11 Integration

Objectives

- To use **numerical methods** to estimate the area under the graph of a function.
- To be able to calculate **definite integrals**.
- To use the definite integral to find the **exact area** under the graph of a function.
- To integrate **polynomial functions**, **exponential functions** and **circular functions**.
- To use integration to determine **areas under curves**.
- To use integration to **solve problems**.

We have used the derivative to find the gradients of tangents to curves, and in turn this has been used in graph sketching. The derivative has also been used to define instantaneous rate of change and to solve problems involving motion in a straight line.

It comes as a surprise that a related idea can be used to determine areas. In this chapter we define an area function A for a given function f on an interval $[a, b]$, and show that the derivative of the area function is the original function f. Hence, you can go from the function f to its area function by a process which can loosely be described as 'undoing' the derivative. This result is so important that it carries the title **fundamental theorem of calculus**.

The result was developed over many centuries: a method for determining areas described in the last section of this chapter is due to Archimedes. The final result was brought together by both Leibniz and Newton in the seventeenth century. The wonder of it is that the two seemingly distinct ideas – calculation of areas and calculation of gradients – were shown to be so closely related.

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11A Estimating the area under a graph

Consider the graph of a function f. We want to find the area under the graph. For now we'll assume that the graph $y = f(x)$ is always *above the x-axis*, and we will estimate the area between the graph $y = f(x)$ and the x-axis. We set left and right endpoints and estimate the area between those endpoints.

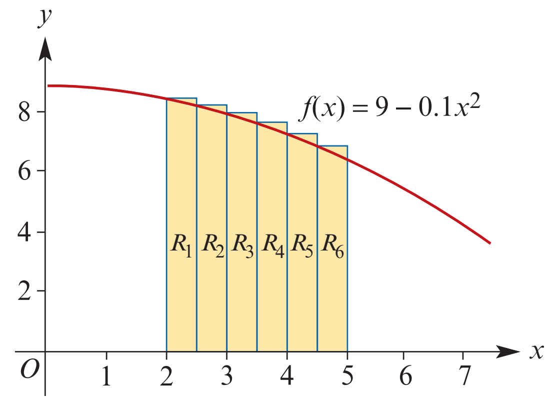
Below is the graph of $f(x) = 9 - 0.1x^2$. We consider three methods for determining the area under this graph between $x = 2$ and $x = 5$. The **trapezium rule** is mentioned in the Study Design.

The left-endpoint method

We first find an approximation for the area under the graph between $x = 2$ and $x = 5$ by dividing the region into rectangles as illustrated. The width of each rectangle is 0.5.

Areas of rectangles:

- $R_1 = 0.5 \times f(2.0) = 0.5 \times 8.60 = 4.30$
- $R_2 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$
- $R_3 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$
- $R_4 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$
- $R_5 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$
- $R_6 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$



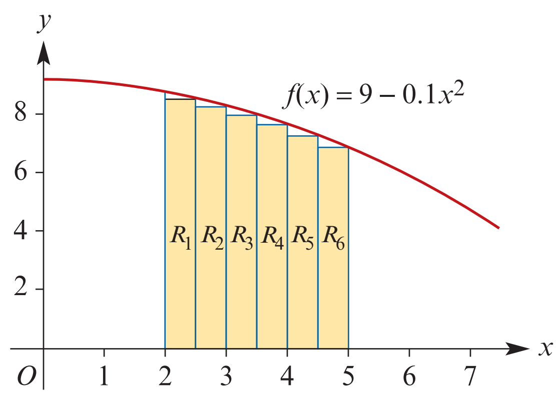
The sum of the areas of the rectangles is 23.62 square units. This is called the **left-endpoint estimate** for the area under the graph.

The left-endpoint estimate will be larger than the actual area for a graph that is decreasing over the interval, and smaller than the actual area for a graph that is increasing.

The right-endpoint method

Areas of rectangles:

- $R_1 = 0.5 \times f(2.5) = 0.5 \times 8.38 = 4.19$
- $R_2 = 0.5 \times f(3.0) = 0.5 \times 8.10 = 4.05$
- $R_3 = 0.5 \times f(3.5) = 0.5 \times 7.78 = 3.89$
- $R_4 = 0.5 \times f(4.0) = 0.5 \times 7.40 = 3.70$
- $R_5 = 0.5 \times f(4.5) = 0.5 \times 6.98 = 3.49$
- $R_6 = 0.5 \times f(5.0) = 0.5 \times 6.50 = 3.25$



The sum of the areas of the rectangles is 22.67 square units.

This is called the **right-endpoint estimate** for the area under the graph.

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For f decreasing over $[a, b]$: left-endpoint estimate $\geq$ true area $\geq$ right-endpoint estimate
For f increasing over $[a, b]$: left-endpoint estimate $\leq$ true area $\leq$ right-endpoint estimate

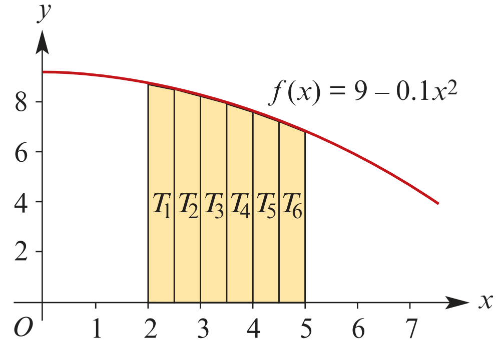
The trapezium rule

For this method we work with trapeziums instead of rectangles.

The area of a trapezium is $\frac{1}{2}(a + b)h$, where a and b are the lengths of the two parallel sides and h is their distance apart.

Areas of trapeziums:

- $T_1 = \frac{1}{2}[f(2.0) + f(2.5)] \times 0.5 = 4.24375$
- $T_2 = \frac{1}{2}[f(2.5) + f(3.0)] \times 0.5 = 4.11875$
- $T_3 = \frac{1}{2}[f(3.0) + f(3.5)] \times 0.5 = 3.96875$
- $T_4 = \frac{1}{2}[f(3.5) + f(4.0)] \times 0.5 = 3.79375$
- $T_5 = \frac{1}{2}[f(4.0) + f(4.5)] \times 0.5 = 3.59375$
- $T_6 = \frac{1}{2}[f(4.5) + f(5.0)] \times 0.5 = 3.36875$



The sum of the areas is 23.0875 square units.

This is called the **trapezium estimate** for the area under the graph.

We can see that the trapezium estimate for this example can also be calculated as

$$\frac{1}{2}\Big[f(2) + 2f(2.5) + 2f(3) + 2f(3.5) + 2f(4) + 2f(4.5) + f(5)\Big] \times 0.5$$

It is clear that, if narrower strips are chosen, we obtain an estimate that is closer to the true value. This is time-consuming to do by hand, but a computer program or spreadsheet makes the process quite manageable. A discussion of the algorithms for the trapezium estimate is undertaken in the Algorithms and pseudocode section of Chapter 12.

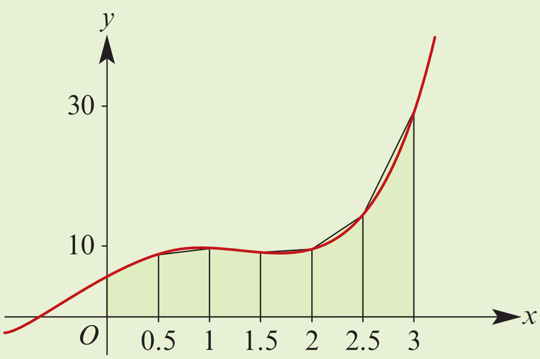
The trapezium estimate is the average of the left-endpoint and right-endpoint estimates.

Example 1

Find the sum of the areas of the shaded trapeziums to approximate the area under the graph of

$$f(x) = (x - 2)(x + 2)(x - 1)^2 + 10$$

between $x = 0$ and $x = 3$.



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Solution

We use the trapezium rule with trapeziums of width 0.5:

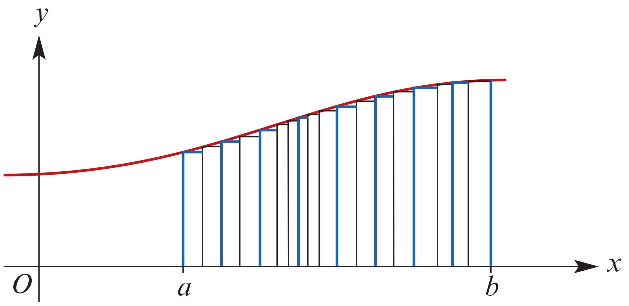
$$\text{Area} = \frac{1}{2}\Big[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\Big] \times 0.5$$

$$= \frac{1}{4}\big(6 + 2 \times 9.0625 + 2 \times 10 + 2 \times 9.5625 + 2 \times 10 + 2 \times 15.0625 + 30\big)$$

$$= 35.84375$$

If f is a continuous function such that $f(x)$ is positive for all x in the interval $[a, b]$, and if the interval $[a, b]$ is partitioned into arbitrarily small subintervals, then the **area** under the curve between $x = a$ and $x = b$ can be defined by this limiting process.

The diagram on the right shows rectangles formed from a partition. The rectangles can be of varying width, but in the limit the width of all the rectangles must approach zero.



The definite integral

Suppose that f is continuous function on a closed interval $[a, b]$ and that $f(x)$ is positive for all x in this interval. Then the area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by

$$\int_a^b f(x)\,dx$$

The function f is called the integrand, and a and b are the lower and upper limits of the integral.

By using summation notation (discussed in Appendix B), this limiting process can be expressed as

$$\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{i=1}^{n} f(x_i^*)\,\delta x_i$$

where the interval $[a, b]$ is partitioned into n subintervals, with the ith subinterval of length δx_i and containing x_i^*, and $\delta x = \max\{\,\delta x_i : i = 1, 2, \dots, n\,\}$.

The trapezium rule can be used to obtain an approximation for the definite integral. For Example 1 we can describe this approximation. for the definite integral as

$$\int_0^3 f(x)\,dx \approx \frac{1}{4}\Big[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + 2f(2) + 2f(2.5) + f(3)\Big]$$

and in general

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$

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For a linear function or a piecewise-defined function with linear components, the area under the graph may be found using geometric techniques.

Example 2

Evaluate each of the following by using an area formula:

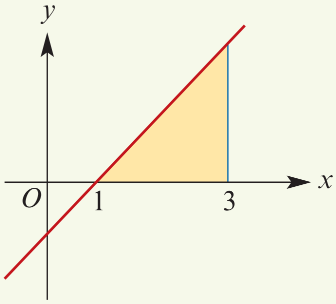
a $\int_1^3 x - 1\,dx$ **b** $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx$ **c** $\int_1^2 x + 1\,dx$

Solution

a Area of triangle $= \frac{1}{2} \times 2 \times 2$

$= 2$ square units

Therefore $\int_1^3 x - 1\,dx = 2$

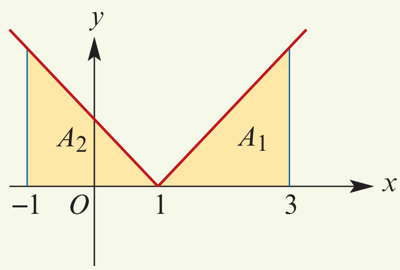


b Area $= A_1 + A_2$

$= 2 + \frac{1}{2} \times 2 \times 2$

$= 4$ square units

Therefore $\int_1^3 (x-1)\,dx + \int_{-1}^1 (1-x)\,dx = 4$

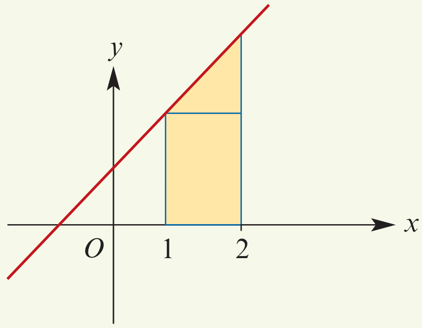


c The required region is a trapezium.

Area $= \frac{1}{2} \times 1 \times (2 + 3)$

$= \frac{5}{2}$ square units

Therefore $\int_1^2 x + 1\,dx = \frac{5}{2}$



In all of our examples we have considered areas of regions above the x-axis. In Section 11E we will consider regions below the x-axis. The convention is that these areas will have a negative sign assigned to them. The trapezium rule can be used in these situations too and you will find some questions in the review section of this chapter to illustrate the use of this convention and its use in approximating definite integrals.

A calculus method for determining areas will be introduced in Section 11E.

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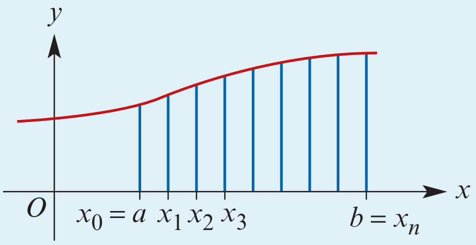
Summary 11A

Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \dots, [x_{n-1}, x_n]$ as illustrated.



Estimates for the area under the graph of $y = f(x)$ between $x = a$ and $x = b$:

- **Left-endpoint method**

$$L_n = \frac{b-a}{n}\left[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\right]$$

- **Right-endpoint method**

$$R_n = \frac{b-a}{n}\left[f(x_1) + f(x_2) + \cdots + f(x_n)\right]$$

- **Trapezium rule**

$$T_n = \frac{b-a}{2n}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

These methods are not limited to situations in which the graph is either increasing or decreasing for the whole interval. They may be used to determine the area under the curve for any continuous function on an interval $[a, b]$.

- **Exact area**

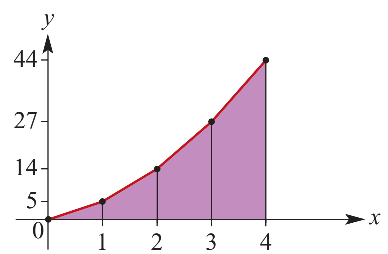
  Let f be a continuous function on a closed interval $[a, b]$ such that $f(x)$ is positive for all $x \in [a, b]$. The exact area under the graph of $y = f(x)$ from $x = a$ to $x = b$ is called the **definite integral** of $f(x)$ from $x = a$ to $x = b$, and is denoted by $\int_a^b f(x)\,dx$.

Exercise 11A

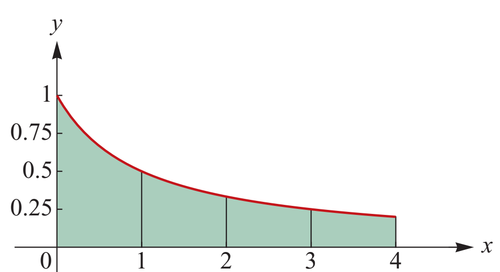
We use the trapezium rule in all of the questions of this exercise

Example 1

1 Find the sum of the areas of the shaded trapeziums to approximate the area under the curve $y = 2x^2 + 3x$ between $x = 0$ and $x = 4$.



2 A part of the graph of $y = \dfrac{1}{x+1}$ is shown. Use the trapezium rule to approximate the area of the shaded region.



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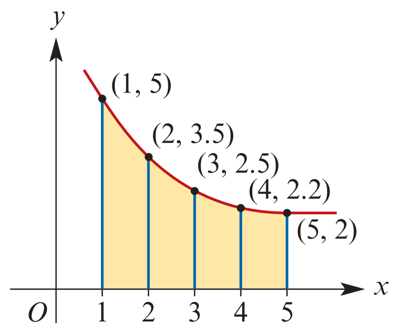
3 Use the trapezium rule to approximate the area of the shaded region.



4 Using the trapezium rule calculate an approximation to the area under the graph of $y = x(3 - x)$ between $x = 0$ and $x = 3$ using strips of width:

a 0.5 **b** 0.2

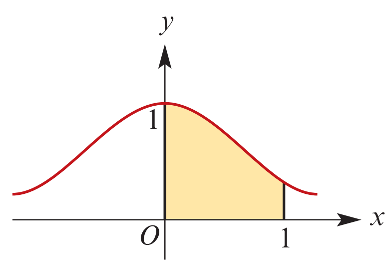
5 A table of values is given for the rule $y = f(x)$.

x	0	1	2	3	4	5	6	7	8	9	10
y	3	3.5	3.7	3.8	3.9	3.9	4.0	4.0	3.7	3.3	2.9

Using the trapezium rule estimate the area enclosed by the graph of $y = f(x)$, the lines $x = 0$ and $x = 10$, and the x-axis.

6 The graph is that of $y = \dfrac{1}{1 + x^2}$. It is known that the area of the shaded region is $\dfrac{\pi}{4}$.

Apply the trapezium rule with strips of width 0.25, and hence find an approximate value for π.



7 Use the trapezium rule to find an approximate value for the area under the graph of:

a $y = 2^x$ between $x = 0$ and $x = 2$, using intervals of width 0.5

b $y = \dfrac{1}{\sqrt{1 - x^2}}$ between $x = 0$ and $x = 0.9$, using intervals of width 0.1.

8 An engineer takes soundings at intervals of 3 metres across a river 30 metres wide to obtain the data in the following table. Use the trapezoidal rule to find an approximate value for the area of the cross-section of the river's channel.

Distance from bank in metres	0	3	6	9	12	15	18	21	24	27	30
Depth of sounding in metres	1	2	3	4	5	5	6	4	4	2	2

Example 2

9 Evaluate each of the following by using an area formula:

a $\int_2^5 x - 2\,dx$ **b** $\int_{-1}^2 (2 - x)\,dx + \int_2^5 (x - 2)\,dx$ **c** $\int_1^2 2x + 1\,dx$

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11B Antidifferentiation: indefinite integrals

Later in this chapter, we will see how to find the exact area under a graph using the technique of 'undoing' the derivative. In this section, we formalise the idea of 'undoing' a derivative.

