# 19 Revision of Chapters 1–18

Revision

## 19A Technology-free questions

1 Let $f(x) = x^2 + 6$ and $g(x) = 3x + 1$. Write down the rule of $f(g(x))$.

2 For the simultaneous linear equations

$$kx + 3y = 0$$
$$4x + (k + 2)y = 0$$

where $k$ is a real constant, find the value(s) of $k$ for which there are infinitely many solutions.

3 Find the equation of the image of the graph of $y = \frac{1}{x}$ under the transformation defined by the rule $(x, y) \to (2x, -3y)$ and describe a sequence of transformations that maps the graph of $y = \frac{1}{x}$ onto its image.

4 **a** Let $f(x) = (5x^3 - 3x)^7$. Find $f'(x)$. **b** Let $f(x) = 2xe^{4x}$. Evaluate $f'(0)$.

5 **a** Differentiate $x^2 \log_e(2x)$ with respect to $x$. **b** For $f(x) = \dfrac{\sin x}{2x + 1}$, find $f'\left(\dfrac{\pi}{2}\right)$.

6 **a** Let $f(x) = e^{\sin(2x)}$. Find $f'(x)$. **b** Let $f(x) = 3x\tan(2x)$. Evaluate $f'\left(\dfrac{\pi}{3}\right)$.

7 Find the general solution to the equation $\sin(2x) - \cos(2x) = 0$.

8 Let $f\colon [-\pi, \pi] \to \mathbb{R}$, $f(x) = 4\sin\left(2\left(x + \dfrac{\pi}{6}\right)\right)$.

**a** Write down the amplitude and period of the function $f$.

**b** Sketch the graph of the function $f$. Label the axis intercepts and the endpoints with their coordinates.

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**9** Sketch the graph of $f\colon [-1, \infty) \setminus \{2\} \to \mathbb{R}$, $f(x) = 1 - \dfrac{4}{x-2}$. Label all axis intercepts, and label each asymptote with its equation.

**10** For the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 5e^{x-1} - 3$:

**a** find the rule for the inverse function $f^{-1}$

**b** find the domain of the inverse function $f^{-1}$.

**11** Solve the equation $\cos\left(\dfrac{5x}{2}\right) = \dfrac{1}{2}$ for $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

**12** Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 5x^2$. Show that $g(u+v) + g(u-v) = 2(g(u) + g(v))$.

**13** Find the average value of $y = e^x$ over the interval $[0, 4]$.

**14** The graph of $y = ax^3 + bx + c$ has intercepts $(0, 6)$ and $(-2, 0)$ and has a turning point where $x = -1$.

**a** Find the value of $c$.

**b** Write down two simultaneous equations in $a$ and $b$ from the given information.

**c** Hence find the values of $a$ and $b$.

**15** Let $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3 - e^{2x}$.

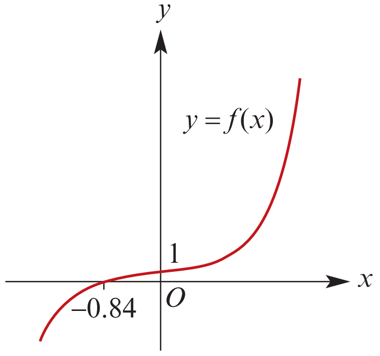
**a** Find the rule and domain of the function $g^{-1}$.

**b** Sketch the graph of $y = g(g^{-1}(x))$ for its maximal domain.

**16** The graph of the piecewise-defined function

$$f(x) = \begin{cases} -2x^4 + 1 & \text{if } x \le 0 \\ 2x^4 + 1 & \text{otherwise} \end{cases}$$

is shown.




**a** Draw the graph of the derivative function $f'$.

**b** Write down a rule for the derivative function.

**17** Find an antiderivative of $\dfrac{1}{1-3x}$ with respect to $x$, for $x < \dfrac{1}{3}$.

**18** Let $f\colon \mathbb{R} \setminus \{\frac{1}{2}\} \to \mathbb{R}$ where $f(x) = \dfrac{3}{2x-1} + 3$. Find $f^{-1}$, the inverse function of $f$.

**19** Solve the equation $\tan(2x) = -\sqrt{3}$ for $x \in \left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right) \cup \left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$.

**20** Let $X$ be a normally distributed random variable with a mean of 84 and a standard deviation of 6. Let $Z$ be the standard normal random variable.

**a** Find the probability that $X$ is greater than 84.

**b** Use the result that $\Pr(Z < 1) = 0.84$ to find the probability that $78 < X < 90$.

**c** Find the probability that $X < 78$ given that $X < 84$.

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**21** The probability density function of a random variable $X$ is given by

$$f(x) = \begin{cases} \dfrac{x}{24} & \text{if } 1 \le x \le 7 \\ 0 & \text{otherwise} \end{cases}$$

**a** Find $\Pr(X < 3)$.

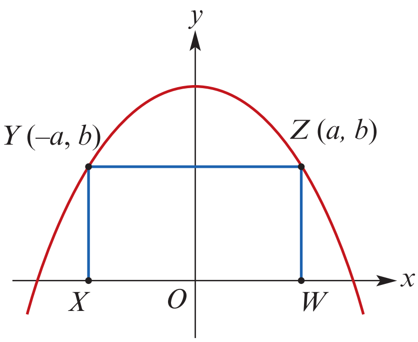
**b** If $b \in [1, 7]$ and $\Pr(X \ge b) = \dfrac{3}{8}$, find $b$.

**22** A tangent to the graph of $y = x^{\frac{1}{3}}$ has equation $y = \dfrac{1}{3}x + a$. Find the value(s) of $a$.

**23** A rectangle $XYZW$ has two vertices on the $x$-axis and the other two vertices on the graph of $y = 16 - 4x^2$, as shown in the diagram.

**a** Find the area, $A$, of rectangle $XYZW$ in terms of $a$.

**b** Find the maximum value of $A$ and the value of $a$ for which this occurs.




**24** Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = -3x^2 + 2bx + 9$ with $\int_{-1}^{3} f(x)\,dx = 32$. Find the value of $b$.

**25** Simone has either a sandwich or pasta for lunch. If she has a sandwich, the probability that she has a sandwich again the next day is 0.6. If she has pasta, the probability that she has pasta again the next day is 0.3. Suppose that Simone has a sandwich for lunch on a Monday.

**a** What is the probability that she has pasta for lunch on the following Wednesday?

**b** If she has a sandwich for lunch on Wednesday, what is the probability that she had also had a sandwich for lunch on Tuesday?

**26** A player in a game of chance can win \$0, \$1, \$2 or \$3. The amount won, \$$X$, is a random variable with probability distribution given by:

| $x$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\Pr(X = x)$ | 0.6 | 0.2 | 0.15 | 0.05 |

**a** Find the mean of $X$.

**b** What is the probability that a player wins the same amount from two games?

**27** Every Thursday night, Chris either goes to the gym or goes for a run. If he goes to the gym one Thursday, the probability that he goes to the gym the next Thursday is 0.5. If he goes for a run one Thursday, the probability that he goes for a run the next Thursday is 0.6. If Chris goes to the gym one Thursday, what is the probability that he goes for a run on exactly two of the next three Thursdays?

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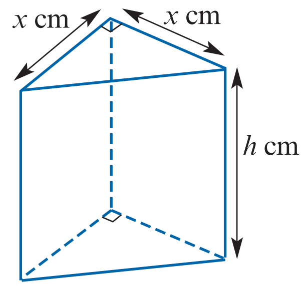
**28** A brick is made in the shape of a right triangular prism. The triangular end is a right-angled isosceles triangle, with the equal sides of length $x$ cm. The height of the brick is $h$ cm. The volume of the brick is 2000 cm$^3$.




**a** Find an expression for $h$ in terms of $x$.

**b** Show that the total surface area, $A$ cm$^2$, of the brick is given by $A = \dfrac{4000\sqrt{2} + 8000}{x} + x^2$.

**c** Find the value of $x^3$ if the brick has minimum surface area.

**29** It is known that in a certain population 10% of people prefer dark chocolate to milk chocolate. A random sample of 10 people was selected from the population.

