Anti-Differentiation by Recognition

Anti-Differentiation and Integration by Recognition

Given the derivative of a function ($F'(x) = f(x)$) (often involving the chain, product, or quotient rule), we can then anti-differentiate back to the original function ($\int f(x)dx = F(x) + c$). We can use this to also find the anti-derivative of a multiple of this derivative or a simple sum function involving this derivative. This is most useful for anti-differentiating product or quotient functions.

Two methods of doing this include:

- building up from the anti-derivative that is being asked to the given derivative, and
- starting from the given derivative and working towards the anti-derivative that is being asked.

Example VCAA 2006 Sample Exam 1 Question 8c

The derivative of $\log_e(x^2+1)$ is $\dfrac{2x}{x^2+1}$. Hence find an anti-derivative of $\dfrac{x}{x^2+1}$.

Method 1: Build up to the given derivative

$$\int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{x^2+1}\,\mathrm{d}x$$

$$= \frac{1}{2}\log_e(x^2+1)$$

Method 2: Start from the given derivative

$$\int \frac{2x}{x^2+1}\,\mathrm{d}x = \log_e(x^2+1)$$

$$\Rightarrow \int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\log_e(x^2+1)$$

Example Modified VCAA 2010 Exam 1 Question 9b

Let $f\colon R^+ \to R, f(x) = x\log_e(x)$. The derivative of $x^2\log_e(x)$ is $2x\log_e(x) + x$.
Using the given derivative, an antiderivative of $f(x)$ is

Method 1: Build up to the given derivative

$$\int x\log_e(x)\,\mathrm{d}x = \frac{1}{2}\int 2x\log_e(x)\,\mathrm{d}x$$

$$= \frac{1}{2}\int (2x\log_e(x) + x - x)\mathrm{d}x$$

$$= \frac{1}{2}\int (2x\log_e(x) + x)dx + \frac{1}{2}\int -x\,\mathrm{d}x$$

$$= \frac{1}{2}x^2\log_e(x) - \frac{1}{4}x^2$$

Method 2: Start from the given derivative

$$\int (2x\log_e(x) + x)\mathrm{d}x = x^2\log_e(x)$$

$$\Rightarrow \int 2x\log_e(x)\,\mathrm{d}x = x^2\log_e(x) - \int x\mathrm{d}x$$

$$\Rightarrow 2\int x\log_e(x)\,\mathrm{d}x = x^2\log_e(x) - \frac{1}{2}x^2$$

$$\Rightarrow \int x\log_e(x)\,\mathrm{d}x = \frac{1}{2}x^2\log_e(x) - \frac{1}{4}x^2$$

Example

Given that $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x^2+1}{e^x}\right) = -\dfrac{(x-1)^2}{e^x}$, evaluate $\displaystyle\int_0^4 \frac{(x-1)^2}{2e^x}\mathrm{d}x$

Method 1: Build up to the given derivative

$$\int_0^4 \frac{(x-1)^2}{2e^x}\mathrm{d}x = -\frac{1}{2}\int_0^4 -\frac{(x-1)^2}{e^x}\mathrm{d}x$$

$$= -\frac{1}{2}\left[\frac{x^2+1}{e^x}\right]_0^4$$

Method 2: Start from the given derivative

$$\int_0^4 -\frac{(x-1)^2}{e^x}\mathrm{d}x = \left[\frac{x^2+1}{e^x}\right]_0^4$$

$$\Rightarrow \int_0^4 \frac{(x-1)^2}{2e^x}\mathrm{d}x = -\frac{1}{2}\left[\frac{x^2+1}{e^x}\right]_0^4$$

$$\int_0^4 \frac{(x-1)^2}{2e^x}\mathrm{d}x = -\frac{1}{2}\left[\frac{x^2+1}{e^x}\right]_0^4 = -\frac{1}{2}\left(\frac{4^2+1}{e^4} - \frac{0^2+1}{e^0}\right) = -\frac{1}{2}\left(\frac{17}{e^4} - 1\right) = \frac{1}{2} - \frac{17}{2e^4}$$