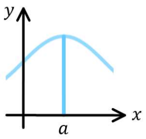
Properties of Anti-Derivatives and Integration

Properties of Definite Integrals

Area of a Line

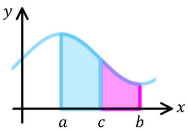
The area under a single point is 0.



$$\int_a^a f(x)\mathrm{d}x = 0$$

Dissection

The area between a and b is the area between a and c and the area between c and b, $a < c < b$.



$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

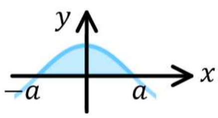
Example VCAA 2018 NHT Exam 2 Question 15

If $\int_{-3}^{2} f(x)\mathrm{d}x = -8$ and $\int_{2}^{3} f(x)\mathrm{d}x = 10$, the value of $\int_{-3}^{3} f(x)\mathrm{d}x$ is

$$\int_{-3}^{3} f(x)\mathrm{d}x = \int_{-3}^{2} f(x)\mathrm{d}x + \int_{2}^{3} f(x)\mathrm{d}x = -8 + 10 = 2$$

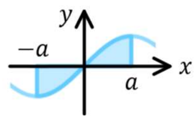
Symmetry

For even functions $f(x) = f(-x)$ (symmetrical around the y-axis)



$$\int_{-a}^{a} f(x)\mathrm{d}x = 2\int_{0}^{a} f(x)\mathrm{d}x$$

For odd functions $f(x) = -f(-x)$ (symmetrical around the origin)



$$\int_{-a}^{a} f(x)\mathrm{d}x = 0$$

Direction

The area from right to left is the negative of the area from left to right.

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a)$$

$$= -\big(F(a) - F(b)\big) = -\int_b^a f(x)\mathrm{d}x$$

Addition and Subtraction

The area below $f + g$ is the sum of the area below f and the area below g.

$$\int_a^b \big(f(x) \pm g(x)\big)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x \pm \int_a^b g(x)\mathrm{d}x$$

Example VCAA 2017 NHT Exam 1 Question 2b

Evaluate $\int_1^2 \left(3x^2 + \frac{4}{x^2}\right)\mathrm{d}x$.

$$\int_1^2 \left(3x^2 + \frac{4}{x^2}\right)\mathrm{d}x = \int_1^2 3x^2\mathrm{d}x + \int_1^2 \frac{4}{x^2}\mathrm{d}x = [x^3]_1^2 + \left[-\frac{4}{x}\right]_1^2 = (2^3 - 1^3) + \left(-\frac{4}{2} - \left(-\frac{4}{1}\right)\right)$$

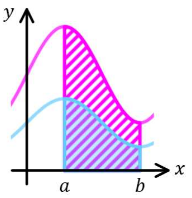
$$= (8 - 1) + (-2 + 4) = 7 + 2 = 9$$

$$\int_1^2 \left(3x^2 + \frac{4}{x^2}\right)\mathrm{d}x = \left[x^3 - \frac{4}{x}\right]_1^2 = \left(2^3 - \frac{4}{2}\right) - \left(1^3 - \frac{4}{1}\right) = (8 - 2) - (1 - 4) = 6 - (-3) = 9$$

Integrals of Transformed Functions

Vertical Dilation

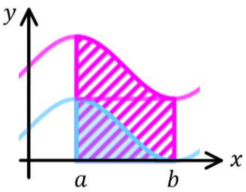
The area under $k \times f$ is k times the area under f.



$$\int_a^b kf(x)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$$

Vertical Translation

The area of a rectangle is added to or subtracted from the integral



$$\int_a^b (f(x)+c)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b c\mathrm{d}x$$
$$= \int_a^b f(x)\mathrm{d}x + c(b-a)$$

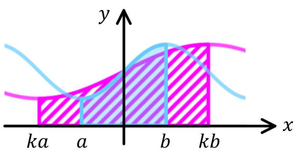
Example VCAA 2003 Exam 1 Question 14

If $\int_1^4 f(x)\mathrm{d}x = 2$, then $\int_1^4 (2f(x)+3)\mathrm{d}x$ is equal to

$$\int_1^4 (2f(x)+3)\mathrm{d}x = 2\int_1^4 f(x)\mathrm{d}x + \int_1^4 3\mathrm{d}x = 2(2) + 3[x]_1^4 = 4 + 3(4-1) = 4 + 9 = 13$$

Horizontal Dilation

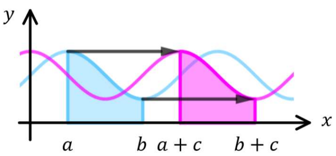
The area between ka and kb under $f\left(\frac{x}{k}\right)$ is equal to k times the area between a and b under $f(x)$



$$\int_{ka}^{kb} f\left(\frac{x}{k}\right)\mathrm{d}x = k\int_a^b f(x)\mathrm{d}x$$

Horizontal Translation

The area between $a+c$ and $b+c$ under $f(x-c)$ is equal to the area between a and b under $f(x)$



$$\int_{a+c}^{b+c} f(x-c)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

Example VCAA 2019 NHT Exam 2 Question 15

The area bounded by the graph of $y = f(x)$, the line $x = 2$, the line $x = 8$ and the x-axis is $3\log_e(13)$.

The value of $\int_4^{10} 3f(x-2)\mathrm{d}x$ is $\int_2^8 3f(x)\mathrm{d}x = 3\int_2^8 f(x)\mathrm{d}x = 3 \times 3\log_e(13) = 9\log_e(13)$

Example VCAA 2010 Exam 2 Question 20

Let f be a differentiable function defined for all real x, where $f(x) \geq 0$ for all $x \in [0, a]$.

If $\int_0^a f(x)\mathrm{d}x = a$, then $2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)\mathrm{d}x$ is equal to

$$2\int_0^{5a}\left(f\left(\frac{x}{5}\right)+3\right)\mathrm{d}x = 2\left(5\int_0^a f(x)\mathrm{d}x + \int_0^{5a} 3\mathrm{d}x\right) = 2\big((5a-0) + 3(5a-0)\big) = 40a$$