Exponentials and Logarithms by First Principles

Exponential Base e

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = \lim_{h\to 0}\left(\frac{e^{(x+h)} - e^x}{h}\right) = \lim_{h\to 0}\left(\frac{e^x e^h - e^x}{h}\right) = \lim_{h\to 0}\left(\frac{e^x\left(e^h - 1\right)}{h}\right) = \lim_{h\to 0}(e^x) \times \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right)$$

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = \lim_{h\to 0}(1+h)^{\frac{1}{h}}, \qquad h = \frac{1}{n}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x \lim_{h\to 0}\left(\frac{1}{h}\left(\left(\lim_{h\to 0}(1+h)^{\frac{1}{h}}\right)^h - 1\right)\right) = e^x \lim_{h\to 0}\left(\frac{\lim_{h\to 0}(1+h) - 1}{h}\right)$$

$$= e^x \lim_{h\to 0}\left(\frac{1+h-1}{h}\right) = e^x \lim_{h\to 0}\left(\frac{h}{h}\right) = e^x \lim_{h\to 0}(1)$$

$$= e^x$$

Natural Logarithm

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \lim_{h\to 0}\left(\frac{\log_e(x+h) - \log_e(x)}{h}\right) = \lim_{h\to 0}\left(\frac{1}{h}\log_e\left(\frac{(x+h)}{x}\right)\right) = \lim_{h\to 0}\left(\frac{1}{h}\log_e\left(\frac{x+h}{x}\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\log_e\left(1 + \frac{h}{x}\right)\right)$$

Let $u = \frac{h}{x}$, $\qquad h = ux, \qquad u \to 0$ as $h \to 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \lim_{u\to 0}\left(\frac{1}{ux}\log_e(1+u)\right) = \frac{1}{x}\lim_{u\to 0}\left(\log_e(1+u)^{\frac{1}{u}}\right) = \frac{1}{x}\log_e(e) = \frac{1}{x}$$