Proof of the Fundamental Theorem of Calculus

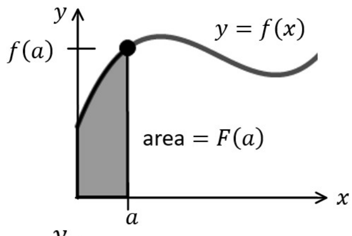
Fundamental Theorem of Calculus

Finding the area bound by a graph is the converse of finding the gradient of a graph.
That is, integration and differentiation are opposites.

Rate of Change of the Area

Let $F(a) = \int_0^a f(x)\mathrm{d}x$.

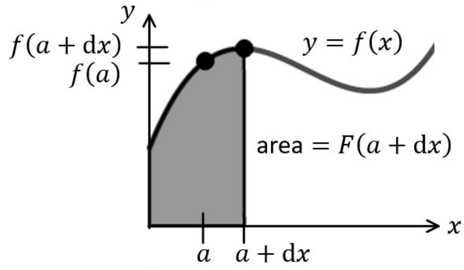
That is, the area bound by the curve $y = f(x)$, the x- and y-axes, and the line $x = a$.



After a small change in the x-value, $\mathrm{d}x$, the area is

$$\int_0^{a+dx} f(x)\mathrm{d}x = F(a + \mathrm{d}x).$$

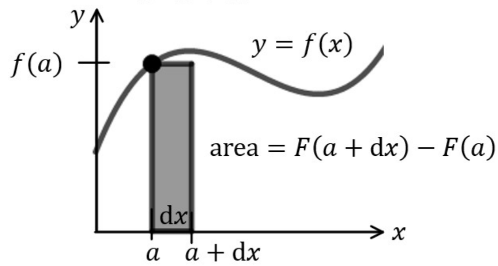
That is, an increase in area of $F(a + \mathrm{d}x) - F(a)$.
However, this increase can also be written as the area of the rectangle with width $\mathrm{d}x$ and height $f(a)$.
That is, $f(a)\mathrm{d}x$.



The rate of change the area is growing at with respect to the width is

$$\frac{\Delta\text{area}}{\Delta\text{width}} = \frac{F(a + dx) - F(a)}{\mathrm{d}x} = \frac{f(a)\mathrm{d}x}{\mathrm{d}x} = f(a),$$

which is the value on the curve $y = f(x)$ at $x = a$.



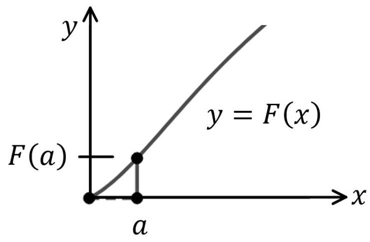
Since we integrated a function, then we got back to the original function by differentiating it, differentiation is the converse of integration.

That is, $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^x y(t)\mathrm{d}t\right) = y(x)$.

Area under the Rate of Change Curve

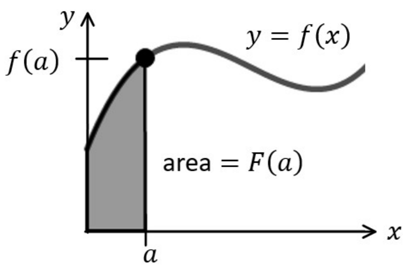
Let $f(a) = \frac{\mathrm{d}}{\mathrm{d}x}(F(x))\bigg|_{x=a}$.

That is, where $F(0) = 0$, the rate of change of F with respect to x at $x = a$.



The amount of change in F from $x = 0$ to $x = a$ is given by the limiting sum of the products of the rate of change of F and small changes in the x-value, Δx, $\int_0^a f(x)\mathrm{d}x$.

However, this amount can also written as the difference in the y-values of $y = F(x)$, $F(a) - F(0) = F(a)$.



The area bound by the curve $y = f(x)$, the x- and y-axes, and the line $x = a$ is $\int_0^a f(x)\mathrm{d}x = F(a)$
which is the amount of change of F from $x = 0$ to $x = a$.

Since we differentiated a function, then we got back to the original function by integrating it, integration is the converse of differentiation.

That is, $\int_0^x \frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x = y(x)$