**a** What is the expected number of people who prefer dark chocolate in the sample?

**b** Suppose that the random variable $\hat{P}$ represents the proportion of people in the sample who prefer dark chocolate.

**i** List the possible values that $\hat{P}$ might take.

**ii** Find $\Pr(\hat{P} < 0.2)$. Express you answer in the form $\dfrac{ab^9}{c^{10}}$ where $a$, $b$ and $c \in Z^+$.

**30** An experiment consists of six independent trials. Each trial results in either a success or a failure. The probability of success in a trial is $p$. If $\Pr(X = 5 \mid X \geq 5) = \dfrac{14}{15}$, find the value of $p$.

**31** The random variable $X$ is normally distributed with mean 50 and standard deviation 5. If $\Pr(46 < X < 54) = q$, find $\Pr(X < 54)$ in terms of $q$.

## 19B Multiple-choice questions

**1** The simultaneous linear equations

$$mx - 2y = 0$$
$$6x - (m + 4)y = 0$$

where $m$ is a real constant, have a unique solution provided

**A** $m \in \{-6, 2\}$ **B** $m \in \mathbb{R} \setminus \{-6, 2\}$ **C** $m \in \{-2, 6\}$
**D** $m \in \mathbb{R} \setminus \{-2, 6\}$ **E** $m \in \mathbb{R} \setminus \{0\}$

**2** The general solution to the equation $\sin(2x) = 1$ is, where $n$ is an integer,

**A** $x = n\pi + \dfrac{\pi}{4}$ **B** $x = \dfrac{n\pi}{2} + (-1)^n \dfrac{\pi}{2}$

**C** $x = 2n\pi + \dfrac{\pi}{4}$ or $x = 2n\pi - \dfrac{\pi}{4}$ **D** $x = \dfrac{n\pi}{4} + (-1)^n \dfrac{\pi}{4}$

**E** $x = n\pi + \dfrac{\pi}{4}$ or $x = 2n\pi + \dfrac{\pi}{4}$

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**3** Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 5x + 1 & \text{if } x \geq -\frac{4}{5} \\ -5x - 7 & \text{if } x < -\frac{4}{5} \end{cases}$$

Which of the following statements is *not* true about this function?

**A** The graph of $f$ is continuous everywhere.
**B** The graph of $f'$ is continuous everywhere.
**C** $f(x) \geq -3$, for all values of $x$.
**D** $f'(x) = 5$, for all $x > 0$.
**E** $f'(x) = -5$, for all $x < -2$.

**4** Let $k = \int_{-6}^{-2} \left(\frac{2}{x}\right) dx$. Then $e^k$ is equal to

**A** $\log_e 3$ **B** $1$ **C** $\frac{1}{9}$ **D** $9$ **E** $\frac{1}{3}$

**5** The average value of the function with rule $f(x) = \log_e(x + 2)$ over the interval $[-1, 3]$ is

**A** $\frac{-1}{5}$ **B** $\log_e 6$ **C** $\frac{\log_e 5}{4}$ **D** $\frac{5\log_e 5 - 4}{4}$ **E** $\frac{5\log_e 5 - 3\log_e 3 - 4}{4}$

**6** The average value of the function $y = \sin(2x)$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

**A** $\frac{2}{\pi}$ **B** $\frac{\pi}{2}$ **C** $0.5$ **D** $0$ **E** $\pi$

**7** The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T(x, y) = (3x + 5, y + 1)$$

The equation of the image of the curve $y = x^2$ under $T$ is

**A** $2y = (x - 5)^2 + 2$ **B** $3y = (x - 5)^2 + 3$ **C** $9y = (x + 5)^2 - 9$
**D** $9y = (x - 5)^2 + 9$ **E** $y = \left(\frac{x}{5} - 5\right)^2 + 2$

**8** If $f(x) = e^{3x}$, for all real $x$, and $[f(x)]^3 = f(y)$, then $y$ is equal to

**A** $e^{9x}$ **B** $9x$ **C** $3x$ **D** $9x^3$ **E** $(3x)^3$

**9** The continuous random variable $X$ has a probability density function given by

$$f(x) = \begin{cases} \sin(2x) & \text{if } 0 \leq x \leq \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

The value of $a$ such that $\Pr(X > a) = 0.25$ is closest to

**A** $0.25$ **B** $0.75$ **C** $1.04$ **D** $1.05$ **E** $1.09$

**10** The function $f$ is a probability density function, with rule

$$f(x) = \begin{cases} 1 + 2e^{\frac{x}{k}} & \text{if } 0 \leq x \leq 2k \\ 0 & \text{otherwise} \end{cases}$$

Hence $k$ is equal to

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**A** $\frac{1}{2}e^{-2}$ **B** $1 + e^2$ **C** $e^{-2}$ **D** $1 - e^{-2}$ **E** $1$

**11** The random variable $X$ has a normal distribution with a mean of 8 and a standard deviation of 0.25. If $Z$ has the standard normal distribution, then the probability that $X$ is less than 7.5 is equal to

**A** $\Pr(Z > 2)$ **B** $\Pr(Z < -1.5)$ **C** $\Pr(Z < 1)$ **D** $\Pr(Z \geq 1.5)$ **E** $\Pr(Z < -4)$

**12** The graph of $y = 2kx - 2$ intersects the graph of $y = x^2 + 12x$ at two points for

**A** $k = 12$ **B** $k > 6 + \sqrt{2}$ or $k < 6 - \sqrt{2}$
**C** $4 < k < 7$ **D** $5 < k < 7$ **E** $6 - \sqrt{2} < k < 6 + \sqrt{2}$

**13** The set of solutions to the equation $e^{4x} - 7e^{2x} + 12 = 0$ is

**A** $\{3, 4\}$ **B** $\{-4, -3\}$ **C** $\{-2, -\sqrt{3}, \sqrt{3}, 2\}$
**D** $\{\log_e \sqrt{3}, \log_e 2\}$ **E** $\{-\log_e \sqrt{3}, \log_e \sqrt{3}, \log_e 2\}$

**14** The graph of the function $f\colon [0, \infty) \to \mathbb{R}$, where $f(x) = 7x^{\frac{3}{2}}$, is reflected in the $x$-axis and then translated 3 units to the right and 4 units down. The equation of the new graph is

**A** $y = 7(x-3)^{\frac{3}{2}} + 4$ **B** $y = -7(x-3)^{\frac{3}{2}} - 4$ **C** $y = -7(x+3)^{\frac{3}{2}} - 1$
**D** $y = -7(x-4)^{\frac{3}{2}} + 3$ **E** $y = 7(x-4)^{\frac{3}{2}} + 3$

**15** If a random variable $X$ has probability density function given by

$$f(x) = \begin{cases} \frac{1}{8}x & \text{if } 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

then $\mathrm{E}(X)$ is equal to

**A** $\frac{1}{2}$ **B** $1$ **C** $\frac{8}{3}$ **D** $\frac{16}{3}$ **E** $2$

**16** The function $f\colon [a, \infty) \to \mathbb{R}$ with rule $f(x) = \log_e((x-2)^4)$ will have an inverse function if

**A** $a \geq 3$ **B** $a \leq -2$ **C** $a < 2$ **D** $a \geq 0$ **E** $a \geq -1$

**17** The transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ which maps the curve with equation $y = e^x$ to the curve with equation $y = e^{(2x+4)} - 3$ could have rule $T(x, y) =$

**A** $(0.5x - 2, y - 3)$ **B** $(0.5x + 2, y + 3)$ **C** $(x - 4, 2y + 3)$
**D** $(x - 2, 2y + 3)$ **E** $(2x - 4, y - 3)$

**18** Assume that $f'(x) = g'(x)$ with $f(1) = 2$ and $g(x) = -xf(x)$. Then $f(x) =$

**A** $g(x) + 4x + 4$ **B** $g'(x) + 4$ **C** $g(x) + 4x$ **D** $\frac{4 - 4x}{x + 1}$ **E** $g(x) + 4$

**19** Two events $A$ and $B$ are independent. If $\Pr(A) = 5\Pr(B) - 0.1$, and $\Pr(A \cup B) = 0.7025$, then $\Pr(A)$ is equal to

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**A** 0.10 **B** 0.15 **C** 0.25 **D** 0.35 **E** 0.65

