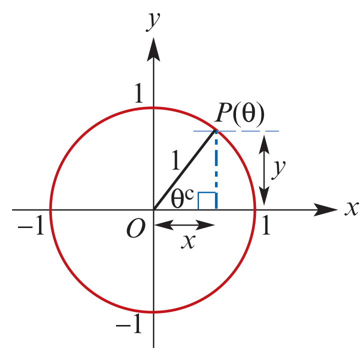
- **Sine and cosine functions**

x-coordinate of $P(\theta)$ on unit circle:

$$x = \cos\theta, \qquad \theta \in \mathbb{R}$$

y-coordinate of $P(\theta)$ on unit circle:

$$y = \sin\theta, \qquad \theta \in \mathbb{R}$$



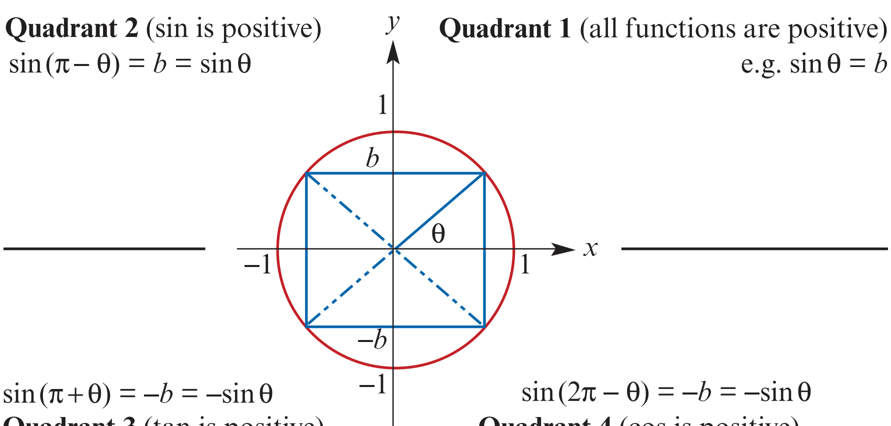
- **Tangent function**

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \quad \text{for } \cos\theta \neq 0$$

- **Symmetry properties of circular functions**

Quadrant 2 (sin is positive)
$\sin(\pi - \theta) = b = \sin\theta$

Quadrant 1 (all functions are positive)
e.g. $\sin\theta = b$



$\sin(\pi + \theta) = -b = -\sin\theta$
Quadrant 3 (tan is positive)

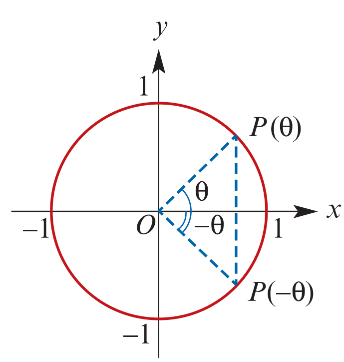
$\sin(2\pi - \theta) = -b = -\sin\theta$
Quadrant 4 (cos is positive)

Negative angles:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = \frac{-\sin\theta}{\cos\theta} = -\tan\theta$$



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Complementary angles:

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$$

$$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$$

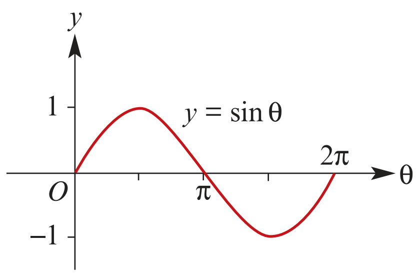
- **Pythagorean identity**

$$\cos^2\theta + \sin^2\theta = 1$$

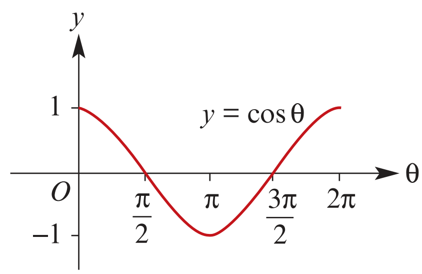
- **Exact values of circular functions**

