Derivatives of Combinations of Functions

Chain Rule: Derivative of Composition of Functions

The derivative of a composite function is the product of the derivative of the outside function and the derivative of the inside function.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f[g(x)]) = f'[g(x)]g'(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

Example VCAA 2006 Exam 1 Question 3a

Let $f(x) = e^{\cos(x)}$. Find $f'(x)$. Let $g(x) = \cos(x)$ and $h(x) = e^x$ so $f(x) = h\big(g(x)\big)$

$g'(x) = -\sin(x)$, $h'(x) = e^x$, $f'(x) = h'\big(g(x)\big)g'(x) = -\sin(x)\, e^{\cos(x)}$

Applying Chain Rule Generally to Functions

By using the chain rule, we can determine the derivatives of the transformations of functions or more generally the derivative of the composition of each function.
Becoming familiar with these forms will make finding more complicated derivatives easier to do.

Transformed

$$\frac{\mathrm{d}}{\mathrm{d}x}\big((ax+b)^n\big) = an(ax+b)^{n-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(e^{ax+b}\big) = ae^{ax+b}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(ax+b)) = \frac{a}{ax+b}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(ax+b)) = a\cos(ax+b)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos(ax+b)) = -a\sin(ax+b)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(ax+b)) = \frac{a}{\cos^2(ax+b)}$$

Composites

$$\frac{\mathrm{d}}{\mathrm{d}x}([f(x)]^n) = n[f(x)]^{n-1}f'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(e^{f(x)}\big) = f'(x)e^{f(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e[f(x)]) = \frac{f'(x)}{f(x)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin[f(x)]) = f'(x)\cos[f(x)]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos[f(x)]) = -f'(x)\sin[f(x)]$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan[f(x)]) = \frac{f'(x)}{\cos^2[f(x)]} = f'(x)\sec^2[f(x)]$$

Example VCAA 2001 Exam 1 Question 14

The derivative of $\log_e(2x)$ with respect to x is $\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(2x)) = \frac{1}{2x} \times \frac{\mathrm{d}}{\mathrm{d}x}(2x) = \frac{1}{2x} \times 2 = \frac{1}{x}$

Example VCAA 2002 Exam 1 Question 14

If $y = \log_e(\cos(2x))$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(2x)} \times \frac{\mathrm{d}}{\mathrm{d}x}(\cos(2x)) = -\frac{2\sin(x)}{\cos(2x)} = -2\tan(2x)$

Example VCAA 2008 Exam 1 Question 1a

Let $y = (3x^2 - 5x)^5$. $\frac{\mathrm{d}y}{\mathrm{d}x} = 5(3x^2 - 5x)^4 \times \frac{\mathrm{d}}{\mathrm{d}x}(3x^2 - 5x) = 5(3x^2 - 5x)^4(6x - 5)$

Example VCAA 2012 Exam 2 Question 2bi

Let $f: R \setminus \{2\} \to R, f(x) = \frac{1}{2x-4} + 3$. $f'(x) = -1 \times (2x-4)^{-2} \times \frac{\mathrm{d}}{\mathrm{d}x}(2x-4) + 0 = -\frac{2}{(2x-4)^2}$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}(2^x) = \frac{\mathrm{d}}{\mathrm{d}x}\big(e^{\log_e(2)x}\big) = \log_e(2)\, e^{\log_e(2)x} = \log_e(2)\, 2^x$$

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_2(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\log_e(x)}{\log_e(2)}\right) = \frac{1/x}{\log_e(2)} = \frac{1}{\log_e(2)\, x}$$

Sum and Difference Rules: Derivative of Sums and Differences of Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$$

Example VCAA 2011 Exam 2 Question 3ai

Consider the function $f: R \to R, f(x) = 4x^3 + 5x - 9, \quad f'(x) = 12x^2 + 5$

Example VCAA 2000 Exam 2 Question 4c

Consider the function $x = -2\cos(2\pi t) + \sin(8\pi t) + 3$.
Use calculus to find the rate of change of x with respect to t when $t = 2$.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -2 \times -2\pi\sin(2\pi t) + 8\pi\cos(8\pi t) = 4\pi\sin(2\pi t) + 8\pi\cos(8\pi t)$$

At $t = 2$, $\quad \frac{\mathrm{d}x}{\mathrm{d}t} = 4\pi\sin(2\pi \times 2) + 8\pi\cos(8\pi \times 2) = 4\pi \times 0 + 8\pi \times 1 = 8\pi$

Product Rule: Derivative of Products of Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)] = g(x)f'(x) + f(x)g'(x)$$

Let $u = f(x)$ and $v = g(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = v\frac{\mathrm{d}u}{\mathrm{d}x} + u\frac{\mathrm{d}v}{\mathrm{d}x} = uv\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = uv\left(\frac{u'}{u} + \frac{v'}{v}\right)$$

Example VCAA 2000 Exam 1 Question 17

If $f(x) = e^{-x}(x^3 - 4)$ then $f'(x)$ is

$$f'(x) = (x^3 - 4)\frac{\mathrm{d}}{\mathrm{d}x}(e^{-x}) + e^{-x}\frac{d}{dx}(x^3 - 4) = -e^{-x}(x^3 - 4) + e^{-x}(3x^2) = e^{-x}(-x^3 + 3x^2 + 4)$$

$$f'(x) = e^{-x}(x^3 - 4)\left(\frac{-e^{-x}}{e^{-x}} + \frac{3x^2}{x^3 - 4}\right) = -e^{-x}(x^3 - 4) + e^{-x}(3x^2) = e^{-x}(-x^3 + 3x^2 + 4)$$

Example VCAA 2010 Exam 1 Question 9a

Find the derivative of $x^2\log_e(x)$.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2\log_e(x)) = x^2\log_e(x)\left(\frac{2x}{x^2} + \frac{1/x}{\log_e(x)}\right) = 2x\log_e(x) + x^2 \times \frac{1}{x} = 2x\log_e(x) + x$$

Quotient Rule: Derivative of Quotients of Functions

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Let $u = f(x)$ and $v = g(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} = \frac{u}{v}\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} - \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}x}\right) = \frac{u}{v}\left(\frac{u'}{u} - \frac{v'}{v}\right)$$

Example VCAA 2007 Exam 1 Question 1

Let $f(x) = \dfrac{x^3}{\sin(x)}$. Find $f'(x)$.

$$f'(x) = \frac{\sin(x)\frac{\mathrm{d}}{\mathrm{d}x}(x^3) - x^3\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x))}{(\sin(x))^2} = \frac{3x^2\sin(x) - x^3\cos(x)}{\sin^2(x)}$$

$$f'(x) = \frac{x^3}{\sin(x)}\left(\frac{3x^2}{x^3} - \frac{\cos(x)}{\sin(x)}\right) = \frac{3x^2}{\sin(x)} - \frac{x^3\cos(x)}{\sin^2(x)} = \frac{3x^2\sin(x) - x^3\cos(x)}{\sin^2(x)}$$

Example VCAA 2015 Exam 1 Question 1bi

Let $f(x) = \dfrac{\log_e(x)}{x^2}$. Find $f'(x)$.

$$f'(x) = \frac{\log_e(x)}{x^2}\left(\frac{1/x}{\log_e(x)} - \frac{2x}{x^2}\right) = \frac{1/x}{x^2} - \frac{2x\log_e(x)}{x^4} = \frac{1 - 2\log_e(x)}{x^3}$$