**20** The random variable $X$ has the following probability distribution.

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $\Pr(X = x)$ | $a$ | $b$ | 0.6 |

If the mean of $X$ is 1.6, then

**A** $a = 0.3$ and $b = 0.7$ **B** $a = 0.2$ and $b = 0.2$ **C** $a = 0.4$ and $b = 0.4$
**D** $a = 0.1$ and $b = 0.3$ **E** $a = 0$ and $b = 0.4$

**21** An approximate 90% confidence interval for the proportion of people in a certain city who are vegetarian was found to be (0.1723,0.3277). The size of the sample which was used to obtain this confidence interval is closest to

**A** $n = 21$ **B** $n = 56$ **C** $n = 84$ **D** $n = 112$ **E** $n = 134$

**22** The simultaneous equations $(m - 4)x + 6y = 6$ and $2x + (m - 3)y = 2m - 10$ have no solution for

**A** $m \in \mathbb{R} \setminus \{0, 7\}$ **B** $m \in \mathbb{R} \setminus \{0\}$ **C** $m \in \mathbb{R} \setminus \{7\}$ **D** $m = 7$ **E** $m = 0$

*Questions 23 and 24 are based on the following information:*

An exit poll of 1000 randomly selected voters found that 520 favoured candidate A.

**23** An approximate 95% confidence interval for the proportion of voters in favour of candidate A is

**A** (0.484, 0.546) **B** (0.422, 0.618) **C** (0.494, 0.546)
**D** (0.489, 0.551) **E** (0.479, 0.561)

**24** On the basis of this confidence interval, what would be your prediction for the result of the election?

**A** predict a win for candidate A
**B** predict a loss for candidate A
**C** too close to make any prediction
**D** cannot tell as we do not know the number of candidates
**E** none of the above

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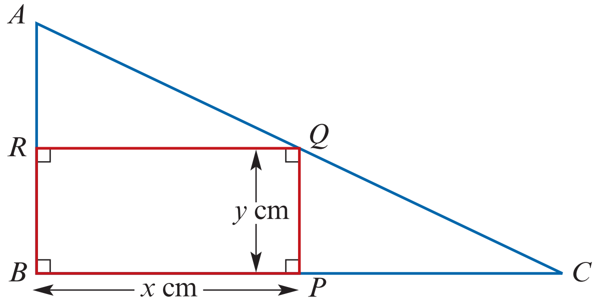
## 19C Extended-response questions

**1** **a** **i** Find the coordinates of the stationary point for the curve with equation

$$y = \frac{16x^3 + 4x^2 + 1}{2x^2}$$

**ii** Determine the nature of this stationary point.

**b** The right-angled triangle $ABC$ shown in the diagram has side lengths $AB = 5$ cm and $AC = 13$ cm.
The rectangle $BPQR$ is such that its vertices $P$, $Q$ and $R$ lie on the line segments $BC$, $CA$ and $AB$ respectively.




**i** Given that $BP = x$ cm and $PQ = y$ cm, show that $y = \dfrac{60 - 5x}{12}$.

**ii** Find the area of the rectangle, $A$ cm$^2$, in terms of $x$.

**iii** Find the maximum value of this area as $x$ varies.

**2** A theoretical model of the relationship between two variables, $x$ and $y$, predicts the values given in the table.

| $x$ | 0 | 1 | 3 |
|---|---|---|---|
| $y$ | 6 | 0 | 0 |

**a** An equation of the form $y = k(x - p)(x - q)$ is suggested, where $p$, $q$ and $k$ are constants and $p < q$.
Use the information in the table to find $p$, $q$ and $k$.

**b** A series of experiments is carried out to test this model. The values of $y$ when $x = 0, 1, 3$ are found to be as predicted. But when $x = 2$, the value of $y$ is found to be 2. After further discussion, a new model is proposed with an equation of the form

$$y = m(x - p)^2(x - q)$$

where $p$ and $q$ have the values already calculated and $m$ is a constant.

**i** Find the value of $m$.

**ii** Obtain the equation of this new model in the form $y = ax^3 + bx^2 + cx + d$.

**iii** Sketch the graph of $y$ against $x$. State the coordinates of the stationary points and the nature of each of these points.

**3** A curve $C$ has equation $y = ax - x^2$, where $a$ is a positive constant.

**a** Sketch $C$, showing clearly the coordinates of the axis intercepts.

**b** Calculate the area of the finite region bounded by $C$ and the $x$-axis, giving your answer in terms of $a$.

**c** The lines $x = \frac{1}{3}a$ and $x = \frac{2}{3}a$ intersect $C$ at the points $A$ and $B$ respectively.

**i** Find, in terms of $a$, the $y$-coordinates of $A$ and $B$.

**ii** Calculate the area of the finite region bounded by $C$ and the straight line $AB$, giving your answer in terms of $a$.

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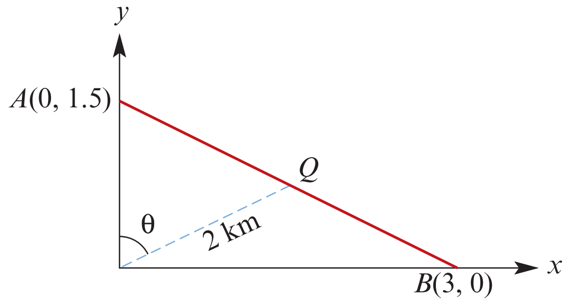
**4** **a** Find the equation of the straight line joining the points $A(0, 1.5)$ and $B(3, 0)$.




**b** Let $y = \sin\theta + 2\cos\theta$.

**i** Find $\dfrac{dy}{d\theta}$.

**ii** Solve the equation $\dfrac{dy}{d\theta} = 0$ for $\theta$, where $0° \le \theta \le 90°$.

**iii** State the coordinates of the stationary point of $y = \sin\theta + 2\cos\theta$, where $0° \le \theta \le 90°$.

**iv** It can be shown that $\sin\theta + 2\cos\theta$ can be written in the form $r\sin(\theta + \alpha)$. Use the result of **iii** and the fact that $y = 2$ when $\theta = 0$ to find the values of $r$ and $\alpha$.

**v** Use addition of ordinates and the result of **iv** to sketch the graph of $y$ against $\theta$ for $0° \le \theta \le 90°$.

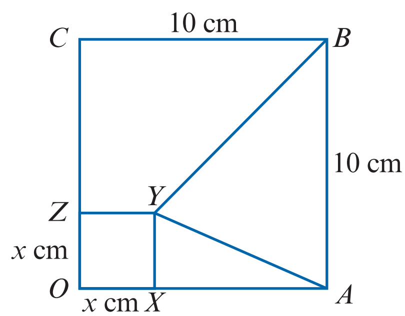
**c** The figure shows a map of a region of wetland. The units of the coordinates are kilometres, and the $y$-axis points due north. A walker leaves her car somewhere on the straight road between $A$ and $B$. She walks in a straight line for a distance of 2 km to a monument at the origin $O$. While she is looking at the monument, a fog comes down, and so she cannot see her way back to her car. She needs to work out the bearing on which she should walk.

**i** Write down the coordinates of a point $Q$ which is 2 km from $O$ on a bearing of $\theta$.

**ii** Show that, for $Q$ to be on the road between $A$ and $B$, the angle $\theta$ must satisfy the equation $2\sin\theta + 4\cos\theta = 3$.

**iii** Use the result of **b iv** to solve this equation for $\theta$, where $0° \le \theta \le 90°$.

**5** A square piece of card $OABC$, of side length 10 cm, is cut into four pieces by removing a square $OXYZ$ of side length $x$ cm as shown, and then cutting out the triangle $ABY$.




**a** **i** Find $A$ cm$^2$, the sum of the areas of $OXYZ$ and $ABY$, in terms of $x$.

**ii** Find the domain of the function which determines this area.

**iii** Sketch the graph of the function, with domain determined in **ii**.

**iv** State the minimum value of this area.

**b** **i** Find the rule for the function of $x$ which represents the area of triangle $AXY$.

**ii** Sketch the graph of this function for a suitable domain.

**c** Find the ratio of the areas of the four pieces when the area of triangle $AXY$ is a maximum.

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**6** The number of people unemployed in a particular population can be modelled by the function

$$f(t) = 1000(t^2 - 10t + 44)e^{\frac{-t}{10}}$$

where $t$ is the number of months after January 2012 and $0 \le t \le 35$.