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	1	0
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	1	0	undefined

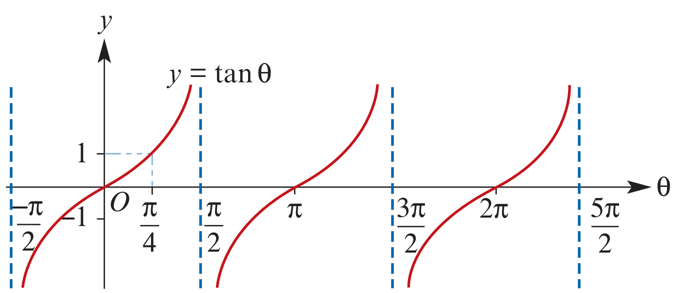
- **Graphs of circular functions**



Amplitude = 1
Period = 2π



Amplitude = 1
Period = 2π



Amplitude is undefined
Period = π

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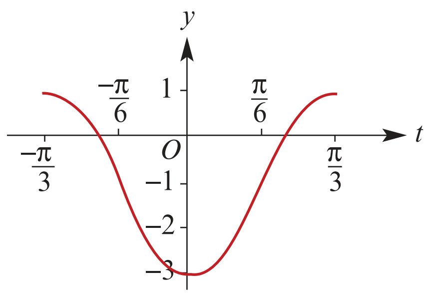
Review

- Transformations of sine and cosine graphs:
  $y = a\sin n(t \pm \varepsilon) \pm b$ and $y = a\cos n(t \pm \varepsilon) \pm b$

  e.g. $y = 2\cos 3\left(t + \frac{\pi}{3}\right) - 1$

  - Amplitude, $a = 2$
  - Period $= \frac{2\pi}{n} = \frac{2\pi}{3}$
  - The graph is the same shape as $y = 2\cos(3t)$ but is translated $\frac{\pi}{3}$ units in the negative direction of the t-axis and 1 unit in the negative direction of the y-axis.

- Transformations of the graph of $y = \tan t$
  e.g. $y = a\tan n(t + \varepsilon) + b$, where $n > 0$
  - Period $= \frac{\pi}{n}$
  - Asymptotes: $t = \frac{(2k+1)\pi}{2n} - \varepsilon$, where $k \in \mathbb{Z}$

- **Solution of trigonometric equations**
  e.g. Solve $\cos x^\circ = -0.7$ for $x \in [0, 360]$.
  First look at the 1st quadrant: If $\cos \alpha^\circ = 0.7$, then $\alpha = 45.6$.
  Since $\cos x^\circ$ is negative for $P(x^\circ)$ in the 2nd and 3rd quadrants, the solutions are

  $$x = 180 - 45.6 = 134.4 \quad \text{and} \quad x = 180 + 45.6 = 225.6$$

- **General solution of trigonometric equations**
  - For $a \in [-1, 1]$, the general solution of the equation $\cos x = a$ is
    $$x = 2n\pi \pm \cos^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
  - For $a \in \mathbb{R}$, the general solution of the equation $\tan x = a$ is
    $$x = n\pi + \tan^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$
  - For $a \in [-1, 1]$, the general solution of the equation $\sin x = a$ is
    $$x = 2n\pi + \sin^{-1}(a) \quad \text{or} \quad x = (2n+1)\pi - \sin^{-1}(a), \quad \text{where } n \in \mathbb{Z}$$

Technology-free questions

1 Solve each of the following equations for $x \in [-\pi, 2\pi]$:

a $\sin x = \frac{1}{2}$ **b** $2\cos x = -1$ **c** $2\cos x = \sqrt{3}$ **d** $\sqrt{2}\sin x + 1 = 0$
e $4\sin x + 2 = 0$ **f** $\sin(2x) + 1 = 0$ **g** $\cos(2x) = \frac{-1}{\sqrt{2}}$ **h** $2\sin(3x) - 1 = 0$

