9

Differentiation

Objectives

- To understand the concept of **limit**.
- To understand the definition of **differentiation**.
- To understand and use the notation for the **derivative** of a polynomial function.
- To find the **gradient** of a tangent to the graph of a polynomial function by calculating its derivative.
- To understand and use the **chain rule**.
- To differentiate **rational powers**.
- To differentiate **exponential functions** and **natural logarithmic functions**.
- To differentiate **circular functions**.
- To understand and use the **product rule** and the **quotient rule**.
- To deduce the **graph of the derivative** from the graph of a function.

It is believed that calculus was discovered independently in the late seventeenth century by two great mathematicians: Isaac Newton and Gottfried Leibniz. Like most scientific breakthroughs, the discovery of calculus did not arise out of a vacuum. In fact, many mathematicians and philosophers going back to ancient times made discoveries relating to calculus.

In this chapter, we review some of the important ideas and results that have been introduced in earlier studies of calculus. We introduce the chain rule, the product rule and the quotient rule, along with the differentiation of exponential, logarithmic and circular functions.

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9A The derivative

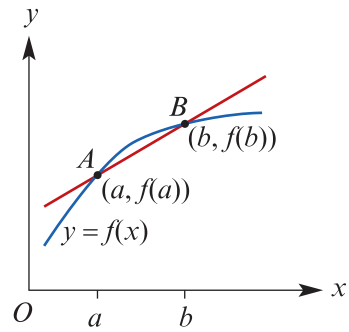
We begin this chapter by recalling the definition of average rate of change from Mathematical Methods Units 1 & 2.

Average rate of change

For any function $y = f(x)$, the **average rate of change** of y with respect to x over the interval $[a, b]$ is the gradient of the line through the two points $A(a, f(a))$ and $B(b, f(b))$.

That is:

$$\text{average rate of change} = \frac{f(b) - f(a)}{b - a}$$



Example 1

Find the average rate of change of the function with rule $f(x) = x^2 - 2x + 5$ as x changes from 1 to 5.

Solution

$$\text{Average rate of change} = \frac{\text{change in } y}{\text{change in } x}$$

$f(1) = (1)^2 - 2(1) + 5 = 4$

$f(5) = (5)^2 - 2(5) + 5 = 20$

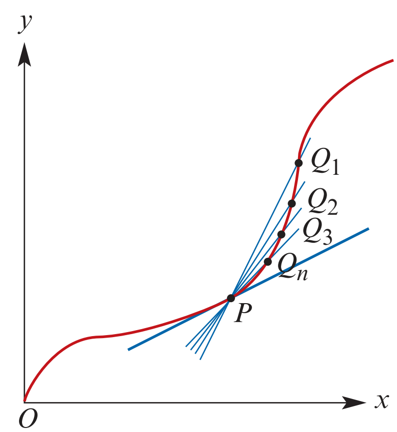
$$\text{Average rate of change} = \frac{20 - 4}{5 - 1} = 4$$

The tangent to a curve at a point

We first recall that a **chord** of a curve is a line segment joining points P and Q on the curve. A **secant** is a line through points P and Q on the curve.

The **instantaneous rate of change** at P can be defined by considering what happens when we look at a sequence of secants $PQ_1, PQ_2, PQ_3, \dots, PQ_n, \dots$, where the points Q_i get closer and closer to P.

Here we first focus our attention on the gradient of the tangent at P.



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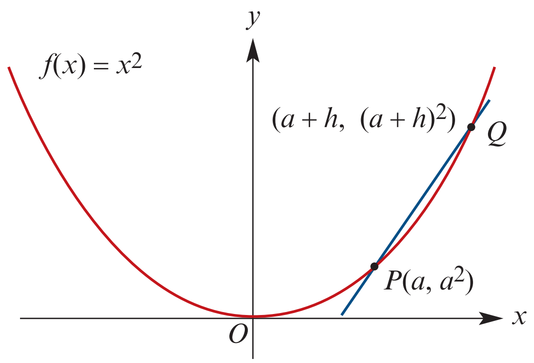
Consider the function $f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$.

The gradient of the secant PQ shown on the graph is

$$\text{gradient of } PQ = \frac{(a+h)^2 - a^2}{a+h-a}$$
$$= \frac{a^2 + 2ah + h^2 - a^2}{h}$$
$$= 2a + h$$



The limit of $2a + h$ as h approaches 0 is $2a$, and so the gradient of the tangent at P is said to be $2a$.

Note: This also can be interpreted as the instantaneous rate of change of f at $(a, f(a))$.

The straight line that passes through the point P and has gradient $2a$ is called the **tangent** to the curve at P.

It can be seen that there is nothing special about a here. The same calculation works for any real number x. The gradient of the tangent to the graph of $y = x^2$ at any point x is $2x$.

We say that the **derivative of x^2 with respect to x is $2x$**, or more briefly, we can say that the **derivative of x^2 is $2x$**.

Limit notation

The notation for the limit of $2x + h$ as h approaches 0 is

$$\lim_{h \to 0} (2x + h)$$

The derivative of a function with rule $f(x)$ may be found by:

1. finding an expression for the gradient of the line through $P(x, f(x))$ and $Q(x + h, f(x + h))$
2. finding the limit of this expression as h approaches 0.

Example 2

Consider the function $f(x) = x^3$. By first finding the gradient of the secant through $P(2, 8)$ and $Q(2 + h, (2 + h)^3)$, find the gradient of the tangent to the curve at the point $(2, 8)$.

Solution

$$\text{Gradient of } PQ = \frac{(2+h)^3 - 8}{2+h-2}$$
$$= \frac{8 + 12h + 6h^2 + h^3 - 8}{h}$$
$$= \frac{12h + 6h^2 + h^3}{h}$$
$$= 12 + 6h + h^2$$

The gradient of the tangent line at $(2, 8)$ is $\lim_{h \to 0} (12 + 6h + h^2) = 12$.

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The following example provides practice in determining limits.

Example 3

Find:

a $\lim_{h \to 0} (22x^2 + 20xh + h)$

b $\lim_{h \to 0} \dfrac{3x^2h + 2h^2}{h}$

c $\lim_{h \to 0} 12x$

d $\lim_{h \to 0} 4$

Solution

a $\lim_{h \to 0} (22x^2 + 20xh + h) = 22x^2$

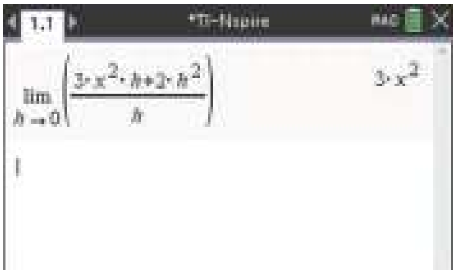
b $$\lim_{h \to 0} \frac{3x^2h + 2h^2}{h} = \lim_{h \to 0} (3x^2 + 2h)$$
$$= 3x^2$$

c $\lim_{h \to 0} 12x = 12x$

d $\lim_{h \to 0} 4 = 4$

Using the TI-Nspire

To calculate a limit, use menu > **Calculus** > **Limit** and complete as shown.



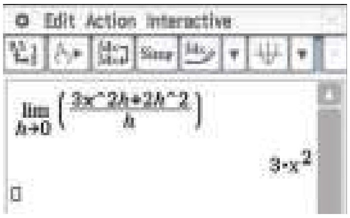
Note: The limit template can also be accessed from the 2D-template palette.
When you insert the limit template, you will notice a superscript field (small box) on the template – generally this will be left empty.

Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression
$$\frac{3x^2h + 2h^2}{h}$$

Note: Use h from the Var keyboard.

- Select lim from the Math2 keyboard .
- Enter h and 0 in the spaces provided as shown and tap EXE.



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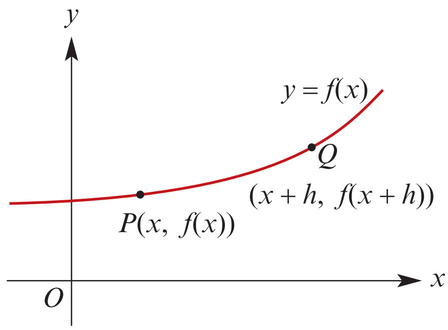
Definition of the derivative

In general, consider the graph $y = f(x)$ of a function $f \colon \mathbb{R} \to \mathbb{R}$.

$$\text{Gradient of secant } PQ = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h}$$



The gradient of the tangent to the graph of $y = f(x)$ at the point $P(x, f(x))$ is the limit of this expression as h approaches 0.

Derivative of a function

The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Warning: This definition of the derivative assumes that the limit exists. For polynomial functions, such limits always exist. But it is not true that for every function you can find the derivative at every point of its domain. This is discussed further in Sections 9L and 9M.

Differentiation by first principles

Determining the derivative of a function by evaluating the limit is called **differentiation by first principles**.

Example 4

Find $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for each of the following:

a $f(x) = 3x^2 + 2x + 2$

b $f(x) = 2 - x^3$

Solution

a
$$\begin{aligned}\frac{f(x+h) - f(x)}{h} &= \frac{3(x+h)^2 + 2(x+h) + 2 - (3x^2 + 2x + 2)}{h}\\ &= \frac{3x^2 + 6xh + 3h^2 + 2x + 2h + 2 - 3x^2 - 2x - 2}{h}\\ &= \frac{6xh + 3h^2 + 2h}{h}\\ &= 6x + 3h + 2\end{aligned}$$

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Therefore

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (6x+3h+2) = 6x+2$$

b
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{2-(x+h)^3-(2-x^3)}{h} \\ &= \frac{2-(x^3+3x^2h+3xh^2+h^3)-2+x^3}{h} \\ &= \frac{-3x^2h-3xh^2-h^3}{h} \\ &= -3x^2-3xh-h^2\end{aligned}$$

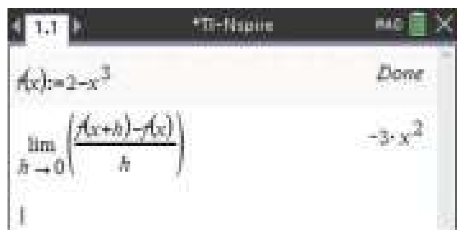
Therefore

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} (-3x^2-3xh-h^2) = -3x^2$$

Using the TI-Nspire

- Assign the function $f(x)$ as shown.
- Use (menu) > **Calculus** > **Limit** or the 2D-template palette (template key), and complete as shown.

Note: The assign command := is accessed using (ctrl) (template key) (:=). Define can also be used.



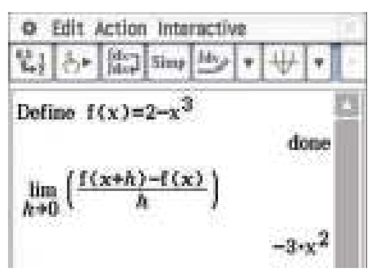
Using the Casio ClassPad

- In $\sqrt{\alpha}$ Main, enter and highlight the expression $2-x^3$. Select **Interactive** > **Define** and tap OK.
- Now enter and highlight the expression

$$\frac{f(x+h)-f(x)}{h}$$

Note: Select f from the (Math3) keyboard and x, h from the (Var) keyboard.

- Select (lim) from the (Math2) keyboard.
- Enter h and 0 in the spaces provided as shown and tap (EXE).



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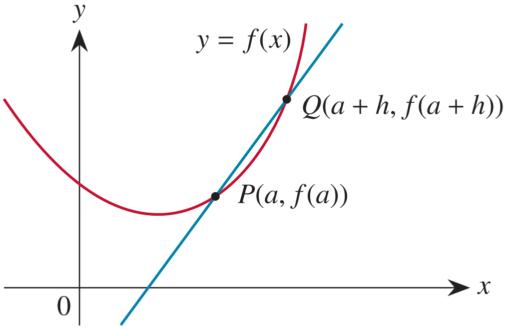
Approximating the value of the derivative

From the definition of the derivative, we can see that

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

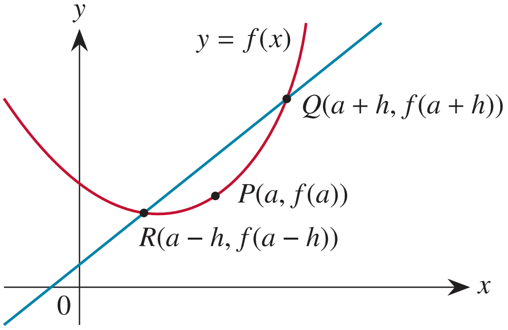
for a small value of h. This is the gradient of the secant through points $P(a, f(a))$ and $Q(a+h, f(a+h))$.



We can often obtain a better approximation by using

$$f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

for a small value of h. This is the gradient of the secant through points $R(a-h, f(a-h))$ and $Q(a+h, f(a+h))$.



This is sometimes called the **central difference approximation**. In Exercise 9A, you are asked to compare these two approximations for several functions. See the **Algorithms ad pseudocode** section of Chapter 12 for a further discussion.

Summary 9A

- The **derivative** of the function f is denoted f' and is defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.
- The value of the derivative of f at $x = a$ can be approximated by

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} \quad \text{or} \quad f'(a) \approx \frac{f(a+h) - f(a-h)}{2h}$$

for a small value of h.

Exercise 9A

Example 1

1 Find the average rate of change of the function with rule $f(x) = -x^2 + 2x + 1$ as x changes from -1 to 4.

2 Find the average rate of change of the function with rule $f(x) = 6 - x^3$ as x changes from -1 to 1.

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Example 2

3 For the curve with equation $y = x^2 + 5x$:

a Find the gradient of the secant though points P and Q, where P is the point $(2, 14)$ and Q is the point $(2 + h, (2 + h)^2 + 5(2 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(2, 14)$.

Example 3

4 Find:

a $\lim\limits_{h \to 0} \dfrac{4x^2h^2 + xh + h}{h}$

b $\lim\limits_{h \to 0} \dfrac{2x^3h - 2xh^2 + h}{h}$

c $\lim\limits_{h \to 0} (40 - 50h)$

d $\lim\limits_{h \to 0} 5h$

e $\lim\limits_{h \to 0} 5$

f $\lim\limits_{h \to 0} \dfrac{30h^2x^2 + 20h^2x + h}{h}$

g $\lim\limits_{h \to 0} \dfrac{3h^2x^3 + 2hx + h}{h}$

h $\lim\limits_{h \to 0} 3x$

i $\lim\limits_{h \to 0} \dfrac{3x^3h - 5x^2h^2 + xh}{h}$

j $\lim\limits_{h \to 0} (6x - 7h)$

5 For the curve with equation $y = x^3 - x$:

a Find the gradient of the chord PQ, where P is the point $(1, 0)$ and Q is the point $(1 + h, (1 + h)^3 - (1 + h))$.

b From the result of **a**, find the gradient of the tangent to the curve at the point $(1, 0)$.

6 If $f(x) = x^2 - 2$, simplify $\dfrac{f(x + h) - f(x)}{h}$. Hence find the derivative of $x^2 - 2$.

7 Let P and Q be points on the curve $y = x^2 + 2x + 5$ at which $x = 2$ and $x = 2 + h$ respectively. Express the gradient of the line PQ in terms of h, and hence find the gradient of the tangent to the curve $y = x^2 + 2x + 5$ at $x = 2$.

Example 4

8 For each of the following, find $f'(x)$ by finding $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$:

a $f(x) = 5x^2$

b $f(x) = 3x + 2$

c $f(x) = 5$

d $f(x) = 3x^2 + 4x + 3$

9 Consider the following two approximations for $f'(a)$, where h is small:

i $f'(a) \approx \dfrac{f(a + h) - f(a)}{h}$

ii $f'(a) \approx \dfrac{f(a + h) - f(a - h)}{2h}$

Compare these two approximations for each of the following:

a $f(x) = x^2 + 2x + 2, \ a = 2$

b $f(x) = x^2 + 6x + 7, \ a = 2$

c $f(x) = x^3 + 3x^2 + 2, \ a = 2$

d $f(x) = x^3 + 2x - 4, \ a = 2$

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9B Rules for differentiation

The derivative of x^n where n is a positive integer

Differentiating from first principles gives the following:

- For $f(x) = x$, $f'(x) = 1$.
- For $f(x) = x^2$, $f'(x) = 2x$.
- For $f(x) = x^3$, $f'(x) = 3x^2$.

This suggests the following general result:

> For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$

We can prove this result using the binomial theorem, which is discussed in Appendix A. The proof is not required to be known.

Proof Let $f(x) = x^n$, where $n \in \mathbb{N}$ with $n \geq 2$.

Then
$$\begin{aligned} f(x+h) - f(x) &= (x+h)^n - x^n \\ &= x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n - x^n \\ &= {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \\ &= nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n \end{aligned}$$

and so
$$\begin{aligned} \frac{f(x+h) - f(x)}{h} &= \frac{1}{h}\left(nx^{n-1}h + {}^nC_2 x^{n-2}h^2 + \cdots + {}^nC_{n-1} x h^{n-1} + h^n\right) \\ &= nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1} \end{aligned}$$

Thus
$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + {}^nC_{n-1} x h^{n-2} + h^{n-1}\right) \\ &= nx^{n-1} \end{aligned}$$

The derivative of a polynomial function

The following results are very useful when finding the derivative of a polynomial function.

- **Constant function:** If $f(x) = c$, then $f'(x) = 0$.
- **Multiple:** If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
  For example: if $f(x) = 5x^2$, then $f'(x) = 5(2x) = 10x$.
- **Sum:** If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
  For example: if $f(x) = x^2 + 2x$, then $f'(x) = 2x + 2$.
- **Difference:** If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
  For example: if $f(x) = x^2 - 2x$, then $f'(x) = 2x - 2$.

You will meet rules for the derivatives of products and quotients later in this chapter.

The process of finding the derivative function is called **differentiation**.

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Example 5

Find the derivative of $x^5 - 2x^3 + 2$, i.e. differentiate $x^5 - 2x^3 + 2$ with respect to x.

Solution

Let $\quad f(x) = x^5 - 2x^3 + 2$

Then $\quad f'(x) = 5x^4 - 2(3x^2) + 0$

$\qquad\qquad = 5x^4 - 6x^2$

Explanation

We use the following results:

- the derivative of x^n is nx^{n-1}
- the derivative of a number is 0
- the multiple, sum and difference rules.

Example 6

Find the derivative of $f(x) = 3x^3 - 6x^2 + 1$ and thus find $f'(1)$.

Solution

Let $\quad f(x) = 3x^3 - 6x^2 + 1$

Then $\quad f'(x) = 3(3x^2) - 6(2x) + 0$

$\qquad\qquad = 9x^2 - 12x$

$\therefore \quad f'(1) = 9 - 12 = -3$

Using the TI-Nspire

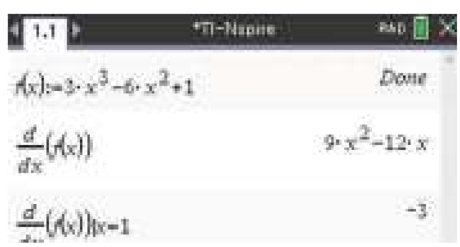
For Example 5:

- Use menu > **Calculus** > **Derivative** and complete as shown.

Note: The derivative template can also be accessed from the 2D-template palette. Alternatively, using shift – will paste the derivative template to the screen.

For Example 6:

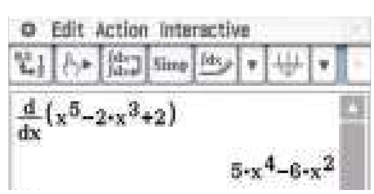
- Assign the function $f(x)$ as shown.
- Use menu > **Calculus** > **Derivative** to differentiate as shown.
- To evaluate the derivative at $x = 1$, use menu > **Calculus** > **Derivative at a Point**.