**a** Use this function to find an expression for:

**i** the rate of increase of the number unemployed

**ii** the rate of increase of this rate of increase.

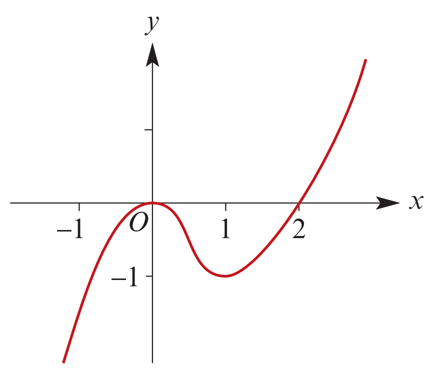
**b** Find the values of $t$ for which:

**i** the number unemployed was increasing

**ii** the rate of increase of the number unemployed was going down

**iii** the number unemployed was increasing and the rate of increase of the number unemployed was going down.

**7** The graph of $y = f(x)$ is shown.




**a** Sketch the graph of:

**i** $y = 2f(x)$ **ii** $y = f(2x)$

**iii** $y = f(-x)$ **iv** $y = -f(x)$

**v** $y = f(x + 2)$

**b** Explain why $f$ does not have an inverse function.

**c** **i** Sketch the graph of the function $g: (2, \infty) \to \mathbb{R}, g(x) = f(x)$.

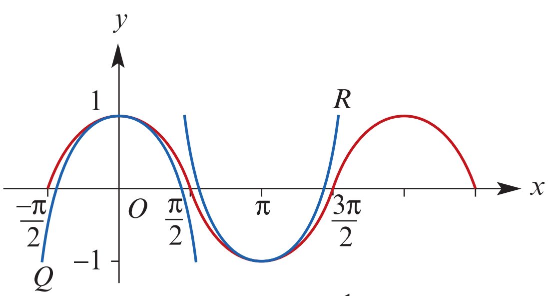
**ii** Sketch the graph of $g^{-1}$.

**d** **i** Given that $g: (2, \infty) \to \mathbb{R}$ where $g(x) = x^2(x - 2)$, calculate the gradient of the graph of $y = g(x)$ at the point $(3, 9)$.

**ii** Hence find the gradient of $y = g^{-1}(x)$ at the point $(9, 3)$.

**8** The diagram shows part of the graph of $y = \cos x$ and the graphs of two quadratic functions, denoted by $Q$ and $R$, which approximate to the cosine function around $x = 0$ and $x = \pi$ respectively.

The equation of $Q$ is $y = 1 - \frac{1}{2}x^2$.




**a** **i** Find an estimate of $\cos 0.1$ by using the approximation $y = 1 - \frac{1}{2}x^2$.

**ii** Find an approximation for the solution to the equation $\cos x = 0.98$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, by solving the quadratic equation $1 - \frac{1}{2}x^2 = 0.98$.

**b** **i** The graph $Q$ can be transformed into $R$ by a reflection in the $x$-axis, followed by a translation. Use this fact to find an equation for the graph $R$.

**ii** Estimate the value of $\cos 3$ using this approximation.

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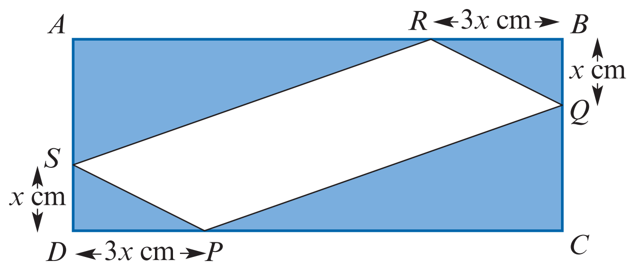
**9** In the figure, $ABCD$ is a rectangle with $AB = 30$ cm and $AD = 10$ cm. The shaded portions are cut away, leaving the parallelogram $PQRS$, where $BQ = SD = x$ cm and $RB = DP = 3x$ cm.



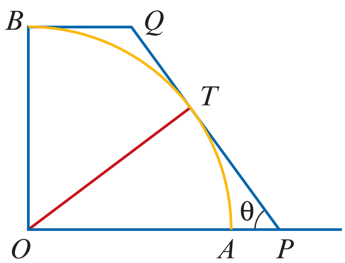

**a** Find the area, $S$ cm$^2$, of the parallelogram in terms of $x$.

**b** Find the allowable values of $x$.

**c** Find the value of $x$ for which $S$ is a maximum.

**d** Sketch the graph of $S$ against $x$ for a suitable domain.

**10** In the figure, $OAB$ is a quadrant of a circle of radius 1 unit. The line segment $OA$ is extended to a point $P$. From $P$, a tangent to the quadrant is drawn, touching it at $T$ and meeting another tangent, $BQ$, at $Q$. Let $\angle OPQ = \theta$.




**a** **i** Find the length $OP$ as a function of $\theta$.

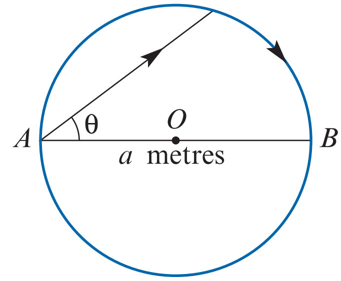
**ii** Find the length $BQ$ as a function of $\theta$.

**b** Show that the area, $S$, of trapezium $OPQB$ is given by $\dfrac{2 - \cos\theta}{2\sin\theta}$.

**c** Show that $\dfrac{dS}{d\theta} = \dfrac{2 - 4\cos\theta}{4\sin^2\theta}$.

**d** Find the minimum value of $S$ and the distance $AP$ when $S$ is a minimum.

**11** A dog is at point $A$ on the edge of a circular lake of diameter $a$ metres, and she wishes to reach her owner who is at the diametrically opposite point $B$. The dog can swim at $\frac{1}{2}$ m/s and run at 1 m/s.




**a** If she swims in a direction making an angle of $\theta$ with $AB$ and then runs round the edge of the lake to $B$, show that the time taken, $T$ s, is given by

$$T = a(\theta + 2\cos\theta)$$

**b** On one set of axes, sketch the graphs of $y = 200\theta$ and $y = 400\cos\theta$ for $0 \le \theta \le \dfrac{\pi}{2}$. Using addition of ordinates, sketch the graph of

$$y = 200(\theta + 2\cos\theta)$$

(Find the maximum value of $y$ for $0 \le \theta \le \dfrac{\pi}{2}$ by finding $\dfrac{dy}{d\theta}$ and then solving the equation $\dfrac{dy}{d\theta} = 0$.)

**c** Sketch the graph of $T = a(\theta + 2\cos\theta)$ for $0 \le \theta \le \dfrac{\pi}{2}$ and state the minimum value of $T$.

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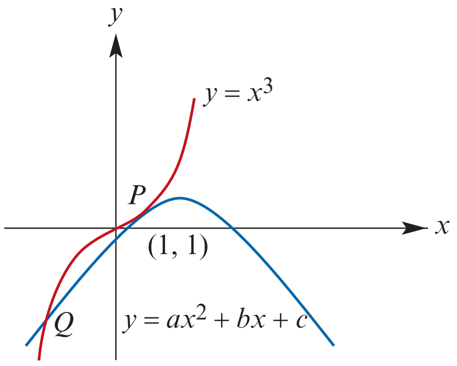
**12** **a** **i** Show that, if $f(x) = (x - 1)g(x)$ and $f'(x) = (x - 1)h(x)$, where $g(x)$ and $h(x)$ are polynomials, then $(x - 1)$ must be a factor of $g(x)$.

**ii** Let $F(x) = x^3 - kx^2 - (3 - 2k)x - (k - 2)$. Show that $F(1) = F'(1) = 0$.

**iii** Using the results of **i** and **ii**, solve the equation $F(x) = 0$.

**b** The parabola $y = ax^2 + bx + c$ and the cubic $y = x^3$ touch at $P(1, 1)$ (and have the same gradient at this point). The curves also meet at $Q$.