2 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $f(x) = \sin(3x)$ **b** $f(x) = 2\sin(2x) - 1$ **c** $g(x) = 2\sin(2x) + 1$
d $f(x) = 2\sin\left(x - \frac{\pi}{4}\right)$ **e** $f(x) = 2\sin\left(\frac{\pi x}{3}\right)$ **f** $h(x) = 2\cos\left(\frac{\pi x}{4}\right)$

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3 Solve each of the following equations for $x \in [0, 360]$:

a $\sin x^\circ = 0.5$ **b** $\cos(2x)^\circ = 0$ **c** $2\sin x^\circ = -\sqrt{3}$

d $\sin(2x + 60)^\circ = \dfrac{-\sqrt{3}}{2}$ **e** $2\sin(\frac{1}{2}x)^\circ = \sqrt{3}$

4 Sketch the graph of each of the following, showing one cycle. Clearly label axis intercepts.

a $y = 2\sin\left(x + \dfrac{\pi}{3}\right) + 2$ **b** $y = -2\sin\left(x + \dfrac{\pi}{3}\right) + 1$ **c** $y = 2\sin\left(x - \dfrac{\pi}{4}\right) + \sqrt{3}$

d $y = -3\sin x$ **e** $y = \sin\left(x - \dfrac{\pi}{6}\right) + 3$ **f** $y = 2\sin\left(x - \dfrac{\pi}{2}\right) + 1$

5 Sketch, on the same set of axes, the curves $y = \cos x$ and $y = \sin(2x)$ for the interval $0 \le x \le 2\pi$, labelling each curve carefully. State the number of solutions in this interval for each of the following equations:

a $\sin(2x) = 0.6$ **b** $\sin(2x) = \cos x$ **c** $\sin(2x) - \cos x = 1$

6 Sketch on separate axes for $0^\circ \le x^\circ \le 360^\circ$:

a $y = 3\cos x^\circ$ **b** $y = \cos(2x)^\circ$ **c** $y = \cos(x - 30)^\circ$

7 Solve each of the following for $x \in [-\pi, \pi]$:

a $\tan x = \sqrt{3}$ **b** $\tan x = -1$

c $\tan(2x) = -1$ **d** $\tan(2x) + \sqrt{3} = 0$

8 Solve the equation $\sin x = \sqrt{3}\cos x$ for $x \in [-\pi, \pi]$.

9 The graphs of $y = a\cos x$ and $y = \sin x$, where a is a real constant, have a point of intersection at $x = \dfrac{\pi}{6}$.

a Find the value of a.

b If $x \in [-\pi, \pi]$, find the x-coordinate(s) of the other point(s) of intersection of the two graphs.

10 Find the general solution for each of the following:

a $\sin(2x) = -1$ **b** $\cos(3x) = 1$ **c** $\tan x = -1$

Multiple-choice questions

1 The period of the graph of $y = 3\sin\left(\dfrac{1}{2}x - \pi\right) + 4$ is

A π **B** 3 **C** 4π **D** $\pi + 4$ **E** 2π

2 The range of the graph of $y = f(x)$, where $f(x) = 5\cos\left(2x - \dfrac{\pi}{3}\right) - 7$, is

A $[-12, -2]$ **B** $[-7, 7]$ **C** $(-2, 5)$ **D** $[-2, 5]$ **E** $[-2, 12]$

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3 The equation of the image of the graph of $y = \sin x$ under a transformation of a dilation of factor $\frac{1}{2}$ from the y-axis followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis is

A $y = \sin\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \sin\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\sin\left(x - \frac{\pi}{4}\right)$

D $y = \sin\left(2x - \frac{\pi}{4}\right)$ **E** $y = \sin 2\left(x - \frac{\pi}{4}\right)$

4 The function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = a\sin(bx) + c$, where a, b and c are positive constants, has period