Using the Casio ClassPad

For Example 5:

- In Main $\sqrt{\alpha}$, enter and highlight the expression $x^5 - 2x^3 + 2$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.



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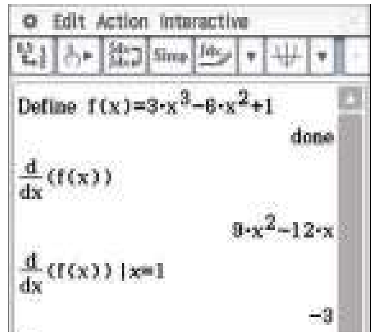
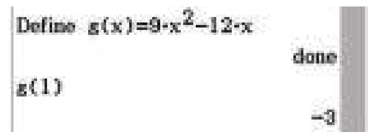
For Example 6:

- Define the function $f(x) = 3x^3 - 6x^2 + 1$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK; this will give the derivative only.
- To find the value of the derivative at $x = 1$, tap the stylus at the end of the entry line. Select | from the Math3 keyboard and type $x = 1$. Then tap EXE.



- Alternatively, define the derivative as $g(x)$ and find $g(1)$.



Finding the gradient of a tangent line

We discussed the tangent line at a point on a graph in Section 9A. We recall the following:

> The **tangent line** to the graph of the function f at the point $(a, f(a))$ is defined to be the line through $(a, f(a))$ with gradient $f'(a)$.

Example 7

For the curve determined by the rule $f(x) = 3x^3 - 6x^2 + 1$, find the gradient of the tangent line to the curve at the point $(1, -2)$.

Solution

Now $f'(x) = 9x^2 - 12x$ and so $f'(1) = 9 - 12 = -3$.

The gradient of the tangent line at the point $(1, -2)$ is -3.

Alternative notations

It was mentioned in the introduction to this chapter that the German mathematician Gottfried Leibniz was one of the two people to whom the discovery of calculus is attributed. A form of the notation he introduced is still in use today.

Leibniz notation

An alternative notation for the derivative is the following:

> If $y = x^3$, then the derivative can be denoted by $\dfrac{dy}{dx}$, and so we write $\dfrac{dy}{dx} = 3x^2$.

In general, if y is a function of x, then the derivative of y with respect to x is denoted by $\dfrac{dy}{dx}$.

Similarly, if z is a function of t, then the derivative of z with respect to t is denoted $\dfrac{dz}{dt}$.

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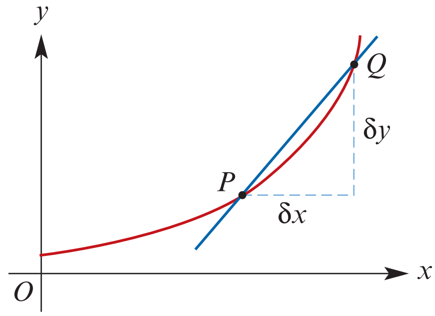
Warning: In Leibniz notation, the symbol d is not a factor and cannot be cancelled.

This notation came about because, in the eighteenth century, the standard diagram for finding the limiting gradient was labelled as shown:

- δx means a small difference in x
- δy means a small difference in y

where δ (delta) is the lowercase Greek letter d.



Example 8

a If $y = t^2$, find $\frac{dy}{dt}$.   **b** If $x = t^3 + t$, find $\frac{dx}{dt}$.   **c** If $z = \frac{1}{3}x^3 + x^2$, find $\frac{dz}{dx}$.

Solution

a
$$y = t^2$$
$$\frac{dy}{dt} = 2t$$

b
$$x = t^3 + t$$
$$\frac{dx}{dt} = 3t^2 + 1$$

c
$$z = \frac{1}{3}x^3 + x^2$$
$$\frac{dz}{dx} = x^2 + 2x$$

Example 9

a For $y = (x+3)^2$, find $\frac{dy}{dx}$.

b For $z = (2t-1)^2(t+2)$, find $\frac{dz}{dt}$.

c For $y = \frac{x^2 + 3x}{x}$, find $\frac{dy}{dx}$.

d Differentiate $y = 2x^3 - 1$ with respect to x.

Solution

a First write $y = (x+3)^2$ in expanded form:

$$y = x^2 + 6x + 9$$
$$\therefore \quad \frac{dy}{dx} = 2x + 6$$

b Expanding:

$$\begin{aligned} z &= (4t^2 - 4t + 1)(t + 2) \\ &= 4t^3 - 4t^2 + t + 8t^2 - 8t + 2 \\ &= 4t^3 + 4t^2 - 7t + 2 \end{aligned}$$
$$\therefore \quad \frac{dz}{dt} = 12t^2 + 8t - 7$$

c First simplify:

$$y = x + 3 \quad (\text{for } x \neq 0)$$
$$\therefore \quad \frac{dy}{dx} = 1 \quad (\text{for } x \neq 0)$$

d
$$y = 2x^3 - 1$$
$$\therefore \quad \frac{dy}{dx} = 6x^2$$

Operator notation

'Find the derivative of $2x^2 - 4x$ with respect to x' can also be written as 'find $\frac{d}{dx}(2x^2 - 4x)$'.

In general: $\frac{d}{dx}(f(x)) = f'(x)$.

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Example 10

Find:

a $\frac{d}{dx}(5x - 4x^3)$ **b** $\frac{d}{dz}(5z^2 - 4z)$ **c** $\frac{d}{dz}(6z^3 - 4z^2)$

Solution

a $\frac{d}{dx}(5x - 4x^3)$
$= 5 - 12x^2$

b $\frac{d}{dz}(5z^2 - 4z)$
$= 10z - 4$

c $\frac{d}{dz}(6z^3 - 4z^2)$
$= 18z^2 - 8z$

Example 11

For each of the following curves, find the coordinates of the points on the curve at which the gradient of the tangent line at that point has the given value:

a $y = x^3$, gradient = 8 **b** $y = x^2 - 4x + 2$, gradient = 0

c $y = 4 - x^3$, gradient = -6

Solution

a $y = x^3$ implies $\frac{dy}{dx} = 3x^2$

$\therefore \quad 3x^2 = 8$

$\therefore \quad x = \pm\sqrt{\frac{8}{3}} = \frac{\pm 2\sqrt{6}}{3}$

The points are $\left(\frac{2\sqrt{6}}{3}, \frac{16\sqrt{6}}{9}\right)$ and $\left(\frac{-2\sqrt{6}}{3}, \frac{-16\sqrt{6}}{9}\right)$.

b $y = x^2 - 4x + 2$ implies $\frac{dy}{dx} = 2x - 4$

$\therefore \quad 2x - 4 = 0$

$\therefore \quad x = 2$

The only point is $(2, -2)$.

c $y = 4 - x^3$ implies $\frac{dy}{dx} = -3x^2$

$\therefore \quad -3x^2 = -6$

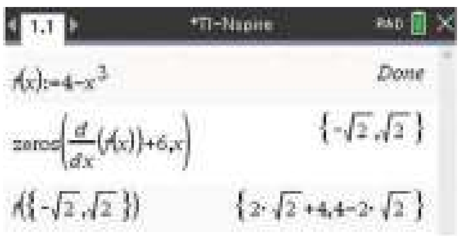
$\therefore \quad x^2 = 2$

$\therefore \quad x = \pm\sqrt{2}$

The points are $\left(2^{\frac{1}{2}}, 4 - 2^{\frac{3}{2}}\right)$ and $\left(-2^{\frac{1}{2}}, 4 + 2^{\frac{3}{2}}\right)$.

Using the TI-Nspire

- Assign the function $f(x)$ as shown.
- Use menu > **Algebra** > **Zeros** and menu > **Calculus** > **Derivative** to solve the equation $\frac{d}{dx}(f(x)) = -6$.
- Substitute in $f(x)$ to find the y-coordinates.



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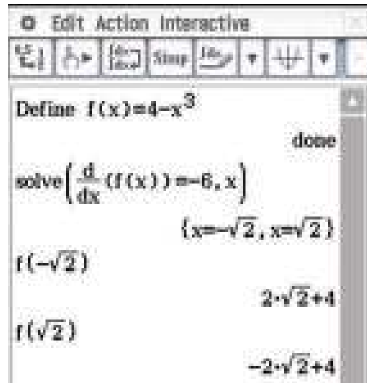
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, enter and highlight the expression $4 - x^3$.
- Go to **Interactive** > **Define** and tap OK.
- In the next entry line, type and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and tap OK.
- Type $= -6$ after $\frac{d}{dx}(f(x))$. Highlight the equation and use **Interactive** > **Equation/Inequality** > **solve**.
- Enter $f(-\sqrt{2})$ and $f(\sqrt{2})$ to find the required y-values.



An angle associated with the gradient of a curve at a point

The gradient of a curve at a point is the gradient of the tangent at that point. A straight line, the tangent, is associated with each point on the curve.

If α is the angle a straight line makes with the positive direction of the x-axis, then the gradient, m, of the straight line is equal to $\tan\alpha$. That is, $m = \tan\alpha$.

For example, if $\alpha = 135°$, then $\tan\alpha = -1$ and so the gradient is -1.

Example 12

Find the coordinates of the points on the curve with equation $y = x^2 - 7x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = -2x + 6$.

Solution

a $\frac{dy}{dx} = 2x - 7$

$2x - 7 = 1$ (as $\tan 45° = 1$)

$2x = 8$

$\therefore\ x = 4$

$y = 4^2 - 7 \times 4 + 8 = -4$

The coordinates are $(4, -4)$.

b The line $y = -2x + 6$ has gradient -2.

$2x - 7 = -2$

$2x = 5$

$\therefore\ x = \frac{5}{2}$

The coordinates are $\left(\frac{5}{2}, -\frac{13}{4}\right)$.

Increasing and decreasing functions

We have discussed strictly increasing and strictly decreasing functions in previous chapters:

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

We have the following very important results.

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If $f'(x) > 0$, for all x in the interval, then the function is strictly increasing.
(Think of the tangents at each point – they each have positive gradient.)

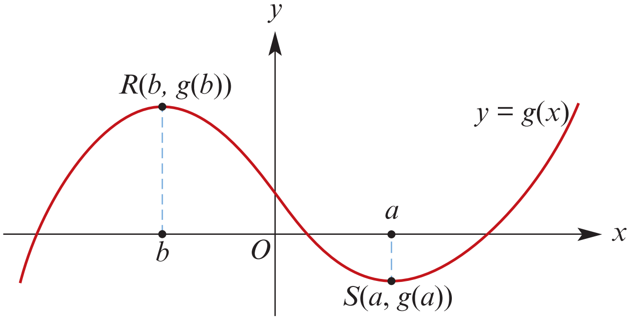
If $f'(x) < 0$, for all x in the interval, then the function is strictly decreasing.
(Think of the tangents at each point – they each have negative gradient.)

Warning: The function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is strictly increasing, but $f'(0) = 0$. This means that *strictly increasing does not imply* $f'(x) > 0$.

Sign of the derivative

Gradients of tangents can, of course, be negative or zero. They are not always positive.

At a point $(a, g(a))$ on the graph of $y = g(x)$, the gradient of the tangent is $g'(a)$.



Some features of the graph shown are:

- For $x < b$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.
- For $x = b$, the gradient of the tangent is zero, i.e. $g'(b) = 0$.
- For $b < x < a$, the gradient of any tangent is negative, i.e. $g'(x) < 0$.
- For $x = a$, the gradient of the tangent is zero, i.e. $g'(a) = 0$.
- For $x > a$, the gradient of any tangent is positive, i.e. $g'(x) > 0$.

Note: This function g is strictly decreasing on the open interval (b, a), but it is also strictly decreasing on the closed interval $[b, a]$. Similarly, the function g is strictly increasing on the intervals $[a, \infty)$ and $(-\infty, b]$.

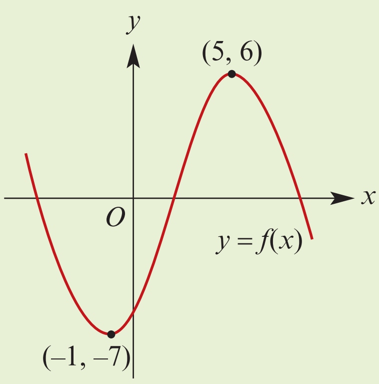
Example 13

For the graph of $f : \mathbb{R} \to \mathbb{R}$, find:

a $\{ x : f'(x) > 0 \}$

b $\{ x : f'(x) < 0 \}$

c $\{ x : f'(x) = 0 \}$



Solution

a $\{ x : f'(x) > 0 \} = \{ x : -1 < x < 5 \} = (-1, 5)$

b $\{ x : f'(x) < 0 \} = \{ x : x < -1 \} \cup \{ x : x > 5 \} = (-\infty, -1) \cup (5, \infty)$

c $\{ x : f'(x) = 0 \} = \{-1, 5\}$

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Summary 9B

- For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where $n = 1, 2, 3, \ldots$
- Constant function: If $f(x) = c$, then $f'(x) = 0$.
- Multiple: If $f(x) = k\,g(x)$, where k is a constant, then $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- Sum: If $f(x) = g(x) + h(x)$, then $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of the sum is the sum of the derivatives.
- Difference: If $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.
  That is, the derivative of the difference is the difference of the derivatives.
- Angle of inclination of tangent
  - A straight line, the tangent, is associated with each point on a smooth curve.
  - If α is the angle that a straight line makes with the positive direction of the x-axis, then the gradient of the line is given by $m = \tan\alpha$.
- Increasing and decreasing functions
  - A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
  - A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
  - If $f'(x) > 0$ for all x in the interval, then the function is strictly increasing.
  - If $f'(x) < 0$ for all x in the interval, then the function is strictly decreasing.

Skill-sheet

Exercise 9B

1 For each of the following, find the derivative with respect to x:

Example 5

a x^5 **b** $4x^7$ **c** $6x$
d $5x^2 - 4x + 3$ **e** $4x^3 + 6x^2 + 2x - 4$ **f** $5x^4 + 3x^3$
g $-2x^2 + 4x + 6$ **h** $6x^3 - 2x^2 + 4x - 6$

Example 6

2 For each of the following, find the derivative of $f(x)$ and thus find $f'(1)$:

a $f(x) = 2x^3 - 5x^2 + 1$ **b** $f(x) = -2x^3 - x^2 - 1$
c $f(x) = x^4 - 2x^3 + 1$ **d** $f(x) = x^5 - 3x^3 + 2$

Example 7

3 **a** For the curve determined by the rule $f(x) = 2x^3 - 5x^2 + 2$, find the gradient of the tangent line to the curve at the point $(1, -1)$.

b For the curve determined by the rule $f(x) = -2x^3 - 3x^2 + 2$, find the gradient of the tangent line to the curve at the point $(2, -26)$.

Example 8

4 **a** If $y = t^3$, find $\dfrac{dy}{dt}$.

b If $x = t^3 - t^2$, find $\dfrac{dx}{dt}$.

c If $z = \dfrac{1}{4}x^4 + 3x^3$, find $\dfrac{dz}{dx}$.

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Example 9

5 For each of the following, find $\frac{dy}{dx}$:

a $y = -2x$

b $y = 7$

c $y = 5x^3 - 3x^2 + 2x + 1$

d $y = \frac{2}{5}(x^3 - 4x + 6)$

e $y = (2x + 1)(x - 3)$

f $y = 3x(2x - 4)$

g $y = \frac{10x^7 + 2x^2}{x^2}, \ x \neq 0$

h $y = \frac{9x^4 + 3x^2}{x}, \ x \neq 0$

Example 10

6 Find:

a $\frac{d}{dx}(2x^2 - 5x^3)$

b $\frac{d}{dz}(-2z^2 - 6z)$

c $\frac{d}{dz}(6z^3 - 4z^2 + 3)$

d $\frac{d}{dx}(-2x - 5x^3)$

e $\frac{d}{dz}(-2z^2 - 6z + 7)$

f $\frac{d}{dz}(-z^3 - 4z^2 + 3)$

Example 11

7 Find the coordinates of the points on the curves given by the following equations at which the gradient has the given value:

a $y = 2x^2 - 4x + 1$, gradient $= -6$

b $y = 4x^3$, gradient $= 48$

c $y = x(5 - x)$, gradient $= 1$

d $y = x^3 - 3x^2$, gradient $= 0$

Example 12

8 Find the coordinates of the points on the curve with equation $y = 2x^2 - 3x + 8$ at which the tangent line:

a makes an angle of 45° with the positive direction of the x-axis

b is parallel to the line $y = 2x + 8$.

9 Find the value of x such that the tangent line to the curve $f(x) = x^2 - x$ at $(x, f(x))$:

a makes an angle of 45° with the positive direction of the x-axis

b makes an angle of 135° with the positive direction of the x-axis

c makes an angle of 60° with the positive direction of the x-axis

d makes an angle of 30° with the positive direction of the x-axis

e makes an angle of 120° with the positive direction of the x-axis.

10 For each of the following, find the angle that the tangent line to the curve $y = f(x)$ makes with the positive direction of the x-axis at the given point:

a $y = x^2 + 3x, \ (1, 4)$

b $y = -x^2 + 2x, \ (1, 1)$

c $y = x^3 + x, \ (0, 0)$

d $y = -x^3 - x, \ (0, 0)$

e $y = x^4 - x^2, \ (1, 0)$

f $y = x^4 - x^2, \ (-1, 0)$

11 **a** Differentiate $y = (2x - 1)^2$ with respect to x.

b For $y = \frac{x^3 + 2x^2}{x}$, $x \neq 0$, find $\frac{dy}{dx}$.

c Given that $y = 2x^3 - 6x^2 + 18x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} > 0$ for all x.

d Given that $y = \frac{x^3}{3} - x^2 + x$, find $\frac{dy}{dx}$. Hence show that $\frac{dy}{dx} \geq 0$ for all x.

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12 At the points on the following curves corresponding to the given values of x, find the y-coordinate and the gradient:

a $y = x^2 + 2x + 1, \ x = 3$
b $y = x^2 - x - 1, \ x = 0$
c $y = 2x^2 - 4x, \ x = -1$
d $y = (2x + 1)(3x - 1)(x + 2), \ x = 4$
e $y = (2x + 5)(3 - 5x)(x + 1), \ x = 1$
f $y = (2x - 5)^2, \ x = 2\frac{1}{2}$

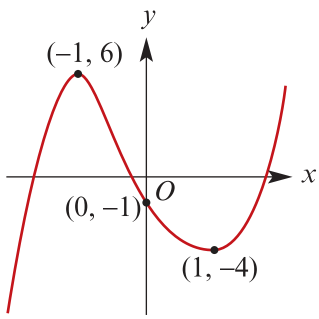
13 For the function $f(x) = 3(x - 1)^2$, find the value(s) of x for which:

a $f(x) = 0$
b $f'(x) = 0$
c $f'(x) > 0$
d $f'(x) < 0$
e $f'(x) = 10$
f $f(x) = 27$

Example 13

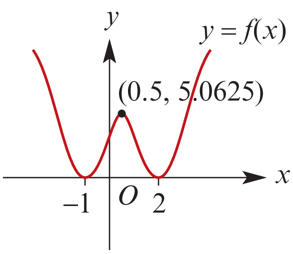
14 For the graph of $y = h(x)$ illustrated, find:

a $\{x : h'(x) > 0\}$
b $\{x : h'(x) < 0\}$
c $\{x : h'(x) = 0\}$



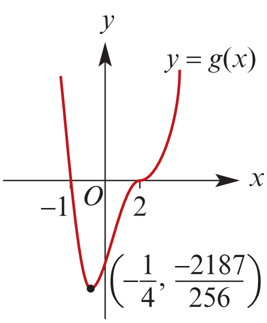
15 For the graph of $y = f(x)$ shown, find:

a $\{x : f'(x) > 0\}$
b $\{x : f'(x) < 0\}$
c $\{x : f'(x) = 0\}$



16 For the graph of $y = g(x)$ shown, find:

a $\{x : g'(x) > 0\}$
b $\{x : g'(x) < 0\}$
c $\{x : g'(x) = 0\}$



17 Find the coordinates of the points on the parabola $y = x^2 - 4x - 8$ at which:

a the gradient is zero
b the tangent is parallel to $y = 2x + 6$
c the tangent is parallel to $3x + 2y = 8$.

