Derivatives of Combinations by First Principles

Sum and Difference Rules

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f(x)\pm g(x)\right)=\lim_{h\to 0}\left(\frac{f(x+h)\pm g(x+h)-\left(f(x)\pm g(x)\right)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)\pm\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=f'(x)\pm g'(x)$$

Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f[g(x)]\right)=\lim_{h\to 0}\left(\frac{f[g(x+h)]-f[g(x)]}{h}\right)=\lim_{h\to 0}\left(\frac{\left(f[g(x+h)]-f[g(x)]\right)\left(g(x+h)-g(x)\right)}{h\left(g(x+h)-g(x)\right)}\right)$$

$$=\lim_{h\to 0}\left(\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\frac{g(x+h)-g(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f[g(x+h)]-f[g(x)]}{g(x+h)-g(x)}\right)\times\lim_{h\to 0}\left(\frac{g(x+h)-g(x)}{h}\right)=f'[g(x)]g'(x)$$

Product Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}[f(x)g(x)]=\lim_{h\to 0}\left(\frac{f(x+h)g(x+h)-f(x)g(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(f(x)g(x+h)\times\left(\frac{f(x+h)}{hf(x)}-\frac{g(x)}{hg(x+h)}\right)\right)$$

$$=\lim_{h\to 0}\left(f(x)g(x+h)\right)\times\lim_{h\to 0}\left(\left(\frac{f(x+h)}{hf(x)}-\frac{1}{h}\right)+\left(\frac{1}{h}-\frac{g(x)}{hg(x+h)}\right)\right)$$

$$=f(x)g(x)\lim_{h\to 0}\left(\left(\frac{f(x+h)}{hf(x)}-\frac{f(x)}{hf(x)}\right)+\left(\frac{g(x+h)}{hg(x+h)}-\frac{g(x)}{hg(x+h)}\right)\right)$$

$$=f(x)g(x)\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{hf(x)}+\frac{g(x+h)-g(x)}{hg(x+h)}\right)$$

$$=f(x)g(x)\left(\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}\right)=g(x)f'(x)+f(x)g'(x)$$

Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g(x)}\right]=\lim_{h\to 0}\left(\frac{f(x+h)/g(x+h)-f(x)/g(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(x)}{g(x+h)}\times\frac{f(x+h)/f(x)-g(x+h)/g(x)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{f(x)}{g(x+h)}\right)\times\lim_{h\to 0}\left(\frac{f(x+h)}{hf(x)}-\frac{g(x+h)}{hg(x)}\right)=\frac{f(x)}{g(x)}\lim_{h\to 0}\left(\left(\frac{f(x+h)}{hf(x)}-\frac{1}{h}\right)-\left(\frac{g(x+h)}{hg(x)}-\frac{1}{h}\right)\right)$$

$$=\frac{f(x)}{g(x)}\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{hf(x)}-\frac{g(x+h)-g(x)}{hg(x)}\right)$$

$$=\frac{f(x)}{g(x)}\left(\frac{f'(x)}{f(x)}-\frac{g'(x)}{g(x)}\right)=\frac{f'(x)}{g(x)}-\frac{f(x)g'(x)}{[g(x)]^2}=\frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$$