Power Rule by First Principles

Natural Exponents

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = \lim_{h\to 0}\left(\frac{(x+h)^n - x^n}{h}\right) = \lim_{h\to 0}\left(\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 \ldots + h^n - x^n}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 \ldots + h^n}{h}\right) = \lim_{h\to 0}\left(nx^{n-1} + \binom{n}{2}x^{n-2}h \ldots + h^{n-1}\right)$$

$$= nx^{n-1}$$

Negative Exponents

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{-n}) = \lim_{h\to 0}\left(\frac{(x+h)^{-n} - x^{-n}}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h}\left(\frac{1}{(x+h)^n} - \frac{1}{x^n}\right)\right) = \lim_{h\to 0}\left(\frac{x^n - (x+h)^n}{h(x+h)^n x^n}\right)$$

$$= \lim_{h\to 0}\left(\frac{x^n - \left(x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 \ldots + h^n\right)}{h(x-h)^n x^n}\right) = \lim_{h\to 0}\left(-\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 \ldots + h^n}{h(x-h)^n x^n}\right)$$

$$= -\lim_{h\to 0}\left(\frac{nx^{n-1} + \binom{n}{2}x^{n-2}h \ldots + h^{n-1}}{(x-h)^n x^n}\right) = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1} = \frac{-n}{x^{n+1}}$$

Positive Rational Exponents

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{m}{n}}\right) = \lim_{h\to 0}\left(\frac{(x+h)^{\frac{m}{n}} - x^{\frac{m}{n}}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(x+h)^{\frac{m}{n}} - x^{\frac{m}{n}}}{h} \times \frac{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\frac{(x+h)^m - x^m}{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\frac{x^m + \binom{m}{1}x^{m-1}h + \binom{m}{2}x^{m-2}h^2 \ldots + h^m - x^m}{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\frac{mx^{m-1}h + \binom{m}{2}x^{m-2}h^2 \ldots + h^m}{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$= \lim_{h\to 0}\left(\frac{mx^{m-1} + \binom{m}{2}x^{m-2}h \ldots + h^{m-1}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1} + \left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2} + \left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$= \frac{mx^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1} + \left(x^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right) + \cdots + \left(x^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-1} + \left(x^{\frac{m}{n}}\right)^{n-1}} = \frac{mx^{m-1}}{n\left(x^{\frac{m}{n}}\right)^{n-1}} = \frac{mx^{m-1}}{nx^{\frac{m(n-1)}{n}}}$$

$$= \frac{m}{n}x^{m-1-\left(\frac{m(n-1)}{n}\right)} = \frac{m}{n}x^{\frac{mn-n-mn+m}{n}} = \frac{m}{n}x^{\frac{m-n}{n}} = \frac{m}{n}x^{\frac{m}{n}-1}$$

Negative Rational Exponents

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{-\frac{m}{n}}\right)=\lim_{h\to 0}\left(\frac{(x+h)^{-\frac{m}{n}}-x^{-\frac{m}{n}}}{h}\right)=\lim_{h\to 0}\left(\frac{1}{h}\left(\frac{1}{(x-h)^{\frac{m}{n}}}-\frac{1}{x^{\frac{m}{n}}}\right)\right)=\lim_{h\to 0}\left(\frac{x^{\frac{m}{n}}-(x-h)^{\frac{m}{n}}}{h(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\right)$$

$$=\lim_{h\to 0}\left(\frac{x^{\frac{m}{n}}-(x+h)^{\frac{m}{n}}}{h(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\times\frac{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$=\lim_{h\to 0}\left(\frac{1}{h(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\frac{x^{m}-(x+h)^{m}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$=\lim_{h\to 0}\left(\frac{1}{h(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\frac{x^{m}-\left(x^{m}+\binom{m}{1}x^{m-1}h+\binom{m}{2}x^{m-2}h^{2}\ldots+h^{m}\right)}{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$=\lim_{h\to 0}\left(-\frac{1}{h(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\frac{mx^{m-1}h+\binom{m}{2}x^{m-2}h^{2}\ldots+h^{m}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$=-\lim_{h\to 0}\left(\frac{1}{(x-h)^{\frac{m}{n}}x^{\frac{m}{n}}}\frac{mx^{m-1}+\binom{m}{2}x^{m-2}h\ldots+h^{m-1}}{\left((x+h)^{\frac{m}{n}}\right)^{n-1}+\left((x+h)^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left((x+h)^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-2}+\left(x^{\frac{m}{n}}\right)^{n-1}}\right)$$

$$=-\frac{1}{x^{\frac{m}{n}}x^{\frac{m}{n}}}\frac{mx^{m-1}}{\left(x^{\frac{m}{n}}\right)^{n-1}+\left(x^{\frac{m}{n}}\right)^{n-2}\left(x^{\frac{m}{n}}\right)+\cdots+\left(x^{\frac{m}{n}}\right)\left(x^{\frac{m}{n}}\right)^{n-1}+\left(x^{\frac{m}{n}}\right)^{n-1}}=-\frac{mx^{m-1}}{x^{\frac{2m}{n}}n\left(x^{\frac{m}{n}}\right)^{n-1}}$$

$$=-\frac{mx^{m-1}}{nx^{\frac{2m+m(n-1)}{n}}}=-\frac{m}{n}x^{m-1-\left(\frac{2m+m(n-1)}{n}\right)}=-\frac{m}{n}x^{\frac{mn-n-2m-mn+m}{n}}=-\frac{m}{n}x^{\frac{-m-n}{n}}=-\frac{m}{n}x^{-\frac{m}{n}-1}$$