18 **a** Show that $f : \mathbb{R} \to \mathbb{R}, f(x) = x^3$ is a strictly increasing function for $\mathbb{R}$ by showing that $f'(x) > 0$, for all non-zero x, and showing that, if $b > 0$, then $f(b) > f(0)$ and, if $0 > b$, then $f(0) > f(b)$.

b Show that $f : \mathbb{R} \to \mathbb{R}, f(x) = -x^3$ is a strictly decreasing function for $\mathbb{R}$.

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19 **a** Show that $f: [0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is a strictly increasing function.

b Show that $f: (-\infty, 0] \to \mathbb{R}$, $f(x) = x^2$ is a strictly decreasing function.

20 For the function $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - x - 12$, show that the largest interval for which f is strictly increasing is $[\frac{1}{2}, \infty)$.

21 For each of the following, find the largest interval for which the function is strictly decreasing:

a $y = x^2 + 2x$ **b** $y = -x^2 + 4x$ **c** $y = 2x^2 + 3$ **d** $y = -2x^2 + 6x$

9C Differentiating x^n where n is a negative integer

In the previous sections we have seen how to differentiate polynomial functions. In this section we add to the family of functions that we can differentiate. In particular, we will consider functions which involve linear combinations of powers of x, where the indices may be negative integers.

e.g. $f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x^{-1}$

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = 2x + x^{-1}$

$f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = x + 3 + x^{-2}$

Note: We have reintroduced function notation to emphasise the need to consider domains.

Example 14

Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$. Find $f'(x)$ by first principles.

Solution

The gradient of secant PQ is given by

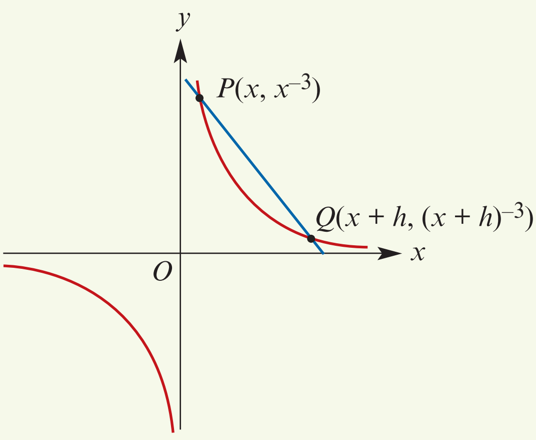
$$\frac{(x+h)^{-3} - x^{-3}}{h}$$

$$= \frac{x^3 - (x+h)^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{x^3 - (x^3 + 3x^2h + 3xh^2 + h^3)}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2h - 3xh^2 - h^3}{(x+h)^3 x^3} \times \frac{1}{h}$$

$$= \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3}$$



So the gradient of the curve at P is given by

$$\lim_{h \to 0} \frac{-3x^2 - 3xh - h^2}{(x+h)^3 x^3} = \frac{-3x^2}{x^6} = -3x^{-4}$$

Hence $f'(x) = -3x^{-4}$.

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We are now in a position to state the generalisation of the result we found in Section 9B. This result can be proved by again using the binomial theorem.

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

When n is positive, we take the domain of f to be $\mathbb{R}$, and when n is negative, we take the domain of f to be $\mathbb{R} \setminus \{0\}$.

Example 15

Find the derivative of $x^4 - 2x^{-3} + x^{-1} + 2$, $x \neq 0$.

Solution

If $\quad f(x) = x^4 - 2x^{-3} + x^{-1} + 2 \qquad$ (for $x \neq 0$)

then $\quad f'(x) = 4x^3 - 2(-3x^{-4}) + (-x^{-2}) + 0$

$\qquad = 4x^3 + 6x^{-4} - x^{-2} \qquad$ (for $x \neq 0$)

Example 16

Find the derivative f' of $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = 3x^2 - 6x^{-2} + 1$.

Solution

$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = 3(2x) - 6(-2x^{-3}) + 0$

$\qquad = 6x + 12x^{-3}$

Example 17

Find the gradient of the tangent to the curve determined by the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^2 + \frac{1}{x}$ at the point $(1, 2)$.

Solution

$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = 2x + (-x^{-2})$

$\qquad = 2x - x^{-2}$

Therefore $f'(1) = 2 - 1 = 1$. The gradient of the curve is 1 at the point $(1, 2)$.

Example 18

Show that the derivative of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-3}$ is always negative.

Solution

$f': \mathbb{R} \setminus \{0\} \to \mathbb{R}, \quad f'(x) = -3x^{-4} = \frac{-3}{x^4}$

Since x^4 is positive for all $x \neq 0$, we have $f'(x) < 0$ for all $x \neq 0$.

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Summary 9C

For $f(x) = x^n$, $f'(x) = nx^{n-1}$, where n is a non-zero integer.

For $f(x) = c$, $f'(x) = 0$, where c is a constant.

Exercise 9C

1 **a** Sketch the graph of $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{2}{x^2}$.

b Let P be the point $(1, 2)$ and Q the point $(1 + h, f(1 + h))$. Find the gradient of the secant PQ.

c Hence find the gradient of the tangent to the curve $f(x) = \dfrac{2}{x^2}$ at $(1, 2)$.

Example 14

2 **a** Let $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x - 3}$. Find $f'(x)$ by first principles.

b Let $f: \mathbb{R} \setminus \{-2\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x + 2}$. Find $f'(x)$ by first principles.

3 Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-4}$. Find $f'(x)$ by first principles.
Hint: Remember that $(x + h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$.

Example 15
Example 16

4 Differentiate each of the following with respect to x:

a $3x^{-2} + 5x^{-1} + 6$ **b** $\dfrac{5}{x^3} + 6x^2$ **c** $\dfrac{-5}{x^3} + \dfrac{4}{x^2} + 1$

d $6x^{-3} + 3x^{-2}$ **e** $\dfrac{4x^2 + 2x}{x^2}$

5 Find the derivative of each of the following:

a $\dfrac{2z^2 - 4z}{z^2}$, $z \neq 0$ **b** $\dfrac{6 + z}{z^3}$, $z \neq 0$ **c** $16 - z^{-3}$, $z \neq 0$

d $\dfrac{4z + z^3 - z^4}{z^2}$, $z \neq 0$ **e** $\dfrac{6z^2 - 2z}{z^4}$, $z \neq 0$ **f** $\dfrac{6}{x} - 3x^2$, $x \neq 0$

Example 17

6 Find the gradient of the tangent to each of the following curves at the stated point:

a $y = x^{-2} + x^3$, $x \neq 0$, at $(2, 8\frac{1}{4})$ **b** $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(4, \frac{1}{2})$

c $y = x^{-2} - \dfrac{1}{x}$, $x \neq 0$, at $(1, 0)$ **d** $y = x(x^{-1} + x^2 - x^{-3})$, $x \neq 0$, at $(1, 1)$

Example 18

7 Show that the derivative of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = -2x^{-5}$ is always positive.

8 Find the x-coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

9 Given that the curve $y = ax^2 + \dfrac{b}{x}$ has a gradient of -5 at the point $(2, -2)$, find the values of a and b.

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10 Find the gradient of the curve $y = \dfrac{2x - 4}{x^2}$ at the point where the curve crosses the x-axis.

11 The gradient of the curve $y = \dfrac{a}{x} + bx^2$ at the point $(3, 6)$ is 7. Find the values of a and b.

12 For the curve with equation $y = \dfrac{5}{3}x + kx^2 - \dfrac{8}{9}x^3$, calculate the possible values of k such that the tangents at the points with x-coordinates 1 and $-\frac{1}{2}$ are perpendicular.

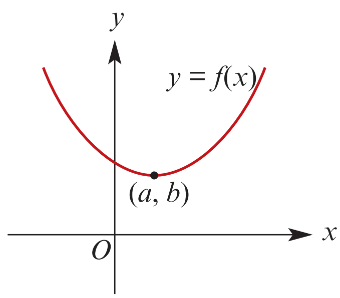
9D The graph of the derivative function

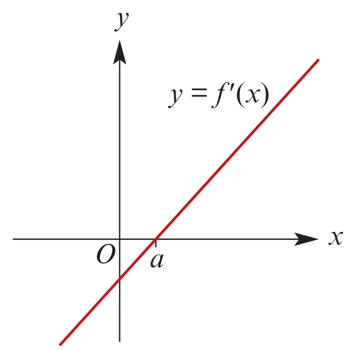
First consider the quadratic function with rule $y = f(x)$ shown in the graph on the left. The vertex is at the point with coordinates (a, b).

- For $x < a$, $f'(x) < 0$.
- For $x = a$, $f'(x) = 0$.
- For $x > a$, $f'(x) > 0$.

The graph of the derivative function with rule $y = f'(x)$ is therefore as shown on the right.

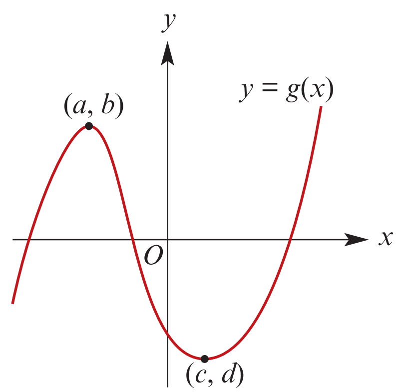
The derivative f' is known to be linear as f is quadratic.



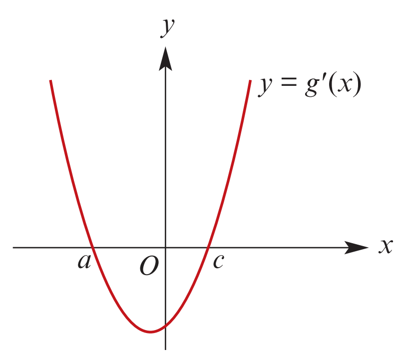


Now consider the cubic function with rule $y = g(x)$ shown in the graph.

- For $x < a$, $g'(x) > 0$.
- For $x = a$, $g'(x) = 0$.
- For $a < x < c$, $g'(x) < 0$.
- For $x = c$, $g'(x) = 0$.
- For $x > c$, $g'(x) > 0$.



The graph of the derivative function with rule $y = g'(x)$ is therefore as shown to the right. The derivative g' is known to be quadratic as g is cubic.



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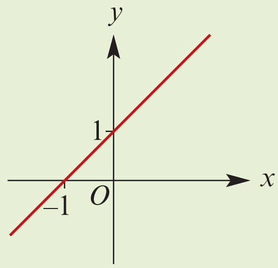
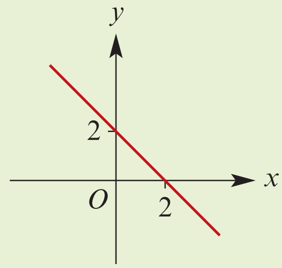
Example 19

Sketch the graph of the derivative function for each of the functions of the graphs shown:

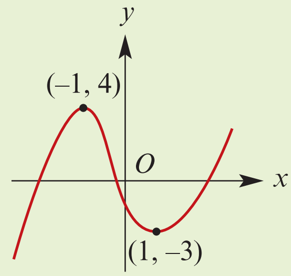
a



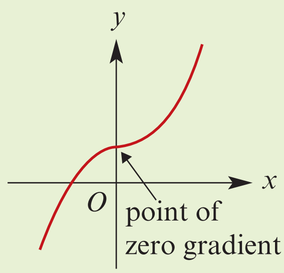
b



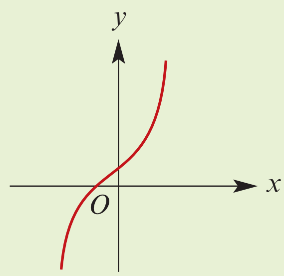
c



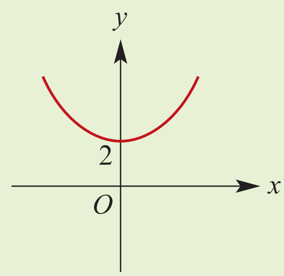
d



e



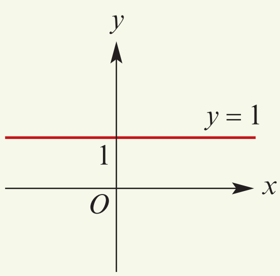
f



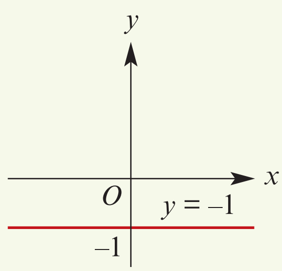
Solution

Note: Not all features of the graphs are known.

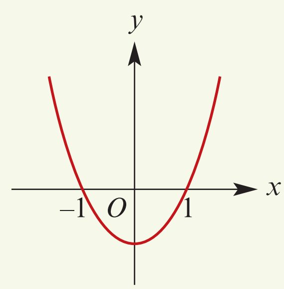
a



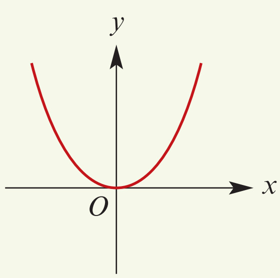
b



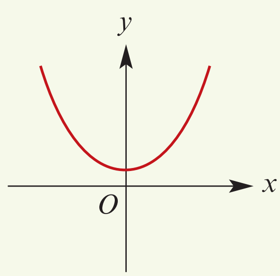
c



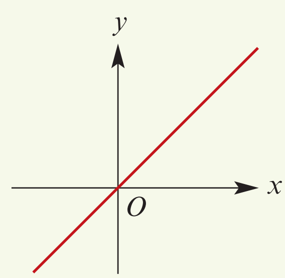
d



e



f



For some functions f, there are values of x for which the derivative $f'(x)$ is not defined. We will consider differentiability informally here and more formally in Section 9M.

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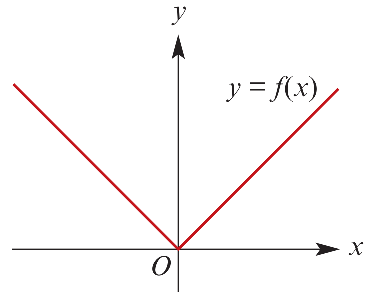
Consider the function $f: \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



Now consider the gradient of the secant through the points $(0, 0)$ and $(h, f(h))$ on the graph of $y = f(x)$:

$$\text{gradient} = \frac{f(0+h) - f(0)}{h} = \begin{cases} \dfrac{h}{h} & \text{for } h > 0 \\ \dfrac{-h}{h} & \text{for } h < 0 \end{cases}$$

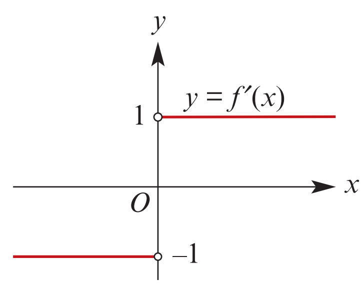
$$= \begin{cases} 1 & \text{for } h > 0 \\ -1 & \text{for } h < 0 \end{cases}$$

The gradient does not approach a unique value as $h \to 0$, and so we say $\lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist. The function f is not differentiable at $x = 0$.

The gradient of the curve $y = f(x)$ is -1 to the left of 0, and 1 to the right of 0. Therefore the derivative function $f' : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is given by

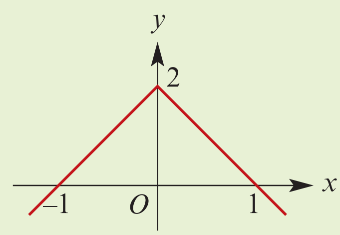
$$f'(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

The graph of f' is shown on the right.



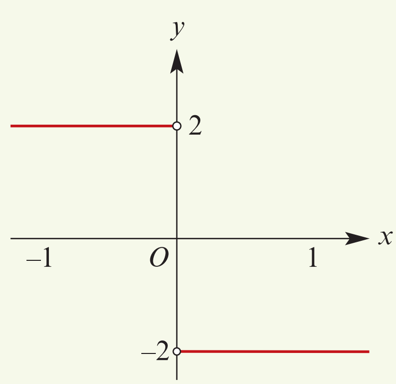
Example 20

Draw a sketch graph of f' where the graph of f is as illustrated. Indicate where f' is not defined.



Solution

The derivative does not exist at $x = 0$,
i.e. the function is not differentiable at $x = 0$.



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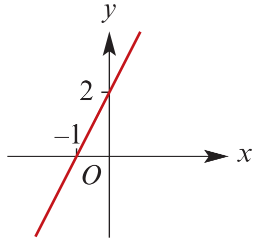
Exercise 9D