A a **B** b **C** $\frac{2\pi}{a}$ **D** $\frac{2\pi}{b}$ **E** $\frac{b}{2\pi}$

5 The equation $3\sin x - 1 = b$, where b is a positive real number, has one solution in the interval $(0, 2\pi)$. The value of b is

A 2 **B** 0.2 **C** 3 **D** 5 **E** 6

6 The range of the function $f: \left[0, \frac{2\pi}{9}\right) \to \mathbb{R}$, $f(x) = \cos(3x) - 1$ is

A $(-1, 0)$ **B** $(-2, 0]$ **C** $\left(-\frac{3}{2}, 0\right]$ **D** $\left[-\frac{3}{2}, 0\right)$ **E** $[-2, 0)$

7 Let $f(x) = p\cos(5x) + q$ where $p > 0$. Then $f(x) \le 0$ for all values of x if

A $q \ge 0$ **B** $-p \le q \le p$ **C** $p \le -q$ **D** $p \ge q$ **E** $-q \le p$

8 The vertical distance of a point on a wheel from the ground as it rotates is given by $D(t) = 3 - 3\sin(6\pi t)$, where t is the time in seconds. The time in seconds for a full rotation of the wheel is

A $\frac{1}{6\pi}$ **B** $\frac{1}{3}$ **C** 6π **D** $\frac{1}{3\pi}$ **E** 3

9 Let $f: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ where $f(x) = \cos(3x) - 2$. The graph of f is transformed by a reflection in the x-axis followed by a dilation of factor 3 from the y-axis. The resulting graph is defined by

A $g: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = 6 - 3\cos(3x)$ **B** $g: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = 3\cos x - 6$

C $g: \left[0, \frac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = 2 - \cos x$ **D** $g: \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$, $g(x) = \cos(-x) - 2$

E $g: \left[0, \frac{3\pi}{2}\right] \to \mathbb{R}$, $g(x) = \cos(9x) + 1$

10 The equation of the image of $y = \cos x$ under a transformation of a dilation of factor 2 from the x-axis, followed by a translation of $\frac{\pi}{4}$ units in the positive direction of the x-axis is

A $y = \cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$ **B** $y = \cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$ **C** $y = 2\cos\left(x + \frac{\pi}{4}\right)$

D $y = 2\sin\left(x - \frac{\pi}{4}\right)$ **E** $y = 2\cos\left(x - \frac{\pi}{4}\right)$

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11 A sequence of transformations which takes the graph of $y = \cos x$ to the graph of $y = -2\cos\left(\frac{x}{3}\right)$ is

A a dilation of factor $\frac{1}{3}$ from the x-axis, followed by dilation of factor $\frac{1}{2}$ from the y-axis, followed by a reflection in the x-axis

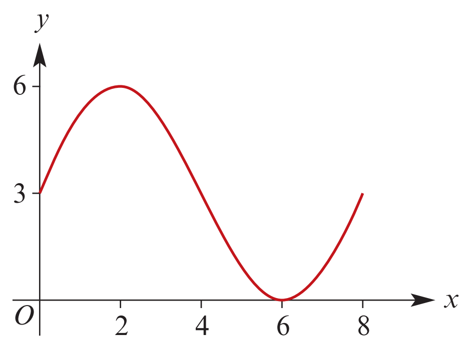
B a dilation of factor $\frac{1}{2}$ from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the y-axis

C a dilation of factor 2 from the x-axis, followed by dilation of factor 3 from the y-axis, followed by a reflection in the x-axis

D a dilation of factor 3 from the x-axis, followed by dilation of factor 2 from the y-axis, followed by a reflection in the x-axis

E a dilation of factor 2 from the x-axis, followed by dilation of factor $\frac{1}{3}$ from the y-axis, followed by a reflection in the x-axis

