# Derivatives using Product, Quotient, and Chain Rules

## Quotient Rule using Product Rule and Chain Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \times \frac{1}{g(x)}\right) = f'(x) \times \frac{1}{g(x)} + f(x) \times \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = \frac{\mathrm{d}}{\mathrm{d}x}([g(x)]^{-1}) = -[g(x)]^{-2}g'(x) = -\frac{g'(x)}{[g(x)]^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = f'(x) \times \frac{1}{g(x)} + f(x) \times -\frac{g'(x)}{[g(x)]^2} = \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{[g(x)]^2} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

## Derivatives of Inverses using Chain Rule

$$f\big(g(x)\big) = x \;\Rightarrow\; f^{-1}(x) = g(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(f\big(g(x)\big)\Big) = \frac{\mathrm{d}}{\mathrm{d}x}(x) \;\Rightarrow\; f'\big(g(x)\big) \times g'(x) = 1 \;\Rightarrow\; g'(x) = \frac{1}{f'\big(g(x)\big)} = \frac{1}{f'\big(f^{-1}(x)\big)}$$

## Derivative of Root Functions

Let $y = \sqrt[n]{x} = x^{\frac{1}{n}}, \qquad x = y^n$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{n}}\right) = \frac{1}{\frac{d}{dy}(y^n)} = \frac{1}{ny^{n-1}} = \frac{1}{n\left(x^{\frac{1}{n}}\right)^{n-1}} = \frac{1}{nx^{\left(1-\frac{1}{n}\right)}} = \frac{1}{nx^{-\left(\frac{1}{n}-1\right)}} = \frac{1}{n}x^{\frac{1}{n}-1}$$

## Derivative of Exponentials

Let $y = e^x, \qquad x = \log_e(y)$

$$\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}y}(\log_e(y))} = \frac{1}{\frac{1}{y}} = y = e^x$$

## Derivative of Logarithms

Let $y = \log_e(x), \qquad x = e^y$

$$\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \frac{1}{\frac{\mathrm{d}}{\mathrm{d}y}(e^y)} = \frac{1}{e^y} = \frac{1}{e^{\log_e(x)}} = \frac{1}{x}$$

## Derivative of Negative Power Functions using Quotient Rule

Let $y = \frac{1}{x^n} = x^{-n} = (x^n)^{-1}$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x^n}\right) = \frac{\frac{\mathrm{d}}{\mathrm{d}x}(1) \times x^n - \frac{\mathrm{d}}{\mathrm{d}x}(x^n) \times 1}{(x^n)^2} = \frac{0 - nx^{n-1}}{x^{2n}} = -\frac{nx^{n-1}}{x^2} = -nx^{n-1-2n} = -nx^{-n-1}$$

## Derivative of Rational Power Functions using Chain Rule

Let $y = x^{\frac{m}{n}} = \left(x^{\frac{1}{n}}\right)^m$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{m}{n}}\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\left(x^{\frac{1}{n}}\right)^m\right) = m\left(x^{\frac{1}{n}}\right)^{m-1} \times \frac{1}{n}x^{\frac{1}{n}-1} = \frac{m}{n}\left(x^{\frac{m-1}{n}}\right)\left(x^{\frac{1-n}{n}}\right) = \frac{m}{n}x^{\frac{m-1+1-n}{n}} = \frac{m}{n}x^{\frac{m}{n}-1}$$

## Derivative of Tangent using Quotient Rule

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan(x)) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin(x)}{\cos(x)}\right) = \frac{\cos(x)\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) - \frac{\mathrm{d}}{\mathrm{d}x}(\cos(x))\sin(x)}{\cos^2(x)}$$

$$= \frac{\cos(x)\cos(x) - -\sin(x)\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$