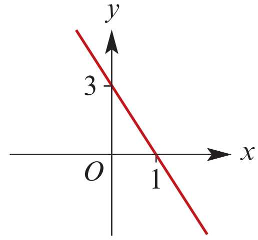
Example 19

1 Sketch the graph of the derivative function for each of the following functions:

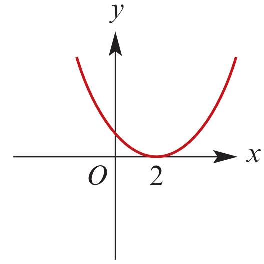
a



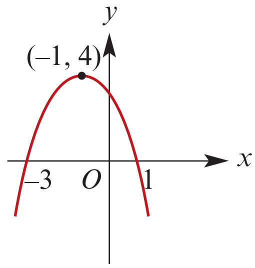
b



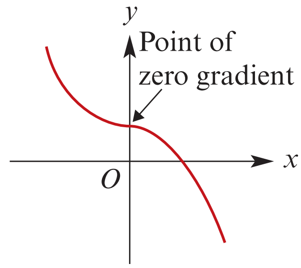
c



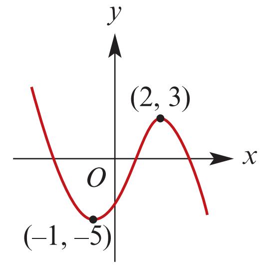
d



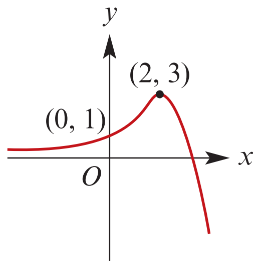
e



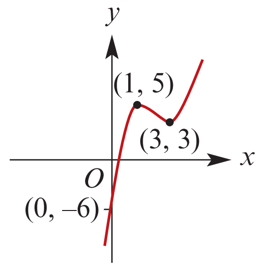
f



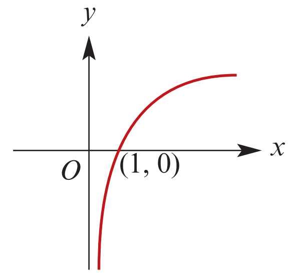
g



h



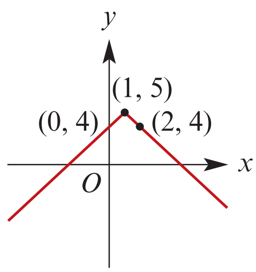
i



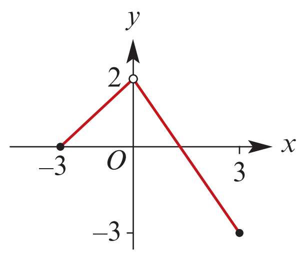
Example 20

2 Sketch the graph of the derivative function for each of the following functions:

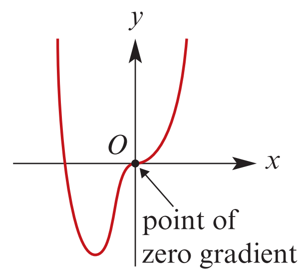
a



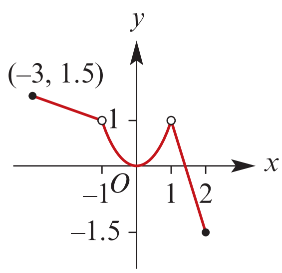
b



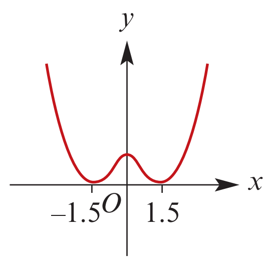
c



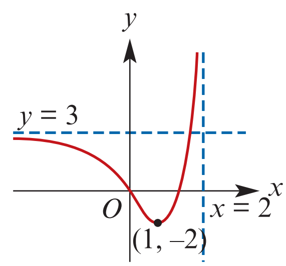
d



e



f



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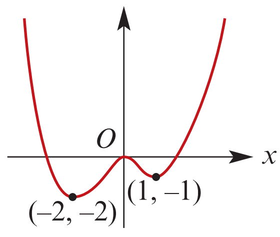
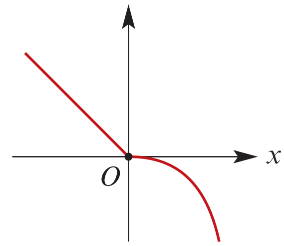
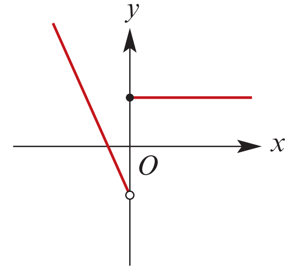
g



h

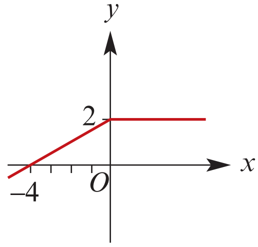


i

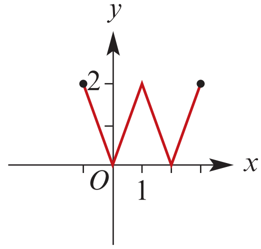


3 Match the graphs of the functions **a**–**f** with the graphs of their derivatives **A**–**F**:

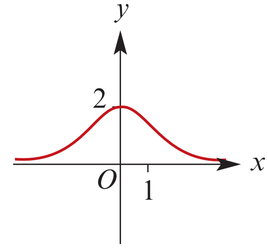
a



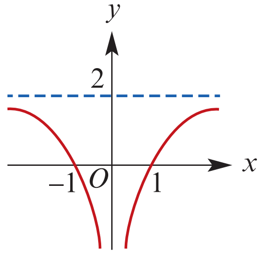
b



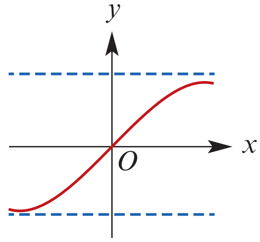
c



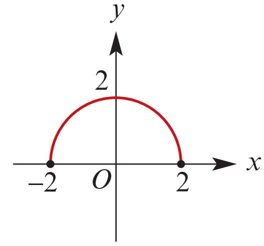
d



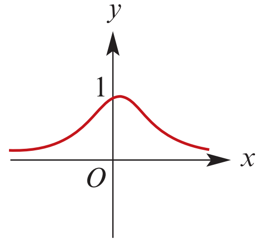
e



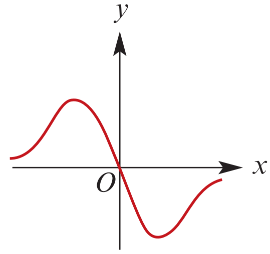
f



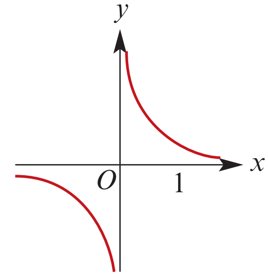
A



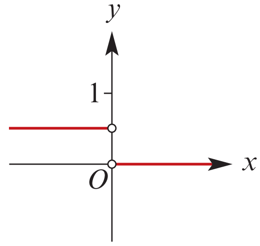
B



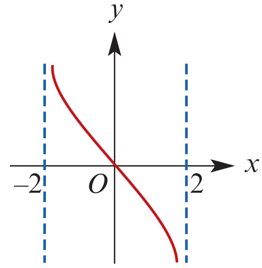
C



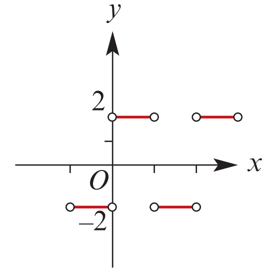
D



E



F



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4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = (x^2 - 2x)^2$.

b Using the same screen, plot the graph of $y = f'(x)$. (Do not attempt to determine the rule for $f'(x)$ first.)

c Use a calculator to determine $f'(x)$ for:

i $x = 0$ **ii** $x = 2$ **iii** $x = 1$ **iv** $x = 4$

d For $0 \le x \le 1$, find the value of x for which:

i $f(x)$ is a maximum **ii** $f'(x)$ is a maximum.

5 For $f(x) = \dfrac{x^3}{3} - x^2 + x + 1$, plot the graphs of $y = f(x)$ and $y = f'(x)$ on the same screen. Comment.

6 For $g(x) = x^3 + 2x + 1$, plot the graphs of $y = g(x)$ and $y = g'(x)$ on the same screen. Comment.

7 **a** For $h(x) = x^4 + 2x + 1$, plot the graphs of $y = h(x)$ and $y = h'(x)$ on the same screen.

b Find the value(s) of x such that:

i $h(x) = 3$ **ii** $h'(x) = 3$

9E The chain rule

An expression such as $q(x) = (x^3 + 1)^2$ may be differentiated by expanding and then differentiating each term separately. This method is a great deal more tiresome for an expression such as $q(x) = (x^3 + 1)^{30}$.

We can express $q(x) = (x^3 + 1)^2$ as the composition of two simpler functions defined by

$$u = g(x) = x^3 + 1 \quad \text{and} \quad y = f(u) = u^2$$

which are ‘chained’ together:

$$x \xrightarrow{g} u \xrightarrow{f} y$$

That is, $q(x) = (x^3 + 1)^2 = f(g(x))$, and so q is expressed as the composition $f \circ g$.

The chain rule gives a method of differentiating such functions.

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\,g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

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Proof To find the derivative of $q = f \circ g$ where $x = a$, consider the secant through the points $(a, f \circ g(a))$ and $(a + h, f \circ g(a + h))$. The gradient of this secant is

$$\frac{f \circ g(a+h) - f \circ g(a)}{h}$$

We carry out the trick of multiplying the numerator and the denominator by $g(a + h) - g(a)$. This gives

$$\frac{f(g(a+h)) - f(g(a))}{h} \times \frac{g(a+h) - g(a)}{g(a+h) - g(a)}$$

provided $g(a + h) - g(a) \neq 0$.

Now write $b = g(a)$ and $b + k = g(a + h)$ so that $k = g(a + h) - g(a)$. The expression for the gradient becomes

$$\frac{f(b+k) - f(b)}{k} \times \frac{g(a+h) - g(a)}{h}$$

The function g is continuous, since its derivative exists, and therefore

$$\lim_{h \to 0} k = \lim_{h \to 0} [g(a+h) - g(a)] = 0$$

Thus, as h approaches 0, so does k. Hence $q'(a) = f'(g(a))\, g'(a)$.

Note that this proof does not hold for a function g such that $g(a + h) - g(a) = 0$ for arbitrarily chosen small h. However, a fully rigorous proof is beyond the scope of this course.

Example 21

Differentiate $y = (4x^3 - 5x)^{-2}$.

Solution

The differentiation is undertaken using both notations:

Let $u = 4x^3 - 5x$

Then $y = u^{-2}$

We have

$$\frac{dy}{du} = -2u^{-3}$$

$$\frac{du}{dx} = 12x^2 - 5$$

Therefore

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= -2u^{-3} \cdot (12x^2 - 5)$$
$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

Let $h(x) = 4x^3 - 5x$

and $g(x) = x^{-2}$

Then $f(x) = g(h(x))$

We have

$$h'(x) = 12x^2 - 5$$
$$g'(x) = -2x^{-3}$$

Therefore

$$f'(x) = g'(h(x))\, h'(x)$$
$$= -2(h(x))^{-3}\, h'(x)$$
$$= -2(4x^3 - 5x)^{-3} \times (12x^2 - 5)$$
$$= \frac{-2(12x^2 - 5)}{(4x^3 - 5x)^3}$$

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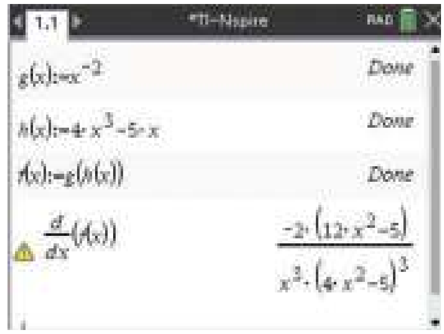
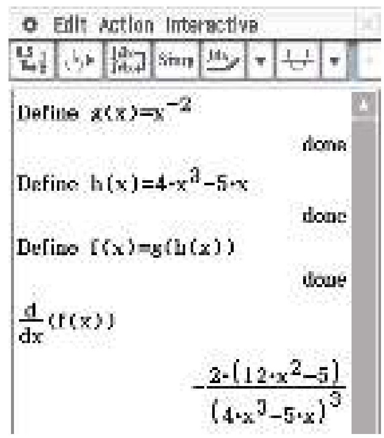
Using the TI-Nspire

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Use (menu) > **Calculus** > **Derivative** and complete as shown.



Using the Casio ClassPad

- Define $g(x)$ and $h(x)$.
- Then define $f(x) = g(h(x))$.
- Find the derivative of $f(x)$.



Example 22

Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^2 + 1}$ at the point $(1, 4)$.

Solution

Let $u = 3x^2 + 1$ then $y = 16u^{-1}$

So $\dfrac{du}{dx} = 6x$ and $\dfrac{dy}{du} = -16u^{-2}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -16u^{-2} \cdot 6x$$

$$= \frac{-96x}{(3x^2 + 1)^2}$$

$\therefore$ At $x = 1$, the gradient is $\dfrac{-96}{16} = -6$.

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Summary 9E

The chain rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $q(x) = f(g(x))$ is differentiable at x and

$$q'(x) = f'(g(x))\, g'(x)$$

Or using Leibniz notation, where $u = g(x)$ and $y = f(u)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Exercise 9E

Example 21

1 Differentiate each of the following with respect to x:

a $(x^2 + 1)^4$ **b** $(2x^2 - 3)^5$

c $(6x + 1)^4$ **d** $(ax + b)^n$

e $(ax^2 + b)^n$ **f** $(1 - x^2)^{-3}$

g $\left(x^2 - \dfrac{1}{x^2}\right)^{-3}$ **h** $(1 - x)^{-1}$

2 Differentiate each of the following with respect to x:

a $(x^2 + 2x + 1)^3$ **b** $(x^3 + 2x^2 + x)^4$

c $\left(6x^3 + \dfrac{2}{x}\right)^4$ **d** $(x^2 + 2x + 1)^{-2}$

Example 22

3 Find the gradient of the tangent to the curve with equation $y = \dfrac{16}{3x^3 + x}$ at the point $(1, 4)$.

4 Find the gradient of the tangent to the curve with equation $y = \dfrac{1}{x^2 + 1}$ at the points $(1, \frac{1}{2})$ and $(-1, \frac{1}{2})$.

5 Given that $f'(x) = \sqrt{3x + 4}$ and $g(x) = x^2 - 1$, find $F'(x)$ where $F(x) = f(g(x))$.

6 Differentiate each of the following with respect to x, giving the answer in terms of $f(x)$ and $f'(x)$:

a $[f(x)]^n$, where n is a positive integer

b $\dfrac{1}{f(x)}$, where $f(x) \neq 0$

7 Find the value of x for which the gradient of the tangent to the curve $y = \dfrac{1}{3x - x^2}$ is equal to 0.

8 Let $h(x) = f(g(x))$. If $g(3) = 4$, $g'(3) = 6$ and $f'(4) = 8$, find $h'(3)$

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9F Differentiating rational powers

Before using the chain rule to differentiate rational powers, we will show how to differentiate $x^{\frac{1}{2}}$ and $x^{\frac{1}{3}}$ by first principles.

Example 23

Differentiate each of the following by first principles:

a $f(x) = x^{\frac{1}{2}},\ x > 0$

b $g(x) = x^{\frac{1}{3}},\ x \neq 0$

Solution

a
$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\\ &= \frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\ &= \frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

$$\therefore\quad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

b We use the identity

$$a^3 - b^3 = (a-b)(a^2+ab+b^2)$$

By observing that $\left(a^{\frac{1}{3}}\right)^3 = a$ and $\left(b^{\frac{1}{3}}\right)^3 = b$, we obtain

$$a - b = \left(a^{\frac{1}{3}} - b^{\frac{1}{3}}\right)\left(a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}\right)$$

and therefore

$$a^{\frac{1}{3}} - b^{\frac{1}{3}} = \frac{a-b}{a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$$

We now have

$$\begin{aligned}\frac{g(x+h)-g(x)}{h} &= \frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h}\\ &= \frac{x+h-x}{h\left((x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}\\ &= \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\end{aligned}$$

Hence

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3x^{\frac{2}{3}}}$$

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Note: We can prove that $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1})$ for $n \geq 2$. We could use this result to find the derivative of $x^{\frac{1}{n}}$ by first principles, but instead we will use the chain rule.

Using the chain rule

If y is a one-to-one function of x, then using the chain rule in the form $\frac{dy}{du} = \frac{dy}{dx} \cdot \frac{dx}{du}$ with $y = u$, we have

$$1 = \frac{dy}{dx} \cdot \frac{dx}{dy}$$

Thus $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} \quad \text{for } \dfrac{dx}{dy} \neq 0$

Now let $y = x^{\frac{1}{n}}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

We have $y^n = x$ and so $\frac{dx}{dy} = ny^{n-1}$. Therefore

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

For $y = x^{\frac{1}{n}}$, $\frac{dy}{dx} = \frac{1}{n}x^{\frac{1}{n}-1}$, where $n \in \mathbb{Z} \setminus \{0\}$ and $x > 0$.

This result may now be extended to rational powers.

Let $y = x^{\frac{p}{q}}$, where $p, q \in \mathbb{Z} \setminus \{0\}$.

Write $y = \left(x^{\frac{1}{q}}\right)^p$. Let $u = x^{\frac{1}{q}}$. Then $y = u^p$. The chain rule yields

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\
&= pu^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= p\left(x^{\frac{1}{q}}\right)^{p-1} \cdot \frac{1}{q}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-\frac{1}{q}}x^{\frac{1}{q}-1} \\
&= \frac{p}{q}x^{\frac{p}{q}-1}
\end{aligned}$$

Thus the result for integer powers has been extended to rational powers. In fact, the analogous result holds for any non-zero real power:

For $f(x) = x^a$, $f'(x) = ax^{a-1}$, where $a \in \mathbb{R} \setminus \{0\}$ and $x > 0$.

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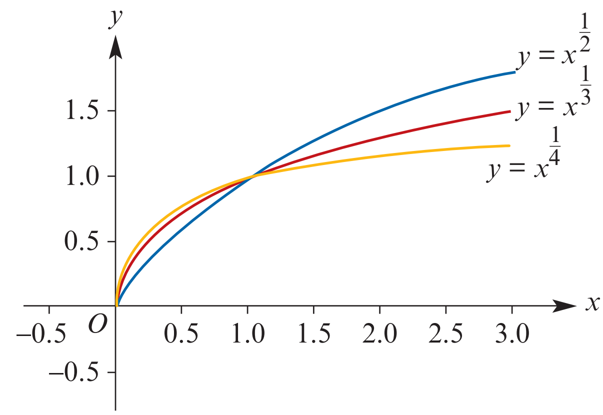
This result is stated for $x > 0$, as $(-3)^{\frac{1}{2}}$ is not defined, although $(-2)^{\frac{1}{3}}$ is defined.

The graphs of $y = x^{\frac{1}{2}}$, $y = x^{\frac{1}{3}}$ and $y = x^{\frac{1}{4}}$ are shown.

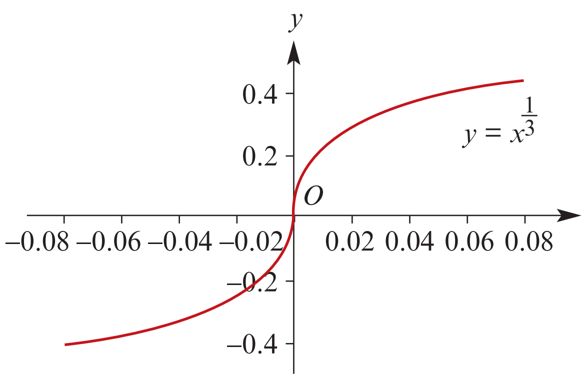
The domain of each has been taken to be $\mathbb{R}^+$.



The figure to the right is the graph of the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$.

Note that the values shown here are $-0.08 \le x \le 0.08$.

From this it can be seen that the tangent to $y = x^{\frac{1}{3}}$ at the origin is on the y-axis.



Example 24

Find the derivative of each of the following with respect to x:

a $2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}$

b $\sqrt[3]{x^2 + 2x}$

Solution

a
$$\frac{d}{dx}\left(2x^{-\frac{1}{5}} + 3x^{\frac{2}{7}}\right)$$
$$= 2\left(\frac{-1}{5}x^{-\frac{6}{5}}\right) + 3\left(\frac{2}{7}x^{-\frac{5}{7}}\right)$$
$$= -\frac{2}{5}x^{-\frac{6}{5}} + \frac{6}{7}x^{-\frac{5}{7}}$$

b
$$\frac{d}{dx}\left(\sqrt[3]{x^2 + 2x}\right)$$
$$= \frac{d}{dx}\left((x^2 + 2x)^{\frac{1}{3}}\right)$$
$$= \frac{1}{3}(x^2 + 2x)^{-\frac{2}{3}}(2x + 2) \qquad \text{(chain rule)}$$
$$= \frac{2x + 2}{3\sqrt[3]{(x^2 + 2x)^2}}$$

Summary 9F

For any non-zero rational number $r = \frac{p}{q}$, if $f(x) = x^r$, then $f'(x) = rx^{r-1}$.

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Exercise 9F

Example 23

1 Differentiate $2x^{\frac{1}{2}}$ by first principles.

Example 24a

2 Find the derivative of each of the following with respect to x:

a $x^{\frac{1}{5}}$ **b** $x^{\frac{5}{2}}$ **c** $x^{\frac{5}{2}} - x^{\frac{3}{2}},\ x > 0$

d $3x^{\frac{1}{2}} - 4x^{\frac{5}{3}}$ **e** $x^{-\frac{6}{7}}$ **f** $x^{-\frac{1}{4}} + 4x^{\frac{1}{2}}$

3 Find the gradient of the tangent to the curve for each of the following at the stated value for x:

a $f(x) = x^{\frac{1}{3}}$ where $x = 27$ **b** $f(x) = x^{\frac{1}{3}}$ where $x = -8$

c $f(x) = x^{\frac{2}{3}}$ where $x = 27$ **d** $f(x) = x^{\frac{5}{4}}$ where $x = 16$

Example 24b

4 Find the derivative of each of the following with respect to x:

a $\sqrt{2x+1}$ **b** $\sqrt{4-3x}$ **c** $\sqrt{x^2+2}$

d $\sqrt[3]{4-3x}$ **e** $\dfrac{x^2+2}{\sqrt{x}}$ **f** $3\sqrt{x}\,(x^2+2x)$

5 **a** Show that $\dfrac{d}{dx}\left(\sqrt{x^2 \pm a^2}\right) = \dfrac{x}{\sqrt{x^2 \pm a^2}}$.

b Show that $\dfrac{d}{dx}\left(\sqrt{a^2 - x^2}\right) = \dfrac{-x}{\sqrt{a^2 - x^2}}$.