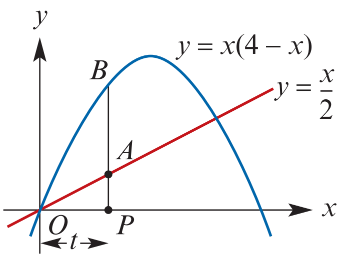

**i** Find $b$ and $c$ in terms of $a$.

**ii** If the coordinates of $Q$ are $(h, k)$, find $h$ in terms of $a$. (Use the result of **a**.)

**iii** If $Q$ has coordinates $(-2, -8)$, find the values of $a$, $b$ and $c$.

**iv** If $Q$ has coordinates $(-3, -27)$, find the values of $a$, $b$ and $c$.

**13** The point $P$ has coordinates $(t, 0)$, where $0 < t < \frac{7}{2}$. The line $PAB$ is parallel to the $y$-axis.




**a** Let $Z$ be the length of $AB$. Find $Z$ in terms of $t$.

**b** Sketch the graph of $Z$ against $t$.

**c** State the maximum value of $Z$ and the value of $t$ for which it occurs.

**14** A study is being conducted of the numbers of male and female children in families in a certain population.

**a** A simple model is that each child in any family is equally likely to be male or female, and that the sex of each child is independent of the sex of any previous children in the family. Using this model, calculate the probability that in a randomly chosen family of four children:

**i** there will be two males and two females

**ii** there will be exactly one female, given that there is at least one female.

**b** An alternative model is that the first child in any family is equally likely to be male or female, but that, for any subsequent children, the probability that they will be of the same sex as the previous child is $\frac{3}{5}$. Using this model, calculate the probability that in a randomly chosen family of four children:

**i** all four will be of the same sex

**ii** no two consecutive children will be of the same sex

**iii** there will be two males and two females.

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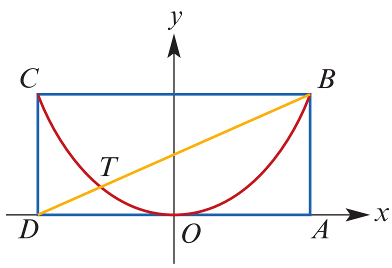
**15** In the figure, $ABCD$ is a rectangle with $OA = OD = a$ and $AB = b$.
The equation of the parabola $BOC$ is $y = kx^2$.




**a** Express $k$ in terms of $a$ and $b$.

**b** If $BD$ cuts the parabola at $T$, find:

**i** the equation of the straight line $BD$

**ii** the coordinates of $T$.

**c** Show that the area bounded by the parabola and the line $BC$ is $\frac{4}{3}ab$ square units.

**d** Let $S_1$ be the area of the region bounded by the line segment $BT$ and the curve $BOT$. Let $S_2$ be the area of the region bounded by the curve $CT$ and the line segments $BC$ and $BT$. Find the ratio $S_1 : S_2$.

**16** A certain type of brass washer is manufactured as follows. A length of brass rod is cut cross-sectionally into pieces of mean thickness 0.25 cm, with a standard deviation of 0.002 cm. These brass slices are then put through a machine that punches out a circular hole of mean diameter 0.5 cm through the middle of the slice, with a standard deviation of 0.05 cm. The thickness of the washers and the diameters of the holes are known to be normally distributed, and do not depend on each other.

**a** Find the probability that a randomly selected washer will:

**i** have a thickness of less than 0.253 cm

**ii** have a thickness of less than 0.247 cm

**iii** have a hole punched with a diameter greater than 0.56 cm

**iv** have a hole punched with a diameter less than 0.44 cm.

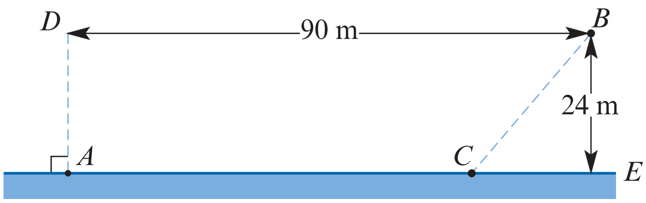
**b** The brass washers are acceptable only if they are between 0.247 cm and 0.253 cm in thickness with a hole of diameter between 0.44 cm and 0.56 cm. Find:

**i** the percentage of washers that are rejected

**ii** the expected number of washers of acceptable thickness in a batch of 1000 washers

**iii** the expected number of washers of acceptable thickness that will be rejected in a batch of 1000 washers.

**17** A ditch is to be dug to connect the points $A$ and $B$ in the figure. The earth on the same side of $AE$ as $B$ is hard, and the earth on the other side is soft.




The cost of digging hard earth is \$200 per metre and soft earth is \$100 per metre. Find the position of point $C$, where the turn is made, that will minimise the cost.

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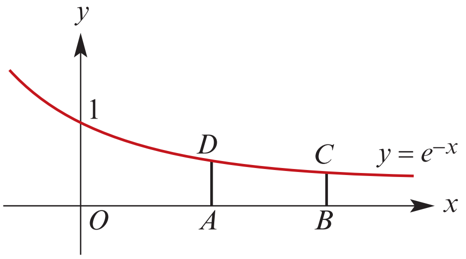
**18** The diagram shows the graph of $y = e^{-x}$. The points $A$ and $B$ have coordinates $(n, 0)$ and $(n + 1, 0)$ respectively, and the points $C$ and $D$ on the curve are such that $AD$ and $BC$ are parallel to the $y$-axis.



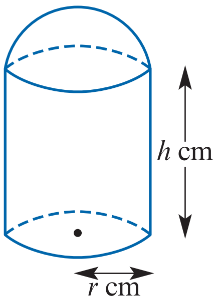

**a** **i** Find the equation of the tangent to $y = e^{-x}$ at the point $D$.

**ii** Find the intercept of the tangent with the $x$-axis.

**b** **i** Find the area of the region $ABCD$.

**ii** The line $BD$ divides the region into two parts. Find the ratio of the areas of these two parts.

**19** A closed capsule is to be constructed as shown in the diagram. It consists of a circular cylinder of height $h$ cm with a flat base of radius $r$ cm. It is surmounted by a hemispherical cap.




**a** **i** Show that the volume of the capsule, $V$ cm$^3$, is given by $V = \frac{\pi r^2}{3}(3h + 2r)$.

**ii** Show that the surface area of the capsule, $S$ cm$^2$, is given by $S = \pi r(2h + 3r)$.

**b** **i** If $V = \pi a^3$, where $a$ is a positive constant, find $h$ in terms of $a$ and $r$.

**ii** Hence find $S$ in terms of $a$ and $r$.

**c** **i** By using addition of ordinates, sketch the graph of $S$ against $r$ for a suitable domain.

**ii** Find the coordinates of the turning point by first finding $\frac{dS}{dr}$, and then solving the equation $\frac{dS}{dr} = 0$ for $r$ and determining the corresponding value of $S$.

**20** A manufacturer sells cylinders whose diameters are normally distributed with mean 3 cm and standard deviation 0.002 cm. The selling price is \$$s$ per cylinder and the cost of manufacture is \$1 per cylinder. A cylinder is returned and the purchase money is refunded if the diameter of the cylinder is found to differ from 3 cm by more than $d$ cm. A returned cylinder is regarded as a total loss to the manufacturer. The probability that a cylinder is returned is 0.25.

**a** Find $d$.

**b** The profit, \$$Q$, per cylinder is a random variable. Give the possible values of $Q$ in terms of $s$, and the probabilities of these values.

**c** Express the mean and standard deviation of $Q$ in terms of $s$.

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**21** The length of a certain species of worm has a normal distribution with mean 20 cm and standard deviation 1.5 cm.

**a** Find the probability that a randomly selected worm has a length greater than 22 cm.

**b** If the lengths of the worms are measured to the nearest centimetre, find the probability that a randomly selected worm has its length measured as 20 cm.

**c** If five worms are randomly selected, find the probability that exactly two will have their lengths measured as 20 cm (to the nearest centimetre).