The derivative of x^2 with respect to x is $2x$. Conversely, given that an unknown expression has derivative $2x$, it is clear that the unknown expression could be x^2. The process of finding a function from its derivative is called **antidifferentiation**.

Now consider the functions $f(x) = x^2 + 1$ and $g(x) = x^2 - 7$.

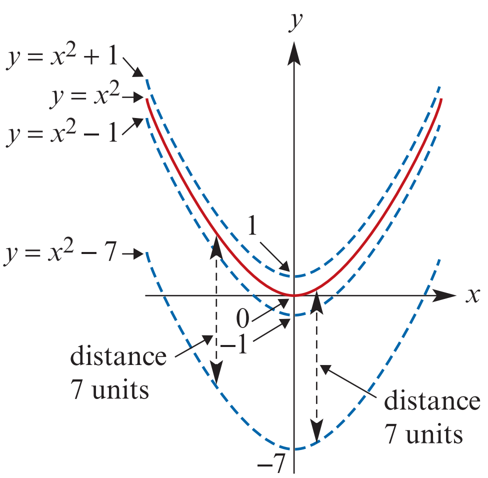
We have $f'(x) = 2x$ and $g'(x) = 2x$. So the two different functions have the same derivative function.

Both $x^2 + 1$ and $x^2 - 7$ are said to be **antiderivatives** of $2x$.

If two functions have the same derivative function, then they differ by a constant. So the graphs of the two functions can be obtained from each other by translation parallel to the y-axis.

The diagram shows several antiderivatives of $2x$.

Each of the graphs is a translation of $y = x^2$ parallel to the y-axis.



Notation

The general antiderivative of $2x$ is $x^2 + c$, where c is an arbitrary real number. We use the notation of Leibniz to state this with symbols:

$$\int 2x\,dx = x^2 + c$$

This is read as 'the **general antiderivative** of $2x$ with respect to x is equal to $x^2 + c$' or as 'the **indefinite integral** of $2x$ with respect to x is $x^2 + c$'.

To be more precise, the indefinite integral is the set of all antiderivatives and to emphasise this we could write:

$$\int 2x\,dx = \{\, f(x) : f'(x) = 2x \,\} = \{\, x^2 + c : c \in \mathbb{R} \,\}$$

This set notation is not commonly used, but it should be clearly understood that there is not a unique antiderivative for a given function. We will not use this set notation, but it is advisable to keep it in mind when considering further results.The reason why the symbol is the same as that used for the definite integral in Section 11A will become evident in Section 11E.

In general:

> If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.

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The antiderivative of x^r where $r \neq -1$

We know that:

$$f(x) = x^3 \text{ implies } f'(x) = 3x^2 \qquad f(x) = x^8 \text{ implies } f'(x) = 8x^7$$

$$f(x) = x^{\frac{3}{2}} \text{ implies } f'(x) = \frac{3}{2}x^{\frac{1}{2}} \qquad f(x) = x^{-4} \text{ implies } f'(x) = -4x^{-5}$$

Reversing this process gives:

$\int 3x^2\,dx = x^3 + c$ where c is an arbitrary constant

$\int 8x^7\,dx = x^8 + c$ where c is an arbitrary constant

$\int \frac{3}{2}x^{\frac{1}{2}}\,dx = x^{\frac{3}{2}} + c$ where c is an arbitrary constant

$\int -4x^{-5}\,dx = x^{-4} + c$ where c is an arbitrary constant

We also have:

$$\int x^2\,dx = \frac{1}{3}x^3 + c \qquad \int x^{\frac{1}{2}}\,dx = \frac{2}{3}x^{\frac{3}{2}} + c \qquad \int x^{-5}\,dx = -\frac{1}{4}x^{-4} + c$$

Generalising, it is seen that:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c, \quad r \in \mathbb{Q} \setminus \{-1\}$$

Note: This result can only be applied for suitable values of x for a given value of r. For example, if $r = \frac{1}{2}$, then $x \in \mathbb{R}^+$ is a suitable restriction. If $r = -2$, we can take $x \in \mathbb{R} \setminus \{0\}$, and if $r = 3$, we can take $x \in \mathbb{R}$.

We also record the following results, which follow immediately from the corresponding results for differentiation:

Sum $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$

Difference $\int f(x) - g(x)\,dx = \int f(x)\,dx - \int g(x)\,dx$

Multiple $\int kf(x)\,dx = k\int f(x)\,dx$, where k is a real number

Example 3

Find the general antiderivative (indefinite integral) of each of the following:

a $3x^5$

b $3x^2 + 4x^{-2} + 3$

Solution

a
$$\int 3x^5\,dx = 3\int x^5\,dx = 3 \times \frac{x^6}{6} + c = \frac{x^6}{2} + c$$

b
$$\int 3x^2 + 4x^{-2} + 3\,dx = 3\int x^2\,dx + 4\int x^{-2}\,dx + 3\int 1\,dx = \frac{3x^3}{3} + \frac{4x^{-1}}{-1} + \frac{3x}{1} + c = x^3 - \frac{4}{x} + 3x + c$$

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Example 4

Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^2}$ **b** $\frac{dy}{dx} = 3\sqrt{x}$ **c** $\frac{dy}{dx} = x^{\frac{3}{4}} + x^{-\frac{3}{4}}$

Solution

a
$$\int \frac{1}{x^2}\,dx = \int x^{-2}\,dx$$
$$= \frac{x^{-1}}{-1} + c$$
$$\therefore \quad y = \frac{-1}{x} + c$$

b
$$\int 3\sqrt{x}\,dx = 3\int x^{\frac{1}{2}}\,dx$$
$$= 3 \times \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$$
$$\therefore \quad y = 2x^{\frac{3}{2}} + c$$

c
$$y = \frac{4}{7}x^{\frac{7}{4}} + 4x^{\frac{1}{4}} + c$$

Given extra information, we can find a unique antiderivative.

Example 5

It is known that $f'(x) = x^3 + 4x^2$ and $f(0) = 0$. Find $f(x)$.

Solution

$$\int x^3 + 4x^2\,dx = \frac{x^4}{4} + \frac{4x^3}{3} + c$$
$$\therefore \quad f(x) = \frac{x^4}{4} + \frac{4x^3}{3} + c$$

As $f(0) = 0$, we have $c = 0$. Hence $f(x) = \frac{x^4}{4} + \frac{4x^3}{3}$.

Example 6

If the gradient of the tangent at a point (x, y) on a curve is given by $2x$ and the curve passes through the point $(-1, 4)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 2x$.

$$\int 2x\,dx = \frac{2x^2}{2} + c = x^2 + c$$
$$\therefore \quad f(x) = x^2 + c$$

But $f(-1) = 4$ and therefore $4 = (-1)^2 + c$.

Hence $c = 3$ and so $f(x) = x^2 + 3$.

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Summary 11B

- Antiderivative of x^r, for $r \in \mathbb{Q} \setminus \{-1\}$:

$$\int x^r \, dx = \frac{x^{r+1}}{r+1} + c$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x) \, dx = \int f(x) \, dx + \int g(x) \, dx$
  - $\int f(x) - g(x) \, dx = \int f(x) \, dx - \int g(x) \, dx$
  - $\int k f(x) \, dx = k \int f(x) \, dx$, where k is a real number

Skill-sheet

Exercise 11B

Example 3

1 Find:

a $\int \frac{1}{2}x^3 \, dx$ **b** $\int 5x^3 - 2x \, dx$ **c** $\int \frac{4}{5}x^3 - 3x^2 \, dx$ **d** $\int (2 - z)(3z + 1) \, dz$

Example 4

2 Find y in terms of x if:

a $\frac{dy}{dx} = \frac{1}{x^3}$ **b** $\frac{dy}{dx} = 4\sqrt[3]{x}$ **c** $\frac{dy}{dx} = x^{\frac{1}{4}} + x^{-\frac{3}{5}}$

3 Find:

a $\int 3x^{-2} \, dx$ **b** $\int 2x^{-4} + 6x \, dx$ **c** $\int 2x^{-2} + 6x^{-3} \, dx$

d $\int 3x^{\frac{1}{3}} - 5x^{\frac{5}{4}} \, dx$ **e** $\int 3x^{\frac{3}{4}} - 7x^{\frac{1}{2}} \, dx$ **f** $\int 4x^{\frac{3}{5}} + 12x^{\frac{5}{3}} \, dx$

Example 5

4 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = 2x - 3$ and $y = 1$ when $x = 1$ **b** $\frac{dy}{dx} = x^3$ and $y = 6$ when $x = 0$

c $\frac{dy}{dx} = x^{\frac{1}{2}} + x$ and $y = 6$ when $x = 4$

5 Find:

a $\int \sqrt{x}(2 + x) \, dx$ **b** $\int \frac{3z^4 + 2z}{z^3} \, dz$ **c** $\int \frac{5x^3 + 2x^2}{x} \, dx$

d $\int \sqrt{x}(2x + x^2) \, dx$ **e** $\int x^2(2 + 3x^2) \, dx$ **f** $\int \sqrt[3]{x}(x + x^4) \, dx$

Example 6

6 A curve with equation $y = f(x)$ passes through the point $(2, 0)$ and $f'(x) = 3x^2 - \frac{1}{x^2}$. Find $f(x)$.

7 Find s in terms of t if $\frac{ds}{dt} = 3t - \frac{8}{t^2}$ and $s = 1\frac{1}{2}$ when $t = 1$.

8 A curve $y = f(x)$ for which $f'(x) = 16x + k$, where k is a constant, has a stationary point at $(2, 1)$. Find:

a the value of k **b** the value of $f(x)$ when $x = 7$.

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11C The antiderivative of $(ax + b)^r$

Case 1: $r \neq -1$

For $f(x) = (ax + b)^{r+1}$, where $r \neq -1$, we can use the chain rule to find

$$f'(x) = a(r + 1)(ax + b)^r$$

Thus it follows that:

$$\int (ax + b)^r \, dx = \frac{1}{a(r + 1)}(ax + b)^{r+1} + c, \quad r \neq -1$$

This result does not hold for $r = -1$.

Example 7

Find the general antiderivative of:

a $(3x + 1)^5$ **b** $(2x - 1)^{-2}$

Solution

a
$$\int (3x + 1)^5 \, dx = \frac{1}{3(5 + 1)}(3x + 1)^6 + c$$
$$= \frac{1}{18}(3x + 1)^6 + c$$

b
$$\int (2x - 1)^{-2} \, dx = \frac{1}{2(-2 + 1)}(2x - 1)^{-1} + c$$
$$= -\frac{1}{2}(2x - 1)^{-1} + c$$

Using the TI-Nspire

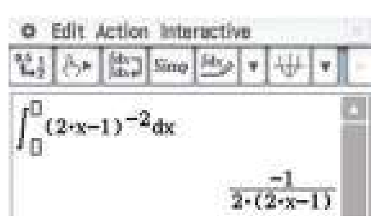
- Use menu > **Calculus** > **Integral** to find the integral of $(2x - 1)^{-2}$.

Note: The integral template can also be accessed using the 2D-template palette [template key] or shift +.

Using the Casio ClassPad

- Enter and highlight the expression $(2x - 1)^{-2}$.
- Select **Interactive** > **Calculation** > $\int$.

Note: The two boxes on the integral allow for definite integrals to be evaluated. This is covered later in the chapter.



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Case 2: $r = -1$

But what happens when $r = -1$? In other words, what is $\int \frac{1}{ax+b}\,dx$?

Remember that $\frac{d}{dx}(\log_e x) = \frac{1}{x}$. Thus $\int \frac{1}{x}\,dx = \log_e x + c$ provided that $x > 0$.

More generally:

For $ax + b > 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c$$

For $x < 0$, we have

$$\frac{d}{dx}\big(\log_e(-x)\big) = \frac{1}{-x} \times (-1) = \frac{1}{x}$$

and so $\int \frac{1}{x}\,dx = \log_e(-x)$.

More generally, for $ax + b < 0$, we have

$$\frac{d}{dx}\big(\log_e(-ax-b)\big) = \frac{1}{-ax-b} \times (-a) = \frac{a}{ax+b}$$

and so $\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(-ax-b)$.

We can summarise these results as:

$$\int \frac{1}{ax+b}\,dx = \begin{cases} \frac{1}{a}\log_e(ax+b) + c & \text{for } ax+b > 0 \\ \frac{1}{a}\log_e(-ax-b) + c & \text{for } ax+b < 0 \end{cases}$$

Example 8

a Find the general antiderivative of $\frac{2}{3x-2}$ for $x > \frac{2}{3}$.

b Find the general antiderivative of $\frac{2}{3x-2}$ for $x < \frac{2}{3}$.

c Given $\frac{dy}{dx} = \frac{3}{x}$ for $x > 0$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

d Given $\frac{dy}{dx} = \frac{3}{x}$ for $x < 0$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

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Solution

a For $x > \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\log_e(3x-2) + c$$
$$= \frac{2}{3}\log_e(3x-2) + c$$

b For $x < \frac{2}{3}$,

$$\int \frac{2}{3x-2}\,dx = \frac{1}{3} \times 2\log_e(2-3x) + c$$
$$= \frac{2}{3}\log_e(2-3x) + c$$

c $y = \int \frac{3}{x}\,dx = 3\log_e x + c$

When $x = 1$, $y = 10$ and so

$$10 = 3\log_e 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\log_e x + 10$.

d $y = \int \frac{3}{x}\,dx = 3\log_e(-x) + c$

When $x = -1$, $y = 10$ and so

$$10 = 3\log_e 1 + c$$
$$10 = 0 + c$$
$$\therefore \quad c = 10$$

Hence $y = 3\log_e(-x) + 10$.

The situation is simplified by using the **absolute value function**, which is not explicitly in the syllabus of Mathematical Methods Units 3 & 4. It is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

For example, $|-2| = |2| = 2$.

For the type of example we are working with here, we have

$$|ax+b| = \begin{cases} ax+b & \text{if } ax+b \geq 0 \\ -ax-b & \text{if } ax+b < 0 \end{cases}$$

Thus:

For $ax + b \neq 0$,

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c$$

Notes:

- We can now deal with parts **a** and **b** of Example 8 simultaneously by writing $\int \frac{2}{3x-2}\,dx = \frac{2}{3}\log_e|3x-2| + c$. For parts **c** and **d**, write $y = 3\log_e|x| + c$.
- Using this notation is recommended as it avoids difficulties and is consistent with the calculators being used.

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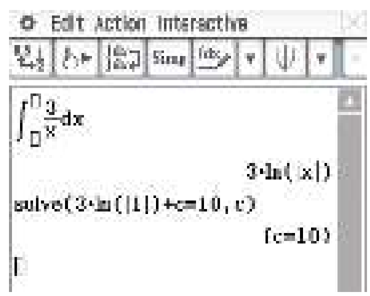
Using the TI-Nspire

- Use menu > **Calculus** > **Integral** to find *an* antiderivative of $\frac{3}{x}$.
- Add c to find the *general* antiderivative of $\frac{3}{x}$.
- Use **solve( )** to determine the value of c as shown.

Using the Casio ClassPad

- Enter and highlight the expression $\frac{3}{x}$.
- Select **Interactive** > **Calculation** > $\int$.
- Note that the ClassPad does not add c to the indefinite integral.
- Copy and paste the answer to the next line. Replace the x with 1 and add c to complete the equation $3 \ln|1| + c = 10$.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is set to c.



Example 9

a Find the general antiderivative of $\frac{2}{2-3x}$.

b Given $\frac{dy}{dx} = \frac{2}{x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

c Given $\frac{dy}{dx} = \frac{2}{x}$ and $y = 10$ when $x = -1$, find an expression for y in terms of x.

Solution

a $\int \frac{2}{2-3x}\, dx = -\frac{1}{3} \times 2 \log_e|2-3x| + c$

$= -\frac{2}{3} \log_e|2-3x| + c$

b $y = \int \frac{2}{x}\, dx = 2\log_e|x| + c$

When $x = 1$, $y = 10$ and so

$10 = 2\log_e|1| + c$

$10 = 0 + c$

$\therefore \quad c = 10$

Hence $y = 2\log_e|x| + 10$.

c $y = \int \frac{2}{x}\, dx = 2\log_e|x| + c$

When $x = -1$, $y = 10$ and so

$10 = 2\log_e|-1| + c$

$10 = 0 + c$

$\therefore \quad c = 10$

Hence $y = 2\log_e|x| + 10$.

The law of exponential change

We are now able to prove the following result from Chapter 5; we do not dwell on the proof as its place is in Specialist Mathematics.

If the rate at which a quantity increases or decreases is proportional to its current value, then the quantity obeys the **law of exponential change**.

Let A be the quantity at time t. Then

$$A = A_0 e^{kt}$$

where A_0 is the initial quantity and k is a constant.

Proof Assume that A is a positive quantity such that $\frac{dA}{dt} = kA$, for some constant $k \neq 0$.

Then A is strictly increasing or A is strictly decreasing, depending on whether $k > 0$ or $k < 0$. Thus A is a one-to-one function of t, and so from Chapter 9 we can write

$$\frac{dt}{dA} = \frac{1}{kA}$$

Antidifferentiating with respect to A gives

$$t = \frac{1}{k}\log_e A + c$$

$$\therefore \quad \log_e A = k(t - c)$$

Thus $A = e^{kt-kc} = e^{-kc}e^{kt}$. We let $A_0 = e^{-kc}$ and we have the result $A = A_0 e^{kt}$.

