Maxima and Minima Problems

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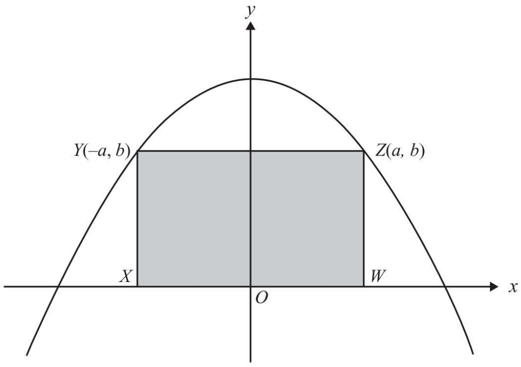
Maxima and minima problems are optimisation problems where we wish to find the largest possible value or the smallest possible value a model can take. To solve these optimisation problems we need to identify maximums and minimums of the graph of the function. The zeroes of the derivative will be when the maximum/minimum occurs. These can be substituted into the original function to determine the maximum/minimum. For derivatives with multiple zeroes, a quick sketch of the original function should identify which is a maximum and which is a minimum. It is important to consider what you are modelling. Real life units cannot be negative and will restrict your domain.

Example VCAA 2006 Exam 1 Question 9

A rectangle $XYZW$ has two vertices, X and W, on the x-axis and the other two vertices, Y and Z, on the graph of $y = 9 - 3x^2$, as shown in the diagram below. The coordinates of Z are (a, b) where a and b are positive real numbers. The area, A, of rectangle $XYZW$ in terms of a is



$A = lw$
$A = 2a \times b, \qquad b = 9 - 3a^2$
$A = 2a(9 - 3a^2)$

The maximum value of A and the value of a for which this occurs when $\dfrac{dA}{da} = 0$

$A = 18a - 6a^3$

$\dfrac{\mathrm{d}A}{\mathrm{d}a} = 18 - 18a^2$

$18 - 18a^2 = 0$
$a^2 = 1$
$a = \pm 1, a > 0$
$\therefore a = 1$

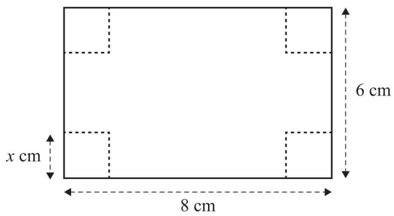
The area when $a = 1$ is

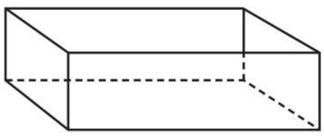
$A = 2(1)(9 - 3(1)^2) = 2(9 - 3) = 2 \times 6$

$= 12$ square units

Example VCAA 2014 Exam 2 Question 15

Zoe has a rectangular piece of cardboard that is 8 cm long and 6 cm wide. Zoe cuts squares of side length x centimetres from each of the corners of the cardboard, as shown in the diagram below. Zoe turns up the sides to form an open box. The value of x for which the volume of the box is a maximum is closest to



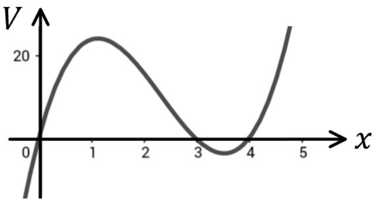


$V = lwh$
$V = x(8 - 2x)(6 - 2x)$
$V = 4x(4 - x)(3 - x)$
$V = 4x^3 - 28x^2 + 48x$

Since $V = 4x(4 - x)(3 - x) \Rightarrow 0 < x < 3$ as lengths and volumes cannot be negative

Maximum volume occurs when $\dfrac{dV}{dx} = 0$

$\dfrac{\mathrm{d}V}{\mathrm{d}x} = 12x^2 - 56x + 48$



$12x^2 - 56x + 48 = 0$

$x \approx 1.1$ or $x \approx 3.6$, reject $x = 3.6$ as it is not in the domain and is also not a maximum: $0 < x < 3$
$\therefore x = 1.1$ cm

Identification of Interval Endpoint Maxima and Minima

It is entirely possible that the maximum or minimum value does not occur at the local maximum or local minimum. Given a particular domain of the function, it may get larger than the local maximum or smaller than the local minimum at the ends of the domain, that is the endpoints.

Local maximums and local minimum are respectively the highest and lowest points in their surrounding region. So if there is a point higher or lower than these points it can only be either another local maximum or local minimum, or an endpoint. If another point, not an endpoint and not a local maximum (or local minimum) were the highest (or lowest point), then it cannot have any points higher (or lower) than it on either side therefore it is actually a local maximum (or local minimum).

To clarify the language:

	Minimum	Maximum
Local	Lowest point around a particular region	Highest point around a particular region
Endpoint	Lowest point of the end of the curve	Highest point of the end of the curve
Absolute	Lowest point on the curve over the domain	Highest point on the curve over the domain

All this means is that you should check the values of the endpoints of the function in case they may be larger than the local minimum or smaller than the local minimum. A quick graph or knowledge of the shape of the graph can help to quickly eliminate endpoints as possible maximums or minimums.

Example

The profit earned from selling a particular toy after t months of sales is given by
$P(t) = 3t^3 - 50t^2 + 209t$

The maximum and minimum amount of sales earned in the first year and when they occurred are

Local maximum and local minimum occur when $P'(t) = 0$
$P'(t) = 9t^2 - 10t + 209$

$$P'(t) = 0 = 9t^2 - 100t + 209, \qquad t = \frac{50 \pm \sqrt{619}}{9} \approx 2.79, 8.32$$

$$P\left(\frac{50 + \sqrt{619}}{9}\right) \approx 5.55, \qquad P\left(\frac{50 - \sqrt{619}}{9}\right) \approx 259.06$$

Interval endpoints: $t = 0$ and $t = 12$
$P(0) = 0, \qquad P(12) = 492$

$\therefore$ Maximum profit earned is \$492 after 12 months. Minimum profit earned is \$0 after 0 months.