6 If $y = (x + \sqrt{x^2+1})^2$, show that $\dfrac{dy}{dx} = \dfrac{2y}{\sqrt{x^2+1}}$.

7 Find the derivative with respect to x of each of the following:

a $\sqrt{x^2+2}$ **b** $\sqrt[3]{x^2-5x}$ **c** $\sqrt[5]{x^2+2x}$

9G Differentiation of e^x

In this section we investigate the derivative of functions of the form $f(x) = a^x$. We will see that Euler's number e has the special property that $f'(x) = f(x)$ where $f(x) = e^x$.

First consider $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2^x$.

To find the derivative of f we recall that:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{2^{x+h} - 2^x}{h} \\ &= 2^x \lim_{h\to 0} \frac{2^h - 1}{h} \\ &= 2^x f'(0) \end{aligned}$$

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We can investigate this limit numerically to find that $f'(0) \approx 0.6931$ and therefore

$$f'(x) \approx 0.6931 \times 2^x$$

Now consider $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3^x$. Then, as for f, it may be shown that $g'(x) = 3^x g'(0)$. We find $g'(0) \approx 1.0986$ and hence

$$g'(x) \approx 1.0986 \times 3^x$$

The question now arises:

> Can we find a number b between 2 and 3 such that, if $f(x) = b^x$, then $f'(0) = 1$ and therefore $f'(x) = b^x$?

Using a calculator or a spreadsheet, we can investigate the limit as $h \to 0$ of $\dfrac{b^h - 1}{h}$, for various values of b between 2 and 3.

This investigation is carried out in the spreadsheet shown on the right.

Start by taking values for b between 2.71 and 2.72 (first table) and finding $f'(0)$ for each of these values. From these results it may be seen that the required value of b lies between 2.718 and 2.719.

The investigation is continued with values of b between 2.718 and 2.719 (second table). From this the required value of b is seen to lie between 2.7182 and 2.7183.

b	$f'(0)$
2.710	0.996949
2.711	0.997318
2.712	0.997686
2.713	0.998055
2.714	0.998424
2.715	0.998792
2.716	0.999160
2.717	0.999528
2.718	0.999896
2.719	1.000264
2.720	1.000632

b	$f'(0)$
2.7180	0.999896
2.7181	0.999933
2.7182	0.999970
2.7183	1.000007
2.7184	1.000043
2.7185	1.000080
2.7186	1.000117
2.7187	1.000154
2.7188	1.000191
2.7189	1.000227
2.7190	1.000264

The required value of b is in fact Euler's number e, which was introduced in Chapter 5.

Our results can be recorded:

For $f(x) = e^x$, $f'(x) = e^x$.

Next consider $y = e^{kx}$ where $k \in \mathbb{R}$. The chain rule can be used to find the derivative:

Let $u = kx$. Then $y = e^u$. The chain rule yields

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= e^u \cdot k \\ &= ke^{kx} \end{aligned}$$

For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

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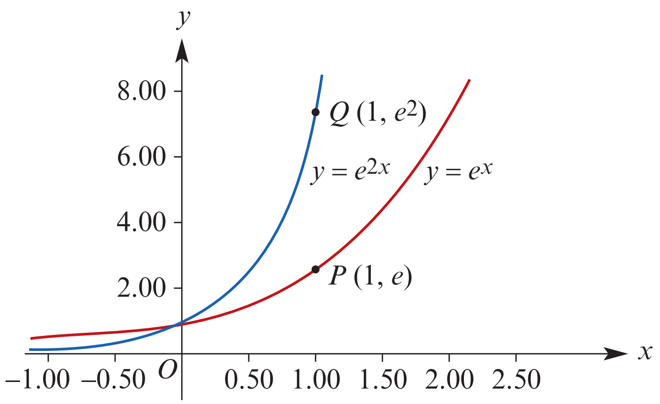
The graph illustrates the case where $k = 2$:

- the gradient of $y = e^x$ at the point $P(1, e)$ is e
- the gradient of $y = e^{2x}$ at the point $Q(1, e^2)$ is $2e^2$.



Example 25

Find the derivative of each of the following with respect to x:

a e^{3x} **b** e^{-2x} **c** e^{2x+1} **d** $\dfrac{1}{e^{2x}} + e^{3x}$

Solution

a Let $y = e^{3x}$. Then $\dfrac{dy}{dx} = 3e^{3x}$.

b Let $y = e^{-2x}$. Then $\dfrac{dy}{dx} = -2e^{-2x}$.

c Let $y = e^{2x+1}$. Then

$$y = e^{2x} \cdot e \quad \text{(index laws)}$$
$$= e \cdot e^{2x}$$
$$\therefore \quad \frac{dy}{dx} = 2e \cdot e^{2x}$$
$$= 2e^{2x+1}$$

d Let $y = \dfrac{1}{e^{2x}} + e^{3x}$. Then

$$y = e^{-2x} + e^{3x}$$
$$\therefore \quad \frac{dy}{dx} = -2e^{-2x} + 3e^{3x}$$

Example 26

Find the derivative of each of the following with respect to x:

a e^{x^2} **b** e^{x^2+4x}

Solution

a Let $y = e^{x^2}$ and $u = x^2$.
Then $y = e^u$ and the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u \cdot 2x$$
$$= 2xe^{x^2}$$

b Let $y = e^{x^2+4x}$ and $u = x^2 + 4x$.
Then $y = e^u$ and the chain rule yields

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= e^u(2x + 4)$$
$$= (2x + 4)\, e^{x^2+4x}$$

In general, for $h(x) = e^{f(x)}$, the chain rule gives $h'(x) = f'(x)\, e^{f(x)}$.

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Example 27

Find the gradient of the tangent to the curve $y = e^{2x} + 4$ at the point:

a $(0, 5)$

b $(1, e^2 + 4)$

Solution

We have $\frac{dy}{dx} = 2e^{2x}$.

a When $x = 0$, $\frac{dy}{dx} = 2$.

The gradient at $(0, 5)$ is 2.

b When $x = 1$, $\frac{dy}{dx} = 2e^2$.

The gradient at $(1, e^2 + 4)$ is $2e^2$.

Example 28

For each of the following, first find the derivative with respect to x. Then evaluate the derivative at $x = 2$, given that $f(2) = 0$, $f'(2) = 4$ and $f'(e^2) = 5$.

a $e^{f(x)}$

b $f(e^x)$

Solution

a Let $y = e^{f(x)}$ and $u = f(x)$. Then $y = e^u$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= e^u f'(x)\\ &= e^{f(x)} f'(x)\end{aligned}$$

When $x = 2$, $\frac{dy}{dx} = e^0 \times 4 = 4$.

b Let $y = f(e^x)$ and $u = e^x$. Then $y = f(u)$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx}\\ &= f'(u) \cdot e^x\\ &= f'(e^x) \cdot e^x\end{aligned}$$

When $x = 2$, $\frac{dy}{dx} = f'(e^2) \cdot e^2 = 5e^2$.

Summary 9G

For $f(x) = e^{kx}$, $f'(x) = ke^{kx}$, where $k \in \mathbb{R}$.

Exercise 9G

Example 25

1 Find the derivative of each of the following with respect to x:

a e^{5x}

b $7e^{-3x}$

c $3e^{-4x} + e^x - x^2$

d $\dfrac{e^{2x} - e^x + 1}{e^x}$

e $\dfrac{4e^{2x} - 2e^x + 1}{2e^{2x}}$

f $e^{2x} + e^4 + e^{-2x}$

Example 26

2 Find the derivative of each of the following with respect to x:

a e^{-2x^3}

b $e^{x^2} + 3x + 1$

c $e^{x^2-4x} + 3x + 1$

d $e^{x^2-2x+3} - x$

e $e^{\frac{1}{x}}$, $x \neq 0$

f $e^{x^{\frac{1}{2}}}$

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Example 27

3 Find the gradient of the tangent to the curve $y = e^{\frac{x}{2}} + 4x$ at the point:

a $(0, 1)$ **b** $(1, e^{\frac{1}{2}} + 4)$

4 Find the gradient of the tangent to the curve $y = e^{x^2+3x} + 2x$ at the point:

a $(0, 1)$ **b** $(1, e^4 + 2)$

Example 28

5 Find the derivative with respect to x of:

a $e^{2f(x)}$ **b** $f(e^{2x})$

6 Find the derivative with respect to x of:

a $(e^{2x} - 1)^4$ **b** $e^{\sqrt{x}}$ **c** $\sqrt{e^x - 1}$

d $e^{x^{\frac{2}{3}}}$ **e** $e^{(x-1)(x-2)}$ **f** e^{e^x}

9H Differentiation of the natural logarithm function

For the function with rule $f(x) = e^x$, we have seen that $f'(x) = e^x$.

This will be used to find the derivative of $g\colon \mathbb{R}^+ \to \mathbb{R}$, $g(x) = \log_e(kx)$ where $k > 0$.

Let $y = \log_e(kx)$ and solve for x:

$$e^y = kx$$

$$\therefore \quad x = \frac{1}{k}e^y$$

From our observation above:

$$\frac{dx}{dy} = \frac{1}{k}e^y$$

Since $e^y = kx$, this gives

$$\frac{dx}{dy} = \frac{kx}{k} = x$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}$$

Let $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(kx)$ where $k > 0$.

Then $f'\colon \mathbb{R}^+ \to \mathbb{R}$, $f'(x) = \dfrac{1}{x}$.

Example 29

Find the derivative of each of the following with respect to x:

a $\log_e(5x)$, $x > 0$ **b** $\log_e(5x + 3)$, $x > \dfrac{-3}{5}$

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Solution

a Let $y = \log_e(5x)$ for $x > 0$.

Then $\dfrac{dy}{dx} = \dfrac{1}{x}$.

Alternatively, let $u = 5x$. Then $y = \log_e u$ and the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 5 = \frac{5}{u} = \frac{1}{x}$$

b Let $y = \log_e(5x + 3)$ for $x > \dfrac{-3}{5}$.

Let $u = 5x + 3$. Then $y = \log_e u$ and the chain rule gives

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times 5 = \frac{5}{u} = \frac{5}{5x+3}$$

In general, if $y = \log_e(ax + b)$ for $x > \dfrac{-b}{a}$, then $\dfrac{dy}{dx} = \dfrac{a}{ax+b}$.

Note: Let $y = \log_e(-x)$, $x < 0$. Using the chain rule with $u = -x$ gives $\dfrac{dy}{dx} = \dfrac{1}{-x} \times (-1) = \dfrac{1}{x}$.

Example 30

Differentiate each of the following with respect to x:

a $\log_e(x^2 + 2)$

b $(\log_e x)^2$, $x > 0$

Solution

a We use the chain rule.

Let $y = \log_e(x^2 + 2)$ and $u = x^2 + 2$.

Then $y = \log_e u$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 2x = \frac{2x}{x^2+2}$$

b We use the chain rule.

Let $y = (\log_e x)^2$ and $u = \log_e x$.

Then $y = u^2$.

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \cdot \frac{1}{x} = \frac{2\log_e x}{x}$$

Summary 9H

- If $y = \log_e(ax + b)$ for $x > \dfrac{-b}{a}$, then $\dfrac{dy}{dx} = \dfrac{a}{ax+b}$.
- If $h(x) = \log_e(f(x))$, then the chain rule gives $h'(x) = \dfrac{f'(x)}{f(x)}$.

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Exercise 9H

Example 29

1 Find the derivative of each of the following with respect to x:

a $y = 2\log_e x$
b $y = 2\log_e(2x)$
c $y = x^2 + 3\log_e(2x)$
d $y = 3\log_e x + \frac{1}{x}$
e $y = 3\log_e(4x) + x$
f $y = \log_e(x + 1)$
g $y = \log_e(2x + 4)$
h $y = \log_e(3x - 1)$
i $y = \log_e(6x - 1)$

Example 30

2 Find the derivative of each of the following with respect to x:

a $y = \log_e(x^3)$
b $y = (\log_e x)^3$
c $y = \log_e(x^2 + x - 1)$
d $y = \log_e(x^3 + x^2)$
e $y = \log_e((2x + 3)^2)$
f $y = \log_e((3 - 2x)^2)$

3 For each of the following, find $f'(x)$:

a $f(x) = \log_e(x^2 + 1)$
b $f(x) = \log_e(e^x)$

4 Find the y-coordinate and the gradient of the tangent to the curve at the point corresponding to the given value of x:

a $y = \log_e x$, $x > 0$, at $x = e$
b $y = \log_e(x^2 + 1)$ at $x = e$
c $y = \log_e(-x)$, $x < 0$, at $x = -e$
d $y = x + \log_e x$ at $x = 1$
e $y = \log_e(x^2 - 2x + 2)$ at $x = 1$
f $y = \log_e(2x - 1)$ at $x = \frac{3}{2}$

5 Find $f'(1)$ if $f(x) = \log_e \sqrt{x^2 + 1}$.

6 Differentiate $\log_e(1 + x + x^2)$.

7 If $f(x) = \log_e(x^2 + 1)$, find $f'(3)$.

8 Given that $f(0) = 2$ and $f'(0) = 4$, find $\frac{d}{dx}\big(\log_e(f(x))\big)$ when $x = 0$.

9I Derivatives of circular functions

In this section we find the derivatives of sin, cos and tan.

The derivative of $\sin(k\theta)$

We first consider the sine function. The following proof uses a trigonometric identity from Specialist Mathematics Units 1 & 2 and is beyond the scope of this course, but it is important to know that the result can be proved.

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If $f: \mathbb{R} \to \mathbb{R}$, $f(\theta) = \sin\theta$, then $f'(\theta) = \cos\theta$.

Proof We use the identity

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

Consider points $P(\theta, \sin\theta)$ and $Q(\theta + h, \sin(\theta + h))$ on the graph of $f(\theta) = \sin\theta$. The gradient of the secant PQ is

$$\begin{aligned}\frac{\sin(\theta + h) - \sin\theta}{h} &= \frac{\sin\theta\cos h + \cos\theta\sin h - \sin\theta}{h} \\ &= \frac{\sin\theta\cdot(\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}\end{aligned}$$

We now consider what happens as $h \to 0$. We use two limit results (the second limit is proved below and the first limit then follows using a trigonometric identity):

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \quad\text{and}\quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

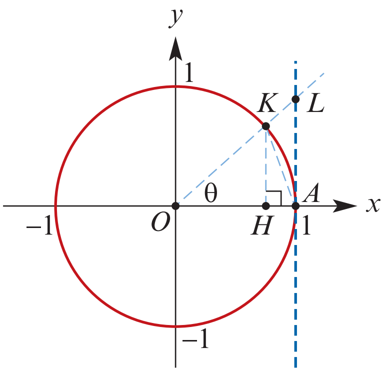
Therefore

$$\begin{aligned}f'(\theta) &= \lim_{h\to 0}\left(\frac{\sin\theta\cdot(\cos h - 1)}{h} + \frac{\cos\theta\sin h}{h}\right) \\ &= \sin\theta\times 0 + \cos\theta\times 1 \\ &= \cos\theta\end{aligned}$$

We now prove the following result.

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof Let K be a point on the unit circle as shown, and let $\angle KOH = \theta$. The coordinates of K are $(\cos\theta, \sin\theta)$. Point H is on the x-axis such that $\angle KHO$ is a right angle.



Draw a tangent to the circle at $A(1, 0)$. The line OK intersects this tangent at $L(1, \tan\theta)$.

The area of sector OAK is $\frac{1}{2}\theta$.

Thus $\quad$ area $\triangle OAK \le \frac{1}{2}\theta \le$ area $\triangle OAL$

i.e. $\quad \frac{1}{2}OA\cdot HK \le \frac{1}{2}\theta \le \frac{1}{2}OA\cdot AL$

This implies that $\sin\theta \le \theta \le \tan\theta$.

For $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta > 0$, and so we can divide both inequalities by $\sin\theta$ to obtain

$$1 \le \frac{\theta}{\sin\theta} \le \frac{1}{\cos\theta}$$

As θ approaches 0, the value of $\cos\theta$ approaches 1, and so $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$.

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We now turn our attention to the function $f(\theta) = \sin(k\theta)$. The graph of $y = \sin(k\theta)$ is obtained from the graph of $y = \sin\theta$ by a dilation of factor $\frac{1}{k}$ from the y-axis (and so this immediately suggests that the gradient will change by a factor of k).

We use the chain rule to determine $f'(\theta)$.

Let $y = \sin(k\theta)$ and let $u = k\theta$. Then $y = \sin u$ and therefore

$$\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} = \cos u \cdot k = k\cos(k\theta)$$

For $f : \mathbb{R} \to \mathbb{R}, \ f(\theta) = \sin(k\theta)$

$f' : \mathbb{R} \to \mathbb{R}, f'(\theta) = k\cos(k\theta)$

The derivative of $\cos(k\theta)$

We next find the derivative of $\cos(k\theta)$. We first note the following:

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad \sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$$

These results will be used in the following way.

Let $y = \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

Now let $u = \frac{\pi}{2} - \theta$. Then $y = \sin u$. The chain rule gives

$$\begin{aligned}\frac{dy}{d\theta} = \frac{dy}{du} \cdot \frac{du}{d\theta} &= \cos u \cdot (-1) \\ &= -\cos\left(\frac{\pi}{2} - \theta\right) \\ &= -\sin\theta\end{aligned}$$

We have the following results:

- For $f : \mathbb{R} \to \mathbb{R}, \ f(\theta) = \cos\theta$

  $f' : \mathbb{R} \to \mathbb{R}, f'(\theta) = -\sin\theta$
- For $f : \mathbb{R} \to \mathbb{R}, \ f(\theta) = \cos(k\theta)$

  $f' : \mathbb{R} \to \mathbb{R}, f'(\theta) = -k\sin(k\theta)$

The derivative of $\tan(k\theta)$

For convenience, we introduce a new function, called **secant**, given by

$$\sec\theta = \frac{1}{\cos\theta}$$

We can write $\sin^n\theta = (\sin\theta)^n$ and $\cos^n\theta = (\cos\theta)^n$.

Here we find the derivative of $\tan\theta$ by first principles. In Section 9K we show another method.