**22** The amount of coal, $P$ tonnes, produced by $x$ miners in one shift is given by the rule:

$$P = \frac{x^2}{90}(56 - x) \quad \text{where } 1 \le x \le 40$$

**a** Find $\frac{dP}{dx}$.

**b** **i** Sketch the graph of $P$ against $x$ for $1 \le x \le 40$.

**ii** State the maximum value of $P$.

**c** Write down an expression in terms of $x$ for the average production per miner in the shift. Denote the average production per miner by $A$ (in tonnes).

**i** Sketch the graph of $A$ against $x$ for $1 \le x \le 40$.

**ii** State the maximum value of $A$ and the value of $x$ for which it occurs.

**23** Consider the family of quadratic functions with rules of the form

$$f(x) = (k+2)x^2 + (6k-4)x + 2$$

where $k$ is an arbitrary constant.

**a** Sketch the graph of $f$ when:

**i** $k = 0$ **ii** $k = -2$ **iii** $k = -4$

**b** Find the coordinates of the turning point of the graph of $y = f(x)$ in terms of $k$. If the coordinates of the turning point are $(a, b)$, find:

**i** $\{k : a > 0\}$ **ii** $\{k : a = 0\}$ **iii** $\{k : b > 0\}$ **iv** $\{k : b < 0\}$

**c** For what values of $k$ is the turning point a local maximum?

**d** By using the discriminant, state the values of $k$ for which:

**i** $f(x)$ is a perfect square

**ii** there are no solutions to the equation $f(x) = 0$.

**24** **a** Find the solution to the equation $e^{2-2x} = 2e^{-x}$.

**b** Let $y = e^{2-2x} - 2e^{-x}$.

**i** Find $\frac{dy}{dx}$. **ii** Solve the equation $\frac{dy}{dx} = 0$.

**iii** State the coordinates of the turning points of $y = e^{2-2x} - 2e^{-x}$.

**iv** Sketch the graph of $y = e^{2-2x} - 2e^{-x}$ for $x \ge 0$.

**c** State the set of values of $k$ for which the equation $e^{2-2x} - 2e^{-x} = k$ has two distinct positive solutions.

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**25** **a** Sketch, on a single clear diagram, the graphs of:

**i** $y = x^2$ **ii** $y = (x + a)^2$ **iii** $y = b(x + a)^2$ **iv** $y = b(x + a)^2 + c$

where $a$, $b$ and $c$ are positive constants with $b > 1$.

**b** Show that $\dfrac{2x^2 + 4x + 5}{x^2 + 2x + 1} = \dfrac{3}{(x + 1)^2} + 2$, for all values except $x = -1$.

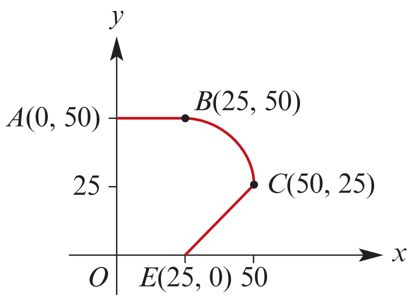
**c** Hence state precisely a sequence of transformations by which the graph of $y = \dfrac{2x^2 + 4x + 5}{x^2 + 2x + 1}$ may be obtained from the graph of $y = \dfrac{1}{x^2}$.

**d** Evaluate $\displaystyle\int_0^1 \frac{2x^2 + 4x + 5}{x^2 + 2x + 1}\, dx$.

**e** Sketch the graphs of $y = \dfrac{1}{x^2}$ and $y = \dfrac{3}{(x + 1)^2} + 2$ on the one set of axes, and indicate the region for which the area has been determined in **d**.

**26** A real-estate agent has a block of land to sell. An $x$–$y$ coordinate grid is placed with the origin at $O$, as shown in the diagram.

The block of land is $OABCE$, where $OA$, $AB$, $CE$ and $EO$ are straight line segments and the curve through points $B$ and $C$ is part of a parabola with equation of the form $y = ax^2 + 4x + c$.




**a** Find the equation of line segments:

**i** $AB$ **ii** $EC$

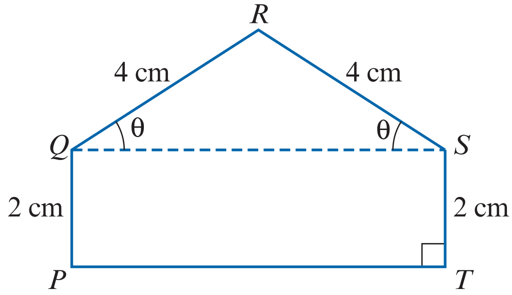
**b** Find the values of $a$ and $c$ and hence find the equation of the parabola through points $B$ and $C$.

**c** Find the area of:

**i** the rectangle $OEBA$ **ii** the region $EBC$ (with boundaries as defined above)

**iii** the block of land.

**27** In the diagram, $PQRST$ is a thin metal plate, where $PQST$ is a rectangle with $PQ = 2$ cm and $QRS$ is an isosceles triangle with $QR = RS = 4$ cm.




**a** Show that the area of the metal plate, $A$ cm$^2$, is given by

$$A = 16(\cos\theta + \cos\theta\sin\theta)$$

for $0 < \theta < \dfrac{\pi}{2}$.

**b** Show that $\dfrac{dA}{d\theta} = 16(1 - \sin\theta - 2\sin^2\theta)$.

**c** Solve the equation $\dfrac{dA}{d\theta} = 0$ for $0 < \theta < \dfrac{\pi}{2}$ by first solving $16(1 - a - 2a^2) = 0$ for $a$.

**d** Hence sketch the graph of $A$ against $\theta$ for $0 < \theta < \dfrac{\pi}{2}$ and state the maximum value of $A$.

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**28** The length of an engine part must be between 4.81 cm and 5.20 cm. In mass production, it is found that 0.8% are too short and 3% are too long. Assume that the lengths are normally distributed.

**a** Find the mean and standard deviation of this distribution.

**b** Each part costs \$4 to produce; those that turn out to be too long are shortened at an extra cost of \$2, and those that turn out to be too short are rejected. Find the expected total cost of producing 100 parts that meet the specifications.

**29** The temperature, $T$°C, of water in a kettle at time $t$ minutes is given by the formula

$$T = \theta + Ae^{-kt}$$

where $\theta$°C is the temperature of the room in which the kettle sits.

**a** Assume that the room is of constant temperature 21°C. At 2:23 p.m., the water in the kettle boils at 100°C. After 10 minutes, the temperature of the water in the kettle is 84°C. Use this information to find the values of $k$ and $A$, giving your answer correct to two decimal places.

**b** At what time will the temperature of the water in the kettle be 70°C?

**c** Sketch the graph of $T$ against $t$ for $t \geq 0$.

**d** Find the average rate of change of temperature for the time interval [0, 10].

**e** Find the instantaneous rate of change of temperature when:

**i** $t = 6$ **ii** $T = 60$

**30** Large batches of similar components are delivered to a company. A sample of five articles is taken at random from each batch and tested. If at least four of the five articles are found to be good, the batch is accepted. Otherwise, the batch is rejected.

**a** If the fraction of defectives in a batch is $\frac{1}{2}$, find the probability of the batch being accepted.

**b** If the fraction of defectives in a batch is $p$, show that the probability of the batch being accepted is given by a function of the form

$$A(p) = (1 - p)^4(1 + bp), \quad 0 \leq p \leq 1$$

and find the value of $b$.