12 Which of the following is likely to be the rule for the graph of the circular function shown?



A $y = 3 + 3\cos\left(\frac{\pi x}{4}\right)$

B $y = 3 + 3\sin\left(\frac{\pi x}{4}\right)$

C $y = 3 + 3\sin(4\pi x)$

D $y = 3 + 3\cos\left(\frac{x}{4}\right)$

E $y = 3 + 3\sin\left(\frac{x}{4}\right)$

Extended-response questions

1 In a tidal river, the time between high tide and low tide is 6 hours. The average depth of water at a point in the river is 4 metres; at high tide the depth is 5 metres.

a Sketch the graph of the depth of water at the point for the time interval from 0 to 24 hours if the relationship between time and depth is sinusoidal and there is high tide at noon.

b If a boat requires a depth of 4 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

c If a boat requires a depth of 3.5 metres of water in order to sail, at what time before noon can it enter the point in the river and by what time must it leave if it is not to be stranded?

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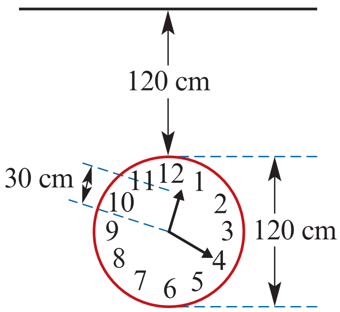
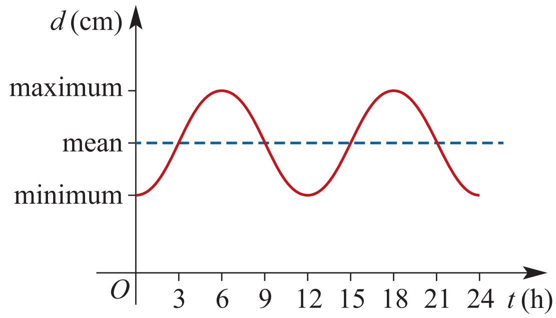
2 A clock hangs 120 cm below a ceiling. The diameter of the clock is 120 cm, and the length of the hour hand is 30 cm. The graph shows the distance from the ceiling to the tip of the hour hand over a 24-hour period.





a What are the values for the maximum, minimum and mean distance?

b An equation that determines this curve is of the form

$$y = A\sin(nt + \varepsilon) + b$$

Find suitable values of A, n, ε and b.

c Find the distance from the ceiling to the tip of the hour hand at:

i 2 a.m.

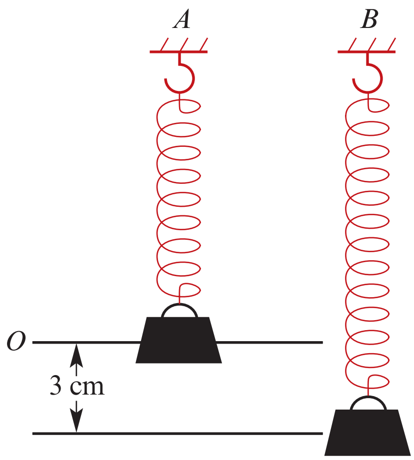
ii 11 p.m.

d Find the times in the morning at which the tip of the hour hand is 200 cm below the ceiling.

3 A weight is suspended from a spring as shown. The weight is pulled down 3 cm from O and released. The vertical displacement from O at time t is described by a function of the form

$$y = a\cos(nt)$$

where y cm is the vertical displacement at time t seconds. The following data were recorded.



t	0	0.5	1
y	−3	3	−3

It was also noted that the centre of the weight went no further than 3 cm from the centre O.

a Find the values of a and n.

b Sketch the graph of y against t.

c Find when the centre of the weight is first:

i 1.5 cm above O

ii 1.5 cm below O.

d When does the weight first reach a point 1 cm below O?

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4 The manager of a reservoir and its catchment area has noted that the inflow of water into the reservoir is very predictable and in fact models the inflow using a function with rule of the form

$$y = a\sin(nt + \varepsilon) + b$$

The following observations were made:

- The average inflow is 100 000 m^3/day.
- The minimum daily inflow is 80 000 m^3/day.
- The maximum daily inflow is 120 000 m^3/day, and this occurs on 1 May ($t = 121$) each year.

a Find the values of a, b and n and the smallest possible positive value for ε.

b Sketch the graph of y against t.

c Find the times of year when the inflow per day is:

i 90 000 m^3/day

ii 110 000 m^3/day

d Find the inflow rate on 1 June.