Summary 11C

- If $r \in \mathbb{Q} \setminus \{-1\}$, then

$$\int (ax+b)^r \, dx = \frac{1}{a(r+1)}(ax+b)^{r+1} + c$$

- For $ax + b > 0$,

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\log_e(ax+b) + c$$

- For $ax + b < 0$,

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\log_e(-ax-b) + c$$

Or more conveniently:

- For $ax + b \neq 0$,

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a}\log_e |ax+b| + c$$

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Skill-sheet

Exercise 11C

1 Find:

Example 7

a $\int (2x-1)^2\,dx$ **b** $\int (2-t)^3\,dt$ **c** $\int (5x-2)^3\,dx$

d $\int (4x-6)^{-2}\,dx$ **e** $\int (6-4x)^{-3}\,dx$ **f** $\int (4x+3)^{-3}\,dx$

g $\int (3x+6)^{\frac{1}{2}}\,dx$ **h** $\int (3x+6)^{-\frac{1}{2}}\,dx$ **i** $\int (2x-4)^{\frac{7}{2}}\,dx$

j $\int (3x+11)^{\frac{4}{3}}\,dx$ **k** $\int \sqrt{2-3x}\,dx$ **l** $\int (5-2x)^4\,dx$

Example 8

2 Find an antiderivative of each of the following:

a $\dfrac{1}{2x},\ x > 0$ **b** $\dfrac{1}{3x+2},\ x > -\dfrac{2}{3}$ **c** $\dfrac{4}{1+4x},\ x > -\dfrac{1}{4}$

d $\dfrac{5}{3x-2},\ x > \dfrac{2}{3}$ **e** $\dfrac{3}{1-4x},\ x < \dfrac{1}{4}$ **f** $\dfrac{3}{2-\frac{x}{2}},\ x < 4$

Example 9

3 Find (using the absolute value function in your answer):

a $\int \dfrac{5}{x}\,dx$ **b** $\int \dfrac{3}{x-4}\,dx$ **c** $\int \dfrac{10}{2x+1}\,dx$

d $\int \dfrac{6}{5-2x}\,dx$ **e** $\int 6(1-2x)^{-1}\,dx$ **f** $\int (4-3x)^{-1}\,dx$

4 Find an antiderivative of each of the following. (Where appropriate, use the absolute value function in your answer.)

a $\dfrac{3x+1}{x}$ **b** $\dfrac{x+1}{x}$ **c** $\dfrac{1}{(x+1)^2}$

d $\dfrac{(x+1)^2}{x}$ **e** $\dfrac{3}{(x-1)^3}$ **f** $\dfrac{1-2x}{x}$

Example 8

Example 9

5 Find y in terms of x for each of the following:

a $\dfrac{dy}{dx} = \dfrac{1}{2x}$ and $y = 2$ when $x = e^2$ **b** $\dfrac{dy}{dx} = \dfrac{2}{5-2x}$ and $y = 10$ when $x = 2$

6 A curve with equation $y = f(x)$ passes through the point $(5+e, 10)$ and $f'(x) = \dfrac{10}{x-5}$. Find the equation of the curve.

7 Find an antiderivative of each of the following. (Where appropriate, use the absolute value function in your answer.)

a $\dfrac{x}{x+1}$ **b** $\dfrac{1-2x}{x+1}$ **c** $\dfrac{2x+1}{x+1}$

8 Given that $\dfrac{dy}{dx} = \dfrac{3}{x-2}$ and $y = 10$ when $x = 0$, find an expression for y in terms of x.

9 Given that $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$ and $y = 10$ when $x = -2$, find an expression for y in terms of x.

10 Given that $\dfrac{dy}{dx} = \dfrac{5}{2-4x}$ and $y = 10$ when $x = 1$, find an expression for y in terms of x.

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11D The antiderivative of e^{kx}

In Chapter 9 we found that, if $f(x) = e^{kx}$, then $f'(x) = ke^{kx}$.

Thus:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

Example 10

Find the general antiderivative of each of the following:

a e^{4x} **b** $e^{5x} + 6x$ **c** $e^{3x} + 2$ **d** $e^{-x} + e^{x}$

Solution

a $\int e^{4x}\,dx = \frac{1}{4}e^{4x} + c$

b $\int e^{5x} + 6x\,dx = \frac{1}{5}e^{5x} + 3x^2 + c$

c $\int e^{3x} + 2\,dx = \frac{1}{3}e^{3x} + 2x + c$

d $\int e^{-x} + e^{x}\,dx = -e^{-x} + e^{x} + c$

Example 11

If the gradient of the tangent at a point (x, y) on a curve is given by $5e^{2x}$ and the curve passes through the point $(0, 7.5)$, find the equation of the curve.

Solution

Let the curve have equation $y = f(x)$. Then $f'(x) = 5e^{2x}$.

$$\int 5e^{2x}\,dx = \frac{5}{2}e^{2x} + c$$

$$\therefore \quad f(x) = \frac{5}{2}e^{2x} + c$$

But $f(0) = 7.5$ and therefore

$$7.5 = \frac{5}{2}e^{0} + c$$

$$= 2.5 + c$$

$$\therefore \quad c = 5$$

Hence $f(x) = \frac{5}{2}e^{2x} + 5$.

Summary 11D

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c, \quad k \neq 0$$

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Skill-sheet

Exercise 11D

Example 10

1 Find the general antiderivative of each of the following:

a e^{6x} **b** $e^{2x} + 3x$ **c** $e^{-3x} + 2x$ **d** $e^{-2x} + e^{2x}$

2 Find:

a $\int e^{2x} - e^{\frac{x}{2}}\, dx$ **b** $\int \frac{e^{2x} + 1}{e^x}\, dx$ **c** $\int 2e^{3x} - e^{-x}\, dx$

d $\int 5e^{\frac{x}{3}} - 2e^{\frac{x}{5}}\, dx$ **e** $\int 3e^{\frac{2x}{3}} - 3e^{\frac{7x}{5}}\, dx$ **f** $\int 5e^{\frac{4x}{3}} - 3e^{\frac{2x}{3}}\, dx$

Example 11

3 Find y in terms of x for each of the following:

a $\frac{dy}{dx} = e^{2x} - x$ and $y = 5$ when $x = 0$

b $\frac{dy}{dx} = \frac{3 - e^{2x}}{e^x}$ and $y = 4$ when $x = 0$

4 Given that $\frac{dy}{dx} = ae^{-x} + 1$ and that when $x = 0$, $\frac{dy}{dx} = 3$ and $y = 5$, find the value of y when $x = 2$.

5 A curve for which $\frac{dy}{dx} = e^{kx}$, where k is a constant, is such that the tangent at $(1, e^2)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

6 A curve for which $\frac{dy}{dx} = -e^{kx}$, where k is a constant, is such that the tangent at $(1, -e^3)$ passes through the origin. Find the gradient of this tangent and hence determine:

a the value of k **b** the equation of the curve.

11E The fundamental theorem of calculus and the definite integral

The integrals that you have learned to evaluate in the previous sections are known as **indefinite integrals** because they are only defined to within an arbitrary constant: for example, we have $\int 3x^2\, dx = x^3 + c$. In general terms, we can write $\int f(x)\, dx = F(x) + c$; that is, the integral of $f(x)$ is $F(x)$ plus a constant, where $F(x)$ is an antiderivative of $f(x)$.

We now resume our consideration of the **definite integral** and investigate its connection with the indefinite integral.

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Signed area

We first look at regions below the x-axis as well as those above the x-axis.

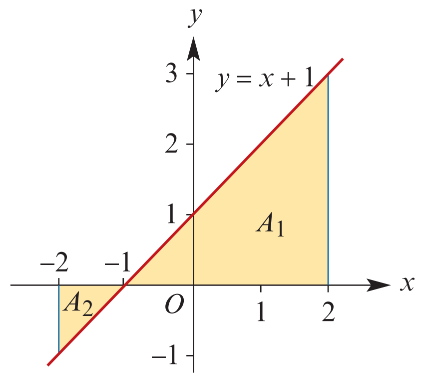
Consider the graph of $y = x + 1$ shown to the right.

$A_1 = \frac{1}{2} \times 3 \times 3 = 4\frac{1}{2}$ (area of a triangle)

$A_2 = \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

The total area is $A_1 + A_2 = 5$.

The **signed area** is $A_1 - A_2 = 4$.

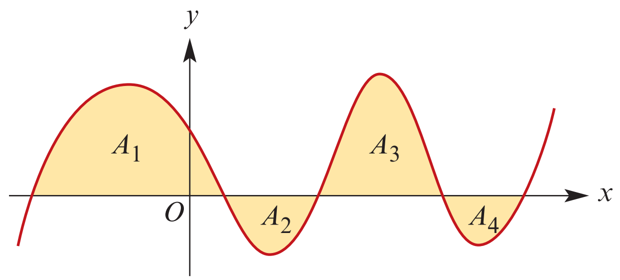


Regions above the x-axis have **positive signed area**.

Regions below the x-axis have **negative signed area**.

The total area of the shaded region is $A_1 + A_2 + A_3 + A_4$.

The signed area of the shaded region is $A_1 - A_2 + A_3 - A_4$.



For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.

In this more general setting, the definite integral can still be determined by a limiting process as discussed in the first section of this chapter.

The fundamental theorem of calculus

The fundamental theorem of calculus provides a connection between the area definition of the definite integral and the antiderivatives discussed previously. An outline of the proof is given in the final section of this chapter.

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

To facilitate setting out, we sometimes write

$$G(b) - G(a) = [G(x)]_a^b$$

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Example 12

Evaluate the definite integral $\int_1^2 x\,dx$.

Solution

We have $\int x\,dx = \frac{1}{2}x^2 + c$ and so

$$\int_1^2 x\,dx = \frac{1}{2}\times 2^2 + c - \left(\frac{1}{2}\times 1^2 + c\right)$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

Note: The arbitrary constant cancels out. Because of this, we ignore it when evaluating definite integrals. We also use the more compact notation $G(b) - G(a) = [G(x)]_a^b$ to help with setting out:

$$\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = \frac{2^2}{2} - \frac{1^2}{2} = \frac{3}{2}$$

Example 13

Evaluate each of the following the definite integrals:

a $\int_2^3 x^2\,dx$ **b** $\int_3^2 x^2\,dx$ **c** $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

Solution

a $\int_2^3 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_2^3$$

$$= \frac{27}{3} - \frac{8}{3}$$

$$= \frac{19}{3}$$

b $\int_3^2 x^2\,dx$

$$= \left[\frac{x^3}{3}\right]_3^2$$

$$= \frac{8}{3} - \frac{27}{3}$$

$$= -\frac{19}{3}$$

c $\int_0^1 x^{\frac{1}{2}} + x^{\frac{3}{2}}\,dx$

$$= \left[\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{5}x^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{3} + \frac{2}{5}$$

$$= \frac{16}{15}$$

Example 14

Evaluate each of the following definite integrals:

a $\int_0^1 2e^{-2x}\,dx$ **b** $\int_0^4 e^{2x} + 1\,dx$ **c** $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx$

Solution

a $\int_0^1 2e^{-2x}\,dx = \left[\frac{2}{-2}e^{-2x}\right]_0^1$

$$= -1(e^{-2\times 1} - e^{-2\times 0})$$

$$= -1(e^{-2} - 1)$$

$$= 1 - e^{-2}$$

b $\int_0^4 e^{2x} + 1\,dx = \left[\frac{1}{2}e^{2x} + x\right]_0^4$

$$= \frac{1}{2}e^8 + 4 - \left(\frac{1}{2}e^0 + 0\right)$$

$$= \frac{1}{2}(e^8 + 7)$$

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c $\int_1^4 2x^{\frac{1}{2}} + e^{\frac{x}{2}}\,dx = \left[\frac{4}{3}x^{\frac{3}{2}} + 2e^{\frac{x}{2}}\right]_1^4$

$= \frac{4}{3} \times 8 + 2e^2 - \left(\frac{4}{3} + 2e^{\frac{1}{2}}\right)$

$= \frac{28}{3} + 2e^2 - 2e^{\frac{1}{2}}$

$= 2\left(\frac{14}{3} + e^2 - e^{\frac{1}{2}}\right)$

Example 15

Evaluate each of the following definite integrals:

a $\int_6^8 \frac{1}{x-5}\,dx$

b $\int_4^5 \frac{1}{2x-5}\,dx$

Solution

a $\int_6^8 \frac{1}{x-5}\,dx = \left[\log_e(x-5)\right]_6^8$

$= \log_e 3 - \log_e 1$

$= \log_e 3$

b $\int_4^5 \frac{1}{2x-5}\,dx = \frac{1}{2}\left[\log_e(2x-5)\right]_4^5$

$= \frac{1}{2}(\log_e 5 - \log_e 3)$

$= \frac{1}{2}\log_e\left(\frac{5}{3}\right)$

Important properties of the definite integral are listed in the summary below.

Summary 11E

- For any continuous function f on an interval $[a, b]$, the **definite integral** $\int_a^b f(x)\,dx$ gives the **signed area** enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$.
- **Fundamental theorem of calculus**

  If f is a continuous function on the interval $[a, b]$, then

  $$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

  where G is any antiderivative of f.
- **Properties of the definite integral**
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

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Skill-sheet

Exercise 11E

1 Evaluate each of the following:

Example 12

Example 13

a $\int_1^2 x^2\,dx$ **b** $\int_{-1}^3 x^3\,dx$ **c** $\int_0^1 x^3 - x\,dx$

d $\int_{-1}^2 (x+1)^2\,dx$ **e** $\int_1^2 \frac{1}{x^2}\,dx$ **f** $\int_1^4 x^{\frac{1}{2}} + 2x^2\,dx$

g $\int_0^2 x^3 + 2x^2 + x + 2\,dx$ **h** $\int_1^4 2x^{\frac{3}{2}} + 5x^3\,dx$

2 Evaluate each of the following:

a $\int_0^1 (2x+1)^3\,dx$ **b** $\int_0^2 (4x+1)^{-\frac{1}{2}}\,dx$ **c** $\int_1^2 (1-2x)^2\,dx$

d $\int_0^1 (3-2x)^{-2}\,dx$ **e** $\int_0^2 (3+2x)^{-3}\,dx$ **f** $\int_{-1}^1 (4x+1)^3\,dx$

g $\int_0^1 \sqrt{2-x}\,dx$ **h** $\int_3^4 \frac{1}{\sqrt{2x-4}}\,dx$ **i** $\int_0^1 \frac{1}{(3+2x)^2}\,dx$

Example 14

3 Evaluate each of the following:

a $\int_0^1 e^{2x}\,dx$ **b** $\int_0^1 e^{-2x} + 1\,dx$ **c** $\int_0^1 2e^{\frac{x}{3}} + 2\,dx$ **d** $\int_{-2}^2 \frac{e^x + e^{-x}}{2}\,dx$

4 Given that $\int_0^4 h(x)\,dx = 5$, evaluate:

a $\int_0^4 2h(x)\,dx$ **b** $\int_0^4 h(x) + 3\,dx$ **c** $\int_4^0 h(x)\,dx$

d $\int_0^4 h(x) + 1\,dx$ **e** $\int_0^4 h(x) - x\,dx$

Example 15

5 **a** Find $\int_0^4 \frac{1}{x-6}\,dx$. **b** Find $\int_2^4 \frac{1}{2x-3}\,dx$. **c** Find $\int_5^6 \frac{3}{2x+7}\,dx$.

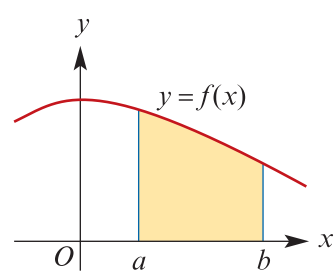
11F Finding the area under a curve

Recall that the definite integral $\int_a^b f(x)\,dx$ gives the net signed area 'under' the curve.

Finding the area of a region

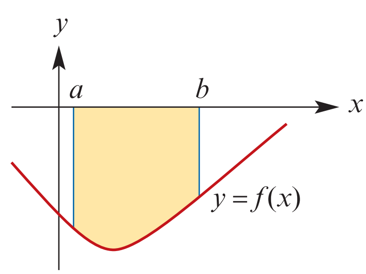
- If $f(x) \ge 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$A = \int_a^b f(x)\,dx$$



- If $f(x) \le 0$ for all $x \in [a, b]$, then the area A of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by

$$A = -\int_a^b f(x)\,dx$$
$$= \int_b^a f(x)\,dx$$



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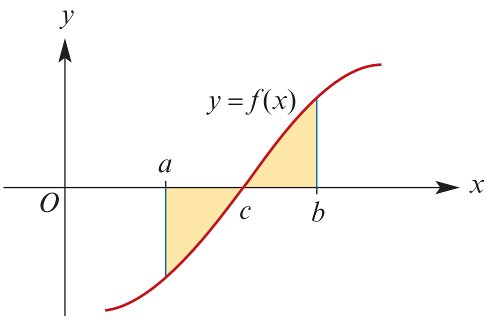
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area A of the shaded region is given by

$$A = \int_c^b f(x)\, dx + \left(-\int_a^c f(x)\, dx\right)$$

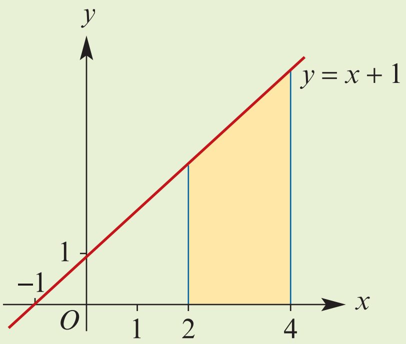


Note: In determining the area 'under' a curve $y = f(x)$, the sign of $f(x)$ in the given interval is the critical factor.

