# Circular Functions by First Principles

## Fundamental Limit

$$\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right) = 1$$

## Sine

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \lim_{h\to 0}\left(\frac{\sin(x+h)-\sin(x)}{h}\right), \qquad \sin(A)-\sin(B) = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\frac{d}{dx}(\sin(x)) = \lim_{h\to 0}\left(\frac{2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h/2}\right) = \lim_{h\to 0}\left(\cos\left(x+\frac{h}{2}\right)\frac{\sin\left(\frac{h}{2}\right)}{h/2}\right)$$

$$= \lim_{h\to 0}\left(\cos\left(x+\frac{h}{2}\right)\right)\times\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{h/2}\right) = \cos(x)$$

## Cosine

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = \lim_{h\to 0}\left(\frac{\cos(x+h)-\cos(x)}{h}\right), \qquad \cos(A)-\cos(B) = -2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = \lim_{h\to 0}\left(\frac{-2\sin\left(\frac{x+h+x}{2}\right)\cos\left(\frac{h}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{-2\sin\left(\frac{2x+h}{2}\right)\cos\left(\frac{h}{2}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2\sin\left(x+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)}{h}\right) = \lim_{h\to 0}\left(\frac{-\sin\left(x+\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)}{h/2}\right) = \lim_{h\to 0}\left(-\sin\left(x+\frac{h}{2}\right)\frac{\cos\left(\frac{h}{2}\right)}{h/2}\right)$$

$$= \lim_{h\to 0}\left(-\sin\left(x+\frac{h}{2}\right)\right)\times\lim_{h\to 0}\left(\frac{\cos\left(\frac{h}{2}\right)}{h/2}\right) = -\sin(x)$$

## Tangent

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \lim_{h\to 0}\left(\frac{\tan(x+h)-\tan(x)}{h}\right), \qquad \tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \lim_{h\to 0}\left(\frac{1}{h}\left(\frac{\tan(x)+\tan(h)}{1-\tan(x)\tan(h)}-\tan(x)\right)\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\frac{\tan(x)+\tan(h)-\tan(x)\,(1-\tan(x)\tan(h))}{1-\tan(x)\tan(h)}\right)$$

$$= \lim_{h\to 0}\left(\frac{1}{h}\frac{\tan(x)+\tan(h)-\tan(x)+\tan^2(x)\tan(h)}{1-\tan(x)\tan(h)}\right) = \lim_{h\to 0}\left(\frac{\tan h\,(1+\tan^2(x))}{h(1-\tan(x)\tan(h))}\right)$$

$$= \lim_{h\to 0}\left(\frac{\sin(h)}{h}\frac{(1+\tan^2(x))/\cos h}{1-\tan(x)\tan(h)}\right) = \lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)\times\lim_{h\to 0}\left(\frac{(1+\tan^2(x))/\cos(h)}{1-\tan(x)\tan(h)}\right) = \frac{(1+\tan^2(x))/1}{1-\tan(x)\times 0}$$

$$= 1+\tan^2(x) = \sec^2(x) = \frac{1}{\cos^2(x)}$$