**c** Sketch the graph of $A$ against $p$ for $0 \leq p \leq 1$. (Using a calculator would be appropriate.)

**d** Find correct to two decimal places:

**i** the value of $p$ for which $A(p) = 0.95$

**ii** the value of $p$ for which $A(p) = 0.05$.

**e** **i** Find $A'(p)$, for $0 \leq p \leq 1$.

**ii** Sketch the graph of $A'(p)$ against $p$.

**iii** For what value of $p$ is $A'(p)$ a minimum?

**iv** Describe what the result of **iii** means.

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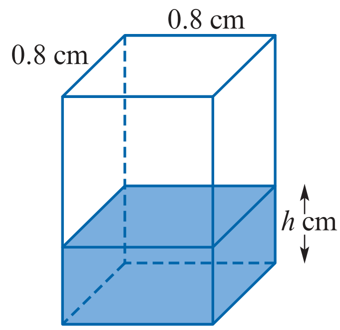
**31** A liquid is contained in a tank which is a cuboid with square cross-section as shown in the diagram. The depth of liquid, $h$ cm, in the tank at time $t$ minutes is given by the function with the rule:

$$h(t) = (4.5 - 0.3t)^3$$




**a** State the depth of the liquid at time $t = 0$.

**b** State the practical domain for the function $h$.

**c** State the rule for the volume, $V$ cm$^3$, of water in the tank at time $t$.

**d** Explain briefly why an inverse function $h^{-1}$ exists and find its rule and domain.

**e** Draw graphs of both $h$ and $h^{-1}$ on the one set of axes.

**32** A machine produces ball-bearings with a mean diameter of 3 mm. It is found that 6.3% of the production is being rejected as below the lower tolerance limit of 2.9 mm, and a further 6.3% is being rejected as above the upper tolerance limit of 3.1 mm. Assume that the diameters are normally distributed.

**a** Calculate the standard deviation of the distribution.

**b** A sample of eight ball-bearings is taken. Find the probability that:

**i** at least one is rejected **ii** two are rejected.

**c** The setting of the machine now 'wanders' such that the standard deviation remains the same, but the mean changes to 3.05 mm.

**i** Calculate the total percentage of the production that will now fall outside the given tolerance limits.

**ii** Find the value of $c$ such that the probability that the diameter lies in the interval $(3.05 - c, 3.05 + c)$ is 0.9.

**33** There is a probability of 0.8 that a boarding student will miss breakfast if he oversleeps. There is a probability of 0.3 that the student will miss breakfast even if he does not oversleep. The student has a probability of 0.4 of oversleeping.

**a** On a random day, what is the probability of:

**i** the student oversleeping and missing breakfast

**ii** the student not oversleeping and still missing breakfast

**iii** the student not missing breakfast?

**b** Given that the student misses breakfast, find the probability that he overslept.

**c** It is found that 10 students in the boarding house have identical probabilities for sleeping in and missing breakfast to the student mentioned above. Find the probability that:

**i** exactly two of the 10 students miss breakfast

**ii** at least one of the 10 students misses breakfast

**iii** at least eight of the students don't miss breakfast.

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**34** **a** On the one set of axes, sketch the graphs of $y = \frac{1}{x}$ and $y = e^x$ for $x > 0$.

**b** Using addition of ordinates, sketch the graph of $y = \frac{1}{x} + e^x$ for $x > 0$.

(Do not attempt at this stage to find the coordinates of the turning points.)

**c** Find $\frac{dy}{dx}$ for $y = \frac{1}{x} + e^x$.

**d** **i** Show that $\frac{dy}{dx} = 0 \Leftrightarrow 2\log_e x = -x$, for $x > 0$.

**ii** Explain why this implies that the local minimum of $y = \frac{1}{x} + e^x$ lies in the interval $(0, 1)$.

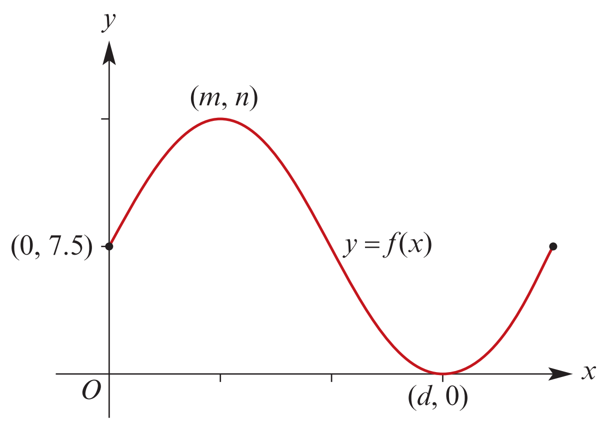
**iii** Using a calculator, show that the point of intersection of the graphs of $y = 2\log_e x$ and $y = -x$ is at $(0.70, -0.70)$, correct to two decimal places.

**iv** Hence find the coordinates of the local minimum of $y = \frac{1}{x} + e^x$, correct to one decimal place.

**35** A section of a creek bank can be modelled by the function:

$$f\colon [0, 50] \to \mathbb{R},\ f(x) = a + b\sin\left(\frac{2\pi x}{50}\right)$$

where units are in metres.




**a** **i** Find the values of $a$, $b$, $d$, $m$ and $n$.

**ii** The other bank of the creek can be modelled by the function $y = f(x) + 4$. Sketch the graph of this new function.

**b** Find the coordinates of the points on the first bank with $y$-coordinate 10.

**c** A particular river has a less severe bend than this creek. It is found that a section of the bank of the river can be modelled by the function:

$$g\colon [0, 250] \to \mathbb{R},\ g(x) = 2f\left(\frac{x}{5}\right)$$

Sketch the graph of this function; label the turning points with their coordinates.

**d** Over the years, the river bank moves. The shape of the bends are maintained, but there is a translation of 10 metres in the positive direction of the $x$-axis.

**i** Give the rule that describes this section of the river bank after the translation (relative to the original axes).

**ii** Sketch the graph of this new function.

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Revision

**36** The continuous random variable $X$ has probability density function $f$ given by

$$f(x) = \begin{cases} k(5-2x) & \text{if } 2 < x \leq \dfrac{5}{2} \\ 0 & \text{otherwise} \end{cases}$$

**a** Find the value of $k$.

**b** **i** Find $\mathrm{E}(X)$.

**ii** Find the median of $X$.

**iii** Find $\sigma$, the standard deviation of $X$, correct to two decimal places.

**c** Find $\Pr(X < \mu - \sigma)$, where $\mu = \mathrm{E}(X)$.

**37** The lifetime, $X$ days, of a particular type of computer component has a probability density function given by

$$f(x) = \begin{cases} k(a-x) & \text{if } 0 < x \leq a \\ 0 & \text{if } x \leq 0 \text{ or } x > a \end{cases}$$

where $k$ and $a$ are positive constants.

**a** Find $k$ in terms of $a$.

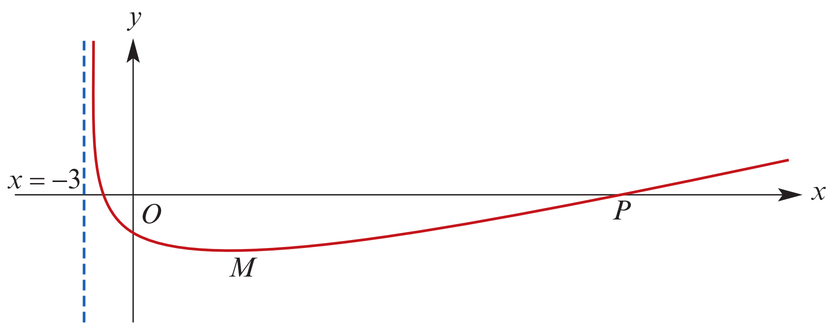
**b** Find the mean, $\mu$, and the variance, $\sigma^2$, of $X$ in terms of $a$.

**c** Find $\Pr(X > \mu + 2\sigma)$.

**d** Find the value of $a$ if the median lifetime is 1000 days.

**38** The diagram shows a sketch graph of

$$y = \frac{x}{10} - \log_e(x+3), \quad x > -3$$




**a** Find the $x$-coordinate of the local minimum at $M$.

**b** Show that the gradient of the curve is always less than $\dfrac{1}{10}$.

**c** Find the equation of the straight line through $M$ with a gradient of $\dfrac{1}{10}$.

**d** **i** Hence show that the value of the $x$-axis intercept at $P$ is greater than $10\log_e 10$.

**ii** Find, correct to three decimal places, the value of the intercept at $P$.

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**39** A particle is moving along a path with equation $y = \sqrt{x^2 + 24}$.

**a** Find $\dfrac{dy}{dx}$.

**b** Find the coordinates of the local minimum of the curve.

**c** Does this rule define an even function?

**d** As $x \to \infty$, $y \to x$ and as $x \to -\infty$, $y \to -x$. Sketch the graph of $y = \sqrt{x^2 + 24}$, showing the asymptotes.