Example 16

a Find the area of the region between the x-axis, the line $y = x + 1$ and the lines $x = 2$ and $x = 4$. Check the answer by working out the area of the trapezium.

b Find the area under the line $y = x + 1$ between $x = -4$ and $x = -2$.



Solution

a Area $= \int_2^4 x + 1\, dx = \left[\frac{x^2}{2} + x\right]_2^4$

$$= \left(\frac{4^2}{2} + 4\right) - \left(\frac{2^2}{2} + 2\right)$$

$$= 12 - 4 = 8$$

The area of the shaded region is 8 square units.

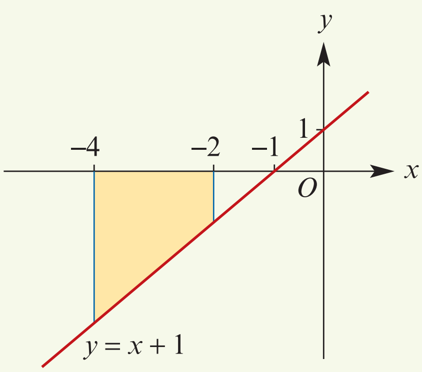
Check: Area of trapezium = average height × base $= \frac{3 + 5}{2} \times 2 = 8$

b Area $= -\int_{-4}^{-2} x + 1\, dx = -\left[\frac{x^2}{2} + x\right]_{-4}^{-2}$

$$= -(0 - 4) = 4$$

The area of the shaded region is 4 square units.

Note: The negative sign is introduced as the integral gives the *signed area* from −4 to −2, which is negative.



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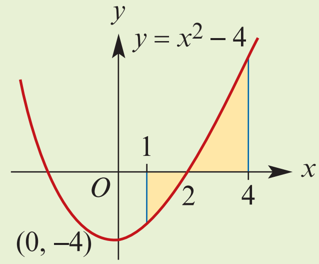
Example 17

Find the exact area of the shaded region.



Solution

$$\text{Area} = \int_2^4 (x^2 - 4)\, dx + - \int_1^2 (x^2 - 4)\, dx$$

$$= \left[\frac{x^3}{3} - 4x\right]_2^4 - \left[\frac{x^3}{3} - 4x\right]_1^2$$

$$= \left(\frac{64}{3} - 16\right) - \left(\frac{8}{3} - 8\right) - \left(\left(\frac{8}{3} - 8\right) - \left(\frac{1}{3} - 4\right)\right)$$

$$= \frac{56}{3} - 8 - \left(\frac{7}{3} - 4\right) = \frac{37}{3}$$

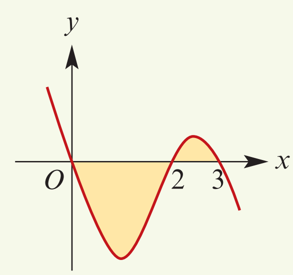
The area is $\frac{37}{3}$ square units.

Example 18

Find the exact area of the regions enclosed by the graph of $y = x(2 - x)(x - 3)$ and the x-axis.

Solution

$$y = x(-x^2 + 5x - 6)$$
$$= -x^3 + 5x^2 - 6x$$



$$\text{Area} = \int_2^3 (-x^3 + 5x^2 - 6x)\, dx + - \int_0^2 (-x^3 + 5x^2 - 6x)\, dx$$

$$= \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_2^3 - \left[\frac{-x^4}{4} + \frac{5x^3}{3} - \frac{6x^2}{2}\right]_0^2$$

$$= \left(\frac{-81}{4} + 45 - 27\right) - \left(-4 + \frac{40}{3} - 12\right) - \left(-4 + \frac{40}{3} - 12\right)$$

$$= \frac{-81}{4} + 18 + 32 - \frac{80}{3}$$

$$= 50 - \frac{243 + 320}{12} = \frac{37}{12}$$

The area is $\frac{37}{12}$ square units.

Note: There is no need to find the coordinates of stationary points.

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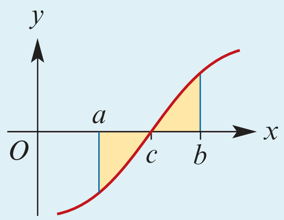
Summary 11F

Finding areas:

- If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
- If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
- If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



Exercise 11F

1 Sketch the graph and find the exact area of the region(s) bounded by the x-axis and the graph of each of the following:

Example 16
Example 17

a $y = 3x^2 + 2$ between $x = 0$ and $x = 1$

b $y = x^3 - 8$ between $x = 2$ and $x = 4$

c $y = 4 - x$ between:

i $x = 0$ and $x = 4$ **ii** $x = 0$ and $x = 6$

Example 18

2 Find the exact area bounded by the x-axis and the graph of each of the following:

a $y = x^2 - 2x$ **b** $y = (4 - x)(3 - x)$ **c** $y = (x + 2)(7 - x)$

d $y = x^2 - 5x + 6$ **e** $y = 3 - x^2$ **f** $y = x^3 - 6x^2$

3 For each of the following, sketch a graph to illustrate the region for which the definite integral gives the area:

a $\int_1^4 2x + 1\,dx$ **b** $\int_0^3 3 - x\,dx$ **c** $\int_0^4 x^2\,dx$

d $\int_{-1}^1 4 - 2x^2\,dx$ **e** $\int_2^4 \sqrt{x}\,dx$ **f** $\int_0^1 (1 - x)(1 + x)^2\,dx$

4 Find the exact area of the region bounded by the curve $y = 3x + 2x^{-2}$, the lines $x = 2$ and $x = 5$ and the x-axis.

5 Sketch the graph of $f(x) = 1 + x^3$ and find the exact area of the region bounded by the curve and the axes.

6 Sketch the graph of $f(x) = 4e^{2x} + 3$ and find the exact area of the region enclosed by the curve, the axes and the line $x = 1$.

7 Sketch the graph of $y = x(2 - x)(x - 1)$ and find the exact area of the region enclosed by the curve and the x-axis.

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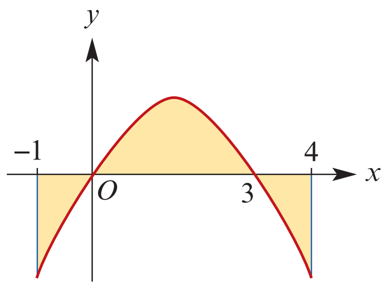
8 **a** Evaluate $\int_{-1}^{4} x(3-x)\,dx$.

b Find the exact area of the shaded region in the figure.

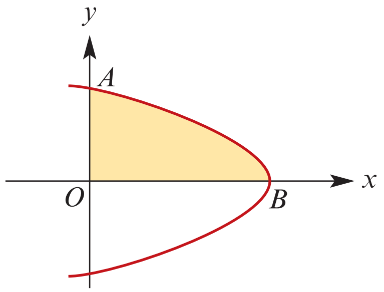


9 **a** In the figure, the graph of $y^2 = 9(1-x)$ is shown. Find the coordinates of A and B.

b Find the exact area of the shaded region by evaluating

$$\int_0^b 1 - \frac{y^2}{9}\,dy$$

for a suitable choice of b.



10 Sketch the graph of $y = \dfrac{1}{2-3x}$ and find the exact area of the region enclosed by the curve, the x-axis and the lines with equations $x = -3$ and $x = -2$.

11 Sketch the graph of $y = 2 + \dfrac{1}{x+4}$ and find the exact area of the region enclosed by the curve, the axes and the line $x = -2$.

12 Let $a > 0$ with $a \neq 1$.

a Show that $a^x = e^{x(\log_e a)}$.

b Hence find the derivative and an antiderivative of a^x.

c Hence, or otherwise, show that the area under the curve $y = a^x$ between the lines $x = 0$ and $x = b$ is $\dfrac{1}{\log_e a}(a^b - 1)$.

11G Integration of circular functions

Recall the following results from Chapter 9:

- If $f(x) = \sin(kx + a)$, then $f'(x) = k\cos(kx + a)$.
- If $g(x) = \cos(kx + a)$, then $g'(x) = -k\sin(kx + a)$.

Thus:

$$\int \sin(kx + a)\,dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\,dx = \frac{1}{k}\sin(kx + a) + c$$

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Example 19

Find an antiderivative of each of the following:

a $\sin\left(3x + \frac{\pi}{4}\right)$ **b** $\frac{1}{4}\sin(4x)$

Solution

a $-\frac{1}{3}\cos\left(3x + \frac{\pi}{4}\right) + c$ **b** $-\frac{1}{16}\cos(4x) + c$

Example 20

Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$ **b** $\int_0^{\frac{\pi}{2}} 2\cos x + 1\,dx$

Solution

a $\int_0^{\frac{\pi}{4}} \sin(2x)\,dx$

$= \left[\frac{-1}{2}\cos(2x)\right]_0^{\frac{\pi}{4}}$

$= \frac{-1}{2}\cos\left(\frac{\pi}{2}\right) - \left(\frac{-1}{2}\cos 0\right)$

$= 0 + \frac{1}{2} = \frac{1}{2}$

b $\int_0^{\frac{\pi}{2}} 2\cos x + 1\,dx$

$= \left[2\sin x + x\right]_0^{\frac{\pi}{2}}$

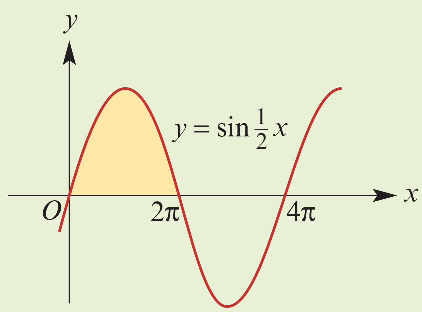
$= 2\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2} - (2\sin 0 + 0)$

$= 2 + \frac{\pi}{2}$

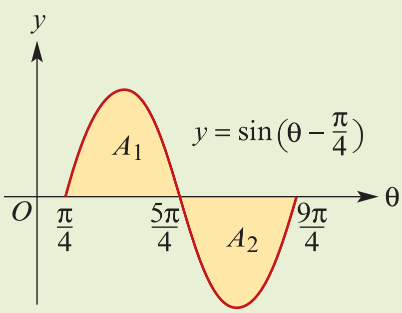
Example 21

Find the exact area of the shaded region for each graph:

a



b



Solution

a Area $= \int_0^{2\pi} \sin\left(\frac{1}{2}x\right) dx$

$= \left[-2\cos\left(\frac{1}{2}x\right)\right]_0^{2\pi}$

$= -2\cos\pi - (-2\cos 0)$

$= 4$

$\therefore$ Area of shaded region is 4 square units.

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b Regions A_1 and A_2 must be considered separately:

$$\text{Area } A_1 = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$$

$$= \left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

$$= -(\cos\pi - \cos 0)$$

$$= 2$$

$$\text{Area } A_2 = -\int_{\frac{5\pi}{4}}^{\frac{9\pi}{4}} \sin\left(\theta - \frac{\pi}{4}\right) d\theta$$

$$= -\left[-\cos\left(\theta - \frac{\pi}{4}\right)\right]_{\frac{5\pi}{4}}^{\frac{9\pi}{4}}$$

$$= \cos(2\pi) - \cos\pi$$

$$= 2$$

∴ Total area of shaded region is 4 square units.

Summary 11G

$$\int \sin(kx + a)\, dx = -\frac{1}{k}\cos(kx + a) + c$$

$$\int \cos(kx + a)\, dx = \frac{1}{k}\sin(kx + a) + c$$

Exercise 11G

Example 19

1 Find an antiderivative of each of the following:

a $\cos(3x)$
b $\sin\left(\frac{1}{2}x\right)$
c $3\cos(3x)$
d $2\sin\left(\frac{1}{2}x\right)$
e $\sin\left(2x - \frac{\pi}{3}\right)$
f $\cos(3x) + \sin(2x)$
g $\cos(4x) - \sin(4x)$
h $-\frac{1}{2}\sin(2x) + \cos(3x)$
i $-\frac{1}{2}\cos\left(2x + \frac{\pi}{3}\right)$
j $\sin(\pi x)$

Example 20

2 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{4}} \sin x\, dx$
b $\int_0^{\frac{\pi}{4}} \cos(2x)\, dx$
c $\int_{-\frac{\pi}{2}}^{\frac{\pi}{4}} \cos\theta\, d\theta$
d $\int_0^{\frac{\pi}{2}} \sin\theta + \cos\theta\, d\theta$
e $\int_0^{\frac{\pi}{2}} \sin(2\theta)\, d\theta$
f $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin(3\theta)\, d\theta$
g $\int_0^{\frac{\pi}{3}} \cos(3\theta) + \sin\left(\theta - \frac{\pi}{3}\right) d\theta$
h $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right) dx$
i $\int_0^{\frac{\pi}{4}} \sin\left(2x - \frac{\pi}{3}\right) dx$
j $\int_0^{\pi} \cos(2x) - \sin\left(\frac{x}{2}\right) dx$

Example 21

3 Calculate the exact area of the region bounded by the curve $y = \sin\left(\frac{1}{2}x\right)$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{2}$.

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4 For each of the following, draw a graph to illustrate the area given by the definite integral and evaluate the integral:

a $\int_0^{\frac{\pi}{4}} \cos x \, dx$ **b** $\int_0^{\frac{\pi}{3}} \sin(2x) \, dx$ **c** $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \cos(2x) \, dx$

d $\int_0^{\frac{\pi}{2}} \cos\theta + \sin\theta \, d\theta$ **e** $\int_0^{\frac{\pi}{2}} \sin(2\theta) + 1 \, d\theta$ **f** $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} 1 - \cos(2\theta) \, d\theta$

5 Find the exact value of each of the following definite integrals:

a $\int_0^{\frac{\pi}{2}} \sin\left(2x + \frac{\pi}{4}\right) dx$ **b** $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{6}\right) dx$

c $\int_0^{\frac{\pi}{3}} \cos\left(3x + \frac{\pi}{3}\right) dx$ **d** $\int_0^{\frac{\pi}{4}} \cos(3\pi - x) \, dx$

6 Sketch the curve $y = 2 + \sin(3x)$ for the interval $0 \le x \le \frac{2\pi}{3}$ and calculate the exact area enclosed by the curve, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

11H Miscellaneous exercises

In this section we look at some further integrals to provide additional practice and introduce new approaches.

Example 22

Let $f(x) = \log_e(x^2 + 1)$.

a Show that $f'(x) = \dfrac{2x}{x^2 + 1}$. **b** Hence evaluate $\int_0^2 \dfrac{x}{x^2 + 1} \, dx$.

Solution

a Let $y = \log_e(x^2 + 1)$ and $u = x^2 + 1$.
Then $y = \log_e u$. By the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u} \cdot 2x$$
$$\therefore \quad f'(x) = \frac{2x}{x^2 + 1}$$

b
$$\int_0^2 \frac{x}{x^2 + 1} \, dx = \frac{1}{2} \int_0^2 \left(\frac{2x}{x^2 + 1}\right) dx$$
$$= \frac{1}{2}\left[\log_e(x^2 + 1)\right]_0^2$$
$$= \frac{1}{2}(\log_e 5 - \log_e 1)$$
$$= \frac{1}{2} \log_e 5$$

Example 23

Let $f(x) = \dfrac{\cos x}{\sin x}$.

a Show that $f'(x) = \dfrac{-1}{\sin^2 x}$. **b** Hence evaluate $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \dfrac{1}{\sin^2 x} \, dx$.

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Solution

a Using the quotient rule:

$$f'(x) = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = \frac{-1}{\sin^2 x}$$

b $$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx = -\left[\frac{\cos x}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\frac{\cos(\frac{\pi}{2})}{\sin(\frac{\pi}{2})} + \frac{\cos(\frac{\pi}{4})}{\sin(\frac{\pi}{4})}$$

$$= 1$$

Example 24

a If $f(x) = x\log_e(kx)$, find $f'(x)$ and hence find $\int \log_e(kx)\,dx$, where k is a positive real constant.

b If $f(x) = x^2\log_e(kx)$, find $f'(x)$ and hence find $\int x\log_e(kx)\,dx$, where k is a positive real constant.

Solution

a $$f'(x) = \log_e(kx) + x \times \frac{1}{x}$$

$$= \log_e(kx) + 1$$

Antidifferentiate both sides of the equation with respect to x:

$$\int f'(x)\,dx = \int \log_e(kx)\,dx + \int 1\,dx$$

$$x\log_e(kx) + c_1 = \int \log_e(kx)\,dx + x + c_2$$

Thus $$\int \log_e(kx)\,dx = x\log_e(kx) - x + c_1 - c_2$$

$$= x\log_e(kx) - x + c$$

b $$f'(x) = 2x\log_e(kx) + x^2 \times \frac{1}{x}$$

$$= 2x\log_e(kx) + x$$

Antidifferentiate both sides of the equation with respect to x:

$$\int f'(x)\,dx = \int 2x\log_e(kx)\,dx + \int x\,dx$$

$$x^2\log_e(kx) + c_1 = \int 2x\log_e(kx)\,dx + \frac{x^2}{2} + c_2$$

Thus $$\int x\log_e(kx)\,dx = \frac{1}{2}x^2\log_e(kx) - \frac{x^2}{4} + c$$

It is not possible to find rules for antiderivatives of all continuous functions: for example, for e^{-x^2}. However, for these functions we can find approximations of definite integrals.

For some functions, a CAS calculator can be used to find exact values of definite integrals where finding an antiderivative by hand is beyond the scope of the course. The following example illustrates this case.