5 The number of hours of daylight at a point on the Antarctic Circle is given approximately by $d = 12 + 12\cos\left(\frac{1}{6}\pi\left(t + \frac{1}{3}\right)\right)$, where t is the number of months that have elapsed since 1 January.

a **i** Find d on 21 June ($t \approx 5.7$).

ii Find d on 21 March ($t \approx 2.7$).

b When will there be 5 hours of daylight?

6 The depth, $D(t)$ m, of water at the entrance to a harbour at t hours after midnight on a particular day is given by $D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right)$, $0 \le t \le 24$.

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Boats that need a depth of w m are permitted to enter the harbour only if the depth of the water at the entrance is at least w m for a continuous period of 1 hour. Find, correct to one decimal place, the largest value of w that satisfies this condition.

7 The depth of water at the entrance to a harbour t hours after high tide is D m, where $D = p + q\cos(rt)^\circ$ for suitable constants p, q, r. At high tide the depth is 7 m; at low tide, 6 hours later, the depth is 3 m.

a Show that $r = 30$ and find the values of p and q.

b Sketch the graph of D against t for $0 \le t \le 12$.

c Find how soon after low tide a ship that requires a depth of at least 4 m of water will be able to enter the harbour.

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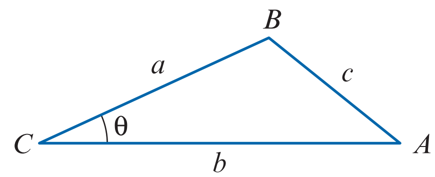
8 The area of a triangle is given by

$$A = \frac{1}{2}ab\sin\theta$$

and the perimeter is given by

$$P = a + b + \sqrt{a^2 + b^2 - 2ab\cos\theta}$$



a For $a = b = 10$ and $\theta = \frac{\pi}{3}$, find:

i the area of the triangle

ii the perimeter of the triangle.

b For $a = b = 10$, find the value(s) of θ for which $A = P$. (Give value(s) correct to two decimal places.)

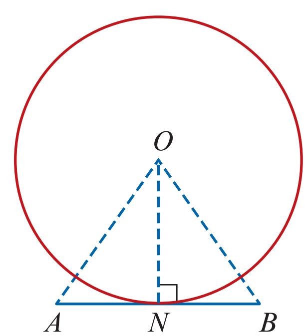
c Show graphically that, if $a = b = 6$, then $P > A$ for all θ.

d Assume $\theta = \frac{\pi}{2}$. If $a = 6$, find the value of b such that $A = P$.

e For $a = 10$ and $b = 6$, find the value(s) of θ for which $A = P$.

f If $a = b$ and $\theta = \frac{\pi}{3}$, find the value of a such that $A = P$.

9 AB is one side of a regular n-sided polygon that circumscribes a circle (i.e. each edge of the polygon is tangent to the circle). The circle has radius 1.



a Show that the area of triangle OAB is $\tan\left(\frac{\pi}{n}\right)$.

b Show that the area, A, of the polygon is given by $A = n\tan\left(\frac{\pi}{n}\right)$.

c Use a calculator to help sketch the graph of $A(x) = x\tan\left(\frac{\pi}{x}\right)$ for $x \geq 3$. Label the horizontal asymptote.

d What is the difference in area of the polygon and the circle when:

i $n = 3$ **ii** $n = 4$ **iii** $n = 12$ **iv** $n = 50$?

e State the area of an n-sided polygon that circumscribes a circle of radius r cm.

f **i** Find a formula for the area of an n-sided regular polygon that can be inscribed in a circle of radius 1.

ii Sketch the graph of this function for $x \geq 3$.

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