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If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Proof Consider points $P(\theta, \tan\theta)$ and $Q(\theta + h, \tan(\theta + h))$ on the graph of $f(\theta) = \tan\theta$. The gradient of the secant PQ is

$$\begin{aligned}\frac{\tan(\theta + h) - \tan\theta}{h} &= \left(\frac{\sin(\theta + h)}{\cos(\theta + h)} - \frac{\sin\theta}{\cos\theta}\right) \times \frac{1}{h} \\ &= \left(\frac{\sin(\theta + h)\cos(\theta) - \cos(\theta + h)\sin(\theta)}{\cos(\theta + h)\cos(\theta)}\right) \times \frac{1}{h} \\ &= \frac{\sin h}{h\cos(\theta + h)\cos(\theta)}\end{aligned}$$

We now consider what happens as $h \to 0$:

$$\lim_{h\to 0}\cos(\theta + h) = \cos\theta \quad \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

Therefore

$$f'(\theta) = \lim_{h\to 0}\left(\frac{\sin h}{h\cos(\theta + h)\cos(\theta)}\right) = \frac{1}{\cos^2\theta} = \sec^2\theta$$

We can use the chain rule to show that, if $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Example 31

Find the derivative with respect to θ of each of the following:

a $\sin(2\theta)$ **b** $\tan(3\theta)$ **c** $\sin^2(2\theta)$

d $\sin^2(2\theta + 1)$ **e** $\cos^3(4\theta + 1)$ **f** $\tan(3\theta^2 + 1)$

Solution

a Let $y = \sin(2\theta)$. Then $\frac{dy}{d\theta} = 2\cos(2\theta)$.

b Let $y = \tan(3\theta)$. Then $\frac{dy}{d\theta} = 3\sec^2(3\theta)$.

c Let $y = \sin^2(2\theta)$ and $u = \sin(2\theta)$.
Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= 2u\cdot 2\cos(2\theta) \\ &= 4u\cos(2\theta) \\ &= 4\sin(2\theta)\cos(2\theta)\end{aligned}$$

d Let $y = \sin^2(2\theta + 1)$ and $u = \sin(2\theta + 1)$.
Then $y = u^2$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= 2u\cdot 2\cos(2\theta + 1) \\ &= 4\sin(2\theta + 1)\cos(2\theta + 1)\end{aligned}$$

e Let $y = \cos^3(4\theta + 1)$ and $u = \cos(4\theta + 1)$.
Then $y = u^3$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= 3u^2\cdot(-4)\sin(4\theta + 1) \\ &= -12\cos^2(4\theta + 1)\sin(4\theta + 1)\end{aligned}$$

f Let $y = \tan(3\theta^2 + 1)$ and $u = 3\theta^2 + 1$.
Then $y = \tan u$. Using the chain rule:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{dy}{du}\cdot\frac{du}{d\theta} \\ &= \sec^2 u\cdot 6\theta \\ &= 6\theta\sec^2(3\theta^2 + 1)\end{aligned}$$

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Example 32

Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of θ:

a $y = \cos\theta$ at $\theta = \frac{\pi}{4}$ and $\theta = \frac{\pi}{2}$ **b** $y = \tan\theta$ at $\theta = 0$ and $\theta = \frac{\pi}{4}$

Solution

a Let $y = \cos\theta$. Then $\frac{dy}{d\theta} = -\sin\theta$.

When $\theta = \frac{\pi}{4}$, we have $y = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$ and $\frac{dy}{d\theta} = -\sin\left(\frac{\pi}{4}\right) = \frac{-1}{\sqrt{2}}$.

So the gradient at $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ is $\frac{-1}{\sqrt{2}}$.

When $\theta = \frac{\pi}{2}$, we have $y = 0$ and $\frac{dy}{d\theta} = -1$. The gradient at $\left(\frac{\pi}{2}, 0\right)$ is -1.

b Let $y = \tan\theta$. Then $\frac{dy}{d\theta} = \sec^2\theta$.

When $\theta = 0$, we have $y = 0$ and $\frac{dy}{d\theta} = 1$. The gradient at $(0, 0)$ is 1.

When $\theta = \frac{\pi}{4}$, we have $y = 1$ and $\frac{dy}{d\theta} = 2$. The gradient at $\left(\frac{\pi}{4}, 1\right)$ is 2.

Summary 9I

- If $f: \mathbb{R} \to \mathbb{R}$, $f(\theta) = \sin(k\theta)$, then $f': \mathbb{R} \to \mathbb{R}$, $f'(\theta) = k\cos(k\theta)$.
- If $f: \mathbb{R} \to \mathbb{R}$, $f(\theta) = \cos(k\theta)$, then $f': \mathbb{R} \to \mathbb{R}$, $f'(\theta) = -k\sin(k\theta)$.
- If $f(\theta) = \tan(k\theta)$, then $f'(\theta) = k\sec^2(k\theta)$.

Exercise 9I

Example 31

1 Find the derivative with respect to x of each of the following:

a $\sin(5x)$ **b** $\cos(5x)$ **c** $\tan(5x)$ **d** $\sin^2 x$ **e** $\tan(3x + 1)$

f $\cos(x^2 + 1)$ **g** $\sin^2\left(x - \frac{\pi}{4}\right)$ **h** $\cos^2\left(x - \frac{\pi}{3}\right)$ **i** $\sin^3\left(2x + \frac{\pi}{6}\right)$ **j** $\cos^3\left(2x - \frac{\pi}{4}\right)$

Example 32

2 Find the y-coordinate and the gradient of the tangent at the points on the following curves corresponding to the given values of x:

a $y = \sin(2x)$ at $x = \frac{\pi}{8}$ **b** $y = \sin(3x)$ at $x = \frac{\pi}{6}$ **c** $y = 1 + \sin(3x)$ at $x = \frac{\pi}{6}$

d $y = \cos^2(2x)$ at $x = \frac{\pi}{4}$ **e** $y = \sin^2(2x)$ at $x = \frac{\pi}{4}$ **f** $y = \tan(2x)$ at $x = \frac{\pi}{8}$

3 For each of the following, find $f'(x)$:

a $f(x) = 5\cos x - 2\sin(3x)$ **b** $f(x) = \cos x + \sin x$

c $f(x) = \sin x + \tan x$ **d** $f(x) = \tan^2 x$

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4 Find the derivative of each of the following. (Change degrees to radians first.)

a $2\cos x°$ **b** $3\sin x°$ **c** $\tan(3x)°$

5 **a** If $y = -\log_e(\cos x)$, find $\frac{dy}{dx}$. **b** If $y = -\log_e(\tan x)$, find $\frac{dy}{dx}$.

6 **a** If $y = e^{2\sin x}$, find $\frac{dy}{dx}$. **b** If $y = e^{\cos(2x)}$, find $\frac{dy}{dx}$.

9J The product rule

In the next two sections, we introduce two more rules for differentiation. The first of these is the **product rule**.

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

For example, consider $F(x) = (x^2 + 3x)(4x + 5)$. Then F is the product of two functions f and g, where $f(x) = x^2 + 3x$ and $g(x) = 4x + 5$. The product rule gives:

$$\begin{aligned} F'(x) &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \\ &= (x^2 + 3x) \cdot 4 + (4x + 5) \cdot (2x + 3) \\ &= 4x^2 + 12x + 8x^2 + 22x + 15 \\ &= 12x^2 + 34x + 15 \end{aligned}$$

This could also have been found by multiplying $x^2 + 3x$ by $4x + 5$ and then differentiating.

The product rule (function notation)

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Proof By the definition of the derivative of F, we have

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x)}{h} \end{aligned}$$

Adding and subtracting $f(x+h)\,g(x)$:

$$\begin{aligned} F'(x) &= \lim_{h \to 0} \frac{f(x+h)\,g(x+h) - f(x)\,g(x) + [f(x+h)\,g(x) - f(x+h)\,g(x)]}{h} \\ &= \lim_{h \to 0} \left[f(x+h) \cdot \left(\frac{g(x+h) - g(x)}{h} \right) + g(x) \cdot \left(\frac{f(x+h) - f(x)}{h} \right) \right] \end{aligned}$$

Since f and g are differentiable, we obtain

$$\begin{aligned} F'(x) &= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \left(\frac{g(x+h) - g(x)}{h} \right) + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right) \\ &= f(x) \cdot g'(x) + g(x) \cdot f'(x) \end{aligned}$$

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We can state the product rule in Leibniz notation and give a geometric interpretation.

The product rule (Leibniz notation)

If $y = uv$, where u and v are functions of x, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

In the following figure, the white region represents $y = uv$ and the shaded region δy, as explained below.

δv	$u\delta v$	$\delta u \delta v$
v	uv	$v\delta u$
	u	δu

$$\begin{aligned}\delta y &= (u + \delta u)(v + \delta v) - uv \\ &= uv + v\delta u + u\delta v + \delta u\delta v - uv \\ &= v\delta u + u\delta v + \delta u\delta v\end{aligned}$$

$$\therefore \quad \frac{\delta y}{\delta x} = v\frac{\delta u}{\delta x} + u\frac{\delta v}{\delta x} + \frac{\delta u}{\delta x}\frac{\delta v}{\delta x}\delta x$$

In the limit, as $\delta x \to 0$, we have

$$\frac{\delta u}{\delta x} = \frac{du}{dx}, \qquad \frac{\delta v}{\delta x} = \frac{dv}{dx} \qquad \text{and} \qquad \frac{\delta y}{\delta x} = \frac{dy}{dx}$$

Therefore

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Example 33

Differentiate each of the following with respect to x:

a $(2x^2 + 1)(5x^3 + 16)$

b $x^3(3x - 5)^4$

Solution

a Let $y = (2x^2 + 1)(5x^3 + 16)$. Let $u = 2x^2 + 1$ and $v = 5x^3 + 16$.

Then $\frac{du}{dx} = 4x$ and $\frac{dv}{dx} = 15x^2$.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} &= u\frac{dv}{dx} + v\frac{du}{dx} \\ &= (2x^2 + 1) \cdot 15x^2 + (5x^3 + 16) \cdot 4x \\ &= 30x^4 + 15x^2 + 20x^4 + 64x \\ &= 50x^4 + 15x^2 + 64x\end{aligned}$$

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b Let $y = x^3(3x-5)^4$. Let $u = x^3$ and $v = (3x-5)^4$.

Then $\dfrac{du}{dx} = 3x^2$ and $\dfrac{dv}{dx} = 12(3x-5)^3$, using the chain rule.

The product rule gives:

$$\begin{aligned}\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} &= 12x^3(3x-5)^3 + (3x-5)^4 \cdot 3x^2 \\ &= (3x-5)^3[12x^3 + 3x^2(3x-5)] \\ &= (3x-5)^3[12x^3 + 9x^3 - 15x^2] \\ &= (3x-5)^3(21x^3 - 15x^2) \\ &= 3x^2(7x-5)(3x-5)^3\end{aligned}$$

Example 34

For $F : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $F(x) = x^{-3}(10x^2-5)^3$, find $F'(x)$.

Solution

Let $f(x) = x^{-3}$ and $g(x) = (10x^2-5)^3$.

Then $f'(x) = -3x^{-4}$ and $g'(x) = 60x(10x^2-5)^2$ using the chain rule.

By the product rule:

$$\begin{aligned}F'(x) &= x^{-3} \cdot 60x(10x^2-5)^2 + (10x^2-5)^3 \cdot (-3x^{-4}) \\ &= (10x^2-5)^2\,[60x^{-2} + (10x^2-5)\cdot(-3x^{-4})] \\ &= (10x^2-5)^2\left(\frac{60x^2 - 30x^2 + 15}{x^4}\right) \\ &= \frac{(10x^2-5)^2(30x^2+15)}{x^4}\end{aligned}$$

Example 35

Differentiate each of the following with respect to x:

a $e^x(2x^2+1)$

b $e^x\sqrt{x-1}$

Solution

a Use the product rule.

Let $y = e^x(2x^2+1)$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x(2x^2+1) + 4xe^x \\ &= e^x(2x^2+4x+1)\end{aligned}$$

b Use the product rule and the chain rule.

Let $y = e^x\sqrt{x-1}$. Then

$$\begin{aligned}\frac{dy}{dx} &= e^x\sqrt{x-1} + \frac{1}{2}e^x(x-1)^{-\frac{1}{2}} \\ &= e^x\sqrt{x-1} + \frac{e^x}{2\sqrt{x-1}} \\ &= \frac{2e^x(x-1) + e^x}{2\sqrt{x-1}} \\ &= \frac{2xe^x - e^x}{2\sqrt{x-1}}\end{aligned}$$

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Example 36

Find the derivative of each of the following with respect to x:

a $2x^2 \sin(2x)$ **b** $e^{2x} \sin(2x+1)$ **c** $\cos(4x)\sin(2x)$

Solution

a Let $y = 2x^2 \sin(2x)$.

Applying the product rule:

$$\frac{dy}{dx} = 4x\sin(2x) + 4x^2\cos(2x)$$

b Let $y = e^{2x}\sin(2x+1)$.

Applying the product rule:

$$\frac{dy}{dx} = 2e^{2x}\sin(2x+1) + 2e^{2x}\cos(2x+1)$$
$$= 2e^{2x}[\sin(2x+1) + \cos(2x+1)]$$

c Let $y = \cos(4x)\sin(2x)$. Then the product rule gives

$$\frac{dy}{dx} = -4\sin(4x)\sin(2x) + 2\cos(2x)\cos(4x)$$

Summary 9J

The product rule

Let $F(x) = f(x) \cdot g(x)$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Exercise 9J

Example 33
Example 34

1 Find the derivative of each of the following with respect to x, using the product rule:

a $(2x^2+6)(2x^3+1)$ **b** $3x^{\frac{1}{2}}(2x+1)$ **c** $3x(2x-1)^3$

d $4x^2(2x^2+1)^2$ **e** $(3x+1)^{\frac{3}{2}}(2x+4)$ **f** $(x^2+1)\sqrt{2x-4}$

g $x^3(3x^2+2x+1)^{-1}$ **h** $x^4\sqrt{2x^2-1}$ **i** $x^2\sqrt[3]{x^2+2x}$

j $x^{-2}(5x^2-4)^3$ **k** $x^{-3}(x^3-4)^2$ **l** $x^3\sqrt[5]{x^3-x}$

Example 35

2 Find $f'(x)$ for each of the following:

a $f(x) = e^x(x^2+1)$ **b** $f(x) = e^{2x}(x^3+3x+1)$

c $f(x) = e^{4x+1}(x+1)^2$ **d** $f(x) = e^{-4x}\sqrt{x+1},\ x \ge -1$

3 For each of the following, find $f'(x)$:

a $f(x) = x\log_e x,\ x > 0$ **b** $f(x) = 2x^2\log_e x,\ x > 0$

c $f(x) = e^x\log_e x,\ x > 0$ **d** $f(x) = x\log_e(-x),\ x < 0$

4 Differentiate each of the following with respect to x:

a x^4e^{-2x} **b** e^{2x+3} **c** $(e^{2x}+x)^{\frac{3}{2}}$ **d** $\frac{1}{x}e^x$

e $e^{\frac{1}{2}x^2}$ **f** $(x^2+2x+2)e^{-x}$

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5 Find each of the following:

a $\frac{d}{dx}\left(e^x f(x)\right)$ **b** $\frac{d}{dx}\left(\frac{e^x}{f(x)}\right)$ **c** $\frac{d}{dx}\left(e^{f(x)}\right)$ **d** $\frac{d}{dx}\left(e^x (f(x))^2\right)$

Example 36

6 Differentiate each of the following with respect to x:

a $x^3 \cos x$ **b** $(1 + x^2) \cos x$ **c** $e^{-x} \sin x$ **d** $6x \cos x$ **e** $\sin(3x) \cos(4x)$

f $\tan(2x) \sin(2x)$ **g** $12x \sin x$ **h** $x^2 e^{\sin x}$ **i** $x^2 \cos^2 x$ **j** $e^x \tan x$

7 For each of the following, find $f'(\pi)$:

a $f(x) = e^x \sin x$ **b** $f(x) = \cos^2(2x)$

8 Given that $f(1) = 2$ and $f'(1) = 4$, find the derivative of $f(x) \log_e(x)$ when $x = 1$.

9K The quotient rule

Let $F(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

For example, if

$$F(x) = \frac{x^3 + 2x}{x^5 + 2}$$

then F can be considered as a quotient of two functions f and g, where $f(x) = x^3 + 2x$ and $g(x) = x^5 + 2$. The quotient rule gives

$$\begin{aligned} F'(x) &= \frac{(x^5 + 2)(3x^2 + 2) - (x^3 + 2x)5x^4}{(x^5 + 2)^2} \\ &= \frac{3x^7 + 6x^2 + 2x^5 + 4 - 5x^7 - 10x^5}{(x^5 + 2)^2} \\ &= \frac{-2x^7 - 8x^5 + 6x^2 + 4}{(x^5 + 2)^2} \end{aligned}$$

The quotient rule (function notation)

Let $F(x) = \frac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$$

Proof The quotient rule can be proved from first principles, but instead we will use the product rule and the chain rule.

We can write $F(x) = f(x) \cdot h(x)$, where $h(x) = [g(x)]^{-1}$. Using the chain rule, we have

$$h'(x) = -[g(x)]^{-2} \cdot g'(x)$$

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Therefore, using the product rule, we obtain

$$\begin{aligned}F'(x) &= f(x)\cdot h'(x) + h(x)\cdot f'(x)\\ &= -f(x)\cdot[g(x)]^{-2}\cdot g'(x) + [g(x)]^{-1}\cdot f'(x)\\ &= [g(x)]^{-2}\big(-f(x)\cdot g'(x) + g(x)\cdot f'(x)\big)\\ &= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}\end{aligned}$$

The quotient rule (Leibniz notation)

If $y = \dfrac{u}{v}$, where u and v are functions of x and $v \neq 0$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

Example 37

Find the derivative of $\dfrac{x-2}{x^2+4x+1}$ with respect to x.

Solution

Let $y = \dfrac{x-2}{x^2+4x+1}$. The quotient rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{x^2+4x+1-(x-2)(2x+4)}{(x^2+4x+1)^2}\\ &= \frac{x^2+4x+1-(2x^2-8)}{(x^2+4x+1)^2}\\ &= \frac{-x^2+4x+9}{(x^2+4x+1)^2}\end{aligned}$$

Example 38

Differentiate each of the following with respect to x:

a $\dfrac{e^x}{e^{2x}+1}$

b $\dfrac{\sin x}{x+1}$, $x \neq -1$

Solution

a Let $y = \dfrac{e^x}{e^{2x}+1}$.

Applying the quotient rule:

$$\begin{aligned}\frac{dy}{dx} &= \frac{(e^{2x}+1)e^x - e^x\cdot 2e^{2x}}{(e^{2x}+1)^2}\\ &= \frac{e^{3x}+e^x-2e^{3x}}{(e^{2x}+1)^2}\\ &= \frac{e^x-e^{3x}}{(e^{2x}+1)^2}\end{aligned}$$

b Let $y = \dfrac{\sin x}{x+1}$ for $x \neq -1$.

Applying the quotient rule:

$$\frac{dy}{dx} = \frac{(x+1)\cos x - \sin x}{(x+1)^2}$$

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Using the quotient rule to find the derivative of tan θ

Let $y = \tan\theta$. We write $y = \dfrac{\sin\theta}{\cos\theta}$ and apply the quotient rule to find the derivative:

$$\begin{aligned}\frac{dy}{d\theta} &= \frac{\cos\theta\cos\theta - \sin\theta\cdot(-\sin\theta)}{(\cos\theta)^2}\\ &= \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta} \qquad \text{(by the Pythagorean identity)}\\ &= \sec^2\theta\end{aligned}$$

Summary 9K

The quotient rule

Let $F(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$. If $f'(x)$ and $g'(x)$ exist, then

$$F'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$$

Exercise 9K

Example 37

1 Find the derivative of each of the following with respect to x:

a $\dfrac{x}{x+4}$ **b** $\dfrac{x^2-1}{x^2+1}$ **c** $\dfrac{x^{\frac{1}{2}}}{1+x}$ **d** $\dfrac{(x+2)^3}{x^2+1}$

e $\dfrac{x-1}{x^2+2}$ **f** $\dfrac{x^2+1}{x^2-1}$ **g** $\dfrac{3x^2+2x+1}{x^2+x+1}$ **h** $\dfrac{2x+1}{2x^3+2x}$

2 Find the y-coordinate and the gradient at the point on the curve corresponding to the given value of x:

a $y = (2x+1)^4x^2$ at $x = 1$ **b** $y = x^2\sqrt{x+1}$ at $x = 0$

c $y = x^2(2x+1)^{\frac{1}{2}}$ at $x = 0$ **d** $y = \dfrac{x}{x^2+1}$ at $x = 1$

e $y = \dfrac{2x+1}{x^2+1}$ at $x = 1$

3 For each of the following, find $f'(x)$:

a $f(x) = (x+1)\sqrt{x^2+1}$ **b** $f(x) = (x^2+1)\sqrt{x^3+1},\ x > -1$

c $f(x) = \dfrac{2x+1}{x+3},\ x \neq -3$

Example 38

4 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{e^x}{e^{3x}+3}$ **b** $f(x) = \dfrac{\cos x}{x+1},\ x \neq -1$ **c** $f(x) = \dfrac{\log_e x}{x+1},\ x > 0$

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5 For each of the following, find $f'(x)$:

a $f(x) = \dfrac{\log_e x}{x}, \quad x > 0$

b $f(x) = \dfrac{\log_e x}{x^2 + 1}, \quad x > 0$

6 Find $f'(x)$ for each of the following:

a $f(x) = \dfrac{e^{3x}}{e^{3x} + 3}$

b $f(x) = \dfrac{e^x + 1}{e^x - 1}$

c $f(x) = \dfrac{e^{2x} + 2}{e^{2x} - 2}$

7 For each of the following, find $f'(\pi)$:

a $f(x) = \dfrac{2x}{\cos x}$

b $f(x) = \dfrac{3x^2 + 1}{\cos x}$

c $f(x) = \dfrac{e^x}{\cos x}$

d $f(x) = \dfrac{\sin x}{x}$

9L Limits and continuity

Limits

It is not the intention of this course to provide a formal introduction to limits. We require only an intuitive understanding of limits and some fairly obvious rules for how to handle them.