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Example 25

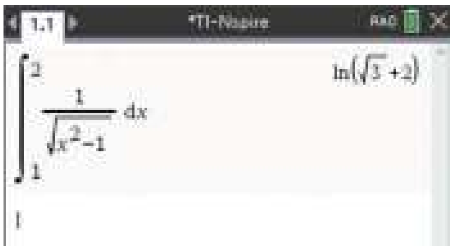
a Find $\int_1^2 \frac{1}{\sqrt{x^2-1}}\,dx$.

b Find $\int_0^{\frac{\pi}{2}} e^x \sin x\,dx$.

Using the TI-Nspire

Use the **Integral** template from the **Calculus** menu and complete as shown.

a

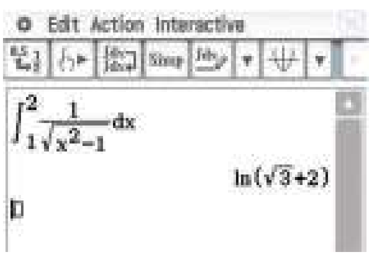


b

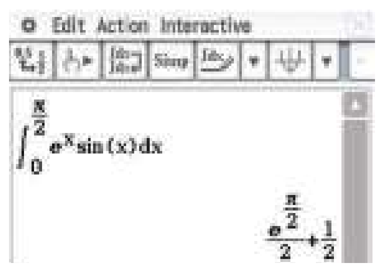
Using the Casio ClassPad

- Enter and highlight the expression $\frac{1}{\sqrt{x^2-1}}$ or the expression $e^x \sin(x)$.
- Go to **Interactive** > **Calculation** > $\int$ and select **Definite**.
- Enter the lower limit and upper limit and tap OK.

a



b



Exercise 11H

1 Find the exact value of each of the following:

a $\int_1^4 \sqrt{x}\,dx$

b $\int_{-1}^1 (1+x)^2\,dx$

c $\int_0^8 \sqrt[3]{x}\,dx$

d $\int_0^{\frac{\pi}{3}} \cos(2x) - \sin\left(\frac{1}{2}x\right)dx$

e $\int_1^2 e^{2x} + \frac{4}{x}\,dx$

f $\int_0^{\frac{\pi}{2}} \sin(2x) + \cos(3x)\,dx$

g $\int_0^{\pi} \sin\left(\frac{x}{4}\right) + \cos\left(\frac{x}{4}\right)dx$

h $\int_0^{\frac{\pi}{2}} 5x + \sin(2x)\,dx$

i $\int_1^4 \left(2+\frac{1}{x}\right)^2 dx$

j $\int_0^1 x^2(1-x)\,dx$

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2 Find the exact area of the region bounded by the graph of $f(x) = \sin x$, the x-axis and the lines $x = 0$ and $x = \frac{\pi}{3}$.

Example 22

Example 23

3 **a** Differentiate $\frac{\sin x}{\cos x}$ and hence find an antiderivative of $\frac{1}{\cos^2 x}$.

b Differentiate $\frac{\cos(2x)}{\sin(2x)}$ and hence find an antiderivative of $\frac{1}{\sin^2(2x)}$.

c Differentiate $\log_e(3x^2 + 7)$ and hence evaluate $\int_0^2 \frac{x}{3x^2 + 7}\, dx$.

d Differentiate $x \sin x$ and hence evaluate $\int_0^{\frac{\pi}{4}} x \cos x \, dx$.

Example 24

4 **a** If $f(x) = x \log_e(2x)$, find $f'(x)$ and hence find $\int \log_e(2x)\, dx$.

b If $f(x) = x^2 \log_e(2x)$, find $f'(x)$ and hence find $\int x \log_e(2x)\, dx$.

c Find the derivatives of $x + \sqrt{1 + x^2}$ and $\log_e(x + \sqrt{1 + x^2})$.

By simplifying your last result if necessary, evaluate $\int_0^1 \frac{1}{\sqrt{1 + x^2}}\, dx$.

5 Find $\frac{d}{dx}(e^{\sqrt{x}})$ and hence evaluate $\int_1^2 \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$.

6 Find $\frac{d}{dx}\left(\sin^3(2x)\right)$ and hence evaluate $\int_0^{\frac{\pi}{4}} \sin^2(2x)\cos(2x)\, dx$.

Example 25

7 Find the value of each of the following definite integrals, correct to two decimal places:

a $\int_0^{20} 10\cos\left(\frac{\pi x}{40}\right) e^{\frac{x}{80}}\, dx$

b $\int_2^5 \frac{e^x}{(x-1)^2}\, dx$

c $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos x}{(x-1)^2}\, dx$

d $\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{100\cos x}{x^2}\, dx$

e $\int_0^{\pi} e^{\left(\frac{x}{10}\right)^2} \sin x \, dx$

f $\int_0^{\frac{\pi}{4}} \cos^3(x)\, e^{-x}\, dx$

8 **a** Show that $\frac{2x+3}{x-1} = 2 + \frac{5}{x-1}$.

b Hence evaluate $\int_2^4 \frac{2x+3}{x-1}\, dx$.

9 **a** Show that $\frac{5x-4}{x-2} = 5 + \frac{6}{x-2}$.

b Hence evaluate $\int_3^4 \frac{5x-4}{x-2}\, dx$.

10 **a** If $y = (1 - \frac{1}{2}x)^8$, find $\frac{dy}{dx}$. Hence, or otherwise, find $\int (1 - \frac{1}{2}x)^7\, dx$.

b If $y = \log_e(\cos x)$ for $\cos x > 0$, find $\frac{dy}{dx}$. Hence evaluate $\int_0^{\frac{\pi}{3}} \tan x \, dx$.

11 Find a function f such that $f'(x) = \sin\left(\frac{1}{2}x\right)$ and $f\left(\frac{4\pi}{3}\right) = 2$.

12 For each of the following, find $f(x)$:

a $f'(x) = \cos(2x)$ and $f(\pi) = 1$

b $f'(x) = \frac{3}{x}$ and $f(1) = 6$

c $f'(x) = e^{\frac{x}{2}}$ and $f(0) = 1$

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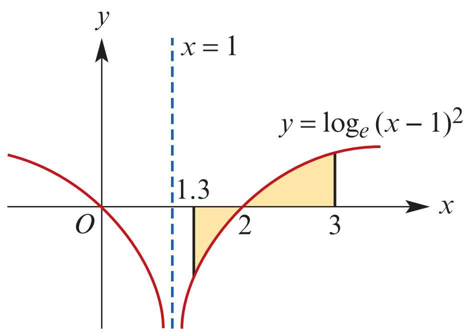
13 Find $\frac{d}{dx}\big(x\sin(3x)\big)$ and hence evaluate $\int_0^{\frac{\pi}{6}} x\cos(3x)\,dx$.

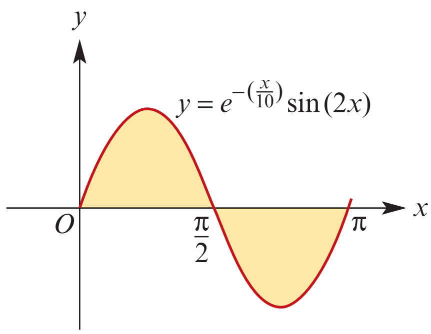
14 The curve with equation $y = a + b\sin\left(\frac{\pi x}{2}\right)$ passes through the points (0, 1) and (3, 3). Find a and b. Find the area of the region enclosed by this curve, the x-axis and the lines $x = 0$ and $x = 1$.

15 For each of the following, find the area of the shaded region correct to three decimal places:

a



b



16 Evaluate $\int_0^{\pi} e^{-\left(\frac{x}{10}\right)}\sin(2x)\,dx$, correct to four decimal places.

17 The gradient of a curve with equation $y = f(x)$ is given by $f'(x) = x + \sin(2x)$ and $f(0) = 1$. Find $f(x)$.

18 Let $f(x) = g'(x)$ and $h(x) = k'(x)$, where $g(x) = (x^2+1)^3$ and $k(x) = \sin(x^2)$. Find:

a $\int f(x)\,dx$ **b** $\int h(x)\,dx$

c $\int f(x) + h(x)\,dx$ **d** $\int -f(x)\,dx$

e $\int f(x) - 4\,dx$ **f** $\int 3h(x)\,dx$

19 Sketch the graph of $y = \frac{2}{x-1} + 4$ and evaluate $\int_2^3 \frac{2}{x-1} + 4\,dx$.
Indicate on your graph the region for which you have determined the area.

20 Sketch the graph of $y = \sqrt{2x-4} + 1$ and evaluate $\int_2^3 \sqrt{2x-4} + 1\,dx$.
Indicate on your graph the region for which you have determined the area.

21 Evaluate each of the following:

a $\int_3^4 \sqrt{x-2}\,dx$ **b** $\int_0^2 \sqrt{2-x}\,dx$

c $\int_0^1 \frac{1}{3x+1}\,dx$ **d** $\int_1^2 \frac{1}{2x-1} + 3\,dx$

e $\int_{2.5}^3 \sqrt{2x-5} - 6\,dx$ **f** $\int_3^4 \frac{1}{\sqrt{x-2}}\,dx$

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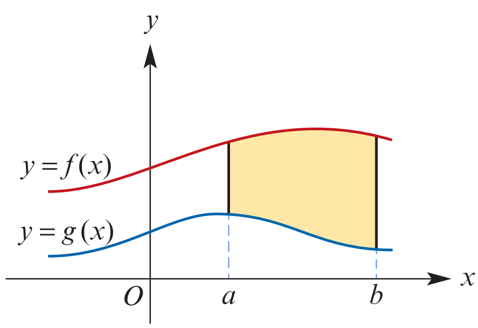
11I The area of a region between two curves

Let f and g be continuous functions on the interval $[a, b]$ such that

$$f(x) \geq g(x) \quad \text{for all } x \in [a, b]$$

Then the area of the region bounded by the two curves and the lines $x = a$ and $x = b$ can be found by evaluating

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$



Example 26

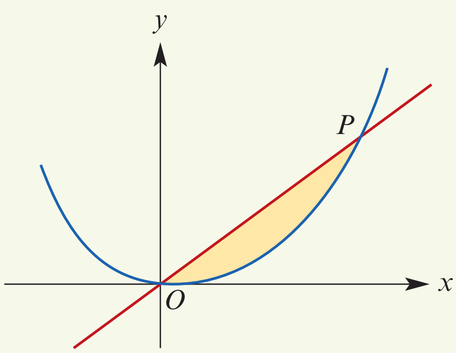
Find the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution

We first find the coordinates of the point P:

$$x^2 = 2x$$
$$x(x-2) = 0$$
$$\therefore \quad x = 0 \text{ or } x = 2$$

Therefore the coordinates of P are $(2, 4)$.



$$\text{Required area} = \int_0^2 2x - x^2\,dx$$
$$= \left[x^2 - \frac{x^3}{3}\right]_0^2$$
$$= 4 - \frac{8}{3} = \frac{4}{3}$$

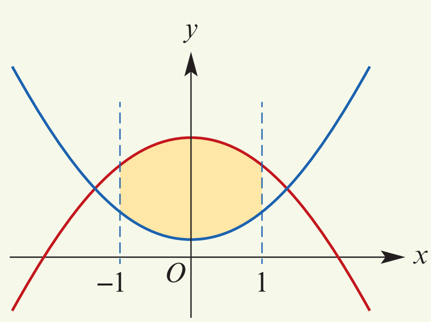
The area is $\frac{4}{3}$ square units.

Example 27

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution

$$\text{Required area} = \int_{-1}^1 4 - x^2 - (x^2 + 1)\,dx$$
$$= \int_{-1}^1 3 - 2x^2\,dx$$
$$= \left[3x - \frac{2x^3}{3}\right]_{-1}^1$$
$$= 3 - \frac{2}{3} - \left(-3 + \frac{2}{3}\right)$$
$$= \frac{14}{3}$$



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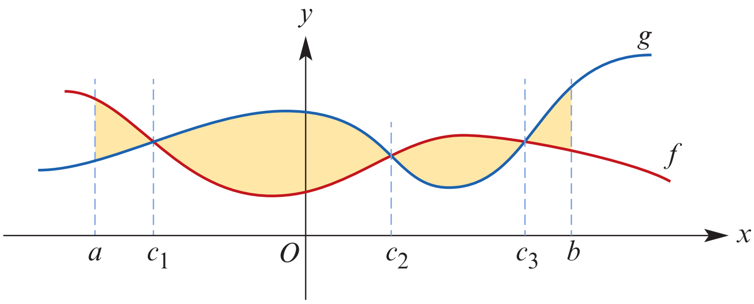
In Examples 26 and 27, the graph of one function is always 'above' the graph of the other for the intervals considered. What happens if the graphs cross?



To find the area of the shaded region, we must consider the intervals $[a, c_1]$, $[c_1, c_2]$, $[c_2, c_3]$ and $[c_3, b]$ separately. Thus, the shaded area is given by

$$\int_a^{c_1} f(x) - g(x)\,dx + \int_{c_1}^{c_2} g(x) - f(x)\,dx + \int_{c_2}^{c_3} f(x) - g(x)\,dx + \int_{c_3}^{b} g(x) - f(x)\,dx$$

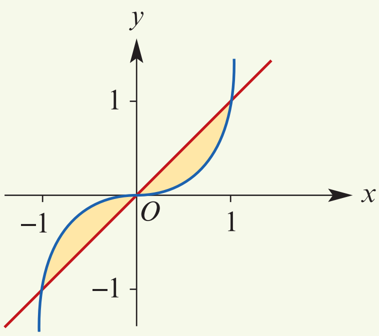
Example 28

Find the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution

The graphs intersect where $f(x) = g(x)$:

$$x^3 = x$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$\therefore\quad x = 0 \text{ or } x = \pm 1$$



We see that:

- $f(x) \geq g(x)$ for $-1 \leq x \leq 0$
- $f(x) \leq g(x)$ for $0 \leq x \leq 1$

Thus the area is given by

$$\int_{-1}^{0} f(x) - g(x)\,dx + \int_0^1 g(x) - f(x)\,dx = \int_{-1}^{0} x^3 - x\,dx + \int_0^1 x - x^3\,dx$$
$$= \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$
$$= -\left(-\frac{1}{4}\right) + \frac{1}{4}$$
$$= \frac{1}{2}$$

The area is $\frac{1}{2}$ square unit.

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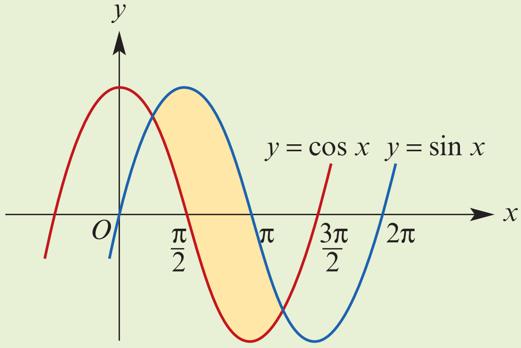
Example 29

Find the area of the shaded region.



Solution

First find the x-coordinates of the two points of intersection.

If $\sin x = \cos x$, then $\tan x = 1$ and so $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x - \cos x \, dx \\ &= \left[-\cos x - \sin x\right]_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \\ &= -\cos\left(\frac{5\pi}{4}\right) - \sin\left(\frac{5\pi}{4}\right) - \left[-\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] \\ &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \\ &= \frac{4}{\sqrt{2}} = 2\sqrt{2}\end{aligned}$$

The area is $2\sqrt{2}$ square units.

Example 30

For the function $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f(x) = \log_e(x+1)$:

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\int_0^{\log_e 2} f^{-1}(x) \, dx$.

c Find the exact value of $\int_0^1 f(x) \, dx$.

