The Definite Integral

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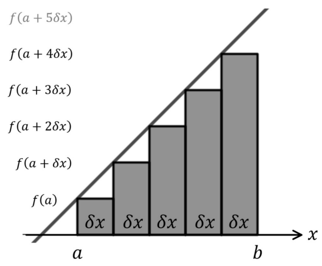
The definite integral can be defined as the exact area between a graph and an axis.

Improving the Accuracy of the Area Calculation



Where $[a, b]$ is partitioned into n equal parts.

The width of all the rectangles is $\delta x = \dfrac{b-a}{n}$.

Alternatively, there are $n = \dfrac{b-a}{\delta x}$ rectangles with width δx over an interval of length a.

Using the left-endpoint approximation the area is approximately $A \approx \sum_{i=0}^{n-1} f(a + \delta x i) \times \delta x$

Using the right-endpoint approximation the area is approximately $A \approx \sum_{i=1}^{n} f(a + \delta x i) \times \delta x$

As more rectangles are used and the width of the rectangles decreases, and the rectangles are more accurate measures of the area under the graph. That is, as $n \to \infty$ and $\delta x \to 0$.

Since we cannot have a width of 0, a limit can be used to find the value of the area precisely:

Left: $A = \lim_{\delta x \to 0} \left(\sum_{i=0}^{n-1} f(a + \delta x i) \times \delta x \right)$

Right: $A = \lim_{\delta x \to 0} \left(\sum_{i=1}^{n} f(a + \delta x i) \times \delta x \right)$

Since $\delta x \to 0$,
For $i = 1$ (or $i = 0$ for right), the input to f approaches a (as $a + 0 \times 1 = a$),
For $i = n - 1 \approx n$ as $n \to \infty$ (or $i = n$ for right), the input to f approaches b (as $a + \delta x \times n = b$).

$$A = \lim_{\delta x \to 0} \left(\sum_{x=a}^{b} f(x) \times \delta x \right)$$

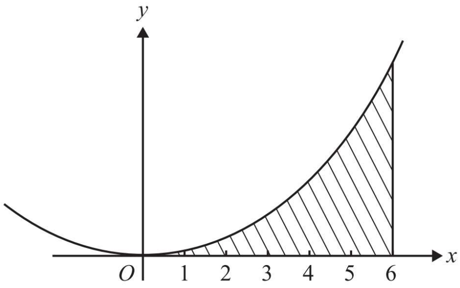
We use Leibniz's notation of an elongated S to denote this limiting sum from 0 to a and write δx as dx

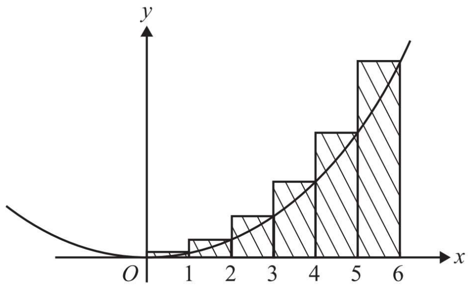
$$A = \int_a^b f(x)\, \mathrm{d}x$$

You can think of this as "the sum ($\int$) of the product of $f(x)$ and little bits of x (dx) for the whole (integral) interval of a to b.

Example Modified VCAA 2011 Exam 2 Question 19

A part of the graph of $f: R \rightarrow R, f(x) = x^2$ is shown below. Zoe finds the approximate area of the shaded region by drawing rectangles as shown in the second diagram.

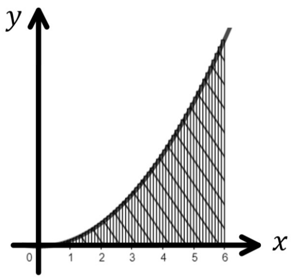


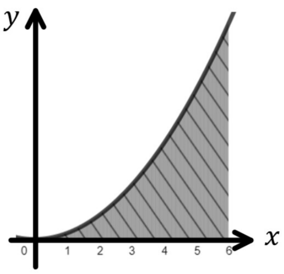


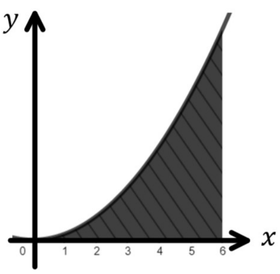
Zoe's Approximation:

$$\sum_{i=1}^{6}(0+1i)^2 \times 1 = \sum_{i=1}^{6} i^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 = 91 \text{ square units}$$

To approximate the exact area better, Zoe decides to use more rectangles with smaller widths:







$$n = \frac{6-0}{0.1} = 60 \qquad n = \frac{6-0}{0.01} = 600 \qquad n = \frac{6-0}{0.001} = 6000$$

$$\sum_{i=1}^{60}(0.1i)^2 \times 0.1 = 0.1(0.1^2 + 0.2^2 + \cdots + 5.9^2 + 6^2) = 73.81$$

$$\sum_{i=1}^{600}(0.01i)^2 \times 0.01 = 0.01(0.01^2 + 0.02^2 + \cdots + 5.99^2 + 6^2) = 72.1801$$

$$\sum_{i=1}^{6000}(0.001i)^2 \times 0.001 = 0.001(0.001^2 + 0.002^2 + \cdots + 5.999^2 + 6^2) = 72.018001$$

As the width approaches zero, the area approximation approaches 72 square units.

$$\lim_{\delta x \to 0}\left(\sum_{i=0}^{\frac{6}{\delta x}}(\delta x i)^2 \times \delta x\right) = 72$$