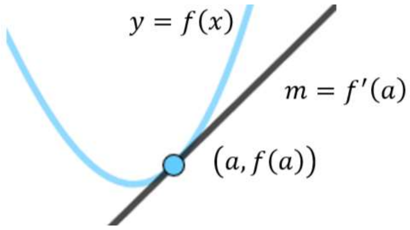
Tangents and Normals using Derivatives

Lines Parallel to a Graph / Tangent to a Graph

A tangent to a graph is line parallel to that graph at a particular point.

Since the tangent is parallel to the graph of a function $y = f(x)$ at a particular point, $x = a$, the gradient of the tangent passing through $(a, f(a))$ is equal to the gradient of the graph at that point.
That is, the gradient of the tangent is equal to the derivative of the function at that point.

$m_{\text{tangent}} = f'(a)$



The equation of a line, $y = m(x - x_1) + y_1$, can then be used to determine the equation of the tangent.

$y = f'(a)(x - a) + f(a)$

Example VCAA 2004 Exam 1 Question 5c

Let $f: R \to R, f(x) = \frac{1}{4}x^3(x - 4)$

The equation of the tangent to the graph of f at the point where $x = 4$ is

$f(x) = \frac{1}{4}x^4 - x^3$

$f(4) = \frac{1}{4}(4)^3(4 - 4) = 0$

$f'(x) = x^3 - 3x^2$

$$\begin{aligned} f'(4) &= (4)^3 - 3(4)^2 \\ &= 64 - 48 \\ &= 16 \end{aligned}$$

The gradient at $(4,0)$ is 16.

$y = 16(x - 4) + 0$

$\Rightarrow y = 16x - 64$

Example

Determine the equation of the tangent to $y = x^3 - 3x + 4$ at $x = 1$

$\frac{dy}{dx} = 3x^2 - 3$, when $x = 1$, $\frac{dy}{dx} = 3(1)^2 - 3 = 0$, $y = (1)^3 - 3(1) + 4 = 2$

The gradient at $(1, 2)$ is 0, so the equation of the tangent is $y = 0(x - 1) + 2 \Rightarrow y = 2$

Example

Determine the equation of the tangent to $y = x^{\frac{1}{3}}$ at $x = 0$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$, when $x = 0$, $\frac{dy}{dx} = \frac{1}{0}$ is undefined, $y = (0)^{\frac{1}{3}} = 0$

The gradient at $(0, 0)$ is undefined, so the equation of the tangent is $x = 0$

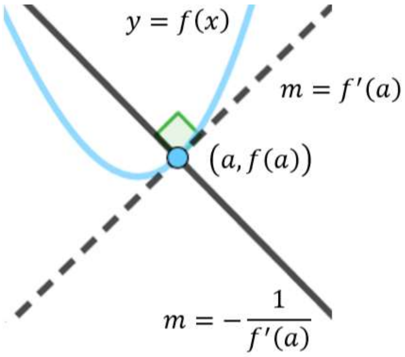
Lines Perpendicular to a Graph / Normal to a Graph

A normal to a graph is a line perpendicular to that graph at a particular point.

Since the normal is perpendicular to the graph of a function $y = f(x)$ at a particular point, $x = a$, the gradient of the normal passing through $(a, f(a))$ is equal to the negative reciprocal of the gradient of the graph at that point.

That is, the gradient of the normal is equal to the negative reciprocal of the derivative of the function at that point.

$$m_{\text{normal}} = -\frac{1}{m_{\text{tangent}}} = -\frac{1}{f'(a)}$$

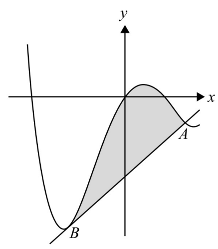


The equation of a line, $y = m(x - x_1) + y_1$, can then be used to determine the equation of the normal.

$$y = -\frac{1}{f'(a)}(x - a) + f(a)$$

Example VCAA 2002 Exam 2 Question 3bi

The diagram shows the curve whose equation is $y = \frac{1}{2}(2x^4 - x^3 - 5x^2 + 3x)$ and the normal to the curve at A, where $x = 1$. Show that the equation of this normal is $y = x - 1.5$.



When $x = 1$, $\quad y = \frac{1}{2}(2 - 1 - 5 + 3) = -\frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2}(8x^3 - 3x^2 - 10x + 3)$, $\quad$ at $x = 1$, $\frac{dy}{dx} = \frac{1}{2}(8 - 3 - 10 + 3) = -1$

The gradient at $\left(1, -\frac{1}{2}\right)$ is -1, the gradient of the normal is $+\frac{1}{1} = 1$.

$$y = (x - 1) - \frac{1}{2} \Rightarrow y = x - 1.5$$

Example

Determine the equation of the normal to $y = x^3 - 3x + 4$ at $x = 1$

$\frac{dy}{dx} = 3x^2 - 3$, $\quad$ when $x = 1$, $\quad \frac{dy}{dx} = 3(1)^2 - 3 = 0$, $\quad y = (1)^3 - 3(1) + 4 = 2$

The gradient at $(1, 2)$ is 0, so the gradient of the normal is undefined.
The equation of the normal is $x = 1$

Example

Determine the equation of the normal to $y = x^{\frac{1}{3}}$ at $x = 0$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$, $\quad$ when $x = 0$, $\quad \frac{dy}{dx} = \frac{1}{0}$ is undefined, $\quad y = (0)^{\frac{1}{3}} = 0$

The gradient at $(0, 0)$ is undefined, so the gradient of the normal is 0.
The equation of the normal is $y = 0(x - 0) + 0 \Rightarrow y = 0$