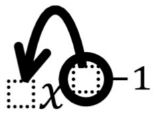
Deriving the Anti-Derivatives of Functions

Anti-Derivatives of Specific Functions

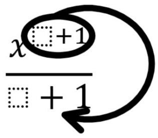
Rules for the anti-derivatives of specific can be found by applying the derivative rule in reverse.

Power Functions



We know that $\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = nx^{n-1}, n \in \mathbb{Q}$

This can be thought of as multiply by the exponent then subtract 1 from the exponent.



Reversing this process: add 1 to the exponent then divide by the increased exponent.

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c, \qquad n \neq -1$$

Transforming to $f(ax+b)$: $\frac{\mathrm{d}}{\mathrm{d}x}\left((ax+b)^{n+1}\right) = a(n+1)(ax+b)^n$. Reversing this process

$$\int (ax+b)^n \, \mathrm{d}x \frac{1}{a(n+1)}(ax+b)^{n+1} + c = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \qquad n \neq -1$$

Hyperbolas

For $n = -1$,

if we were to add 1 to the exponent, the new exponent will be 0, but the derivative of x^0 is not $\frac{1}{x}$.

$$\frac{1}{x} = x^{-1} \quad \Rightarrow \quad x^{-1+1} = x^0 = 1, \qquad \text{but } \frac{\mathrm{d}}{\mathrm{d}x}(1) = 0 \neq \frac{1}{x}$$

However, we know that $\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(x)) = \frac{1}{x}$. Reversing this process: $\int \frac{1}{x} \mathrm{d}x = \log_e(x) + c$.

Since the domain of $\log_e(x)$ is $x > 0$ not $x \in \mathbb{R} \setminus \{0\}$ like it is for $\frac{1}{x}$ the domain of the anti-derivative must be restricted. Depending on whether positive or negative x values are used are required, indicates what domain should be used for the anti-derivative.

If the domain is $x < 0$, the anti-derivative must be $\log_e(-x)$ so that the values of the log are real.

$$\int \frac{1}{x} \, \mathrm{d}x = \log_e(x) + c, \qquad x > 0$$

$$\int \frac{1}{x} \, \mathrm{d}x = \log_e(-x) + c, \qquad x < 0$$

Transformed Hyperbolas

For $\int \frac{1}{kx} \mathrm{d}x$, we write the logarithm in terms of x rather than kx.

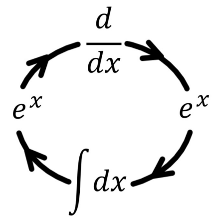
Break the fraction up: $\int \frac{1}{kx} \mathrm{d}x = \int \frac{1}{k}\frac{1}{x} \mathrm{d}x = \frac{1}{k}\int \frac{1}{x} \mathrm{d}x = \frac{1}{k}\log_e(x) + c, x > 0$

Transforming to $f(ax+b)$: $\frac{\mathrm{d}}{\mathrm{d}x}(\log_e(ax+b)) = \frac{a}{ax+b}$. Reversing this process

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a}\log_e(ax+b) + c, x > -\frac{b}{a}$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a}\log_e\left(-(ax+b)\right) + c, x < -\frac{b}{a}$$

Exponential Functions

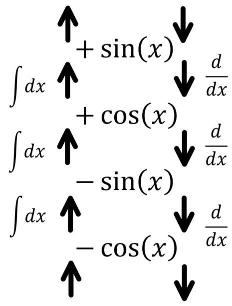


We know that $\frac{\mathrm{d}}{\mathrm{d}x}(e^x) = e^x$. Reversing this process: $\int e^x \, \mathrm{d}x = e^x + c$

Transforming to $f(ax+b)$: $\frac{\mathrm{d}}{\mathrm{d}x}\left(e^{ax+b}\right) = ae^{ax+b}$.

Reversing this process: $\int e^{ax+b} \, \mathrm{d}x = \frac{1}{a}e^{ax+b} + c$

Circular Functions



We know that $\frac{\mathrm{d}}{\mathrm{d}x}(\sin(x)) = \cos(x)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(x)) = -\sin(x)$.

Reversing these processes:

$$\int \sin(x) \, \mathrm{d}x = -\cos(x) + c \qquad \int \cos(x) \, \mathrm{d}x = \sin(x) + c$$

Transforming to $f(ax+b)$:
$\frac{\mathrm{d}}{\mathrm{d}x}(\sin(ax+b)) = a\cos(ax+b)$ and $\frac{\mathrm{d}}{\mathrm{d}x}(\cos(ax+b)) = -a\sin(ax+b)$.

Reversing these processes:

$$\int \sin(ax+b) \, \mathrm{d}x = -\frac{1}{a}\cos(ax+b) + c \qquad \int \cos(ax+b) \, \mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$$