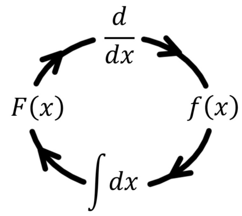
Anti-Differentiation

Anti-Differentiation and Notation

The opposite process of differentiation. The anti-derivative of a function is called the primitive function. It is often denoted with a capital letter. That is, the primitive function of $f(x)$ is $F(x)$.

While there are different ways of representing the anti-derivative of a function such as Lagrange's $f^{(-1)}$ as opposed to f' or $f^{(1)}$, or Euler's $D_x^{-1}f(x)$ as opposed to $D_x f(x)$ Leibniz's notation of $\int y\mathrm{d}x$ as opposed to $\frac{\mathrm{d}y}{\mathrm{d}x}$ has become the standard. Keep in mind that $\mathrm{d}x$ is one term. It is not $d \times x$ and should not be separated as such.



While it is accurate to say that in $\frac{\mathrm{d}y}{\mathrm{d}x}$, dy is being divided by $\mathrm{d}x$ and in $\int y\mathrm{d}x$, y is being multiplied by $\mathrm{d}x$, the main use of dx is to identify what variable you are differentiating or anti-differentiating with respect to.

Anti-Derivative of Sum and Difference of Functions

The anti-derivative of any sum (or difference) of functions is the sum (or difference) of the anti-derivative of each function.

$$\int (f(x) \pm g(x))\mathrm{d}x = \int f(x)\mathrm{d}x \pm \int g(x)\mathrm{d}x$$

VCAA prefers that students write the brackets around all the terms in the integrand (what you are finding the anti-derivative of) for clarity rather than relying on the placement of the dx to indicate the end of the integrand. This should also help you remember to find the anti-derivative each term.

Anti-Differentiation and Families of Curves

Since the derivative of a constant is 0, when anti-differentiating we need to add a generic constant, c, to show that any vertically translated version of the primitive function will have the same derivative. That is, there is a family of curves with the derivative $f'(x)$, they are $f(x) + c$.

if $F'(x) = f(x)$, then $\int f(x)\mathrm{d}x = F(x) + c$

Example VCAA 2002 Exam 1 Question 21

If $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{(2x+1)^{\frac{1}{2}}}$ and c is a real constant, then y is equal to $\int \frac{3}{(2x+1)^{\frac{1}{2}}}\mathrm{d}x = 3(2x+1)^{\frac{1}{2}} + c$

An Anti-Derivative

If the question asks for "an" anti-derivative of a function $f(x)$, any value of c can be chosen, including the generic $+c$ or $+0$. This means you can omit c for these questions.

$\int f(x)\mathrm{d}x = F(x) + c$ $\qquad$ $\int f(x)\mathrm{d}x = F(x) + 72$ $\qquad$ $\int f(x)\mathrm{d}x = F(x)$

Example VCAA 2000 Exam 1 Question 21

An antiderivative of $\frac{1}{3x} + \sin(2x)$, $x > 0$, is $\int \left(\frac{1}{3x} + \sin(2x)\right)\mathrm{d}x = \frac{1}{3}\log_e(x) - \frac{1}{2}\cos(2x)$

Anti-Derivatives of Specific Functions

Power Functions

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1$$

add 1 to the exponent then divide by the increased exponent.

$$\frac{x^{\square+1}}{\square+1}$$

$$\int (ax+b)^n \mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1} + c = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \qquad n \neq -1$$

Example

$$\int (x^4 + x^3 + x^2 + x + 1)\mathrm{d}x = \frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + c$$

Example

$$\int (5-3x)^4 \mathrm{d}x = \frac{(5-3x)^5}{-3(5)} + c = -\frac{(5-3x)^5}{15} + c$$

Example

$$\int \frac{1}{x^2}\mathrm{d}x = \int x^{-2} dx = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c$$

Example

$$\int \sqrt{x+7}\mathrm{d}x = \int (x+7)^{\frac{1}{2}}\mathrm{d}x = \frac{1}{(3/2)}(x+7)^{\frac{3}{2}} + c = \frac{2}{3}(x+7)^{\frac{3}{2}} + c$$

Hyperbolas

$$\int \frac{1}{x}\mathrm{d}x = \int \frac{\mathrm{d}x}{x} = \log_e(x) + c, \qquad x > 0$$

$$\int \frac{1}{x}dx = \int \frac{\mathrm{d}x}{x} = \log_e(-x) + c, \qquad x < 0$$

If the domain is $x < 0$, the anti-derivative must be $\log_e(-x)$ so that the values of the log are real.

$$\int \frac{1}{kx}\mathrm{d}x = \frac{1}{k}\log_e(x) + c, \qquad x > 0$$

$$\int \frac{1}{kx}\mathrm{d}x = \frac{1}{k}\log_e(-x) + c, \qquad x < 0$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\log_e(ax+b) + c, x > -\frac{b}{a}$$

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\log_e\big(-(ax+b)\big) + c, x < -\frac{b}{a}$$

Example

$$\int \frac{1}{2x-3}\mathrm{d}x = \frac{1}{2}\log_e(2x-3) + c, \qquad x > \frac{3}{2}$$

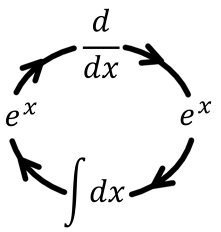
Example

$$\int \frac{1}{2x-3}\mathrm{d}x = \frac{1}{2}\log_e(3-2x) + c, \qquad x < \frac{3}{2}$$

Exponential Functions

$$\int e^x \, \mathrm{d}x = e^x + c$$

$$\int e^{ax+b} \, \mathrm{d}x = \frac{1}{a}e^{ax+b} + c$$



Example

$$\int 6e^{2x}\mathrm{d}x = 6 \times \frac{1}{2}e^{2x} + c = 3e^{2x} + c$$

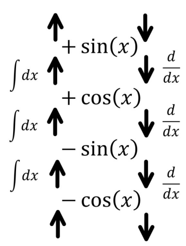
Circular Functions

$$\int \sin(x)\,\mathrm{d}x = -\cos(x) + c$$

$$\int \sin(ax+b)\,\mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c$$

$$\int \cos(x)\,\mathrm{d}x = \sin(x) + c$$

$$\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$$



Example

$$\int 5\sin\left(\frac{\pi}{4} - 3x\right)\mathrm{d}x = 5 \times -\frac{1}{3} \times -\cos\left(\frac{\pi}{4} - 3x\right) + c = \frac{5}{3}\cos\left(\frac{\pi}{4} - 3x\right) + c$$