The notation $\lim\limits_{x \to a} f(x) = p$ says that the limit of $f(x)$, as x approaches a, is p. We can also say: 'As x approaches a, $f(x)$ approaches p.'

This means that we can make the value of $f(x)$ as close as we like to p, provided we choose x-values close enough to a.

We have met a similar idea earlier in the course. For example, we have seen that $\lim\limits_{x \to \infty} f(x) = 4$ for the function with rule $f(x) = \dfrac{1}{x} + 4$. The graph of $y = f(x)$ can get as close as we like to the line $y = 4$, just by taking larger and larger values of x.

As we will see, for many functions (in particular, for polynomial functions), the limit at a particular point is simply the value of the function at that point.

Example 39

Find $\lim\limits_{x \to 2} 3x^2$.

Solution

$\lim\limits_{x \to 2} 3x^2 = 3(2)^2 = 12$

Explanation

As x gets closer and closer to 2, the value of $3x^2$ gets closer and closer to 12.

If the function is not defined at the value for which the limit is to be found, a different procedure is used.

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Example 40

For $f(x) = \dfrac{2x^2 - 5x + 2}{x - 2}$, $x \neq 2$, find $\lim_{x \to 2} f(x)$.

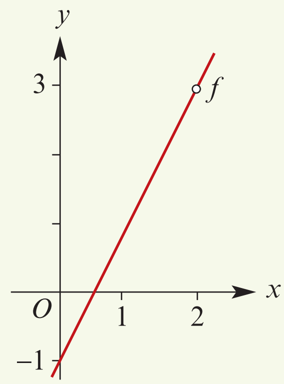
Solution

Observe that

$$\begin{aligned} f(x) &= \frac{2x^2 - 5x + 2}{x - 2} \\ &= \frac{(2x - 1)(x - 2)}{x - 2} \\ &= 2x - 1 \qquad \text{(for } x \neq 2\text{)} \end{aligned}$$

Hence $\lim_{x \to 2} f(x) = 3$.

The graph of $f : \mathbb{R} \setminus \{2\} \to \mathbb{R}$, $f(x) = 2x - 1$ is shown.



We can investigate Example 40 further by looking at the values of the function as we take x-values closer and closer to 2.

Observe that $f(x)$ is defined for $x \in \mathbb{R} \setminus \{2\}$. Examine the behaviour of $f(x)$ for values of x close to 2.

$x < 2$	$x > 2$
$f(1.7) = 2.4$	$f(2.3) = 3.6$
$f(1.8) = 2.6$	$f(2.2) = 3.4$
$f(1.9) = 2.8$	$f(2.1) = 3.2$
$f(1.99) = 2.98$	$f(2.01) = 3.02$
$f(1.999) = 2.998$	$f(2.001) = 3.002$

From the table, it is apparent that, as x takes values closer and closer to 2 (regardless of whether x approaches 2 from the left or from the right), the values of $f(x)$ become closer and closer to 3. That is, $\lim_{x \to 2} f(x) = 3$.

Note that the limit exists, but the function is not defined at $x = 2$.

Algebra of limits

The following important results are useful for the evaluation of limits.

Assume that both $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist.

- **Sum:** $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- **Multiple:** $\lim_{x \to a} kf(x) = k \lim_{x \to a} f(x)$, where k is a given real number.
- **Product:** $\lim_{x \to a} (f(x)\,g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- **Quotient:** $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided $\lim_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

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Example 41

Find:

a $\lim_{h \to 0} (3h + 4)$

b $\lim_{x \to 2} 4x(x + 2)$

c $\lim_{x \to 3} \frac{5x + 2}{x - 2}$

Solution

a $\lim_{h \to 0} (3h + 4) = \lim_{h \to 0} (3h) + \lim_{h \to 0} (4)$

$= 0 + 4$

$= 4$

b $\lim_{x \to 2} 4x(x + 2) = \lim_{x \to 2} (4x) \lim_{x \to 2} (x + 2)$

$= 8 \times 4$

$= 32$

c $\lim_{x \to 3} \frac{5x + 2}{x - 2} = \lim_{x \to 3} (5x + 2) \div \lim_{x \to 3} (x - 2)$

$= 17 \div 1$

$= 17$

Example 42

Find:

a $\lim_{x \to 3} \frac{x^2 - 3x}{x - 3}$

b $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$

c $\lim_{x \to 3} \frac{x^2 - 7x + 10}{x^2 - 25}$

Solution

a $\lim_{x \to 3} \frac{x^2 - 3x}{x - 3} = \lim_{x \to 3} \frac{x(x - 3)}{x - 3} = \lim_{x \to 3} x = 3$

b $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 1)}{x - 2} = \lim_{x \to 2} (x + 1) = 3$

c $\lim_{x \to 3} \frac{x^2 - 7x + 10}{x^2 - 25} = \lim_{x \to 3} \frac{(x - 2)(x - 5)}{(x + 5)(x - 5)} = \frac{\lim_{x \to 3} (x - 2)}{\lim_{x \to 3} (x + 5)} = \frac{1}{8}$

Left and right limits

An idea which is useful in the following discussion is the existence of limits from the left and from the right. This is particularly useful when talking about piecewise-defined functions.

If the value of $f(x)$ approaches the number p as x approaches a from the right-hand side, then it is written as $\lim_{x \to a^+} f(x) = p$.

If the value of $f(x)$ approaches the number p as x approaches a from the left-hand side, then it is written as $\lim_{x \to a^-} f(x) = p$.

The limit as x approaches a exists only if both the limit from the left and the limit from the right exist and are equal. Then $\lim_{x \to a} f(x) = p$.

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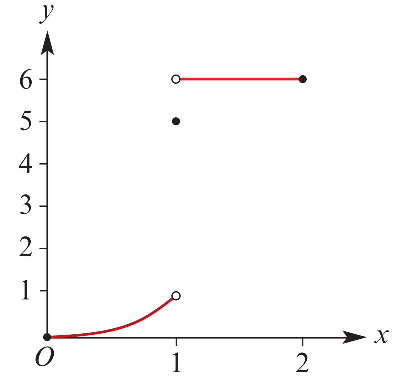
Piecewise-defined function

The following is an example of a piecewise-defined function where the limit does not exist for a particular value.

Let $f(x) = \begin{cases} x^3 & \text{if } 0 \le x < 1 \\ 5 & \text{if } x = 1 \\ 6 & \text{if } 1 < x \le 2 \end{cases}$



It is clear from the graph of f that $\lim_{x\to 1} f(x)$ does not exist. However, if x is allowed to approach 1 from the left, then $f(x)$ approaches 1. On the other hand, if x is allowed to approach 1 from the right, then $f(x)$ approaches 6. Also note that $f(1) = 5$.

Rectangular hyperbola

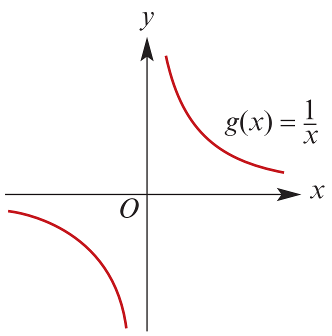
As mentioned at the start of this section, the notation of limits is used to describe the asymptotic behaviour of graphs.

First consider $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x^2}$. Observe that, as x approaches 0 both from the left and from the right, $f(x)$ increases without bound. The limit notation for this is $\lim_{x\to 0} f(x) = \infty$.

Now consider $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = \dfrac{1}{x}$. The behaviour of $g(x)$ as x approaches 0 from the left is different from the behaviour as x approaches 0 from the right.

With limit notation this is written as:

$$\lim_{x\to 0^-} g(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 0^+} g(x) = \infty$$



Now examine this function as the magnitude of x becomes very large. It can be seen that, as x increases without bound through positive values, the corresponding values of $g(x)$ approach zero. Likewise, as x decreases without bound through negative values, the corresponding values of $g(x)$ also approach zero.

Symbolically this is written as:

$$\lim_{x\to \infty} g(x) = 0^+ \qquad \text{and} \qquad \lim_{x\to -\infty} g(x) = 0^-$$

Many functions approach a limiting value or limit as x approaches $\pm\infty$.

Continuity at a point

We only require an intuitive understanding of continuity.

A function with rule $f(x)$ is said to be continuous at $x = a$ if the graph of $y = f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break. Otherwise, there is said to be a discontinuity at $x = a$.

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We can give a more formal definition of continuity using limits. A function f is continuous at the point $x = a$ provided $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ all exist and are equal.

We can state this equivalently as follows:

A function f is **continuous** at the point $x = a$ if the following conditions are met:

- $f(x)$ is defined at $x = a$
- $\lim_{x \to a} f(x) = f(a)$

The function is **discontinuous** at a point if it is not continuous at that point.

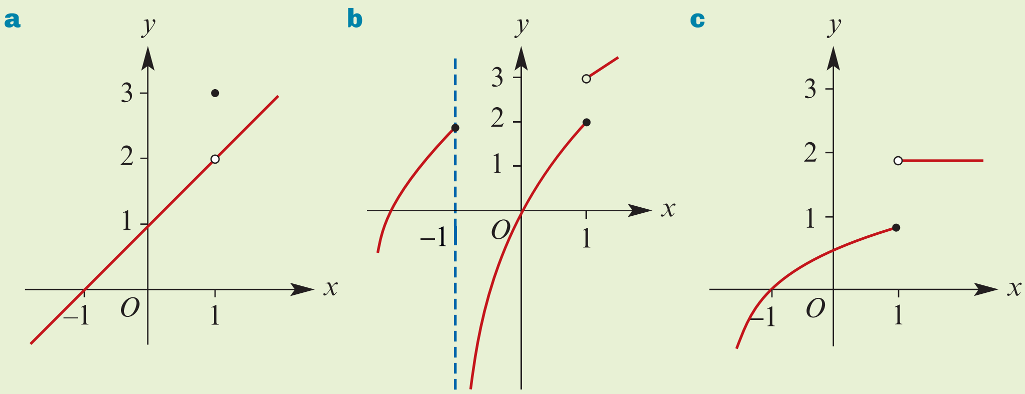
A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere. In contrast, the function

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ 5 & \text{if } x = 1 \\ 6 & \text{if } x > 1 \end{cases}$$

is defined for all real numbers but is not continuous at $x = 1$.

Example 43

State the values for x for which the functions shown below have a discontinuity:



Solution

a Discontinuity at $x = 1$, as $f(1) = 3$ but $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = 2$.

b Discontinuity at $x = -1$, as $f(-1) = 2$ and $\lim_{x \to -1^-} f(x) = 2$ but $\lim_{x \to -1^+} f(x) = -\infty$, and a discontinuity at $x = 1$, as $f(1) = 2$ and $\lim_{x \to 1^-} f(x) = 2$ but $\lim_{x \to 1^+} f(x) = 3$.

c Discontinuity at $x = 1$, as $f(1) = 1$ and $\lim_{x \to 1^-} f(x) = 1$ but $\lim_{x \to 1^+} f(x) = 2$.

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Example 44

For each function, state the values of x for which there is a discontinuity, and use the definition of continuity in terms of $f(a)$, $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ to explain why:

a $f(x) = \begin{cases} 2x & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

c $f(x) = \begin{cases} x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -2x+1 & \text{if } x \geq 0 \end{cases}$

d $f(x) = \begin{cases} x^2+1 & \text{if } x \geq 0 \\ -2x+1 & \text{if } x < 0 \end{cases}$

e $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$

Solution

a $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

b $f(0) = 0$ but $\lim_{x\to 0^-} f(x) = 1$, therefore there is a discontinuity at $x = 0$.

c $f(-1) = -1$ but $\lim_{x\to -1^+} f(x) = 1$, therefore there is a discontinuity at $x = -1$.

$f(0) = 1$ but $\lim_{x\to 0^-} f(x) = 0$, therefore there is a discontinuity at $x = 0$.

d No discontinuity. **e** No discontinuity.

Summary 9L

- A function f is **continuous** at the point $x = a$ if the following conditions are met:
  - $f(x)$ is defined at $x = a$
  - $\lim_{x\to a} f(x) = f(a)$
- A function is **discontinuous** at a point if it is not continuous at that point.
- A function is said to be **continuous everywhere** if it is continuous for all real numbers. All the polynomial functions are continuous everywhere.
- Algebra of limits Assume that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist.
  - $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
    That is, the limit of the sum is the sum of the limits.
  - $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x)$, where k is a given real number.
  - $\lim_{x\to a} (f(x)\,g(x)) = \lim_{x\to a} f(x)\ \lim_{x\to a} g(x)$
    That is, the limit of the product is the product of the limits.
  - $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, provided $\lim_{x\to a} g(x) \neq 0$.
    That is, the limit of the quotient is the quotient of the limits.

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Exercise 9L

Example 41
Example 42

1 Find the following limits:

a $\lim_{x \to 2} 17$ **b** $\lim_{x \to 6} (x - 3)$ **c** $\lim_{x \to \frac{1}{2}} (2x - 5)$

d $\lim_{t \to -3} \dfrac{t + 2}{t - 5}$ **e** $\lim_{t \to 2} \dfrac{t^2 + 2t + 1}{t + 1}$ **f** $\lim_{x \to 0} \dfrac{(x + 2)^2 - 4}{x}$

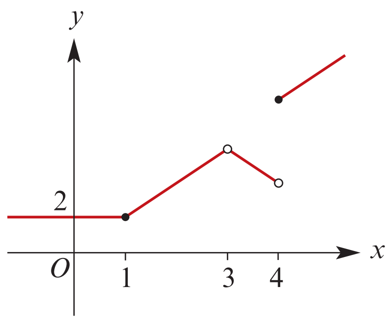
g $\lim_{t \to 1} \dfrac{t^2 - 1}{t - 1}$ **h** $\lim_{x \to 9} \sqrt{x + 3}$ **i** $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

j $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$ **k** $\lim_{x \to 2} \dfrac{3x^2 - x - 10}{x^2 + 5x - 14}$ **l** $\lim_{x \to 1} \dfrac{x^2 - 3x + 2}{x^2 - 6x + 5}$

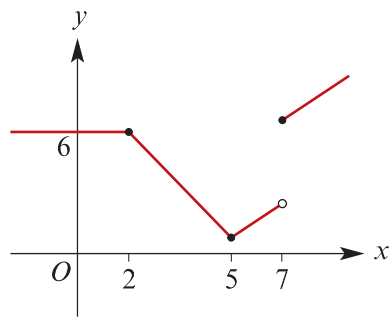
Example 43

2 For each of the following graphs, give the values of x at which a discontinuity occurs. Give reasons.

a



b



Example 44

3 For each of the following functions, state the values of x for which there is a discontinuity and use the definition of continuity in terms of $f(a)$, $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to explain why each stated value of x corresponds to a discontinuity:

a $f(x) = \begin{cases} 3x & \text{if } x \geq 0 \\ -2x + 2 & \text{if } x < 0 \end{cases}$

b $f(x) = \begin{cases} x^2 + 2 & \text{if } x \geq 1 \\ -2x + 1 & \text{if } x < 1 \end{cases}$

c $f(x) = \begin{cases} -x & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 0 \\ -3x + 1 & \text{if } x \geq 0 \end{cases}$

4 The rule of a particular function is given below. For what values of x is the graph of this function continuous?

$$y = \begin{cases} 2, & x < 1 \\ (x - 4)^2 - 9, & 1 \leq x < 7 \\ x - 7, & x \geq 7 \end{cases}$$

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9M When is a function differentiable?

A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

Many of the functions considered in this chapter are differentiable for their implicit domains. However, this is not true for all functions. We noted in Section 9D that $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

is not differentiable at $x = 0$. The gradient is -1 to the left of 0, and 1 to the right of 0.

It was shown in the previous section that some piecewise-defined functions are continuous everywhere. Similarly, some piecewise-defined functions are differentiable everywhere. The smoothness of the 'joins' determines whether this is the case.

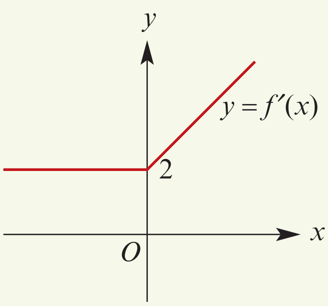
Example 45

For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ 2x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x \geq 0 \\ 2 & \text{if } x < 0 \end{cases}$$



In particular, $f'(0)$ is defined and is equal to 2.

The two sections of the graph of $y = f(x)$ join smoothly at the point $(0, 1)$.

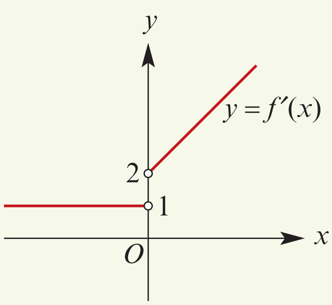
Example 46

For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0 \end{cases}$$

Solution

$$f'(x) = \begin{cases} 2x + 2 & \text{if } x > 0 \\ 1 & \text{if } x < 0 \end{cases}$$



$f'(0)$ is not defined as the limits from the left and right are not equal.

The function f is differentiable for $\mathbb{R} \setminus \{0\}$.

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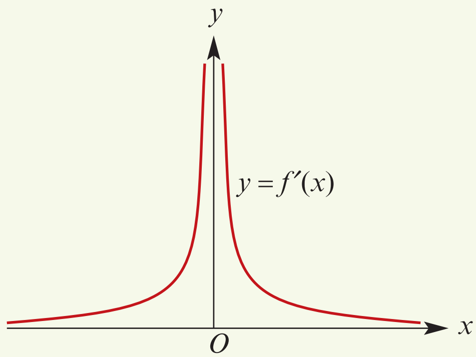
If a function is differentiable at $x = a$, then it is also continuous at $x = a$. But the converse is not true. The function f from Example 46 is continuous at $x = 0$, as $\lim_{x \to 0} f(x) = f(0)$, but f is not differentiable at $x = 0$.

Example 47

For the function with rule $f(x) = x^{\frac{1}{3}}$, state when the derivative is defined and sketch the graph of the derivative function.

Solution

By the rule for differentiating powers, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$. The derivative is not defined at $x = 0$.