Solution

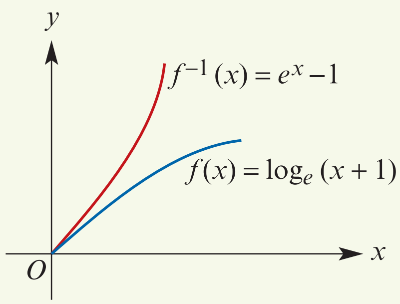
a Let $x = \log_e(y+1)$. Then

$$e^x = y + 1$$

$$\therefore \quad y = e^x - 1$$

Hence the inverse function is

$$f^{-1} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, \ f^{-1}(x) = e^x - 1$$



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b $\int_0^{\log_e 2} f^{-1}(x)\,dx = \int_0^{\log_e 2} e^x - 1\,dx$

$= [e^x - x]_0^{\log_e 2}$

$= e^{\log_e 2} - \log_e 2 - (e^0 - 0)$

$= 2 - \log_e 2 - 1$

$= 1 - \log_e 2$

c Area of rectangle $OABC = \log_e 2$

Area of region $R_1 = \int_0^{\log_e 2} e^y - 1\,dy$

$= [e^y - y]_0^{\log_e 2}$

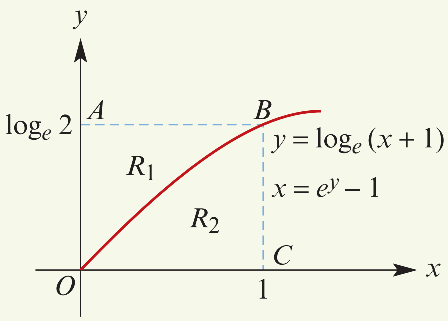
$= 1 - \log_e 2$ (see **b**)

Area of region R_2 = area of rectangle $OABC$ − area of region R_1

$= \log_e 2 - (1 - \log_e 2)$

$= 2\log_e 2 - 1$

$\therefore \int_0^1 f(x)\,dx = 2\log_e 2 - 1$

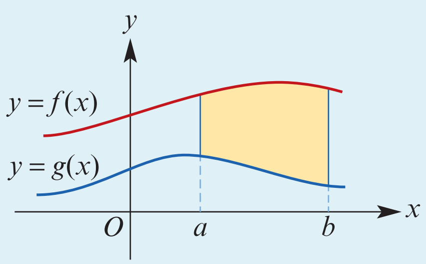


Summary 11I

To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



Exercise 11I

Example 26

1 Find the exact area of the region bounded by the graphs of $y = 12 - x - x^2$ and $y = x + 4$.

Example 27

2 Find the exact area of the region bounded by the graphs of $f(x) = 5 - x^2$ and $g(x) = (x - 1)^2$.

Example 28

3 Find the exact area of the region bounded by the graphs with equations:

a $y = x + 3$ and $y = 12 + x - x^2$

b $y = 3x + 5$ and $y = x^2 + 1$

c $y = 3 - x^2$ and $y = 2x^2$

d $y = x^2$ and $y = 3x$

e $y^2 = x$ and $x - y = 2$

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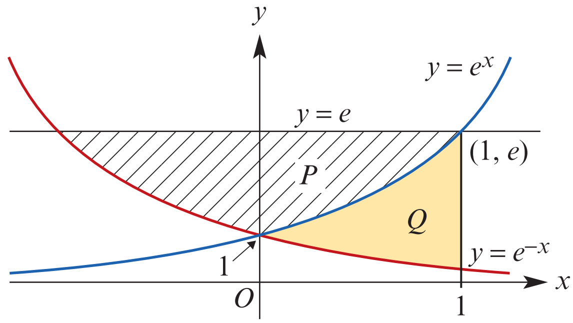
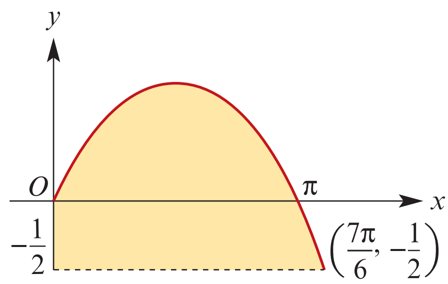
4 **a** Find the area of region P.

b Find the area of region Q.



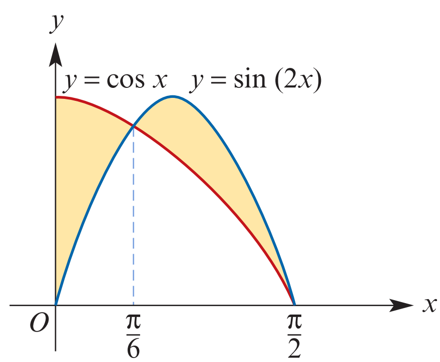
5 The figure shows part of the curve $y = \sin x$. Calculate the area of the shaded region, correct to three decimal places.



Example 29

6 Using the same axes, sketch the curves $y = \sin x$ and $y = \sin(2x)$ for $0 \le x \le \pi$. Calculate the smaller of the two areas enclosed by the curves.

7 Find the area of the shaded region.



8 Find the coordinates of P, the point of intersection of the curves $y = e^x$ and $y = 2 + 3e^{-x}$. If these curves cut the y-axis at points A and B respectively, calculate the area bounded by AB and the arcs AP and BP. Give your answer correct to three decimal places.

Example 30

9 For the function $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \log_e(2x)$:

a Find f^{-1} and sketch the graphs of f and f^{-1} on the one set of axes.

b Find the exact value of the area $\int_0^{\log_e 4} f^{-1}(x)\, dx$.

c Find the exact value of $\int_{\frac{1}{2}}^{2} f(x)\, dx$.

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11J Applications of integration

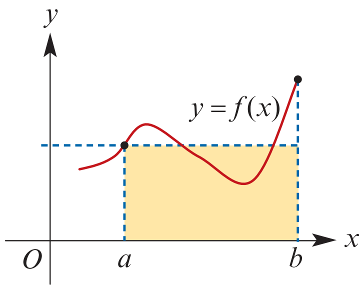
In this section we look at three applications of integration.

Average value of a function

The average value of a function f for an interval $[a, b]$ is defined as:

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

In terms of the graph of $y = f(x)$, the average value is the height of a rectangle having the same area as the area under the graph for the interval $[a, b]$.

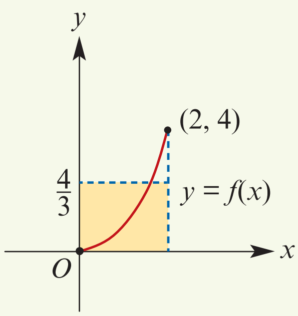


Example 31

Find the average value of $f(x) = x^2$ for the interval $[0, 2]$. Illustrate with a horizontal line determined by this value.

Solution

$$\text{Average} = \frac{1}{2-0}\int_0^2 x^2\,dx$$
$$= \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$
$$= \frac{1}{2}\times\frac{8}{3}$$
$$= \frac{4}{3}$$



Note: Area of rectangle $= \int_0^2 f(x)\,dx$

Rates of change

Given the rate of change of a quantity we can obtain information about how the quantity varies. For example, we have seen that if the velocity of an object travelling in a straight line is given at time t, then the position of the object at time t can be determined using information about the initial position of the object.

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Example 32

The rate of change of temperature with respect to time of a liquid which has been boiled and then allowed to cool is given by $\frac{dT}{dt} = -0.5(T - 30)$, where T is the temperature (°C) at time t (minutes).

a Sketch the graph of $\frac{dT}{dt}$ against T for $T > 30$.

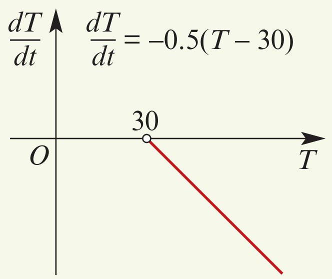
b Sketch the graph of $\frac{dt}{dT}$ against T for $T > 30$.

c **i** Find the area of the region enclosed by the graph of **b**, the x-axis and the lines $T = 35$ and $T = 120$. Give your answer correct to two decimal places.

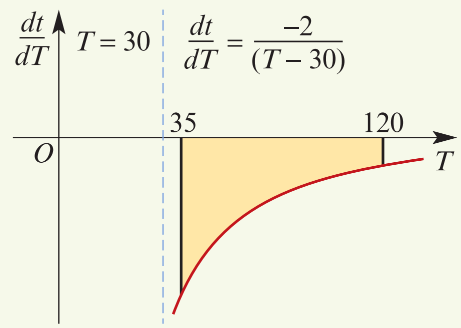
ii What does this area represent?

Solution

a



b



c **i** Area $= -\int_{35}^{120} \frac{-2}{(T-30)}\, dT = 5.78$

ii The area represents the time taken for the liquid to cool from 120°C to 35°C.

Exercise 11J

Example 31

1 Find the average value of each of the following functions for the stated interval:

a $f(x) = x(2 - x),\ x \in [0, 2]$

b $f(x) = \sin x,\ x \in [0, \pi]$

c $f(x) = \sin x,\ x \in \left[0, \frac{\pi}{2}\right]$

d $f(x) = \sin(nx),\ x \in \left[0, \frac{2\pi}{n}\right]$

e $f(x) = e^x + e^{-x},\ x \in [-2, 2]$

2 An object is cooling and its temperature, T°C, after t minutes is given by $T = 50e^{\frac{-t}{2}}$. What is its average temperature over the first 10 minutes of cooling?

3 Find the mean value of $x(a - x)$ from $x = 0$ to $x = a$.

4 A quantity of gas expands according to the law $pv^{0.9} = 300$, where v m^3 is the volume of the gas and p N/m^2 is the pressure.

a What is the average pressure as the volume changes from $\frac{1}{2}$ m^3 to 1 m^3?

b If the change in volume in terms of t is given by $v = 3t + 1$, what is the average

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pressure over the time interval from $t = 0$ to $t = 1$?

Example 32

5 Heat escapes from a storage tank such that the rate of heat loss, in kilojoules per day, is given by

$$\frac{dH}{dt} = 1 + \frac{3}{4}\sin\left(\frac{\pi t}{60}\right), \qquad 0 \le t \le 200$$

where $H(t)$ is the total accumulated heat loss at time t days after noon on 1 April.

a Sketch the graph of $\frac{dH}{dt}$ against t for $0 \le t \le 200$.

b Find the values of t for which the rate of heat loss, i.e. $\frac{dH}{dt}$, is greater than 1.375.

c Find the values of t for which the rate of heat loss reaches its maximum.

d Find the heat lost between:

i $t = 0$ and $t = 120$

ii $t = 0$ and $t = 200$

6 The rate of flow of water from a reservoir is given by $\frac{dV}{dt} = 1000 - 30t^2 + 2t^3$ for $0 \le t \le 15$, where V is measured in millions of litres and t is the number of hours after the sluice gates are opened.

a Find the rate of flow (in million litres per hour) when $t = 0$ and $t = 2$.

b **i** Find the times when the rate of flow is a maximum.

ii Find the maximum flow.

c Sketch the graph of $\frac{dV}{dt}$ against t for $0 \le t \le 15$.

d **i** Find the area beneath the graph between $t = 0$ and $t = 10$.

ii What does this area represent?

7 The population of penguins on an island off the coast of Tasmania is increasing steadily. The rate of growth is given by the function $R\colon [0, \infty) \to \mathbb{R}$, $R(t) = 10\log_e(t + 1)$. The rate is measured in number of penguins per year. The date 1 January 1875 coincides with $t = 0$.

a Find the rate of growth of penguins when $t = 5$, $t = 10$, $t = 100$.

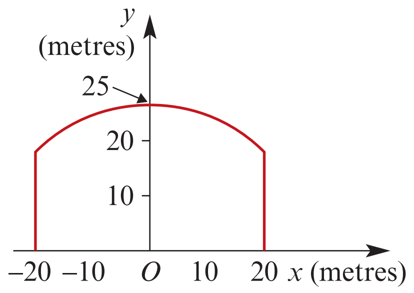
b Sketch the graph of $y = R(t)$.

c Find the inverse function R^{-1}.

d **i** Find the area under the graph of $y = R(t)$ between $t = 0$ and $t = 100$. (Use the inverse function to help find this area.)

ii What does this area represent?

8 The roof of an exhibition hall has the shape of the function $f\colon [-20, 20] \to \mathbb{R}$ where $f(x) = 25 - 0.02x^2$. The hall is 80 metres long. A cross-section of the hall is shown in the figure. An air-conditioning company wishes to find the volume of the hall so that a suitable system may be installed. Find this volume.



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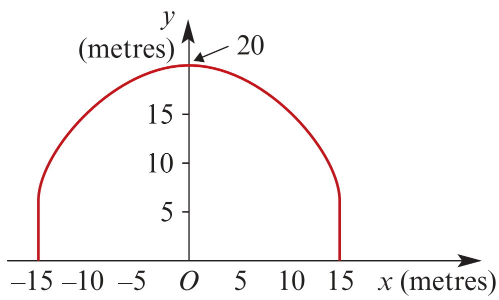
9 An aircraft hangar has the cross-section illustrated. The roof has the shape of the function $f: [-15, 15] \to \mathbb{R}$ where $f(x) = 20 - 0.06x^2$.

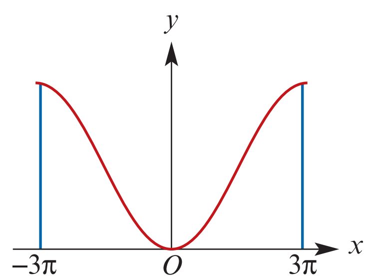


a Find the area of the cross-section.

b Find the volume of the hangar if it is 100 metres long.

10 A long trough with a parabolic cross-section is $1\frac{1}{2}$ metres wide at the top and 2 metres deep. Find the depth of water when the trough is half full.

11 A sculpture has cross-section as shown. The equation of the curve is $y = 3 - 3\cos\left(\frac{x}{3}\right)$ for $x \in [-3\pi, 3\pi]$. All measurements are in metres.



a Find the maximum value of the function and hence the height of the sculpture.

b The sculpture has a flat metal finish on one face, which in the diagram is represented by the region between the curve and the x-axis. Find the area of this region.

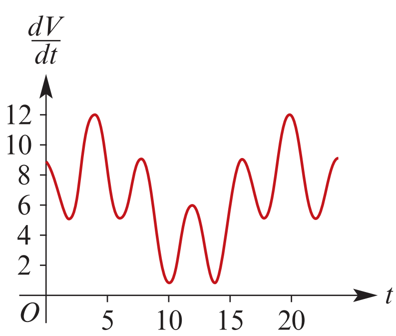
c There is a strut that meets the right side of the curve at right angles and passes through the point $(9, 0)$.

i Find the equation of the normal to the curve where $x = a$.

ii Find, correct to three decimal places, the value of a if the normal passes through $(9, 0)$.

12 The graph shows the number of litres per minute of water flowing through a pipe against the number of minutes since the machine started. The pipe is attached to the machine, which requires the water for cooling.
The curve has equation

$$\frac{dV}{dt} = 3\left[\cos\left(\frac{\pi t}{2}\right) + \sin\left(\frac{\pi t}{8}\right) + 2\right]$$



a What is the rate of flow of water when:

i $t = 0$ **ii** $t = 2$ **iii** $t = 4$?

b Find, correct to three decimal places, the maximum and minimum flow through the pipe.

c Find the volume of water which flows through the pipe in the first 8 minutes.

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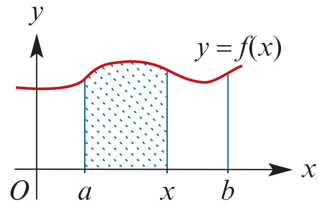
11K The fundamental theorem of calculus

The derivative of the area function

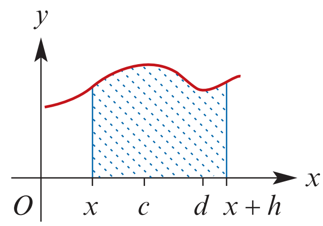
Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(x) \geq 0$ for all $x \in [a, b]$.



We define the function A geometrically by saying that $A(x)$ is the measure of the area under the curve $y = f(x)$ between a and x. We thus have $A(a) = 0$. We will see that $A'(x) = f(x)$, and thus A is an antiderivative of f.

First consider the quotient $\dfrac{A(x+h) - A(x)}{h}$ for $h > 0$.

By our definition of $A(x)$, it follows that $A(x + h) - A(x)$ is the area between x and $x + h$.



Let c be the point in the interval $[x, x+h]$ such that $f(c) \geq f(z)$ for all $z \in [x, x + h]$, and let d be the point in the same interval such that $f(d) \leq f(z)$ for all $z \in [x, x + h]$.

Thus $f(d) \leq f(z) \leq f(c)$ for all $z \in [x, x + h]$.

Therefore $hf(d) \leq A(x + h) - A(x) \leq hf(c)$.

That is, the shaded region has an area less than the area of the rectangle with base h and height $f(c)$ and an area greater than the area of the rectangle with base h and height $f(d)$.

Dividing by h gives

$$f(d) \leq \frac{A(x+h) - A(x)}{h} \leq f(c)$$

As $h \to 0$, both $f(c)$ and $f(d)$ approach $f(x)$.

Thus we have shown that $A'(x) = f(x)$, and therefore A is an antiderivative of f.

Now let G be any antiderivative of f. Since both A and G are antiderivatives of f, they must differ by a constant. That is,

$$A(x) = G(x) + k$$

where k is a constant. First let $x = a$. We then have

$$0 = A(a) = G(a) + k$$

and so $k = -G(a)$.

Thus $A(x) = G(x) - G(a)$, and letting $x = b$ yields

$$A(b) = G(b) - G(a)$$

The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is equal to $G(b) - G(a)$, where G is any antiderivative of f.

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A similar argument could be used if $f(x) \le 0$ for all $x \in [a, b]$, but in this case we must take $A(x)$ to be the negative of the area under the curve. In general:

Fundamental theorem of calculus

If f is a continuous function on an interval $[a, b]$, then

$$\int_a^b f(x)\,dx = G(b) - G(a)$$

where G is any antiderivative of f.

The area as the limit of a sum

Finally, we consider the limit of a sum in a special case. This discussion gives an indication of how the limiting process can be undertaken in general.

Notation

We first introduce a notation to help us express sums. We do this through examples:

$$\sum_{i=1}^{3} i^2 = 1^2 + 2^2 + 3^2$$

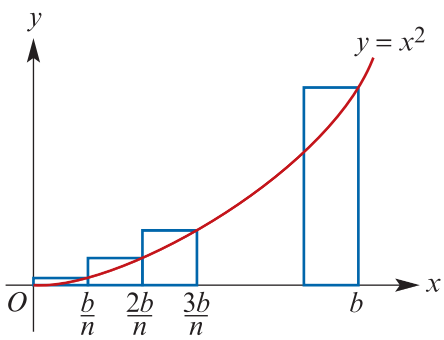
$$\sum_{i=1}^{5} x_i = x_1 + x_2 + x_3 + x_4 + x_5$$

$$\sum_{i=1}^{n} x_i f(x_i) = x_1 f(x_1) + x_2 f(x_2) + x_3 f(x_3) + \cdots + x_n f(x_n)$$

The symbol $\sum$ is the uppercase Greek letter 'sigma', which is used in mathematics to denote *sum*.