**e** Find the equation of the normal to the curve at the point with coordinates $(1, 5)$, and sketch the graph of this normal with the graph of **d**.

**f** When the particle is at the point with coordinates $(5, 7)$, its $y$-coordinate is increasing at a rate of 10 units per second. At what rate is its $x$-coordinate increasing?

**g** Show that

$$\frac{d}{dx}\left(12\log_e\left(\sqrt{x^2 + 24} + x\right) + \frac{x\sqrt{x^2 + 24}}{2}\right) = \sqrt{x^2 + 24}$$

for $x > 0$.

**h** Use this result to find the area of the region bounded by the curve, the $x$-axis and the lines $x = 2$ and $x = 5$.

**40** The boxplot is a display used to describe the distribution of a data set. Located on the boxplot are the minimum, the lower quartile, the median, the upper quartile and the maximum. Boxplots also show outliers. These are values which are more than 1.5 interquartile ranges below the lower quartile or above the upper quartile.

**a** Suppose that a random variable $Z$ is normally distributed with a mean of 0 and a standard deviation of 1.

- **i** Find the value of the median, i.e. find $m$ such that $\Pr(Z \leq m) = 0.5$.
- **ii** Find the value of the lower quartile, i.e. find $q_1$ such that $\Pr(Z \leq q_1) = 0.25$.
- **iii** Find the value of the upper quartile, i.e. find $q_3$ such that $\Pr(Z \leq q_3) = 0.75$.
- **iv** Hence find the interquartile range (IQR) for this distribution.
- **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < Z < q_3 + 1.5 \times \text{IQR})$.
- **vi** What percentage of data values would you expect to be designated as outliers for this distribution?

**b** Suppose that a random variable $X$ is normally distributed with a mean of $\mu$ and a standard deviation of $\sigma$.

- **i** Find the value of the median, i.e. find $m$ such the $\Pr(X \leq m) = 0.5$.
- **ii** Find the value of the lower quartile, i.e. find $q_1$ such that $\Pr(X \leq q_1) = 0.25$.
- **iii** Find the value of the upper quartile, i.e. find $q_3$ such that $\Pr(X \leq q_3) = 0.75$.
- **iv** Hence find the interquartile range (IQR) for this distribution.
- **v** Find $\Pr(q_1 - 1.5 \times \text{IQR} < X < q_3 + 1.5 \times \text{IQR})$.
- **vi** What percentage of data values would you expect to be designated as outliers for this distribution?

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Revision

**41** The random variable $X$ has probability density function given by

$$f(x) = \begin{cases} kx^n & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x < 0 \text{ or } x > 1 \end{cases}$$

where $n$ and $k$ are constants with $n > 0$. Find in terms of $n$:

**a** $k$

**b** $\mathrm{E}(X)$

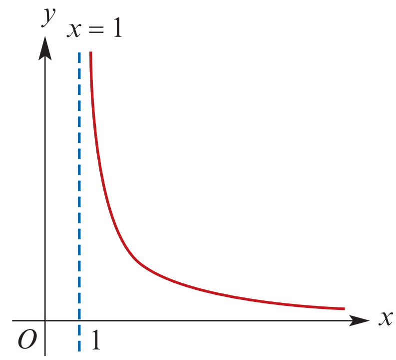
**c** $\mathrm{Var}(X)$

**d** the median of $X$

**42** The diagram shows the graph of the function

$$g\colon (1, \infty) \to \mathbb{R}, \quad g(x) = \frac{1}{x-1}$$

The line segment $AB$ is drawn from the point $A(2, 1)$ to the point $B(b, g(b))$, where $b > 2$.




**a** **i** What is the gradient of $AB$?

**ii** At what value of $x$ between 1 and $b$ does the tangent to the graph of $g$ have the same gradient as $AB$?

**b** **i** Calculate $\int_2^{e+1} g(x)\,dx$.

**ii** Let $c$ be a real number with $1 < c < 2$. Find the exact value of $c$ such that $\int_c^{e+1} g(x)\,dx = 8$.

**c** **i** What is the area of the trapezium bounded by the line segment $AB$, the $x$-axis and the lines $x = 2$ and $x = b$?

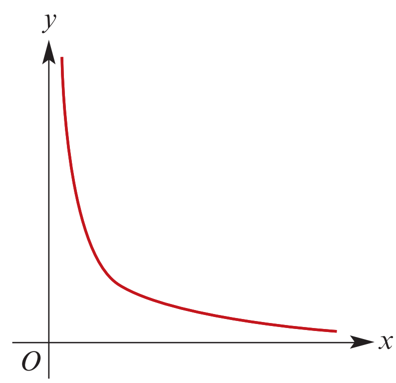
**ii** For what exact value of $b$ does this area equal 8?

**d** Given that $\int_2^{mn+1} g(x)\,dx + \int_2^{\frac{m}{n}+1} g(x)\,dx = 2$, where $n > 0$, find the value of $m$.

**43** The diagram shows the graph of the function

$$f\colon \mathbb{R}^+ \to \mathbb{R}, \quad f(x) = \frac{1}{x^2}$$

The line segment $AB$ is drawn from the point $A(1, 1)$ to the point $B(b, f(b))$, where $b > 1$.




**a** **i** What is the gradient of $AB$?

**ii** At what value of $x$ between 1 and $b$ does the tangent to the graph of $f$ have the same gradient as $AB$?

**b** **i** What is the area, $S(b)$, of the trapezium bounded by the line segment $AB$, the $x$-axis and the lines $x = 1$ and $x = b$?

**ii** For what exact value of $b$ does this area equal $\dfrac{10}{9}$?

**iii** Show that $\int_1^b f(x)\,dx < 1$ for $b > 1$.

**c** Show that the function $D(b) = S(b) - \int_1^b f(x)\,dx$ is strictly increasing for $b > 1$.

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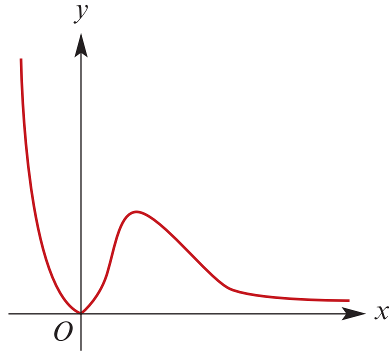
**44** Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^m e^{-nx+n}$, where $m$ and $n$ are positive integers. The graph of $y = f(x)$ is as shown.




**a** Find the coordinates of the stationary point not at the origin in terms of $n$, and state its nature.

**b** Find the coordinates of the point on the graph at which the tangent of $f$ passes through the origin.

**c** Consider the continuous probability density function with rule

$$f(x) = \begin{cases} kx^2 e^{-2x+2} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $k$ is a positive real number.

**i** Find the value of $k$.

**ii** Find $\Pr(X < 1)$, where $X$ is the associated random variable.

**45** Let $X$ be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ke^{-qx} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $q$ is a positive real number.

**a** **i** Find the value of $k$ in terms of $q$.

**ii** Find $\mathrm{E}(X)$ in terms of $q$.

**iii** Find $\mathrm{Var}(X)$ in terms of $q$.

**iv** Show that the median of the distribution is $m = \dfrac{1}{q} \log_e 2$.

**b** Find $\Pr\left(X > \dfrac{1}{q} \log_e 3 \;\middle|\; X > \dfrac{1}{q} \log_e 2\right)$.

**c** The distance, $X$ metres, between flaws in a certain type of yarn is a continuous random variable with probability density function $f(x) = 0.01e^{-0.01x}$ for $x \geq 0$.

**i** Sketch the graph of $y = f(x)$.

**ii** Find the probability, correct to two decimal places, that the distance between consecutive flaws is more than 100 m.

**iii** Find the median value of this distribution, correct to two decimal places.

**46** A coin is tossed 1000 times, and 527 heads observed.

**a** Give a point estimate for $p$, the probability of observing a head when the coin is tossed.

**b** Determine an approximate 95% confidence interval for $p$.

**c** What level of confidence would be given by a confidence interval for $p$ which is half the width of the approximate 95% confidence interval?

**d** What level of confidence would be given by a confidence interval for $p$ which is twice the width of the approximate 95% confidence interval?

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