Explanation

We can also see that $f'(0)$ is not defined from first principles:

$$\frac{f(0+h) - f(0)}{h} = \frac{(0+h)^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h} = \frac{h^{\frac{1}{3}}}{h} = h^{-\frac{2}{3}}$$

But $h^{-\frac{2}{3}} \to \infty$ as $h \to 0$. Thus $\lim_{h \to 0} h^{-\frac{2}{3}}$ does not exist and so $f'(0)$ is not defined.

The function $f(x) = x^{\frac{1}{3}}$ is continuous everywhere, but not differentiable at $x = 0$.

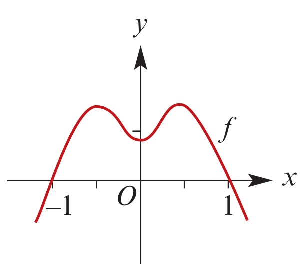
Summary 9M

- A function f is said to be **differentiable** at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.
- If a function is differentiable at a point, then it is also continuous at that point.

Exercise 9M

1 In each of the following figures, the graph of a function f is given. Sketch the graph of f'. Obviously your sketch of f' cannot be exact; but $f'(x)$ should be zero at values of x for which the gradient of f is zero, and $f'(x)$ should be negative where the original graph slopes downwards, and so on.

a



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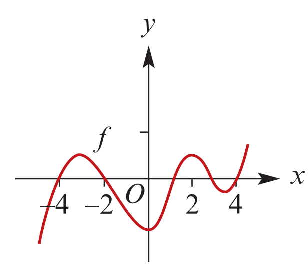

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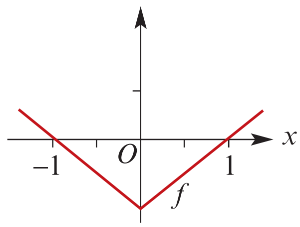

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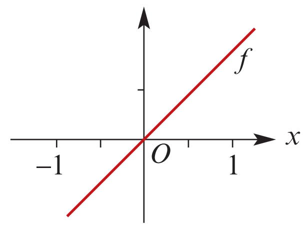

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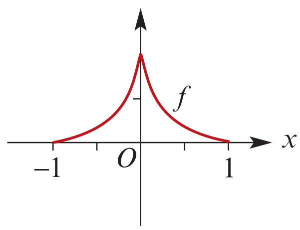

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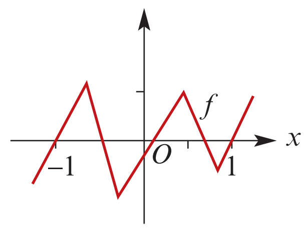

Example 45

2 For the function with following rule, find $f'(x)$ and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 + 3x + 1 & \text{if } x \geq 0 \\ 3x + 1 & \text{if } x < 0 \end{cases}$$

Example 46

3 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 1 \\ -2x + 3 & \text{if } x < 1 \end{cases}$$

4 For the function with the following rule, state the set of values for which the derivative is defined, find $f'(x)$ for this set of values and sketch the graph of $y = f'(x)$:

$$f(x) = \begin{cases} -x^2 - 2x + 1 & \text{if } x \geq -1 \\ -2x + 3 & \text{if } x < -1 \end{cases}$$

Example 47

5 For each of the following, give the set of values for which the derivative is defined, give the derivative and sketch the graph of the derivative function:

a $f(x) = (x - 1)^{\frac{1}{3}}$

b $f(x) = x^{\frac{1}{5}}$

c $f(x) = x^{\frac{2}{3}}$

d $f(x) = (x + 2)^{\frac{2}{5}}$

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Review

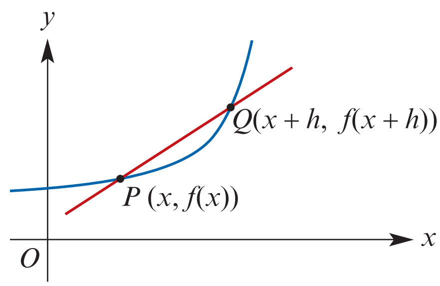
Chapter summary

Assign-ment

Nrich

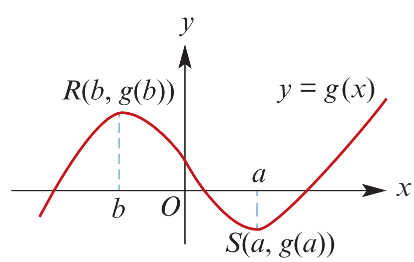
The derivative

- The notation for the limit as h approaches 0 is $\lim_{h \to 0}$.
- For the graph of $y = f(x)$:
  - The gradient of the secant PQ is given by
  $$\frac{f(x+h) - f(x)}{h}$$
  - The gradient of the tangent to the graph at the point P is given by
  $$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$



- The **derivative** of the function f is denoted f' and is defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
- At a point $(a, g(a))$ on the curve $y = g(x)$, the gradient is $g'(a)$.

  For the graph shown:
  - $g'(x) > 0$ for $x < b$ and for $x > a$
  - $g'(x) < 0$ for $b < x < a$
  - $g'(x) = 0$ for $x = b$ and for $x = a$.



Approximations for the derivative

- The value of the derivative of f at $x = a$ can be approximated by $f'(a) \approx \dfrac{f(a+h) - f(a)}{h}$ or $f'(a) \approx \dfrac{f(a+h) - f(a-h)}{2h}$ for a small value of h.

Basic derivatives

$f(x)$	$f'(x)$	
c	0	where c is a constant
x^n	nx^{n-1}	where n is a non-zero integer
x^a	ax^{a-1}	where $a \in \mathbb{R} \setminus \{0\}$
e^{kx}	ke^{kx}	
$\log_e(kx)$	$\frac{1}{x}$	
$\sin(kx)$	$k\cos(kx)$	
$\cos(kx)$	$-k\sin(kx)$	
$\tan(kx)$	$k\sec^2(kx)$	

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Rules for differentiation

- For $f(x) = k\,g(x)$, where k is a constant, $f'(x) = k\,g'(x)$.
  That is, the derivative of a number multiple is the multiple of the derivative.
- For $f(x) = g(x) + h(x)$, $f'(x) = g'(x) + h'(x)$.
  That is, the derivative of a sum is the sum of the derivatives.

- The chain rule
  - If $q(x) = f(g(x))$, then $q'(x) = f'\big(g(x)\big)\,g'(x)$
  - $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$
- The product rule
  - If $F(x) = f(x) \cdot g(x)$, then $F'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$
  - If $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$
- The quotient rule
  - If $F(x) = \dfrac{f(x)}{g(x)}$, then $F'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}$
  - If $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

Algebra of limits

- $\lim\limits_{x \to a} \big(f(x) + g(x)\big) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$
  That is, the limit of the sum is the sum of the limits.
- $\lim\limits_{x \to a} k f(x) = k \lim\limits_{x \to a} f(x)$, where k is a real number.
- $\lim\limits_{x \to a} \big(f(x)\,g(x)\big) = \lim\limits_{x \to a} f(x)\ \lim\limits_{x \to a} g(x)$
  That is, the limit of the product is the product of the limits.
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$, provided $\lim\limits_{x \to a} g(x) \neq 0$.
  That is, the limit of the quotient is the quotient of the limits.

Continuity and differentiability

- A function f is **continuous** at the point $x = a$ if:
  - $f(x)$ is defined at $x = a$
  - $\lim\limits_{x \to a} f(x) = f(a)$
- A function is **discontinuous** at a point if it is not continuous at that point.
- A function f is **differentiable** at the point $x = a$ if $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ exists.

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Review

Technology-free questions

1 For $y = x^2 + 1$:

a Find the average rate of change of y with respect to x over the interval $[3, 5]$.

b Find the instantaneous rate of change of y with respect to x at the point where $x = -4$.

2 Differentiate each of the following with respect to x:

a $x + \sqrt{1 - x^2}$ **b** $\dfrac{4x + 1}{x^2 + 3}$ **c** $\sqrt{1 + 3x}$ **d** $\dfrac{2 + \sqrt{x}}{x}$

e $(x - 9)\sqrt{x - 3}$ **f** $x\sqrt{1 + x^2}$ **g** $\dfrac{x^2 - 1}{x^2 + 1}$ **h** $\dfrac{x}{x^2 + 1}$

i $(2 + 5x^2)^{\frac{1}{3}}$ **j** $\dfrac{2x + 1}{x^2 + 2}$ **k** $(3x^2 + 2)^{\frac{2}{3}}$

3 For each of the following functions, find the gradient of the tangent to the curve at the point corresponding to the given x-value:

a $y = 3x^2 - 4$ at $x = -1$ **b** $y = \dfrac{x - 1}{x^2 + 1}$ at $x = 0$

c $y = (x - 2)^5$ at $x = 1$ **d** $y = (2x + 2)^{\frac{1}{3}}$ at $x = 3$

4 Differentiate each of the following with respect to x:

a $\log_e(x + 2)$ **b** $\sin(3x + 2)$ **c** $\cos\left(\dfrac{x}{2}\right)$ **d** $e^{x^2 - 2x}$

e $\log_e(3 - x)$ **f** $\sin(2\pi x)$ **g** $\sin^2(3x + 1)$ **h** $\sqrt{\log_e x}, \ x > 1$

i $\dfrac{2\log_e(2x)}{x}$ **j** $x^2 \sin(2\pi x)$

5 Differentiate each of the following with respect to x:

a $e^x \sin(2x)$ **b** $2x^2 \log_e x$ **c** $\dfrac{\log_e x}{x^3}$ **d** $\sin(2x)\cos(3x)$

e $\dfrac{\sin(2x)}{\cos(2x)}$ **f** $\cos^3(3x + 2)$ **g** $x^2 \sin^2(3x)$

6 Find the gradient of each of the following curves at the stated value of x:

a $y = e^{2x} + 1, \ x = 1$ **b** $y = e^{x^2+1}, \ x = 0$

c $y = 5e^{3x} + x^2, \ x = 1$ **d** $y = 5 - e^{-x}, \ x = 0$

7 Differentiate each of the following with respect to x:

a e^{ax} **b** e^{ax+b} **c** e^{a-bx} **d** $be^{ax} - ae^{bx}$ **e** $\dfrac{e^{ax}}{e^{bx}}$

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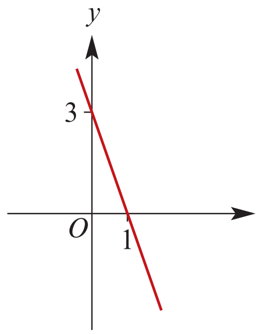
8 Sketch the graph of the derivative function for each of the following functions:

a



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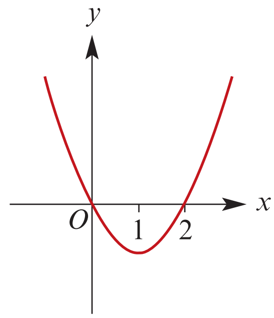

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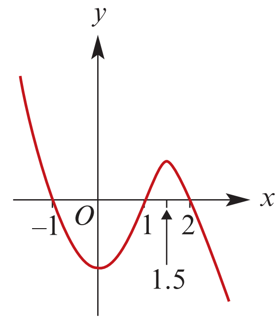

9 Find the derivative of $\left(4x + \frac{9}{x}\right)^2$ and find the values of x at which the derivative is zero.

10 **a** For $y = \frac{2x - 3}{x^2 + 4}$, show that $\frac{dy}{dx} = \frac{8 + 6x - 2x^2}{(x^2 + 4)^2}$.

b Find the values of x for which both y and $\frac{dy}{dx}$ are positive.

11 Find the derivative of each of the following, given that the function f is differentiable for all real numbers:

a $xf(x)$ **b** $\frac{1}{f(x)}$ **c** $\frac{x}{f(x)}$ **d** $\frac{x^2}{[f(x)]^2}$

12 Let $f(x) = 2x^3 - 1$ and $g(x) = \cos x$.

a Find the rule for $f \circ g$. **b** Find the rule for $g \circ f$. **c** Find the rule for $g' \circ f$.

d Find the rule for $(g \circ f)'$. **e** Find $f'\left(g\left(\frac{\pi}{3}\right)\right)$. **f** Find $(f \circ g)'\left(\frac{\pi}{3}\right)$.

13 Let $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

14 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

15 The graph of $y = bx^2 - cx$ crosses the x-axis at the point $(4, 0)$. The gradient at this point is 1. Find the values of b and c.

16 Let $f : \mathbb{R} \to \mathbb{R},\ f(x) = e^{2x} - 16e^x - 36$.

a Solve the equation $f(x) = 0$.

b Find the coordinates of the point on the graph where $f'(x) = 0$.

c Find the values of x for which $f'(x) > 0$.

d Find the average rate of change of $y = f(x)$ with respect to x for the interval $[\log_e 8, \log_e 18]$

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Review

Multiple-choice questions

1 The average rate of change of the function with rule $f(x) = e^x + x^3$ for $x \in [0, 1]$ is

A e **B** $e^3 + 1$ **C** $\dfrac{e^3 + 1}{2}$ **D** $e + 1$ **E** $e^x + 3x^2$

2 If $f: \mathbb{R} \setminus \{7\} \to \mathbb{R}$ where $f(x) = 5 + \dfrac{5}{(7 - x)^2}$, then $f'(x) > 0$ for

A $x \in \mathbb{R} \setminus \{7\}$ **B** $x \in \mathbb{R}$ **C** $x < 7$ **D** $x > 7$ **E** $x > 5$

3 Let $y = f(g(x))$ where $g(x) = 2x^4$. Then $\dfrac{dy}{dx}$ is equal to

A $8x^3 f'(2x^4)$ **B** $8x^2 f(4x^3)$ **C** $8x^4 f(x) f'(x^3)$
D $2f(x) f'(x^3)$ **E** $8x^3$

4 Which of the following is *not true* for the curve of $y = f(x)$ where $f(x) = x^{\frac{1}{3}}$?

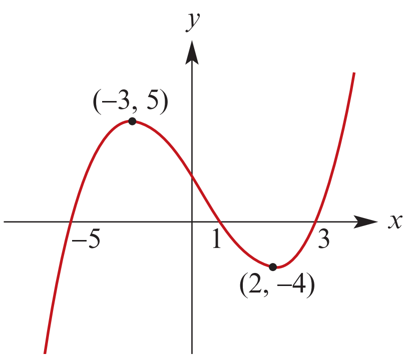
A The gradient is defined for all real numbers.
B The curve passes through the origin.
C The curve passes through the points with coordinates (1, 1) and (−1, −1).
D For $x > 0$, the gradient is positive.
E For $x > 0$, the gradient is decreasing.

5 The graph of the function with rule $y = \dfrac{k}{2(x^3 + 1)}$ has gradient 1 when $x = 1$. The value of k is

A 1 **B** $\dfrac{-8}{3}$ **C** $\dfrac{-1}{2}$ **D** -4 **E** $\dfrac{-1}{4}$

6 For the graph shown, the gradient is positive for

A $-3 < x < 2$
B $-3 \le x \le 2$
C $x < -3$ or $x > 2$
D $x \le -3$ or $x \ge 2$
E $-3 \le x \le 3$



7 For the function $f(x) = 4x(2 - 3x)$, $f'(x) < 0$ for

A $x < \dfrac{1}{3}$ **B** $0 < x < \dfrac{2}{3}$ **C** $x = \dfrac{1}{3}$ **D** $x > \dfrac{1}{3}$ **E** $x = 0, \dfrac{2}{3}$

8 The average rate of change of the function with rule $f(x) = x \log_e x$ for $x \in [2, 4]$ is

A $\dfrac{1}{2}(\log_e(4) - \log_e 2)$ **B** $\dfrac{1}{2}(3 \log_e(2) + 2)$ **C** $\dfrac{1}{2} \log_e(16)$
D $3 \log_e(2)$ **E** $3 \log_e(4)$

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9 The point on the curve defined by the equation $y = (x + 3)(x - 2)$ at which the gradient is -7 has coordinates

A $(-4, 6)$ **B** $(-4, 0)$ **C** $(-3, 0)$ **D** $(-3, -5)$ **E** $(-2, 0)$

10 The function $y = ax^2 - bx$ has zero gradient only for $x = 2$. The x-axis intercepts of the graph of this function are

A $\frac{1}{2}, -\frac{1}{2}$ **B** $0, 4$ **C** $0, -4$ **D** $0, \frac{1}{2}$ **E** $0, -\frac{1}{2}$

11 If $f(x) = \dfrac{4x^4 - 12x^2}{3x - k}$, If $f'(k) = 2$ and $f(5) > 150$ then k is equal to

A -31 **B** -5 **C** -1 **D** 0 **E** 1

12 The functions f and g are differentiable and $g(x) \neq 0$ for all x. Let $h(x) = \dfrac{f(x)}{g(x)}$.
If $f(2) = 3$, $g(2) = 6$, $f'(2) = -4$ and $g'(2) = 8$ then $h'(2)$ is equal to.

A 2 **B** $-\dfrac{1}{2}$ **C** $-\dfrac{4}{3}$ **D** $\dfrac{1}{2}$ **E** $-\dfrac{3}{4}$

13 Let f be a one-to-one differentiable function such that $f(6) = 3$, $f(8) = 6$, $f'(8) = 4$, $f'(6) = 11$. The function g is differentiable and $g(x) = f^{-1}(x)$ for all x. $g'(6)$ is equal to

A $\dfrac{1}{4}$ **B** 1 **C** $\dfrac{1}{6}$ **D** $\dfrac{3}{8}$ **E** $\dfrac{6}{11}$

Extended-response questions

1 **a** For functions f and g, which are defined and differentiable for all real numbers, it is known that:

- $f(1) = 6$, $g(1) = -1$, $g(6) = 7$ and $f(-1) = 8$
- $f'(1) = 6$, $g'(1) = -2$, $f'(-1) = 2$ and $g'(6) = -1$

Find:

i $(f \circ g)'(1)$ **ii** $(g \circ f)'(1)$ **iii** $(fg)'(1)$ **iv** $(gf)'(1)$ **v** $\left(\dfrac{f}{g}\right)'(1)$ **vi** $\left(\dfrac{g}{f}\right)'(1)$

b It is known that f is a cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$. Find the values of a, b, c and d.

2 For a function f, which is differentiable for $\mathbb{R}$, it is known that:

- $f'(x) = 0$ for $x = 1$ and $x = 5$
- $f'(x) > 0$ for $x > 5$ and $x < 1$
- $f'(x) < 0$ for $1 < x < 5$
- $f(1) = 6$ and $f(5) = 1$

a For $y = f(x + 2)$, find the values of x for which:

i $\dfrac{dy}{dx} = 0$ **ii** $\dfrac{dy}{dx} > 0$

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b Find the coordinates of the points on the graph of $y = f(x-2)$ where $\frac{dy}{dx} = 0$.

c Find the coordinates of the points on the graph of $y = f(2x)$ where $\frac{dy}{dx} = 0$.

d Find the coordinates of the points on the graph of $y = f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

e Find the coordinates of the points on the graph of $y = 3f\left(\frac{x}{2}\right)$ where $\frac{dy}{dx} = 0$.

3 Let $f(x) = (x-\alpha)^n(x-\beta)^m$, where m and n are positive integers with $m > n$ and $\beta > \alpha$.

a Solve the equation $f(x) = 0$ for x.

b Find $f'(x)$.

c Solve the equation $f'(x) = 0$ for x.

d **i** If m and n are odd, find the set of values for which $f'(x) > 0$.

ii If m is odd and n is even, find the set of values for which $f'(x) > 0$.

4 Consider the function with rule $f(x) = \frac{x^n}{1+x^n}$, where n is an even positive integer.

a Show that $f(x) = 1 - \frac{1}{x^n+1}$.

b Find $f'(x)$.

c Show that $0 \le f(x) < 1$ for all x.

d State the set of values for which $f'(x) = 0$.

e State the set of values for which $f'(x) > 0$.

f Show that f is an even function.

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