The area under a parabola

Consider the graph of $y = x^2$. We will find the area under the curve from $x = 0$ to $x = b$ using a technique due to Archimedes.



Divide the interval $[0, b]$ into n equal subintervals:

$$\left[0, \frac{b}{n}\right], \left[\frac{b}{n}, \frac{2b}{n}\right], \left[\frac{2b}{n}, \frac{3b}{n}\right], \ldots, \left[\frac{(n-1)b}{n}, b\right]$$

Each subinterval is the base of a rectangle with height determined by the right endpoint of the subinterval.

$$\begin{aligned}\text{Area of rectangles} &= \frac{b}{n}\left[\left(\frac{b}{n}\right)^2 + \left(\frac{2b}{n}\right)^2 + \left(\frac{3b}{n}\right)^2 + \cdots + \left(\frac{nb}{n}\right)^2\right] \\ &= \frac{b}{n}\left(\frac{b^2}{n^2} + \frac{4b^2}{n^2} + \frac{9b^2}{n^2} + \cdots + \frac{n^2b^2}{n^2}\right) \\ &= \frac{b^3}{n^3}(1 + 4 + 9 + \cdots + n^2)\end{aligned}$$

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There is a rule for working out the sum of the first n square numbers:

$$\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$$

$$\begin{aligned}\text{Area of rectangles} &= \frac{b^3}{n^3}\sum_{i=1}^{n} i^2 \\ &= \frac{b^3}{n^3} \times \frac{n}{6}(n+1)(2n+1) \\ &= \frac{b^3}{6n^2}(2n^2+3n+1) \\ &= \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)\end{aligned}$$

As n becomes very large, the terms $\frac{3}{n}$ and $\frac{1}{n^2}$ become very small. We write:

$$\lim_{n\to\infty} \frac{b^3}{6}\left(2+\frac{3}{n}+\frac{1}{n^2}\right) = \frac{b^3}{3}$$

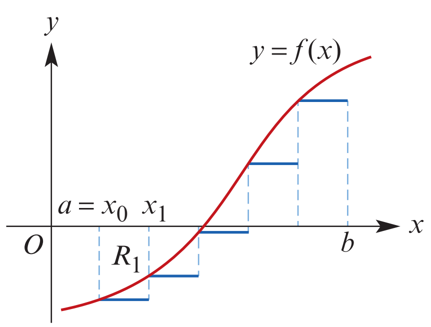
We read this as: the limit of the sum as n approaches infinity is $\frac{b^3}{3}$.

Using n left-endpoint rectangles, and considering the limit as $n \to \infty$, also gives the area $\frac{b^3}{3}$.

The signed area enclosed by a curve

This technique may be applied in general to a continuous function f on an interval $[a, b]$. For convenience, we will consider an increasing function.

Divide the interval $[a, b]$ into n equal subintervals. Each subinterval is the base of a rectangle with its 'height' determined by the left endpoint of the subinterval.



The contribution of rectangle R_1 is $(x_1 - x_0)f(x_0)$. Since $f(x_0) < 0$, the result is negative and so we have found the *signed area* of R_1.

The sum of the signed areas of the rectangles is

$$\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)$$

If the limit as $n \to \infty$ exists, then we can make the following definition:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\left(\frac{b-a}{n}\sum_{i=0}^{n-1} f(x_i)\right)$$

We could also have used the right-endpoint estimate: the left- and right-endpoint estimates will converge to the same limit as n approaches infinity. Definite integrals may be defined as the limit of suitable sums, and the fundamental theorem of calculus holds true under this definition.

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Chapter summary

Assign-ment

Nrich

Review

Antidifferentiation

- To find the general antiderivative:
  If $F'(x) = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is an arbitrary real number.
- Basic antiderivatives:

$$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c \qquad \text{where } r \in \mathbb{Q} \setminus \{-1\}$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(ax+b) + c \qquad \text{for } ax+b > 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e(-ax-b) + c \qquad \text{for } ax+b < 0$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log_e|ax+b| + c \qquad \text{for } ax+b \neq 0$$

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c \qquad \text{where } k \neq 0$$

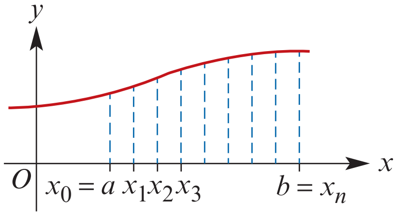
$$\int \sin(kx)\,dx = -\frac{1}{k}\cos(kx) + c \qquad \text{where } k \neq 0$$

$$\int \cos(kx)\,dx = \frac{1}{k}\sin(kx) + c \qquad \text{where } k \neq 0$$

- Properties of antidifferentiation:
  - $\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$
  - $\int k\,f(x)\,dx = k\int f(x)\,dx$, where k is a real number

Integration

- Numerical methods for approximating the area under a graph: Divide the interval $[a, b]$ on the x-axis into n equal subintervals $[a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, b]$.
  - Left-endpoint estimate
    $$L_n = \frac{b-a}{n}\Big[f(x_0) + f(x_1) + \cdots + f(x_{n-1})\Big]$$
  - Right-endpoint estimate
    $$R_n = \frac{b-a}{n}\Big[f(x_1) + f(x_2) + \cdots + f(x_n)\Big]$$



  - Trapezium rule
    $$T_n = \frac{b-a}{2n}\Big[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\Big]$$
- Definite integral The signed area enclosed by the graph of $y = f(x)$ between $x = a$ and $x = b$ is denoted by $\int_a^b f(x)\,dx$.

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Review

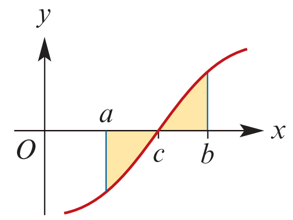
- Fundamental theorem of calculus

  If f is a continuous function on an interval $[a, b]$, then

  $$\int_a^b f(x)\,dx = [G(x)]_a^b = G(b) - G(a)$$

  where G is any antiderivative of f.

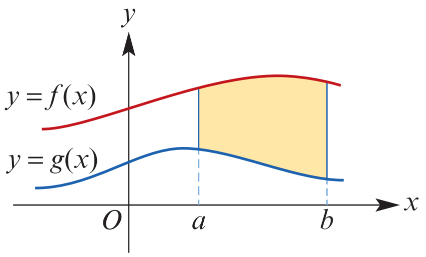
- Finding areas:
  - If $f(x) \geq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$.
  - If $f(x) \leq 0$ for all $x \in [a, b]$, then the area of the region contained between the curve, the x-axis and the lines $x = a$ and $x = b$ is given by $-\int_a^b f(x)\,dx$.
  - If $c \in (a, b)$ with $f(c) = 0$ and $f(x) \geq 0$ for $x \in (c, b]$ and $f(x) \leq 0$ for $x \in [a, c)$, then the area of the shaded region is given by $\int_c^b f(x)\,dx + \left(-\int_a^c f(x)\,dx\right)$.



  - To find the area of the shaded region bounded by the two curves and the lines $x = a$ and $x = b$, use

    $$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b f(x) - g(x)\,dx$$

    where f and g are continuous functions on $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$.



- Properties of the definite integral:
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$
  - $\int_a^a f(x)\,dx = 0$
  - $\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$
  - $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

- The **average value** of a continuous function f for an interval $[a, b]$ is $\dfrac{1}{b-a}\int_a^b f(x)\,dx$.

Technology-free questions

1 Evaluate each of the following definite integrals:

a $\int_2^3 x^3\,dx$

b $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \sin\theta\,d\theta$

c $\int_a^{4a} (a^{\frac{1}{2}} - x^{\frac{1}{2}})\,dx$, where a is a positive constant

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d $\int_1^4 \frac{3}{\sqrt{x}} - 5\sqrt{x} - x^{-\frac{3}{2}}\,dx$

e $\int_0^{\frac{\pi}{4}} \cos(2\theta)\,d\theta$

f $\int_1^e \frac{1}{x}\,dx$

g $\int_0^{\frac{\pi}{2}} \sin 2\left(\theta + \frac{\pi}{4}\right)\,d\theta$

h $\int_0^{\pi} \sin(4\theta)\,d\theta$

2 Find $\int_{-1}^{2} x + 2f(x)\,dx$ if $\int_{-1}^{2} f(x)\,dx = 5$.

3 Find $\int_1^5 f(x)\,dx$ if $\int_0^1 f(x)\,dx = -2$ and $\int_0^5 f(x)\,dx = 1$.

4 Find $\int_3^{-2} f(x)\,dx$ if $\int_{-2}^{1} f(x)\,dx = 2$ and $\int_1^3 f(x)\,dx = -6$.

5 Evaluate $\int_0^2 (x+1)^7\,dx$.

6 Evaluate $\int_0^1 (3x+1)^3\,dx$.

7 Find $\int_0^3 f(3x)\,dx$ if $\int_0^9 f(x)\,dx = 5$.

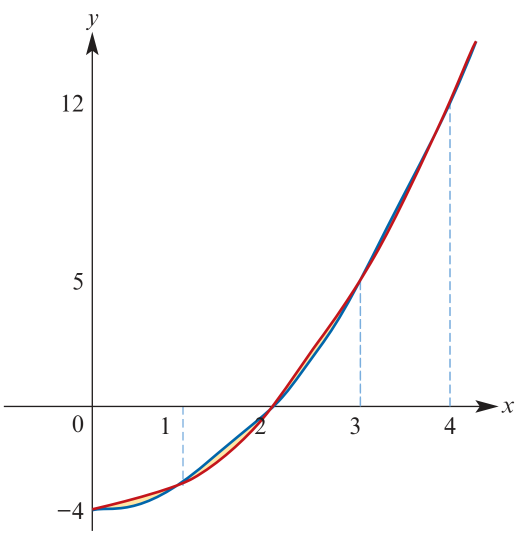
8 Find $\int_0^1 f(3x+1)\,dx$ if $\int_1^4 f(x)\,dx = 5$.

9 **a** Using four intervals use the trapezium rule to approximate $\int_0^4 x^2 - 4\,dx$. (Graph shown opposite)

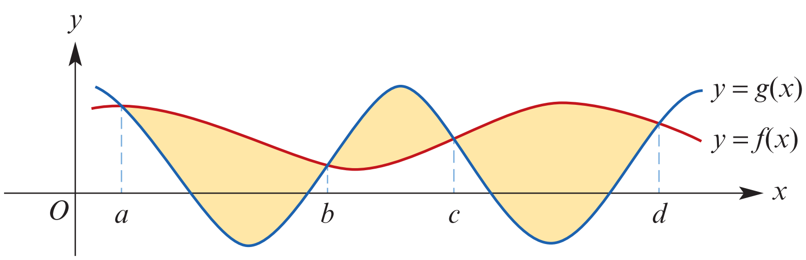
Note: Remember to use signed area for the trapezium and triangle that lie below the x-axis.

b Calculate the exact value of $\int_0^4 x^2 - 4\,dx$.

c Use the trapezium rule to approximate $\int_0^3 x^3 - 8\,dx$. Use 6 strips.



10 Set up a sum of definite integrals that represents the total shaded area between the curves $y = f(x)$ and $y = g(x)$.



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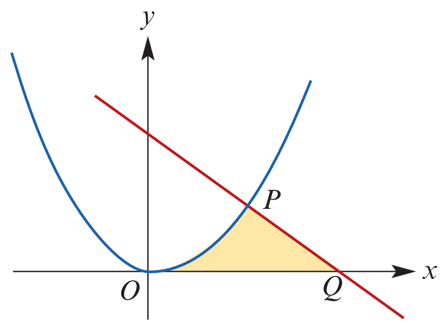
11 The figure shows the curve $y = x^2$ and the straight line $2x + y = 15$. Find:

a the coordinates of P and Q

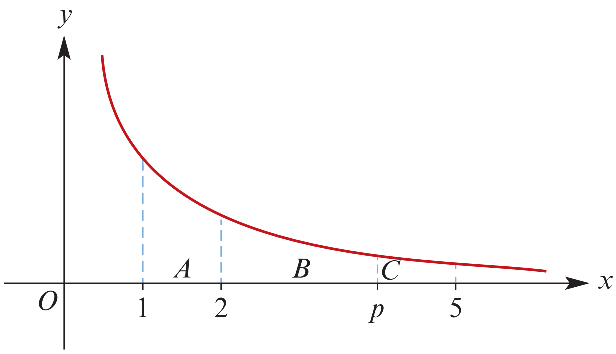
b the area of the shaded region.



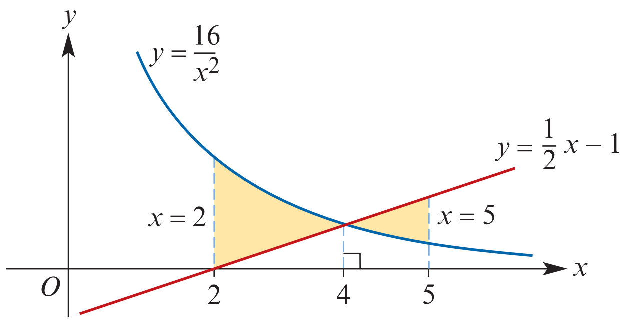
12 The figure shows part of the curve $y = \frac{10}{x^2}$. Find:

a the area of region A

b the value of p for which the regions B and C are of equal area.



13 Find the area of the shaded region.

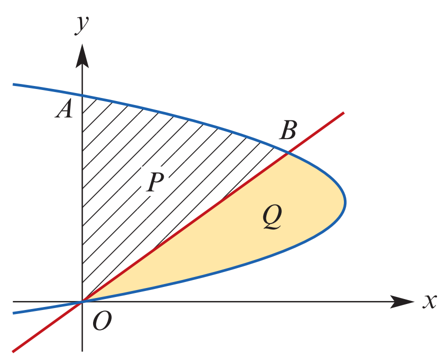


14 The figure shows part of the curve $x = 6y - y^2$ and part of the line $y = x$.

a Find the coordinates of A and B.

b Find the area of region P.

c Find the area of region Q.



15 **a** Sketch the graph of $y = e^x + 1$ and clearly indicate, by shading the region, the area given by the definite integral $\int_0^2 e^x + 1 \, dx$.

b Evaluate $\int_0^2 e^x + 1 \, dx$.

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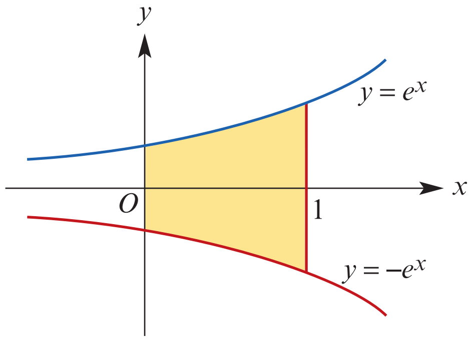
16 **a** Sketch the graphs of $y = e^{-x}$ and $y = e^x$ on the one set of axes and clearly indicate, by shading the region, the area given by $\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx$.

b Evaluate $\int_0^2 e^{-x}\,dx + \int_{-2}^0 e^x\,dx$.

17 **a** Evaluate $\int_0^1 e^x\,dx$.

b By symmetry, find the area of the region shaded in the figure.



18 What is the average square root of the numbers in the interval $[1, 4]$?

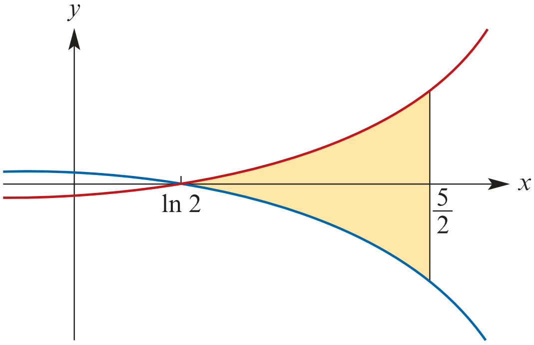
19 Sketch the graph of $f(x) = 2e^{2x} + 3$ and find the area of the region enclosed between the curve, the axes and the line $x = 1$.

20 Sketch the graph of $y = x(x - 2)(x + 1)$ and find the area of the region contained between the graph and the x-axis. (Do not attempt to find the coordinates of the turning points.)

21 Consider the functions

$$f(x) = e^x - 2$$
$$g(x) = -e^x + 2$$

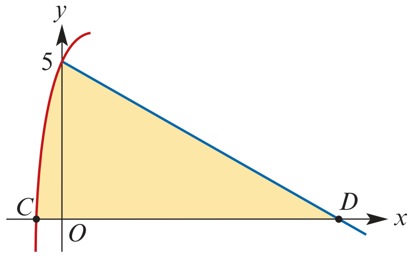
Find the area of the shaded region bounded by the graphs of f and g and the line $x = \frac{5}{2}$.



22 Let $f(x) = 6 - e^{-2x}$. The diagram shows part of the graph of f and also shows the normal to the graph of f at the point $(0, 5)$.

a Find the coordinates of points C and D.

b Find the area of the shaded region.



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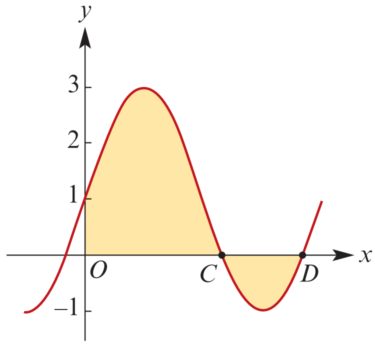
23 Part of the graph of $y = 2\sin(\pi x) + 1$ is shown.

a Find the coordinates of points C and D.

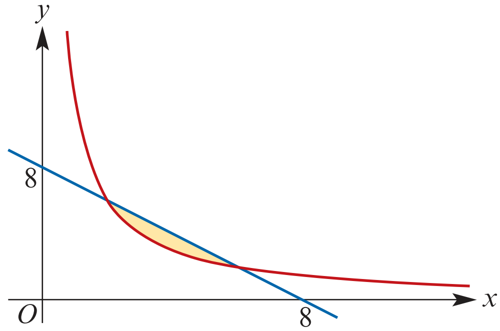
b Find the total area of the shaded regions.



24 The diagram shows the graphs of $f(x) = 8 - x$ and $g(x) = \dfrac{12}{x}$.

a Find the coordinates of the points of intersection of the two graphs.

b Find the area of the shaded region.



25 Evaluate each of the following definite integrals:

a $\int_0^2 e^{-x} + x \, dx$

b $\int_{-2}^{-1} x + \dfrac{1}{x-1} \, dx$

c $\int_0^{\frac{\pi}{2}} \sin x + x \, dx$

d $\int_{-4}^{-5} e^x + \dfrac{1}{2-2x} \, dx$

Multiple-choice questions

1 An equivalent expression for $\int_0^2 3f(x) + 2 \, dx$ is

A $3\int_0^2 f(x)\, dx + 3x$ **B** $3\int_0^2 f(x)\, dx + x$ **C** $3\int_0^2 f(x)\, dx + 4$

D $3f'(x) + 4$ **E** $\int_0^2 f(x)\, dx + 4$

2 If $F(x)$ is an antiderivative of $f(x)$ and $F(3) = 4$, Then $F(5)$ is equal to

A $f'(5)$ **B** $f'(5) + 4$ **C** $\int_3^5 f(x) + 4 \, dx$

D $\int_3^5 f(x)\, dx + 4$ **E** $\int_3^5 f(x)\, dx + 4x$

3 The average value of $f(x) = x^3 - 2x^2$ over the interval $[0, a]$ is $\dfrac{9}{12}$. The value of a is

A $\dfrac{1}{4}$ **B** $\dfrac{7}{12}$ **C** 3 **D** $\dfrac{1}{3}$ **E** -1

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4 An expression using integral notation for the area of the shaded region shown is

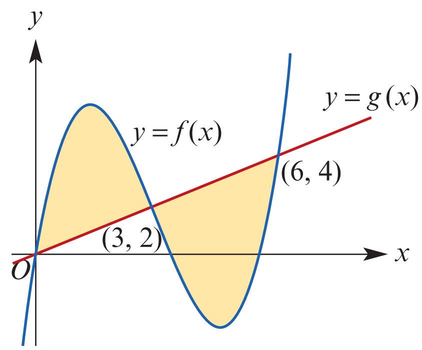
A $\int_0^6 f(x) - g(x)\,dx$

B $\int_0^3 f(x) - g(x)\,dx + \int_3^6 g(x) - f(x)\,dx$

C $\int_0^4 f(x) - g(x)\,dx$

D $\int_0^2 f(x) - g(x)\,dx + \int_2^4 f(x) - g(x)\,dx$

E $\int_0^2 f(x) - g(x)\,dx + \int_2^4 g(x) - f(x)\,dx$



5 An expression for y if $\dfrac{dy}{dx} = \dfrac{ax}{2} + 1$ and $y = 1$ when $x = 0$ is

A $y = \dfrac{ax^2}{4} + x + 1$ **B** $y = a$ **C** $y = ax^2 + x - 1$

D $y = ax^2 + x + a$ **E** $y = ax^2 + ax + a$

6 The function f such that $f'(x) = -6\sin(3x)$ and $f\left(\dfrac{2\pi}{3}\right) = 3$ is given by $f(x) =$

A $-18\cos(3x) + 21$ **B** $-2\cos(2x) + 5$ **C** $-2\sin(3x) + 1$

D $2\cos(3x) + 1$ **E** $2\sin(4x) + 3$

7 If $\int_{-5}^{4} f(x)\,dx = 2$ and $\int_{11}^{4} f(x)\,dx = 6$ then $\int_{-5}^{11} f(x)\,dx$ is equal to

A 0 **B** 2 **C** -4 **D** 8 **E** 4

8 If $\dfrac{dy}{dx} = ae^{-x} + 2$ and when $x = 0$, $\dfrac{dy}{dx} = 5$ and $y = 1$, then when $x = 2$, $y =$

A $-\dfrac{3}{e^2} + 2$ **B** $-\dfrac{3}{e^2} + 4$ **C** $-\dfrac{3}{e^2} + 8$ **D** $3e^2 + 4$ **E** $3e^2 + 8$

9 The rate of flow of water from a tap follows the rule $R(t) = 5e^{-0.1t}$, where $R(t)$ litres per minute is the rate of flow after t minutes. The number of litres, to the nearest litre, which flowed out in the first 3 minutes is

A 0 **B** 5 **C** 13 **D** 50 **E** 153

10 Gardeners have 200m of fence wire. They want to use all of this wire to enclose a rectangular area. Of all the possible rectangular areas they could enclose, what is their average area? (Answer to the nearest integer)

A 1134 m^2 **B** 1147 m^2 **C** 1667 m^2

D 11845 m^2 **E** 2425 m^2

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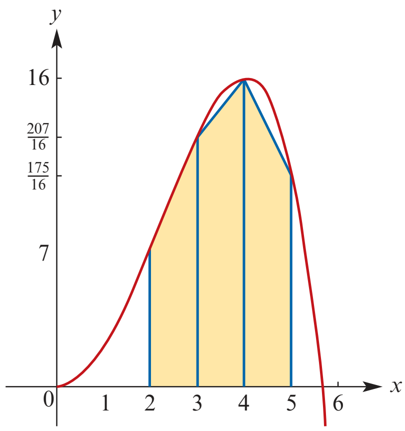
Review

11 Let $f(x) = -\frac{1}{16}x^4 + 2x^2$.

Given that $\int_2^5 f(x)\,dx = \frac{3147}{80}$ and that A is the trapezium estimate for $\int_2^5 f(x)\,dx$ using three strips, $\frac{3147}{80} - A$ is equal to

A $\frac{230}{220}$ **B** $\frac{240}{219}$ **C** $\frac{220}{218}$ **D** $\frac{229}{160}$ **E** $\frac{230}{216}$



12 The graph represents the function $y = \sin x$ where $0 \le x \le 2\pi$. The total area of the shaded regions is

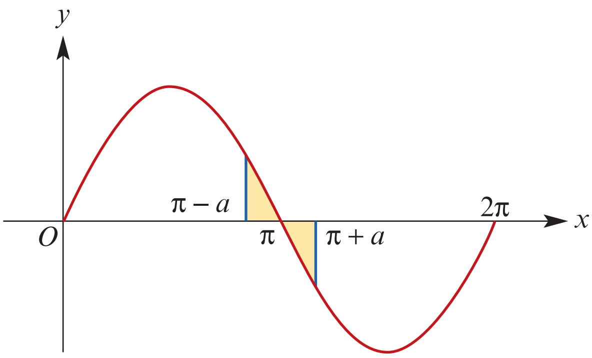
A $\frac{1}{2}\cos a$

B $2\cos a$

C $\frac{1}{2}(1 - \cos a)$

D $2(1 - \cos a)$

E $2\sin^2 a$



Extended-response questions

1 The diagram shows part of the curve with equation

$$y = x - \frac{1}{x^2}$$

The point C has coordinates $(2, 0)$. Find:

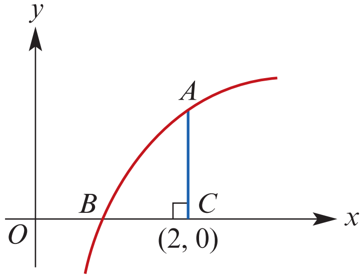
a the equation of the tangent to the curve at point A

b the coordinates of the point T where this tangent meets the x-axis

c the coordinates of the point B where the curve meets the x-axis

d the area of the region enclosed by the curve and the lines AT and BT

e the ratio of the area found in part **d** to the area of the triangle ATC.



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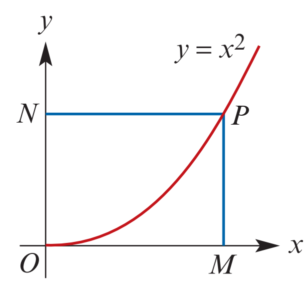
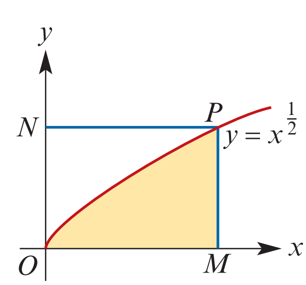
Review

2 **a** In the figure, the point P is on the curve $y = x^2$. Prove that the curve divides the rectangle $OMPN$ into two regions whose areas are in the ratio 2:1.

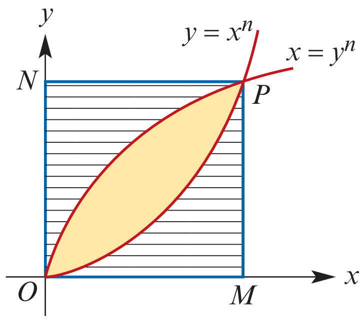


b In the figure, the point P is on the curve $y = x^{\frac{1}{2}}$. Prove that the area of the shaded region is two-thirds the area of the rectangle $OMPN$.



c Consider a point P on the curve $y = x^n$, with PM and PN the perpendiculars from P to the x-axis and the y-axis respectively. Prove that the area of the region enclosed between PM, the x-axis and the curve is equal to $\frac{1}{n+1}$ of the area of the rectangle $OMPN$.

3 **a** Find the area enclosed between the parabolas $y = x^2$ and $y^2 = x$.



b Show that the curves with equations $y = x^n$ and $y^n = x$ intersect at $(1, 1)$, where $n = 1, 2, 3, \dots$.

c Show that the area of the region contained between the curves $y = x^n$ and $y^n = x$ is $\frac{n-1}{n+1}$.

d Find the area of the region indicated by horizontal shading in the diagram.

e Use your result from **c** to find the area of the region between the curves for $n = 10$, $n = 100$ and $n = 1000$.

f Describe the result for n very large.

4 It is thought that the temperature, θ, of a piece of charcoal in a barbecue will increase at a rate $\frac{d\theta}{dt}$ given by $\frac{d\theta}{dt} = e^{2.6t}$, where θ is in degrees and t is in minutes.

a If the charcoal starts at a temperature of 30°C, find the expected temperature of the charcoal after 3 minutes.

b Sketch the graph of θ against t.

c At what time does the temperature of the charcoal reach 500°C?

d Find the average rate of increase of temperature from $t = 1$ to $t = 2$.

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5 It is believed that the velocity of a certain subatomic particle t seconds after a collision will be given by the expression

$$\frac{dx}{dt} = ve^{-t}, \quad v = 5 \times 10^4 \text{ m/s}$$

where x is the distance travelled in metres.

a What is the initial velocity of the particle?

b What happens to the velocity as $t \to \infty$ (i.e. as t becomes very large)?

c How far will the particle travel between $t = 0$ and $t = 20$?

d Find an expression for x in terms of t.

e Sketch the graph of x against t.

6 **a** Differentiate $e^{-3x}\sin(2x)$ and $e^{-3x}\cos(2x)$ with respect to x.

b Hence show that

$$e^{-3x}\sin(2x) + c_1 = -3\int e^{-3x}\sin(2x)\,dx + 2\int e^{-3x}\cos(2x)\,dx$$

and $$e^{-3x}\cos(2x) + c_2 = -3\int e^{-3x}\cos(2x)\,dx - 2\int e^{-3x}\sin(2x)\,dx$$

c Use the two equations from **b** to determine $\int e^{-3x}\sin(2x)\,dx$.

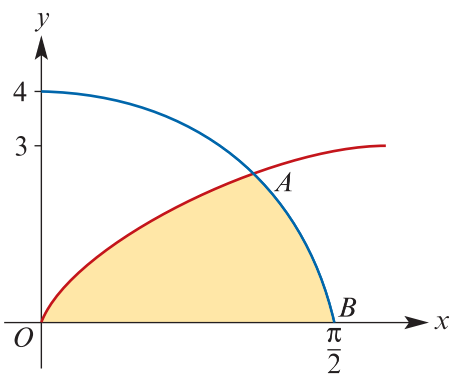
7 The curves $y = 3\sin x$ and $y = 4\cos x$, where $0 \le x \le \frac{\pi}{2}$, intersect at a point A.

a If $x = a$ at the point of intersection of the two curves:

i Find $\tan a$.

ii Hence find $\sin a$ and $\cos a$.

b Hence find the area of the shaded region in the diagram.



8 **a** If $y = x\log_e x$, find $\frac{dy}{dx}$. Hence find the value of $\int_1^e \log_e x\,dx$.

b If $y = x(\log_e x)^n$, where n is a positive integer, find $\frac{dy}{dx}$.

c Let $I_n = \int_1^e (\log_e x)^n\,dx$. For $n > 1$, show that $I_n + nI_{n-1} = e$.

d Hence find the value of $\int_1^e (\log_e x)^3\,dx$.

9 The curves $y^2 = ax$ and $x^2 = by$, where a and b are both positive, intersect at the origin and at the point (r, s). Find r and s in terms of a and b. Prove that the two curves divide the rectangle with corners $(0, 0)$, $(0, s)$, (r, s), $(r, 0)$ into three regions of equal area.

10 **a** Sketch the graph of $f(x) = 2\sin x - 1$ for $\frac{-\pi}{2} \le x \le \frac{\pi}{2}$.

b Evaluate $\int_0^{\frac{\pi}{6}} f(x)\,dx$ and indicate the area given by this integral on the graph of **a**.

c Find the inverse function f^{-1}.

d Evaluate $\int_0^1 f^{-1}(x)\,dx$ and indicate the area given by this integral on the graph of **a**.

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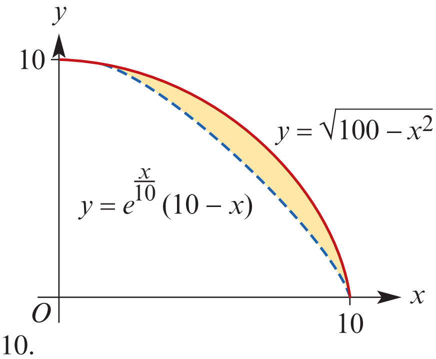
11 A teacher attempts to draw a quarter circle of radius 10 on the white board. However, the first attempt results in a curve with equation $y = e^{\frac{x}{10}}(10 - x)$.

The quarter circle has equation $y = \sqrt{100 - x^2}$.



a Find $\frac{dy}{dx}$ for both functions.

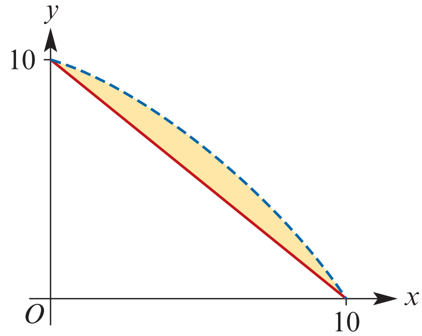
b Find the gradient of each of the functions when $x = 0$.

c Find the gradient of $y = e^{\frac{x}{10}}(10 - x)$ when $x = 10$.

d Find the area of the shaded region correct to two decimal places using a calculator.

e Find the percentage error for the calculation of the area of the quarter circle.

f The teacher draws in a chord from $(0, 10)$ to $(10, 0)$. Find the area of the shaded region using a calculator.



g **i** Use the result that the derivative of $e^{\frac{x}{10}}(10 - x)$ is $-e^{\frac{x}{10}} + \frac{1}{10}e^{\frac{x}{10}}(10 - x)$ to find $\int_0^{10} e^{\frac{x}{10}}(10 - x)\,dx$ by analytic techniques.

ii Find the exact area of the original shaded region and compare it to the answer of **d**.

12 A water-cooling device has a system of water circulation for the first 30 minutes of its operation. The circulation follows the following sequence:

- For the first 3 minutes water is flowing in.
- For the second 3 minutes water is flowing out.
- For the third 3 minutes water is flowing in.

This pattern is continued for the first 30 minutes. The rate of flow of water is given by the function

$$R(t) = 10e^{\frac{-t}{10}} \sin\left(\frac{\pi t}{3}\right)$$

where $R(t)$ litres per minute is the rate of flow at time t minutes. Initially there are 4 litres of water in the device.

a **i** Find $R(0)$. **ii** Find $R(3)$.

b Find $R'(t)$.

c **i** Solve the equation $R'(t) = 0$ for $t \in [0, 12]$.

ii Find the coordinates of the stationary points of $y = R(t)$ for $t \in [0, 12]$.

d Solve the equation $R(t) = 0$ for $t \in [0, 12]$.

e Sketch the graph of $y = R(t)$ for $t \in [0, 12]$.

f **i** How many litres of water flowed into the device for $t \in [0, 3]$?

ii How many litres of water flowed out of the device for $t \in [3, 6]$?

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iii How many litres of water are in the device when $t = 6$? (Remember there are initially 4 litres of water.)

g How many litres of water are there in the device when $t = 30$?

13 **a** Use the identities $\cos(2x) = 2\cos^2 x - 1$ and $\cos(2x) = 1 - 2\sin^2 x$ to show that

$$\frac{1 - \cos(2x)}{1 + \cos(2x)} = \sec^2 x - 1$$

b Hence evaluate $\int_0^{\frac{\pi}{4}} \frac{1 - \cos(2x)}{1 + \cos(2x)}\, dx$.

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